RANDOM FUNCTIONS FROM A POISSON PROCESS

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1. Definition of random functions from a Poisson process

Let $R(t, \tau)$ be an ordinary (nonrandom) function of two variables t and τ, which, for example, might represent time. Let us consider instants distributed at random on the time axis according to the classical Poisson process of density m $(m > 0)$. Let $N_1(t)$ be the number of these instants belonging to the interval $(0, t)$ if $t > 0$, to the interval $(t, 0)$ if $t < 0$. We shall write $N(t) = N_1(t)$ if $t > 0$; $N(t) = -N_1(t)$ if $t < 0$, so that $N(t') - N(t)$ represents the number of instants belonging to (t, t') whatever may be t and t' $(t < t')$.

The following well known results should be recalled:

(a) $N(t)$ is a random function with independent increments, taking integer values, and is almost certainly nondecreasing.

(b) The probability that the number of instants belonging to any finite interval of time be infinite is zero.

(c) The distribution in the time of these instants is stationary.

(d) m is the expectation of the number of instants belonging to any interval of amplitude 1.

(e) The probability that the number of instants belonging to any interval (t, t'), $t < t'$, be equal to n is equal to

$$e^{-m\delta} \frac{(m\delta)^n}{n!},$$

where $\delta = t' - t$.

Calling $\tau_1, \tau_2, \dots, \tau_j, \dots$ the instants of an interval (t_0, t), $(t > \tau_0)$, different applications, for example, the phenomenon of noise in electronics, lead to a consideration of random functions of t defined in the following way:

$$X(t) = \sum_j R(t, \tau_j), \tag{1}$$

which may be written as

$$X(t) = \int_{t_0}^{t} R(t, \tau)\, dN(\tau). \tag{2}$$

More generally, we shall consider

$$X(t) = \int_{-\infty}^{+\infty} R(t, \tau)\, dN(\tau), \tag{3}$$

and we shall denote random functions of this kind, that is random functions from a Poisson process, by P.r.f. The study of these functions has been developed by several authors, in particular, Blanc Lapierre [1], Kac and Hurwitz [3] and Maruyama [2]. But it is possible to extend, to simplify and to complete their results; this is the purpose of our paper.

The first point is to give a precise definition of integrals (2) or (3). They exist only if $R(t, \tau)$ satisfies some conditions which depend on the sense (integral with probability one, integrals in quadratic mean, . . .) attached to the integrals (2), (3).

Let $R(\tau)$ be a function of τ. We first consider the integrals

$$X_{\alpha\beta}=\int_{\alpha}^{\beta}R(\tau)\,dN(\tau)\,, \tag{4}$$

where α and β ($\alpha < \beta$) are any finite, given constants. Denote by assumption H_1 the assumption that $R(\tau)$ is a definite function which is $\mathcal{L}$-measurable on $(-\infty, +\infty)$ and finite, except perhaps on a set of measure zero.

1.1 *Integrals with probability one.* We can give to (4), considered as an integral with probability one, two different meanings.

(a) Except in cases whose probability measure is zero any trial ϵ assigns to N a finite number n of jumps $(\tau_1, \ldots, \tau_n)$ between α and β. It is known that if $N(\beta) - N(\alpha) = n$, the τ_j are independent and follow the uniform law on (α, β). We can consider the sum $\sum_j R(\tau_j)$, and it follows from assumption H_1 that with probability one this sum exists and is finite. Then the formula

$$X_{\alpha\beta}=\sum_j R(\tau_j)\quad\text{for }\epsilon \tag{5}$$

defines a random variable that we may consider to be, by definition, $X_{\alpha\beta}$.

(b) We can define $X_{\alpha\beta}$ as the stochastic limit, if it exists, of a Stieltjes-Riemann sum of the following type: $\sum_{s=1}^{k} R(\xi_s)\,\Delta_s N$, obtained by dividing (α, β) in partial intervals by points $t_0 = \alpha, < \ldots, < t_s < t_{s+1}, \ldots, < t_k = \beta$, putting $\Delta_s N = N(t_s) - N(t_{s-1})$, ξ_s being chosen arbitrarily (not at random) on (t_{s-1}, t_s).[1] The limit is obtained when $\sup (t_s - t_{s-1}) \to 0$. It is necessary that this limit exist and be independent of the division points (t_s), and of the choice of the ξ_s's. Definition (a) is more suitable to the physical origin of the considered integral. Definition (b) is more suitable to the usual definition of stochastic Riemann, or Stieltjes-Riemann integrals, but it requires an additional condition on $R(\xi_s)$, in order that the integral exist with probability one; indeed for a trial ϵ such that $N(\beta) - N(\alpha) = n$, if $\sup (t_s - t_{s-1}) \to 0$, there will be a time when (5) will be confused with

$$\sum_j R(\xi_j)\,, \tag{6}$$

ξ_j denoting a number which tends to τ_j, but in an arbitrary manner. In order that (6) has, with probability one, a limit independent of the choice of the ξ_s's, namely $\sum_j R(\tau_j)$, it is necessary and sufficient that the set of points of (α, β) where $R(\tau)$ is discontinuous be of measure zero. Let H_2 denote the assumption that the points

[1] We may consider choosing the ξ's at random.

of $(-\infty, +\infty)$ at which $R(\tau)$ is not continuous form a set of measure zero (H_2 implies H_1).

It is seen that whatever may be α and β, the integral with probability one (4) exists in the sense (a) if H_1 holds, and in the sense (b) if H_2 holds. Now let H_3 denote the assumption that $\int_{-\infty}^{+\infty} |R(\tau)| d\tau < +\infty$ (this integral is a Lebesgue integral). Let A be a random variable following the uniform law on $(\alpha\beta)$, and consider the random variable $R(A)$. If H_3 holds, the expectation of $R(A)$ exists and is $\mu/\beta - \alpha$, where $\mu = \int_\alpha^\beta R(\tau) d\tau$. It is seen that when $N(\beta) - N(\alpha) = n$, the conditional expectation of $X_{\alpha\beta}$ exists and is $n\mu/\beta - \alpha$; then, the *a priori* expectation of $X_{\alpha\beta}$ exists, and is $m\mu$. This is true with definition (a) as well as with definition (b).

Now let X be the integral with probability one,

$$X = \int_{-\infty}^{+\infty} R(\tau)\, dN(\tau). \tag{7}$$

We will define it as the limit with probability one, if it exists, of $X_{\alpha\beta}$ as α tends to $-\infty$ and β tends to $+\infty$ independently. It follows from a classical theorem that X exists in this sense if H_1 and H_3 (or H_2 and H_3) hold. It is seen that the expectation of X exists and is the limit of the expectation of $X_{\alpha\beta}$, so it is equal to

$$m \int_{-\infty}^{+\infty} R(\tau)\, d\tau.$$

1.2. *Probability law of $X_{\alpha\beta}$ and of X.* In the case in which assumption H_1 (or H_2) holds, let α be a given number and consider $X_{\alpha\beta}$ as a function of the variable β (it is obvious that we can exchange the role of α and β). Obviously $X_{\alpha\beta}$ is a continuous in probability random function of β with independent increments; indeed, if we write $\Delta N = N(\beta + \Delta\beta) - N(\beta)$, $\Delta X = X_{\alpha,\beta+\Delta\beta} - X_{\alpha\beta}$, we get $\Delta X = 0$ as $\Delta N = 0$; but the probability of $\Delta N = 0$ tends to 1 as $\Delta\beta \to 0$. We then have as a result, *that the probability law of $X_{\alpha\beta}$ is an infinitely divisible law and then the probability law of its limit X is also an infinitely divisible law.*

Let $\theta(v)$ be the characteristic function of $R(A)$. The conditional characteristic function of $X_{\alpha\beta}$ as $N(\beta) - N(\alpha) = n$ is obviously $[\theta(v)]^n$. Then the *a priori* characteristic function of $X_{\alpha\beta}$ is

$$\sum_n e^{-m(\beta-\alpha)} \frac{m^n(\beta-\alpha)}{n!} [\theta(v)]^n = \exp\{m(\beta-\alpha)[\theta(v)-1]\}.$$

The logarithm $\psi_{\alpha\beta}(v)$ of the characteristic function of $X_{\alpha\beta}$, is

$$\psi_{\alpha\beta}(v) = m(\beta-\alpha)[\theta(v)-1] = m\int_\alpha^\beta [e^{ivR(\tau)}-1]\, d\tau. \tag{8}$$

From this there results, and it is easy to verify directly, that if not only H_1 (or H_2) holds, but also H_3 holds, the integral

$$\int_\alpha^\beta [e^{ivR(\tau)}-1]\, d\tau$$

converges as α and β tend independently to $-\infty$ and $+\infty$ respectively. Hence, the logarithm $\psi(v)$ of the characteristic function of X is equal to

$$\psi(v) = m \int_{-\infty}^{+\infty} [e^{ivR(\tau)} - 1] \, d\tau . \tag{9}$$

These results will be useful. We see that they permit the computation of the moments of $X_{\alpha\beta}$ and of X, when they exist. In particular the "cumulants" of X, when they exist, are easily computed from the integrals

$$\int_{-\infty}^{+\infty} R^n(\tau) \, d\tau .$$

1.3. *First generalization.* Let $R_1(\tau), R_2(\tau), \ldots, R_k(\tau)$ be k functions such that H_1 (or H_2) and H_3 holds. The k random variables

$$X_j = \int_{-\infty}^{+\infty} R_j(\tau) \, dN(\tau) \quad \text{(integral with probability one)}$$

are not independent. Let $\psi(v_1, \ldots, v_k)$ be the logarithm of the characteristic function of the k-dimensional distribution of the k-dimensional random variable $Z = (X_1, \ldots, X_k)$. Let us set

$$Y = v_1 X_1 + \ldots + v_k X_k .$$

We have $\psi(v_1, \ldots, v_k) = \log \{E[e^{iY}]\}$.

$$Y = \int_{-\infty}^{+\infty} [v_1 R_1(\tau) + \ldots + v_k R_k(\tau)] \, dN(\tau) = \int_{-\infty}^{+\infty} R(\tau) \, dN(\tau) ,$$

where $R(\tau) = v_1 R_1(\tau) + \ldots + v_k R_k(\tau)$.

For any system $(v_1, \ldots, v_k)$ with v_j's finite, $R(\tau)$ satisfies H_1 (or H_2) and H_3; hence, from (9)

$$\psi(v_1, \ldots, v_k) = m \int_{-\infty}^{+\infty} [e^{i[v_1 R_1(\tau) + \ldots + v_k R_k(\tau)]} - 1] \, d\tau . \tag{9'}$$

This expression can be used to compute the moments of Z.

1.4. *Second generalization.* In the integrals (4) or (7) we ascribe, to a jump of $N(\tau)$ at the abscissa τ, a given extent $R(\tau)$. We may imagine that this extent is chosen at random; in other words we may attach to each value of τ a random variable $Y(\tau)$ and consider the integrals

$$X'_{\alpha\beta} = \int_{\alpha}^{\beta} Y(\tau) \, dN(\tau) , \qquad X' = \int_{-\infty}^{+\infty} Y(\tau) \, dN(\tau) .$$

$X'_{\alpha\beta}$ may be defined (definition a) as equal to $\sum_j Y(\tau_j)$ for a trial ϵ, in which the jumps of $N(\tau)$ between α and β occur at the points $\tau_1, \ldots, \tau_j, \ldots$, and X' may be defined as the stochastic limit of $X'_{\alpha\beta}$ as $\alpha \to -\infty$, $\beta \to +\infty$. If the $Y(\tau)$'s are mutually independent, the law of $X'_{\alpha\beta}$ will be again an infinitely divisible law, and also the law of X'; besides it may be sufficient that $Y(\tau)$ be defined except on a set of values of measure zero. Let us make the following assumptions: except perhaps on a set of values of measure zero, $Y(\tau)$ is defined; $E[|Y(\tau)|]$ and $E[Y(\tau)]$ exist and are $\mathcal{L}$-measurable functions of τ; X' exists as the limit almost certainly of $X'_{\alpha\beta}$ and

$$E(X') = m\int_{-\infty}^{+\infty} E[Y(\tau)]\,d\tau .$$

Let us call $\theta(\tau; v)$ the characteristic function of $Y(\tau)$. Assuming that, for any v, $\theta(\tau; v)$ is an $\mathcal{L}$-measurable function of τ, we obtain for the logarithm $\psi'(v)$ of the characteristic function of X'

$$\psi'(v) = m\int_{-\infty}^{+\infty} [\theta(\tau, v) - 1]\,d\tau . \tag{10}$$

As an example let $F(u)$ be a distribution function such that

$$F(0) = 0; \qquad \int_0^{+\infty} u dF(u) = \lambda .$$

We have $\int_a^{+\infty} udF(u) > a[1 - F(a)]$, so that the existence of the expectation λ implies that

$$a[1 - F(a)] \to 0 \quad \text{as} \quad a \to +\infty .$$

Suppose that $Y(\tau)$ be such that

$$Pr[Y(\tau) = 1] = 1 - F(t - \tau),$$

$$Pr[Y(\tau) = 0] = F(t - \tau)$$

when $\tau < t$, and $Y(\tau) = 0$ almost certainly when $\tau \geqq t$; for example X' might represent the number of conversations which are being conducted at time t in a telephone center, if it had no loss. Let $F(u)$ be the distribution of the length of the conversations. We have

$$\theta(\tau, v) = e^{iv} + F(t - \tau)[1 - e^{iv}] \qquad \text{if} \quad \tau < t$$

$$\theta(\tau, v) = 1 \qquad \text{if} \quad \tau \geqq t .$$

It is easy to verify that the above assumptions hold, and that

$$\psi'(v) = m\lambda[e^{iv} - 1],$$

so that the number of conversations conducted follows a Poisson law.

1.5. *Integrals in quadratic mean.* We may also consider the integrals (4) and (7) as integrals in quadratic mean. We shall call H_4 the assumption that $R(\tau)$ and $R^2(\tau)$ are integrable in the sense of Riemann on $(-\infty, +\infty)$. Suppose that H_4 holds (this implies that H_1, H_2 and H_3 hold), and let us consider the Stieltjes-Riemann integral (5). To prove that this integral converges in quadratic mean, we must prove that

$$E\Big[\Big\{\sum_s [R(\xi_s) - R(\xi_s')]\,\Delta_s N\Big\}^2\Big],$$

where ξ_s, ξ_s' are arbitrary numbers on (t_{s-1}, t_s), tends to zero as the supremum of $(t_s - t_{s-1})$ tends to zero.

If we write $u_s = R(\xi_s) - R(\xi_s')$, the considered quantity may be written as

$$m\sum_s u^2(t_s - t_{s-1}) + m^2\Big[\sum_s u_s(t_s - t_{s-1})\Big]^2 - m^2\Big[\sum_s u_s^2(t_s - t_{s-1})^2\Big]$$

which, under assumption H_4, tends to zero. We see also that

$$E\Big[\Big\{\sum_s R(\xi_s)\Delta_s N\Big\}^2\Big] \to m\int_\alpha^\beta R^2(\tau)\,d\tau + m^2\Big[\int_\alpha^\beta R(\tau)\,d\tau\Big]^2 .$$

Then this last quantity is the second moment of $X_{\alpha\beta}$. It follows that if $\alpha \to -\infty$ and $\beta \to +\infty$ (independently each other), $X_{\alpha\beta}$ tends in quadratic mean to X, the second moment of which is

$$m\int_{-\infty}^{+\infty} R^2(\tau)\,d\tau + m^2\Big[\int_{-\infty}^{+\infty} R(\tau)\,d\tau\Big]^2 . \tag{11}$$

Remark. It may be noted that the continuity of $R(\tau)$ is not necessary to obtain the above and following results; nevertheless, we know that a Riemann integrable function is continuous almost everywhere.

1.6. *Another point of view.* Set

$$N^*(t) = N(t) - EN(t) = N(t) - mt .$$

In many applications we have to consider not $X_{\alpha\beta}$ and X, but the quantities

$$X^*_{\alpha,\beta} = \int_\alpha^\beta R(\tau)\,dN^*(\tau) \qquad \text{and} \qquad X^* = \int_{-\infty}^{+\infty} R(\tau)\,dN^*(\tau) . \tag{4'}$$

The integration in quadratic mean leads us to a study of

$$E\Big[\Big(\sum_s u_s\Delta_s N^*\Big)^2\Big] = m\sum_s u_s^2(t_s - t_{s-1}) .$$

If we call H_5, which is much less restrictive than H_4, the assumption that $R^2(\tau)$ is integrable in the sense of Riemann on $(-\infty, +\infty)$, it is easy to see that a sufficient and also necessary condition for the existence in quadratic mean of X^* is that

$$E[X^*] = 0, \qquad E[X^{*2}] = m\int_{-\infty}^{+\infty} R^2(\tau)\,d\tau . \tag{11'}$$

Suppose only that $\int_{-\infty}^{+\infty} R^2(\tau)d\tau$ exists in the sense of Lebesgue (assumption H_5', less restrictive than H_5). $X_{\alpha\beta}$ and then $X^*_{\alpha\beta}$ exist almost certainly [in the sense of (a), p. 2], and we have

$$E[X^*_{\alpha\beta}] = 0, \qquad E[X^{*2}_{\alpha\beta}] = m\int_\alpha^\beta R^2(\tau)\,d\tau .$$

Then X^* exists as the limit in quadratic mean of $X^*_{\alpha\beta}$, and (11′) has a meaning.

1.7. *Conclusion.* Obviously, it is sufficient to apply the above results to solve the problem of the definition and of the existence of the P.r.f. $X(t)$ defined by (3).

2 Convergence to a Laplace law

Suppose that $m \to +\infty$. What are the conditions under which the law of X converges to a Laplace law? That is to say, what are the conditions under which there exists a number a, and a positive number b (depending on m) such that

$$\lim_{m\to+\infty} [\psi(v/b) - iav/b] = -v^2/2 . \tag{12}$$

$\psi(v)$ is given by (9). Suppose that H_4 holds. Differentiating under the integral sign, we get

$$\psi'(v) = im\int_{-\infty}^{+\infty} R(\tau)\, e^{ivR(\tau)} d\tau, \qquad \psi'(0) = im\int_{-\infty}^{+\infty} R(\tau)\, d\tau,$$

$$\psi''(v) = -m\int_{-\infty}^{+\infty} R^2(\tau)\, e^{ivR(\tau)} d\tau, \quad \psi''(0) = -m\int_{-\infty}^{+\infty} R^2(\tau)\, d\tau.$$

We note that the differentiations are justified, that $\psi''(v)$ is a uniformly continuous function of v.

Setting

$$\mu_1 = \int_{-\infty}^{+\infty} R(\tau)\, d\tau, \qquad \mu_2 = \int_{-\infty}^{+\infty} R^2(\tau)\, d\tau,$$

we get

$$\psi(v) = im\mu_1 v - \frac{m}{2}\mu_2 v^2 + v^2\omega(v),$$

where $\omega(v) \to 0$ as $v \to 0$. Excluding the noninteresting case $\mu_2 = 0$, we see that (12) holds when

$$a = m\mu_1; \qquad b = \sqrt{m\mu_2}.$$

Remark. It is easy to extend this result to the case considered in the second generalization above, but we shall consider the case of the first generalization. Assuming that H_4 holds for $R_1(\tau), \ldots, R_k(\tau)$, the same method shows that if

$$\mu_j = \int_{-\infty}^{+\infty} R_j(\tau)\, d\tau, \qquad \mu_{j,s} = \int_{-\infty}^{+\infty} R_j(\tau) R_s(\tau)\, d\tau,$$

the k-dimensional random variable made up of the random variables $\dfrac{X_j - m\mu_j}{\sqrt{m}}$ tends as $m \to +\infty$, to a Laplace variable; indeed, the logarithm of the characteristic function of its limit distribution is

$$-\frac{1}{2}\Bigl(\sum_{j,s} \mu_{j,s} v_j v_s\Bigr).$$

$\sum_{j,s} \mu_{j,s} v_j v_s$ is a definite positive quadratic form of the v_j's if the $R_j(\tau)$'s are linearly independent, because

$$\sum_{j,s} \mu_{j,s} v_j v_s = \int_{-\infty}^{+\infty} \Bigl[\sum_j R_j(\tau)\, v_j\Bigr]^2 d\tau.$$

2.1. *Another point of view.* We may have the convergence to a Laplace law under different conditions. $X_{\alpha\beta}$ being a random function with independent increments, may be considered as the sum of independent random variables. Then, as $\alpha \to -\infty$, $\beta \to +\infty$, it will be asymptotically a Laplace variable, at least under some conditions which may be easily found from the general theory of the convergence of a sum of independent random variables to a Laplace variable. For example, let $R_1(\tau) \ldots R_k(\tau)$ be k functions such that $R_j(\tau)$ and $R_j^2(\tau)$ have finite Riemann integrals on any *finite* interval (α, β), $(j = 1, 2, \ldots, k)$, but

$$\int_{-\infty}^{+\infty} R_j^2(\tau)\, d\tau = +\infty.$$

Let

$$\mu'_j = \int_a^\beta R_j(\tau)\,d\tau\,, \qquad \mu'_{j,s} = \int_a^\beta R_j(\tau)R_s(\tau)\,d\tau\,,$$

and suppose that the limits

$$\lim_{a\to-\infty,\beta\to+\infty} \frac{\mu'_{j,s}}{\sqrt{\mu'_{j,j}\,\mu'_{s,s}}} = \mu_{j,s} \qquad (j,\ s = 1\,,\ldots\,,k)$$

exist (clearly they exist and take the value 1 if $j = s$).

Let S_j be the subset of (a, β) on which

$$|R_j(\tau)| > \epsilon\sqrt{\mu'_{j,j}}\,,$$

and suppose that whatever may be j and ϵ, we have

$$\lim_{a\to-\infty,\beta\to+\infty} \frac{\int_{S_j} R_j^2(\tau)\,d\tau}{\mu'_{jj}} = 0\,.$$

From (9′) it is easy to show that the k-dimensional random variable composed of the set of random variables

$$\frac{\int_a^\beta R_j(\tau)\,dN(\tau) - m\mu'_j}{\sqrt{\mu'_{jj}}}$$

has a distribution which tends, as $a\to-\infty$, $\beta\to+\infty$, to a k-dimensional Laplace distribution whose logarithm has the characteristic function

$$-\frac{1}{2}\Big(\sum_{j,s} \mu_{js}\, v_j\, v_s\Big)\,.$$

A result of the same kind may be given when one of the limits a, β is a constant and the other tends to infinity.

Remark. Above and below we suppose that the functions $R(t, \tau)$, $R(\tau)$, $R_j(\tau)$ are real, but it is not difficult to extend the results to the case where $R(\tau)$, $R_j(\tau)$, $R(t, \tau)$ are complex functions.

2.2. *Application.* It is sufficient to apply the above results to get the main result concerning the P. random functions found by Kac and Hurwitz, Blanc Lapierre, and Maruyama. For example, let us consider the random function $X(t)$ defined by (4′), and suppose that, for any t, $R(t, \tau)$ considered as a function of τ verifies the assumption H'_5.

$$X^*(t) = \int_{-\infty}^{+\infty} R(t,\ \tau)\,dN^*(\tau)$$

defines a random function of t. The expectation of $X^*(t)$ exists and is zero.

Its variance is

$$\sigma^2[X^*(t)] = m\int_{-\infty}^{+\infty} R^2(t,\ \tau)\,d\tau = m\sigma^2(t)\,,$$

where

$$\sigma^2(t) = \int_{-\infty}^{+\infty} R^2(t,\ \tau)\,d\tau\,.$$

It follows that $X^*(t)$ is a random function of second order; its covariance $\Gamma(t, t')$ is found from (9′) by setting $k = 2$, $R_1(\tau) = R_1(t, \tau)$, $R_2(\tau) = R(t', \tau)$ in (9′). Performing the substitution, we have

$$\Gamma(t, t') = m\int_{-\infty}^{+\infty} R(t, \tau)R(t', t)\, d\tau = m\mu_2(t, t'),$$

where

$$\mu_2(t, t') = \int_{-\infty}^{+\infty} R(t, \tau)R(t', \tau)\, d\tau. \tag{13}$$

In the particular case (very important in the theory of noise) of (2), where $R(t, \tau) = R(t - \tau)$ is a function only of $t - \tau$, with $R(t, \tau) = 0$ for $\tau < t_0$ or $\tau > t$, and $R(t, \tau) = R(t - \tau)$ for $t_0 < \tau < t$, assuming $t' > t > t_0$, we get from (13)

$$\mu_2(t, t') = \int_{t_0}^{t} R(t - \tau)R(t' - \tau)\, d\tau = \int_{0}^{t-t_0} R(u)R(t' - t + u)\, du.$$

For $t_0 = -\infty$,

$$\mu_2(t, t') = \int_{-\infty}^{+\infty} R(u)R(t' - t + u)\, du,$$

which depends only on $t' - t$, and in fact only on $|t' - t|$. Besides, under the same conditions $\sigma^2(t)$ is independent of t. This implies that $X(t) = \int_{-\infty}^{+\infty} R(t - \tau)dN(\tau)$ is a stationary random function of the second order. This was obvious, and further it is obvious that $X^*(t)$ is, more precisely, a strictly stationary function.

In practice m is generally very large, and we saw that, as $m \to +\infty$, the temporal law of the process $X^*(t)$, suitably normed, tends to the law of a Laplace process, the correlation function of which is

$$r(h) = \int_{-\infty}^{+\infty} R(u)R(u + h)\, du. \tag{14}$$

So for any function $R(u)$ satisfying H_5', the integral (14) is a positive definite function of h. It is known also that $r(h)$ is a continuous function of h; in other words, $r(h)$ is a characteristic function. Conversely, given any characteristic function $r(h)$, we would like to find a function $R(u)$ so that (14) holds. From a classical theorem [5, p. 91] it is necessary that the spectral function $F(\omega)$ of $r(h)$ has a density function $f(\omega)$ (be absolutely continuous). In this case $R(u)$ exists, and is defined as the Fourier transform of

$$g(\omega) = \sqrt{f(\omega)}.$$

$R(u)$ obtained in that way belongs to L^2, that is to say, satisfies H_5'. Then we can say that every Laplace stationary process with absolutely continuous spectral function may be obtained like the "limit" of a P.r.f. as $m \to +\infty$.

On the other hand, from the above results, it is easy to prove the classical theorem of Khintchine which states a necessary and sufficient condition that a function $r(h)$ be a characteristic function is that it be the limit, uniformly over every finite interval, of a function of the form

$$\int_{-\infty}^{+\infty} R(u)R(u + h)\, dh,$$

where

$$\int_{-\infty}^{+\infty} |R(u)|^2 du = 1$$

(integrals in the sense of Lebesgue).

2.3. *Another application.* The convergence to a Laplace process may be easily obtained for random functions of the following form:

$$X'(t) = \int_{-\infty}^{+\infty} Y(t, \tau)\, dN(\tau),$$

where $Y(t, \tau)$ is a random variable (compare 1.4). Let us take, for example, the case in which $X'(t)$ is the number of conversations in a telephone center without loss, operating since $t = -\infty$ (compare 1.4).

Putting $X'(t) = m\lambda + \sqrt{m\lambda}Z(t)$, we find that the law of $Z(t)$ tends, as $m \to +\infty$, to a stationary Laplace process, the correlation function of which is

$$(15) \quad r(h) = \frac{1}{\lambda}\int_0^{+\infty} [1 - F(u+h)]\, du = \frac{1}{\lambda}\int_h^{+\infty} [1 - F(u)]\, du, \qquad h > 0$$

[if $F(u) = 1 - e^{-u}$, then $r(h) = e^{-|h|}$].

For $h > 0$ this function $r(h)$ is monotonic, nonincreasing, $\lim_{h \to +\infty} r(h) = 0$ [$r(h)$ does not vanish], $\frac{d}{dh} r(h)$ exists.

$$r'(h) = \frac{d}{dh} r(h) = -\frac{1}{\lambda}[1 - F(h)], \qquad h > 0$$

$$r'(+0) = -\frac{1}{\lambda}[1 - F(+0)],$$

and

$$r'(-h) = -r'(h), \qquad h > 0.$$

For $h > 0$, $r'(h)$ is monotonic, negative and $r'(+\infty) = 0$. From these results, it is easy to give a new proof of a theorem of Pólya [6, p. 115].

3. Transforms of a random function

Suppose that the random function $X(t)$ represents a voltage applied to the input of a quadratic detector. It is known that the response voltage is then $[X(t)]^2$; therefore, we are led to the following problem:

Let $V(x)$ be an ordinary function of x defined in $-\infty < x < +\infty$, and let $X(t)$ be a random function, not necessarily a P.r.f., $Y(t) = V[X(t)]$ is a random function. The problem is to study $Y(t)$, knowing $X(t)$. The main problem is generally to find the temporal law of $Y(t)$ from the one of $X(t)$. This last problem has been considered by Blanc Lapierre who proposes to evaluate the moments of the random variable $Y(t)$ and, more generally, those of the n-dimensional random variable $[Y(t_1) \ldots Y(t_n)]$. About this computation we make the following remark: Let us suppose that there exists a function $U(v)$, the Fourier transform of $V(x)$, such that

$$V(x) = \int_{-\infty}^{+\infty} e^{ivx} U(v)\, dv.$$

Let $\phi(t_1, v_1; t_2, v_2; \ldots ; t_n v_n)$ be the characteristic function of the n-dimensional random variable $[X(t_1), \ldots, X(t_n)]$. We may write formally

$$(16)\quad E[Y(t_1)\,Y(t_2)\ldots Y(t_n)]$$

$$=E\{V[X(t_1)]\times V[X(t_2)]\ldots\times V[X(t_n)]\}$$

$$=E\left\{\left[\int_{-\infty}^{+\infty} e^{iv_1X(t_1)}U(v_1)\,dv_1\right]\ldots\times\left[\int_{-\infty}^{+\infty} e^{iv_nX(t_n)}U(v_n)\,dv_n\right]\right\}$$

$$=E\left\{\underbrace{\int\ldots\int}_{n^{th}}\exp i[v_iX(t_1)+\ldots+v_nX(t_n)]\,U(v_1)\times\ldots U(v_n)\,dv_1\ldots dv_n\right\}$$

$$=\underbrace{\int\ldots\int}_{n}\phi(t_1, v_1;\ldots;t_n, v_n)\,U(v_1)\ldots U(v_n)\,dv_1\ldots dv_n.$$

It is easy to extend this result to the case in which the t_j's are not all different. Then we have the formula to compute any moment attached to $Y(t)$, from $U(v)$ and the characteristic $\phi(t_1,v_1, \ldots, t_n, v_n)$ of $X(t)$. It is easy to give sufficient conditions to validate the above formal computation. For example, it is sufficient that $\int_{-\infty}^{+\infty}|U(v)|dv < +\infty$. In particular it is easily seen that

$$E[Y^n(t)] = \underbrace{\int\ldots\int}_{n}\phi(t; v_1, \ldots, v_n)\,U(v_1)\ldots U(v_n)\,dv_1\ldots dv_n.$$

We may compare (16) and the direct formula (17),

$$(17)\quad E[Y(t_1)\ldots Y(t_n)] = \underbrace{\int\ldots\int}_{n} V(x_1)\ldots V(x_n)\,dF(t_1, x_1;\ldots;t_n, x_n),$$

where $F(t_1, x_1; \ldots ; t_n, x_n)$ stands for the distribution function of the n-dimensional random variable $\{X(t_1), \ldots, X(t_n)\}$. (16) is better than (17) whenever $\phi(t_1, v_1; \ldots ; t_n, v_n)$ is better known or simpler than $F(t_1, x_1; \ldots ; t_n, x_n)$, which, in particular, is the case if $X(t)$ is a P.r.f., since $\phi(t_1, v_1; \ldots ; t_n, v_n)$ is then given by (9).

It follows that one may compute the moments of some random variables connected with $Y(t)$, particularly those of the linear functionals of $Y(t)$; for example, one has often to consider random variables such as

$$Z_T = \frac{1}{T}\int_0^T Y(t)\,dt \qquad \text{(stochastic integral)},$$

the temporal mean of $Y(t)$ in $(0, T)$. If this stochastic integral has a meaning, it is easy to construct a method to prove the result obtained by Blanc Lapierre that if $R(t, \tau)$ depends only on $t - \tau$, $X(t) = \int_{-\infty}^{t} R(t-\tau)dN(\tau)$, and $V(x)$ satisfies some wide suitable conditions, then Z_T is asymptotically, as $T \to +\infty$, a Laplace random variable.

According to results of section 2 we conceive that the preceding results can in certain cases permit the study of a random variable of the form

$$(18)\qquad H = \int_a^b V[X_1(t)]\,dt,$$

where $X_1(t)$ is a Laplace process of any kind.

Remark. This leads us to make, incidentally, the following remark. Let $V(x) = x^2$. Kac and Siegert [4] have shown that if $X_1(t)$ is a Laplace stationary process, the correlation function of which is $r(h)$, the characteristic function $\phi(v)$ of H is equal to $[D(2iv)]^{-1/2}$, where $D(\lambda)$ is the Fredholm determinant of the integral equation with kernel $r(t - \tau)$. But this result is not limited to Laplace stationary processes.

Let us suppose that $X_1(t)$ be Laplacian and real, for instance with covariance $\Gamma(t, \tau)$, continuous on the finite interval $[a, b]$. Let (λ_i) be the eigenvalues of the integral equation

$$\lambda \int_a^b \Gamma(t, \tau) f(\tau) d\tau = f(t) + g(t), \tag{19}$$

and let $[f_i(t)]$ be the normed corresponding eigenfunctions

$$\int_a^b f_i(t) f_j(t) dt = \delta_{ij}.$$

Let us set

$$X_i = \int_a^b f_i(t) X_1(t) dt \text{ in quadratic mean}.$$

We know [7, pp. 327–328] that

$$\begin{aligned} E[X_i X_j] &= 0 \quad \text{if} \quad i \neq j \\ &= \frac{1}{\lambda_i} \quad \text{if} \quad i = j. \end{aligned}$$

$$\Gamma(t, \tau) = \sum_i \frac{f_i(t) f_i(\tau)}{\lambda_i} \quad \text{for} \quad a \leqq t, t \leqq b; \tag{20}$$

$X_1(t) = \sum_i f_i(t) X_i$ in quadratic mean for $a \leqq t \leqq b$, so that

$$H = \sum_i X_i^2. \tag{21}$$

In an heuristic manner we can reason in the following way: X_i has for its probability density

$$\frac{1}{\sqrt{2\pi\lambda_i}} e^{-x^2/2\lambda_i},$$

and the X_i's are independent of each other, so that

$$\phi(v) = E[e^{ivH}] = \underbrace{\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty}}_{\infty} \prod_j \frac{1}{\sqrt{2\pi\lambda_i}} \exp\left\{ivx_j^2 - \frac{x_j^2}{2\lambda_j}\right\} \ldots dx_j \ldots,$$

the formal computation of which is immediate and gives

$$\phi(v) = \frac{1}{\sqrt{D(2iv)}}, \tag{22}$$

if $D(\lambda)$ is the Fredholm determinant of the equation (19). This demonstration can be made more rigorous, but not in an immediate manner. Another method, not very elegant, but immediately rigorous, is the following. From (21), we easily get expressions, in terms of the λ_i's, of moments and then of cumulants of H; whence, a development in series of $\log \phi(v)$ with the following form

$$\log \phi(v) = \sum_n i^n \nu_n v^n, \tag{23}$$

where the ν_n's are the cumulants of H. On the other hand, let us call $\Gamma_n(t, \tau)$ the n-th iterated of the kernel $\Gamma(t, \tau)$ [on the interval (a, b)]. We know that

$$\Gamma_n(t, \tau) = \sum_i \frac{f_i(t) f_i(\tau)}{\lambda_i^n}.$$

At last, let us set

$$A_n = \int_a^b \Gamma_n(t, t)\, dt = \sum_i \frac{1}{\lambda_i^n}.$$

We know that (cf. p. 237 in [8])

$$\frac{d}{d\lambda}[\log D(\lambda)] = -\sum_{n=1} A_n \lambda^{n-1},$$

whence a development of $\log [D(2iv)^{-1/2}]$

$$\log [D(2iv)^{-1/2}] = \sum_n 2^n i^n \frac{A_n}{2n} v^n. \tag{24}$$

The identity of the developments (23) and (24) is easily verified; this proves the equality (22). We have supposed a and b finite, but it is easy to give sufficient conditions in order that the result be yet available for a or b or both a and b infinite.

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