CHAPTER 8

INTUITIONISM AND ITS LOGIC

> *"Let those who come after me wonder why I built up these mental constructions and how they can be interpreted in some philosophy; I am content to build them in the conviction that in some way they will contribute to the clarification of human thought."*
>
> L. E. J. Brouwer

8.1. Constructivist philosophy

For a considerable period after the Calculus was discovered by Newton and Leibnitz in the late 17th century, there was controversy and disagreement over its fundamental concepts. Notions of infinitely small quantities, and limits of infinite sequences remained shrouded in mystery, and some of the statements made them look rather strange today (e.g. "A quantity that is increased or decreased infinitely little is neither increased nor decreased" (J. Bernoulli)). The subject acquired a rigorous footing in the 19th century, initially through the development by Cauchy of precise definitions of the concepts of limit and convergence. Later came the "arithmetisation of analysis" by Weierstrass and others, that produced a purely algebraic treatment of the real number system. A significant consequence of this was that analysis began to be separated from its grounding in physical intuition (cf. Weierstrass' proof of the existence of a (counter intuitive?) continuous nowhere-differentiable function). This, along with other factors like the development of non-Euclidean geometry, contributed to the recognition that mathematical structures have an abstract conceptual reality quite independently of the physical world.

Also important during this time was the work of Dedekind and Peano on the number systems. The real numbers were constructed from the rationals, the rationals from the integers, and the integers in turn from the natural numbers. Then the Peano axioms gave an abstract account of the nature of the natural numbers themselves. This kind of reduction contributed to the development of the idea that the whole of mathematics could be presented in one grand axiom system that was itself founded on a few basic notions and principles. This conception has been central to foundational thinking ever since. It takes its extreme form in the "logicist" thesis of Frege and Russell, that mathematics is a part of logic and that mathematical truths are derivable from purely logical principles. It appears also in the work of Hilbert, who attempted to axiomatise mathematics, and prove the consistency of these axioms by finitary methods.

By the time Cantor appeared on the scene it was recognised that references to the infinite, as in "the sequence n^2 tends to infinity as n tends to infinity", could be taken as picturesque articulations of precise, albeit complex, statements about properties of real numbers ("for all ε there exists a δ..." etc.) Cantors set theory transcended this by treating the actual infinite as an object of mathematical investigation. An infinite collection became a "thing-in-itself" that could serve as an *element* of some other collection. The notion of number was extended from the finite to the infinite by the development of a theory of "transfinite" cardinal and ordinal numbers, whose arithmetic involved operations on infinite sets. Cantor's attitude was that as long as statements are grammatically correct and deductions logically sound, such statements have conceptual significance even if they go beyond our basic intuitions about finite numbers and collections.

The theory of sets has been enormously successful, but it has not been without its critics. Leopold Kronecker, well known for having said "God made the integers, all the rest is the work of man", rejected the notions of infinite set and irrational number as being mystical, not mathematical. He maintained that the logical correctness of a theory does not imply the existence of the entities it purported to describe. They remain devoid of any significance unless they can be actually *produced.* Numbers, and operations on them, must, said Kronecker, be "intuitively founded". Definitions and proofs must be "constructive" in a quite literal sense. The definition must show explicitly how to construct the object defined, using objects already known to exist. In classical mathematics an "existence proof" often proceeds by showing that the assumption of the non-existence of an entity of a certain kind leads to contradiction. From the constructivist stand-point this is not a proof of existence at all, since the latter, to be legitimate, must explicitly exhibit the particular object in question. Kronecker believed that the natural numbers could be given such a foundation, but not so for the reals. He actually attempted to rewrite parts of mathematics from this viewpoint.

The conception of things as being "built-up" from already given entities appears also in the reaction of Henri Poincaré to the paradoxes of set theory. He took the view that the source of contradiction lay in the use of *impredicative* definitions. These are circular, self-referential definitions that specify an object X by reference to sets whose own existence depends on that of X. Poincaré held that such definitions were inadmissible and that a set could not be specified until each of its elements had been specified. Thus one half of Russell's paradox (§1.1) consists in showing that $R \in R$. So, on this view, the definition of R is circular, since it can only be given if R has already been defined. Poincaré maintained that mathematics should be founded on the natural number system and developed without impredicative definitions. Thus the Russell class R would not even arise as an object of legitimate study. As it turns out a great deal more would disappear, as significant parts of the classical analysis of the real number system depend on impredicative definitions.

The constructivist attitude, reflected in the views of Kronecker and Poincaré, finds its most spirited expression in the philosophy of Intuitionism, pioneered by the Dutch mathematician L. E. J. Brouwer at the beginning of this century. Brouwer rejected non-constructive arguments, and the conception of infinite collections as things-in-themselves. But he went further than this, to deny traditional logic as a valid representation of mathematical reasoning. We have already noted that the so-called "argument by contradiction" (α is true, because otherwise a contradiction would follow) is constructively unacceptable in existence proofs. But to Brouwer it is not an acceptable principle of argument at all. The same goes for the law of excluded middle, $\alpha \vee {\sim}\alpha$.

Now the classical account of truth as examined in Chapter 6 regards a proposition as being always either true or false, whether we happen to know which is the case. Moreover ${\sim}\alpha$ is true provided only that α is false. Thus "$\alpha \vee {\sim}\alpha$" can be interpreted as saying "either α is true or false" and this last sentence is true on the classical theory. To the intuitionist however a statement is the record of a construction. Asserting the truth of α amounts to saying "I have made a (mental) construction of that which α describes". Likewise ${\sim}\alpha$ records a construction, one that demonstrates that α cannot be the case. From this view, the law of excluded middle has the reading:

> "either I have constructively demonstrated α, or I have constructively demonstrated that α is false."

Now if we take α to be some undecided statement, like Fermat's Last Theorem, then $\alpha \vee {\sim}\alpha$ is not true on this reading. The Theorem has not been shown to be either true, or false, at the present time.

Thus according to Brouwer we cannot assert "α is true" or "α is false" unless we constructively know which is the case. To say that α is not true means only that I have not at this time constructed α, which is not the same as saying α is false. I may well find a construction tomorrow.

The argument by contradiction mentioned earlier can be classically formalised by the tautology ${\sim}{\sim}\alpha \supset \alpha$. To prove α, show that it cannot be that α is false, i.e. show ${\sim}{\sim}\alpha$ is true, and then conclude that α holds. Now the intuitionist account of implication is that to assert the truth of $\alpha \supset \beta$ is to assert "I have developed a construction which when appended to a construction for α yields a construction for β". But then to show that it is contradictory to assume a certain thing does not exist (${\sim}{\sim}\alpha$) does not itself amount to producing that thing (α). Hence ${\sim}{\sim}\alpha \supset \alpha$ is not valid under the constructive interpretation.

Brouwer's view of the history of logic is that the logical laws were obtained by abstraction of the structure of mathematical deductions at a time when the latter were concerned with the world of the finite. These principles of logic were then ascribed an a priori independent existence. Because of this they have been indiscriminately applied to all subsequent developments, including manipulation of infinite sets. Thus contemporary mathematics is based on and uses procedures that are only valid in a more restricted domain. To obtain genuine mathematical knowledge and determine what the correct modes of reasoning are we must go back to the original source of mathematical truth.

Brouwer maintained that this source is found in our primary intuitions about mathematical objects. For him mathematics is an *activity* – autonomous, self-sufficient, and not dependent on language. The essence of this activity lies in mental acts performed by the mathematician – mental constructions of intuitive systems of entities. Language is secondary, and serves only to communicate mathematical understanding. It arises by the formation of verbal parallels of mathematical thinking. This language is then analysed and from that develops formal languages and axiom systems.

Thus logic analyses the structure of the language that parallels mathematical thought. None of this linguistic activity is however to be regarded as part of mathematics itself. It has practical functions in describing and communicating, but is not prerequisite to the activity of performing mental constructions. The essential content of mathematics remains intuitive, not formal.

Having rejected classical mathematics and logic, Brouwer erected in its place a positive and vigorous philosophy of his own. He distinguished what he called the "two acts" of intuitionism. The first act, which demarcates mathematics as a languageless activity, is an intuitive construction in the mind of "two-ness" – the distinction of one thing and then another in time. Our direct awareness of two states of mind – one succeeding the other, lies at the heart of our intuition of objects. The second act recognises the prospect of repetition of a construction once completed. By such iteration we are lead to an infinitely proceeding sequence. Thus with the first act of distinguishing two states of awareness, and the second act of repeating this process, we obtain a linear series, and the sequence of natural numbers emerges as a product of our primary intuitive awareness. There is no such thing to the intuitionist as an actual completed infinite collection. However, by the generation of endlessly proceeding sequences we are lead to a mathematics of the *potentially* infinite, as embodied in the notion of constructions which, although finite at any given stage, can be continued in an unlimited fashion.

From these ideas Brouwer and his followers have built up an extensive treatment of constructive mathematics which is not merely a subsystem of the classical theory, but has a character and range of concepts all of its own, and is the subject of current research interest. The reader may find out more about it in Heyting [66] (cf. also Bishop [67] for a constructive approach even "stricter" than Brouwer's). Another introductory reference is Dummett [77].

8.2. Heyting's calculus

In 1930 an event occurred that greatly enhanced the general understanding of intuitionism. Arend Heyting produced an axiomatic system of propositional logic which was claimed to generate as theorems precisely those sentences that are valid according to the intuitionistic conception of truth. This system is based on the same language PL as used in Chapter 6. Its axioms are the forms I–XI of the CL axioms (i.e. it has all the CL axioms except $\alpha \vee {\sim}\alpha$). Its sole rule of inference is *Detachment*. We shall refer to this system as IL.

Of course the intuitionist only accepts formal systems as imperfect tools for description and communication. He leaves open the possibility that his intuitive deliberations will one day reveal as yet unheard of principles of reasoning. According to Heyting, "in principle it is impossible to set up a formal system which would be equivalent to intuitionist mathematics . . . it can never be proved with mathematical rigour that the system of axioms really embraces every valid method of proof." Nonetheless the investigation of the system IL has proven invaluable in uncovering connections between intuitionistic principles and aspects of topology, recursive functions and computability, models of set theory (forcing), sheaves, and now category theory. Whatever status one attaches to the constructivist view of mathematical reality, there is no doubt that Brouwer's efforts have lead to the elucidation of a significant area of human thought.

Amongst the tautologies that are not IL-theorems are $\alpha \vee {\sim}\alpha$, ${\sim}{\sim}\alpha \supset \alpha$, ${\sim}\alpha \vee {\sim}{\sim}\alpha$. On the other hand $\alpha \supset {\sim}{\sim}\alpha$, ${\sim}{\sim}{\sim}\alpha \supset {\sim}\alpha$, and ${\sim}{\sim}(\alpha \vee {\sim}\alpha)$ *are* derivable. None of the connectives $\sim$, $\wedge$, $\vee$, $\supset$ are definable in terms of each other in IL.

The demonstration of such things is facilitated by the use of a semantical theory that links to IL-derivability. There are several of these available – topological, algebraic, and set-theoretic. The topological aspects of intuitionist logic were discovered independently by Alfred Tarski [38] and Marshall Stone [37]. There it is shown that the open sets of a topological space form an "algebra of sets" in which there are operations satisfying laws corresponding to the axioms of IL. This theme was taken up by J. C. C. McKinsey and Tarski in their study of the algebra of topology [44, 46]. This work involved *closure algebras*, which are BA's with an additional operator whose properties are abstracted from the operation of forming the closure of a set in a topological space. Within a closure algebra there is a special set of elements possessing operations $\sqcap$, $\sqcup$, $\Rightarrow$, $\neg$ obeying intuitionistic principles. McKinsey and Tarski singled these algebras out for special attention, gave an independent axiomatisation of them, and dubbed them *Brouwerian* algebras. Subsequently in [48] they showed that the class of Brouwerian algebras characterises IL in the same way that the class of Boolean algebras characterises CL.

The McKinsey–Tarski approach to algebraic semantics is dual to the one used in §6.5 (an IL-theorem is always assigned 0, rather than 1, etc.). To facilitate comparison with what we have already done we shall discuss, not Brouwerian algebras, but their duals, which are known as

8.3. Heyting algebras

To define these algebras we need to extend our concept of *least upper bound* to sets, rather than just pairs of elements.

If A is a subset of a lattice $\mathbf{L}=(L,\sqsubseteq)$, then $x\in L$ is an *upper bound* of A, denoted $A\sqsubseteq x$, if $y\sqsubseteq x$ whenever $y\in A$. If moreover $x\sqsubseteq z$ whenever $A\sqsubseteq z$, then x is a *least upper bound* (l.u.b.) of A.

EXERCISE 1. A has at most one l.u.b.

EXERCISE 2. Define the notion of g.l.b. of A. □

We say that x is the *greatest element* of A if x is a l.u.b. of A and also a member of A. Thus A *has* a greatest element precisely when one of its members is a l.u.b. of A.

EXERCISE 3. A g.l.b. of A is the greatest element of the set of lower bounds of A.

EXERCISE 4. Define the *least* element of A. □

Now in the powerset lattice $(\mathscr{P}(D),\subseteq)$, $-A$ is the greatest element disjoint from A. That is, $-A$ is disjoint from A, $A\cap -A=\emptyset$, and whenever $A\cap B=\emptyset$, then $B\subseteq -A$. This description of complements can be set out in any lattice and sometimes it leads to a non-Boolean operation. Hence it is given a different name, as follows:

If $\mathbf{L}=(L,\sqsubseteq)$ is a lattice with a zero 0, and $a\in L$, then $b\in L$ is the *pseudo-complement* of a iff b is the greatest element of L disjoint from a, i.e. b is the greatest element of the set $\{x\in L\colon a\sqcap x=0\}$. If every member of L has a pseudo-complement, $\mathbf{L}$ is a *pseudo-complemented lattice.*

Using these definitions it is not hard to verify the

EXERCISE 5. b is the pseudo-complement of a precisely when it satisfies the condition:

$$\text{for all}\quad x\in L,\qquad x\sqsubseteq b\quad\text{iff}\quad a\sqcap x=0.$$ □

EXAMPLE 1. $(\mathscr{P}(D),\subseteq)$: $-A$ is the pseudo-complement of A.

EXAMPLE 2. $\mathbf{B}=(B,\sqsubseteq)$: in any BA,

$$x\sqsubseteq a'\quad\text{iff}\quad a\sqcap x=0\qquad\text{(cf. Exercise 6.4.2)}$$

so the Boolean complement is always a pseudo-complement.

EXAMPLE 3. $(L_M, \subseteq)$: In the lattice of left ideals of monoid $\mathbf{M}$, $\neg B = \{m: \omega_m(B) = \emptyset\}$ is the pseudo-complement of B. (why is $C \subseteq \neg B$ iff $B \cap C = \emptyset$?)

EXAMPLE 4. $(\Theta, \subseteq)$: In the lattice of open sets of a topological space, $U \in \Theta$ has a pseudo-complement, namely $(-U)^\circ$, the *interior* of $-U$ (i.e. the largest open subset of the complement of U). We have $V \subseteq (-U)^\circ$ iff $U \cap V = \emptyset$, for all open V.

EXAMPLE 5. **Sub**(d): In Sub(d), for any topos, $-f: -a \rightarrowtail d$ is the pseudo-complement of $f: a \rightarrowtail d$.

PROOF: We have to show that

$$g \subseteq -f \quad \text{iff} \quad f \cap g \simeq 0_d.$$

Now if $g \subseteq -f$, then by lattice properties, $f \cap g \subseteq f \cap -f \simeq 0_d$ (Theorem 7.2.3), and so $f \cap g \simeq 0_d$.

Conversely suppose $f \cap g \simeq 0_d$. Then the top square of

$$\begin{tikzcd} 0 \arrow[r, "0_b"] \arrow[d] & b \arrow[d, tail, "g"] \\ a \arrow[r, tail, "f"] \arrow[d] & d \arrow[d, "\chi_f"] \\ 1 \arrow[r, "\top"] & \Omega \end{tikzcd}$$

is a pullback. But so is the bottom square, hence the PBL gives the outer rectangle as a pullback. By the Ω-axiom then,

$$\chi_f \circ g = \chi_{0_b} = \bot \circ !_b \qquad \text{(Exercise 5.4.3)}$$

Thus

$$\neg \circ \chi_f \circ g = \neg \circ \bot \circ !_b = \top \circ !_b.$$

But $\top \circ !_b = \chi_g \circ g$ (Ω-axiom) and $\neg \circ \chi_f = \chi_{-f}$, so altogether we have

$$\chi_{-f} \circ g = \chi_g \circ g.$$

But then Lemma 1 of §7.5 gives

$$-f \cap g \simeq g \cap g \simeq g.$$

Hence $g \simeq -f \cap g \subseteq -f$, as required. □

EXAMPLE 6. *Germs.* The collection $\Theta/{\sim_i}=\{[U]_i: U$ open in $I\}$ of germs of open sets at i (cf. the definition of Ω in **Top**(I)) is a pseudo-complemented lattice in which

$$0=[\emptyset]_i, \qquad \text{the germ of } \emptyset$$
$$[U]_i\sqcap[V]_i=[U\cap V]_i$$
$$[U]_i\sqcup[V]_i=[U\cup V]_i$$

and the pseudo-complement of $[U]_i$ is $[(-U)^0]_i$ (i.e. we have the standard quotient lattice construction).

These operations yield the associated truth functions in **Top**(I). There, $\neg:\Omega\to\Omega$ is the function from $\hat{I}$ to $\hat{I}$ taking the germ of U at i to the germ of $(-U)^0$ at i. The conjunction and disjunction arrows from $\Omega\times\Omega$ to Ω are the above meet and join operations acting on each stalk. □

The notion of pseudo-complement can be generalised by replacing the zero 0 by some other element b of the lattice, to obtain the pseudo-complement of *a relative to b*. This, if it exists, is the greatest element of the set $\{x: a\sqcap x\sqsubseteq b\}$. In other words the pseudo-complement of a relative to b is the greatest element c such that $a\sqcap c\sqsubseteq b$. It is readily seen that

EXERCISE 6. c is the pseudo-complement of a relative to b precisely when it satisfies

$$\text{for all } x, \quad x\sqsubseteq c \quad \text{iff} \quad a\sqcap x\sqsubseteq b. \qquad \square$$

EXAMPLE 1. $(\mathscr{P}(D),\subseteq)$: $-A\cup B$ is the pseudo-complement of A relative to B.

EXAMPLE 2. $\mathbf{B}=(B,\sqsubseteq)$: In any **BA**, (Lemma 2(2), §7.5)

$$x\sqsubseteq a'\sqcup b \quad \text{iff} \quad a\sqcap x\sqsubseteq b.$$

EXAMPLE 3. $(L_M,\subseteq)$: $B\Rightarrow C=\{m:\omega_m(B)\subseteq\omega_m(C)\}$ has

$$X\subseteq B\Rightarrow C \quad \text{iff} \quad B\cap X\subseteq C, \qquad \text{all left ideals } X.$$

EXAMPLE 4. $(\Theta,\subseteq)$: The pseudo-complement of U relative to V is $(-U\cup V)^0$, the largest open subset of $-U\cup V$.

$$\text{Whenever } W \text{ is open}, \quad W\subseteq(-U\cup V)^0 \quad \text{iff} \quad U\cap W\subseteq V.$$

EXAMPLE 5. (**Sub**(d)): Theorem 1 of §7.5 states that

$$h \subseteq f \mapsto g \quad \text{iff} \quad f \cap h \subseteq g,$$

hence $\mapsto$ is an operation of relative pseudo-complementation.

EXAMPLE 6. *Germs.* In the lattice $\Theta/\sim_i$ of germs of open sets at i, $[(-U \cup V)^0]_i$ provides $[U]_i$ with a pseudo-complement relative to $[V]_i$. This operation, acting on each stalk, yields the truth-arrow $\Rightarrow : \Omega \times \Omega \to \Omega$ in the topos **Top**(I). □

In a general lattice **L**, the pseudo-complement of a relative to b, when it exists, will be denoted $a \Rightarrow b$. If $a \Rightarrow b$ exists for every a and b in **L**, we will say that **L** is a *relatively pseudo-complemented* (r.p.c.) lattice.

The theory of r.p.c. lattices is thoroughly discussed in Rasiowa–Sikorski [63] and Rasiowa [74]. We list here some basic facts which the reader may care to treat as

Exercises

If **L** is a r.p.c. lattice:

EXERCISE 7. **L** has a unit 1, and for each $a \in L$, $a \Rightarrow a = 1$.

EXERCISE 8. $a \sqsubseteq b$ iff $a \Rightarrow b = 1$.

EXERCISE 9. $b \sqsubseteq a \Rightarrow b$.

EXERCISE 10. $a \sqcap (a \Rightarrow b) = a \sqcap b \sqsubseteq b$.

EXERCISE 11. $(a \Rightarrow b) \sqcap b = b$.

EXERCISE 12. $(a \Rightarrow b) \sqcap (a \Rightarrow c) = a \Rightarrow (b \sqcap c)$.

EXERCISE 13. $(a \Rightarrow b) \sqsubseteq ((a \sqcap c) \Rightarrow (b \sqcap c))$.

EXERCISE 14. if $b \sqsubseteq c$ then $a \Rightarrow b \sqsubseteq a \Rightarrow c$.

EXERCISE 15. $(a \Rightarrow b) \sqcap (b \Rightarrow c) \sqsubseteq (a \Rightarrow c)$.

EXERCISE 16. $(a \Rightarrow b) \sqcap (b \Rightarrow c) \sqsubseteq (a \sqcup b) \Rightarrow c$.

EXERCISE 17. $a \Rightarrow (b \Rightarrow c) \sqsubseteq (a \Rightarrow b) \Rightarrow (a \Rightarrow c)$. □

The definition of r.p.c. lattice does not require the presence of a zero. A *Heyting algebra* (**HA**) is, by definition, an r.p.c. lattice that has a zero 0. If $\mathbf{H}=(H, \sqsubseteq)$ is a Heyting algebra, we define $\neg: H \to H$ by $\neg a = a \Rightarrow 0$. Then $\neg a$ is the l.u.b. of $\{x: a \sqcap x = 0\}$, i.e. $\neg a$ is the pseudo-complement of a.

Again the reader may consult Rasiowa and Sikorski [63] for details of the

Exercises

In any **HA** $\mathbf{H}=(H, \sqsubseteq)$:

EXERCISE 18. $\neg 1 = \neg(a \Rightarrow a) = 0$.

EXERCISE 19. $\neg 0 = 1$, and if $\neg a = 1$, then $a = 0$.

EXERCISE 20. $a \sqsubseteq \neg\neg a$.

EXERCISE 21. $(a \Rightarrow b) \sqsubseteq (\neg b \Rightarrow \neg a)$.

EXERCISE 22. $\neg a = \neg\neg\neg a$.

EXERCISE 23. $a \sqcap \neg a = 0$.

EXERCISE 24. $\neg(a \sqcup b) = \neg a \sqcap \neg b$.

EXERCISE 25. $\neg a \sqcup \neg b \sqsubseteq \neg(a \sqcap b)$.

EXERCISE 26. $\neg a \sqcup b \sqsubseteq a \Rightarrow b$.

EXERCISE 27. $\neg\neg(a \sqcup \neg a) = 1$.

EXERCISE 28. $\neg a \sqsubseteq (a \Rightarrow b)$.

EXERCISE 29. $(a \Rightarrow b) \sqcap (a \Rightarrow \neg b) = \neg a$. □

The six major examples of this section are all Heyting algebras. In the case of the topos **Top**(I) of sheaves over a topological space we can now describe Ω as a topological bundle of Heyting algebras, indexed by I, each of them a quotient of the **HA** of open sets in I.

Now that we know Sub(d) to be an **HA** we can return to the assertion of §7.2 that Sub(d) is a distributive lattice. The point is simply that every r.p.c. lattice is distributive. A proof may be found in Rasiowa and Sikorski, p. 59.

Now in a **BA**, the complement satisfies $x=(x')'$. The analogous property does not occur in all **HA**'s. In our example $\mathbf{M_2}$, in $\mathrm{Sub}(\Omega)$ we have $\top \not\simeq -\bot$, since $\top$ corresponds to the subset $\{2\}$ of L_2, while $-\bot$ corresponds to $\{2,\{0\}\}$ (the character of $-\bot$ is $\neg\circ\chi_\bot=\neg\circ\neg$, which is the function f_Ω of §5.4). Since $\bot\simeq -\top$ in general, we get in $\mathbf{M_2}$ that $\top\not\simeq --\top$.

In the general **HA** we always have $x\sqsubseteq\neg\neg x$, but possibly not $\neg\neg x\sqsubseteq x$ (corresponding to $\sim\sim\alpha\supset\alpha$ not being an IL-theorem). Indeed the situation is as follows:

EXERCISE 30. If an **HA H** satisfies $\neg\neg x\sqsubseteq x$, all $x\in H$, then **H** is a Boolean algebra, i.e. $\neg x$ is an actual complement of x. (Hint: use Exercise 27.) □

In CL, α is logically equivalent to $\sim\sim\alpha$, as reflected in the fact that $x=\neg\neg x$ in 2. In the internal logic of **Set** this means that

$$\begin{tikzcd} 2 \arrow[r, "\neg"] \arrow[dr, "\mathrm{id}_2"'] & 2 \arrow[d, "\neg"] \\ & 2 \end{tikzcd}$$

commutes, i.e. $\neg\circ\neg=\mathrm{id}_2$. The analogous diagram does not commute in all topoi, e.g. in $\mathbf{M_2}$, $\neg\circ\neg$ is the function f_Ω of §5.4 that has output 2 for input $\{0\}$, hence $\neg\circ\neg\neq\mathbf{1}_\Omega$. These deliberations are brought together in

THEOREM 1. *In any topos $\mathscr{E}$ the following are equivalent*

(1) *$\mathscr{E}$ is Boolean*

(2) *In* $\mathrm{Sub}(\Omega)$, $\top\simeq -\bot$

(3) $\neg\circ\neg=\mathbf{1}_\Omega$.

PROOF. (1) implies (2): In general $\bot\cap\top\simeq 0_\Omega$ as shown by the pullback

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & 1 \arrow[d, "\bot"] \\ 1 \arrow[r, "\top"] & \Omega \end{tikzcd}$$

defining $\bot$. But if $\mathscr{E}$ is Boolean, $\bot\cup\top\simeq\mathbf{1}_\Omega$ (cf. §7.3), so that $\top$ is the unique complement of $\bot$ and hence is the pseudo-complement $-\bot$.

(2) implies (3): If $\top \simeq -\bot$, then $\chi_\top = \chi_{-\bot}$, i.e.

$$\mathbf{1}_\Omega = \neg \circ \chi_\bot = \neg \circ \neg.$$

(3) implies (1): Let f be a subobject of d. Then

$$\begin{aligned} \chi_{--f} &= \neg \circ \neg \circ \chi_f \\ &= \chi_f, \quad \text{if} \quad \neg \circ \neg = \mathbf{1}_\Omega \end{aligned}$$

so $--f \simeq f$, making Sub(d) a **BA** by the last exercise. □

Algebraic semantics

If $\mathbf{H} = (H, \sqsubseteq)$ is a Heyting algebra (also known as a *pseudo-Boolean* algebra) then an **H**-*valuation* is a function $V: \Phi_0 \to H$. This may be extended to all sentences using joins $\sqcup$, meets $\sqcap$, relative pseudo-complements $\Rightarrow$, and pseudo-complements $\neg$, to "interpret" the connectives $\vee$, $\wedge$, $\supset$, $\sim$, exactly as for **BA**-valuations in §6.5. A sentence α is **H**-*valid* when $V(\alpha) = 1$ for every **H**-valuation V. α is **HA**-*valid* if valid in every Heyting algebra. We have the following characterisation result:

α is **HA**-*valid iff* $\vdash_{\mathrm{IL}} \alpha$.

The "soundness" part of this consists in showing that the axioms I–XI are **HA**-valid and that *Detachment* preserves this property. For the latter observe by Exercise 8 above that if $V(\alpha) = V(\alpha \supset \beta) = 1$ then $V(\alpha) \sqsubseteq V(\beta)$ so $V(\beta) = 1$. The validity of I–XI is given by various other of the Exercises in combination with 8, e.g. 15. for Axiom IV, 16. for IX, 29. for XI etc.

The completeness of IL with respect to **HA**-validity can be shown by the Lindenbaum algebra method of the Exercise 2 in §6.5. The relation

$$\alpha \sim_{\mathrm{IL}} \beta \quad \text{iff} \quad \vdash_{\mathrm{IL}} \alpha \supset \beta \quad \text{and} \quad \vdash_{\mathrm{IL}} \beta \supset \alpha$$

is an equivalence on Φ. The Lindenbaum algebra for IL is $\mathbf{H}_{\mathrm{IL}} = (\Phi/\sim_{\mathrm{IL}}, \sqsubseteq)$ where

$$[\alpha] \sqsubseteq [\beta] \quad \text{iff} \quad \vdash_{\mathrm{IL}} \alpha \supset \beta$$

$\mathbf{H}_{\mathrm{IL}}$ is an **HA** with $\sqcap$, $\sqcup$ as in the Boolean case, and

$$[\alpha] \Rightarrow [\beta] = [\alpha \supset \beta]$$

$$\neg[\alpha] = [\sim \alpha]$$

The valuation $V(\alpha) = [\alpha]$ can be used to show

$$\vdash_{\mathrm{IL}} \alpha \quad \text{iff} \quad \mathbf{H}_{\mathrm{IL}} \models \alpha,$$

hence any **HA**-valid sentence will be $\mathbf{H}_{\mathrm{IL}}$-valid and so an IL-theorem.

Now the Ω-axiom, through the assignment of χ_f to f establishes, (§4.2) a bijection

$$\mathrm{Sub}(d) \cong \mathscr{E}(d, \Omega)$$

which transfers the **HA** structure of Sub(d) to $\mathscr{E}(d, \Omega)$. Indeed the partial ordering on the latter was described in §7.2 (Theorem 1, Corollary): $\chi_f \sqsubseteq \chi_g$ precisely when $\langle \chi_f, \chi_g \rangle$ factors through $e: \circledleq \rightarrowtail \Omega \times \Omega$. The Heyting operations on $\mathscr{E}(d, \Omega)$ are given by application of the truth-arrows. Thus the lattice meet operation in $\mathscr{E}(d, \Omega)$ assigns to two arrows $f, g: d \rightrightarrows \Omega$, the arrow $f \frown g = \frown \circ \langle f, g \rangle$, the join assigns to them $f \smile g = \smile \circ \langle f, g \rangle$ and so on. The *definition* of the operations $\cap$, $\cup$ etc. on Sub(d) shows that algebraically the two structures look the same, i.e. Sub(d) and $\mathscr{E}(d, \Omega)$ are isomorphic **HA**'s, from which one sees that they validate the same sentences.

The link between topos semantics and the present theory is that in any $\mathscr{E}$, we have

$$\mathscr{E} \models \alpha \quad \textit{iff} \quad \mathscr{E}(1, \Omega) \models \alpha \quad \textit{iff} \quad \mathrm{Sub}(1) \models \alpha$$

(which clarifies further the situation described in Theorem 2 of §7.4).

Thus topos validity in $\mathscr{E}$ amounts to **HA**-validity in the **HA**'s $\mathscr{E}(1, \Omega)$ and Sub(1). The point is that an $\mathscr{E}$-valuation is the same thing as an $\mathscr{E}(1, \Omega)$-valuation, and that $\mathscr{E}$-validity and $\mathscr{E}(1, \Omega)$-validity come to the same thing, since the unit of the **HA** $\mathscr{E}(1, \Omega)$ is $\top: 1 \to \Omega$. This provides the basis of Exercise 2 of §6.7, viz

$$\mathbf{Bn}(I) \models \alpha \quad \textit{iff} \quad (\mathscr{P}(I), \subseteq) \models \alpha,$$

since the truth-values in **Bn**(I) are "essentially" subsets of I. Recalling further that truth-values in **Top**(I) are essentially *open* subsets of I we find that

$$\mathbf{Top}(I) \models \alpha \quad \text{iff} \quad (\Theta, \subseteq) \models \alpha,$$

i.e. validity in the topos of sheaves over I is equivalent to **HA**-validity in the algebra of open subsets of I.

Soundness for $\mathscr{E}$-validity. *If* $\vdash_{\mathrm{IL}} \alpha$ *then* $\mathscr{E} \models \alpha$, *for all topoi* $\mathscr{E}$.

Proof. If α is an IL-theorem then α is **HA**-valid. In particular then, $\mathscr{E}(1, \Omega) \models \alpha$, and so $\mathscr{E} \models \alpha$, by the above. □

Exercise 31. Give an algebraic reason why bivalent topoi always validate $\alpha \vee {\sim}\alpha$. □

Exponentials

The condition $x \sqsubseteq a \Rightarrow b$ iff $a \sqcap x \sqsubseteq b$ means that in an r.p.c. lattice, when considered as a poset category, there is a bijective correspondence between arrows $x \to (a \Rightarrow b)$ and arrows $a \sqcap x \to b$ (either one, or no, arrows in each case). This is reminiscent (§3.16) of the situation in a category $\mathscr{C}$ with exponentiation where there is a bijection $\mathscr{C}(x, b^a) \cong \mathscr{C}(x \times a, b)$. Now in a lattice $a \sqcap x = x \sqcap a$ is the product $x \times a$, and indeed in an r.p.c. lattice $a \Rightarrow b$ provides the exponential b^a. The evaluation arrow $ev : b^a \times a \to b$ is the unique arrow $(a \Rightarrow b) \sqcap a \to b$, which exists by Exercise 10 above. Conversely, exponentials provide relative pseudo-complements, and we find that categorially a *Heyting algebra is no more nor less than a Cartesian closed and finitely co-complete poset.*

The approach we have used in eliciting the **HA** structure of Sub(d) differs from the original method, as described in Freyd [72]. There, $\mid\!\Rightarrow$ is obtained via the Fundamental Theorem, and some complex machinery that we have not even begun to consider (limit preserving functors). The aim is to show that Sub(d) as a poset is Cartesian closed, since exponentials in posets provide r.p.c.'s. By using the truth-arrow $\Rightarrow$ to define $\mid\!\Rightarrow$ we have, apart from showing how the logic of $\mathscr{E}$ determines its subobject behaviour, come in an easier fashion to exactly the same point. For, as Lemma 2(1) of §7.5 indicates, a lattice can be relatively pseudo-complemented in one and only one way.

EXERCISE 32. Show that any chain (linearly ordered poset) with a maximum 1 is r.p.c., with

$$p \Rightarrow q = \begin{cases} 1 & \text{if } p \sqsubseteq q \\ q & \text{otherwise.} \end{cases}$$

(This is the origin of Example 2, §3.16).

EXERCISE 33. Distinguish between, say, $\top \mid\!\Rightarrow \top$ and $\top \Rightarrow \top$ in Sub(Ω) (this is why the special symbol "$\mid\!\Rightarrow$" is being used). □

8.4. Kripke semantics

In 1965 Saul Kripke published a new formal semantics for intuitionistic logic in which PL-sentences are interpreted as subsets of a poset. This theory arose as a sequel to a semantical analysis that Kripke had developed for modal logic. Briefly, modal logic is concerned with the concept of *necessity*, and on the propositional level uses the language PL enriched by a connective whose interpretation is "it is necessarily the case that". The appropriate algebraic "models" here are **BA**'s with an additional operation for this new connective. There is a particular modal axiom system, known as *S4*, that is characterised algebraically by the class of closure algebras. McKinsey and Tarski [48] used this fact to develop a translation of PL-sentences into modal sentences in such a way that IL-theorems correspond to *S4*-theorems. The mechanism of this translation when seen in the light of the Kripke models for *S4*, leads to a new way of giving formal "meaning" to IL sentences.

One attractive feature of the new theory is that its structures, apart from being generally more tractable than the algebraic ones, have an informal interpretation that accords well with the intuitionistic account of the nature of validity. In the latter, truth is temporally conditioned. A sentence is not true or false per se, as in classical logic, but is only so at certain times, i.e. those times at which it has been constructively determined. Now each moment of time is associated with a particular *stage*, or *state* of knowledge. This comprises all the facts that have been constructively established at that time. Sentences then true are so in view of the existing state of knowledge. We thus speak of sentences as being "true at a certain stage" or "true at a certain state of knowledge". The collection of all states of knowledge is ordered by its temporal properties. We speak of one state as coming after, or being later than, another state in time. A sentence true at a certain stage will be held to be true at all later (future) stages. This embodies the idea that constructive knowledge, once established, exists forever more. Having proven α, we cannot later show α to be false.

Now the temporal ordering of states is a partial ordering, not necessarily linear. The states we consider do not always follow one another in a linear sequence because they are *possible* states of knowledge, not just those that do actually occur. Thus at the present moment we may look to the future and contemplate two possible states of knowledge, one in which Fermat's Last Theorem is determined to be true, and one in which it is shown false. These states are incompatible with each other, so in view of the "persistence of truth in time" they cannot be connected by the ordering of states. We cannot proceed from the present to one, and then the other.

Altogether then, the collection of possible states of knowledge is a poset under the ordering of time. A sentence corresponds to a particular subset of this poset, consisting of the states at which the sentence is true.

In view of the persistence of truth in time, this set has a special property: given a particular state in the set, all states in the future of that state belong to the set as well. With these ideas in mind we move to the formal details of Kripke's semantics.

Let $\mathbf{P}=(P,\sqsubseteq)$ be a poset (also called a *frame* in this context). A set $A\subseteq P$ is *hereditary* in $\mathbf{P}$ if it is closed "upwards" under $\sqsubseteq$, i.e. if we have that

$$\text{whenever}\quad p\in A \quad\text{and}\quad p\sqsubseteq q,\quad\text{then}\quad q\in A.$$

The collection of hereditary subsets of $\mathbf{P}$ will be denoted $\mathbf{P}^+$. A $\mathbf{P}$-*valuation* is a function $V:\Phi_0\to\mathbf{P}^+$, assigning to each π_i an *hereditary* subset $V(\pi_i)\subseteq P$. A *model based on* $\mathbf{P}$ is a pair $\mathcal{M}=(\mathbf{P}, V)$, where V is a $\mathbf{P}$-valuation. This notion formally renders the intuitive ideas sketched above. P is a collection of stages of knowledge temporally ordered by $\sqsubseteq$. $V(\pi_i)$ is the set of stages at which π_i is true. The requirement that $V(\pi_i)$ be hereditary formalises the "persistence in time of truth". We now extend the notion of truth at a particular stage to all sentences. The expression "$\mathcal{M}\models_p\alpha$" is to be read "$\alpha$ is true in $\mathcal{M}$ at p", and is defined inductively as follows:

(1) $\mathcal{M}\models_p\pi_i$ iff $p\in V(\pi_i)$
(2) $\mathcal{M}\models_p\alpha\wedge\beta$ iff $\mathcal{M}\models_p\alpha$ and $\mathcal{M}\models_p\beta$
(3) $\mathcal{M}\models_p\alpha\vee\beta$ iff either $\mathcal{M}\models_p\alpha$ or $\mathcal{M}\models_p\beta$
(4) $\mathcal{M}\models_p\sim\alpha$ iff for all q with $p\sqsubseteq q$, not $\mathcal{M}\models_q\alpha$
(5) $\mathcal{M}\models_p\alpha\supset\beta$ iff for all q with $p\sqsubseteq q$, if $\mathcal{M}\models_q\alpha$ then $\mathcal{M}\models_q\beta$.

Thus at stage p, $\sim\alpha$ is true if α is never established at any later stage, and $\alpha\supset\beta$ is true if β holds at all later stages that α is true at.

α is *true* (holds) in the model $\mathcal{M}$, denoted $\mathcal{M}\models\alpha$, if $\mathcal{M}\models_p\alpha$ for every $p\in P$. α is *valid* on the frame $\mathbf{P}$, $\mathbf{P}\models\alpha$, if α is true in every model $\mathcal{M}=(\mathbf{P}, V)$ based on $\mathbf{P}$.

"$\mathcal{M}\not\models_p\alpha$" will abbreviate "not $\mathcal{M}\models_p\alpha$". Similarly "$\mathbf{P}\not\models\alpha$".

Example. Let $\mathbf{P}$ be $\mathbf{2}=(\{0,1\},\leqslant)$ ($0\leqslant 1$ as usual). Take a V with $V(\pi)=\{1\}$ (which is hereditary). Then with $\mathcal{M}=(\mathbf{2}, V)$ we have by (1), $\mathcal{M}\not\models_0\pi$. But $\mathcal{M}\models_1\pi$ and $0\leqslant 1$ so by (4), $\mathcal{M}\not\models_0\sim\pi$. Thus by (3), $\mathcal{M}\not\models_0\pi\vee\sim\pi$, so the law of excluded middle is not valid on this frame. Notice also $\mathcal{M}\not\models_1\sim\pi$, hence $\mathcal{M}\models_0\sim\sim\pi$. Since $0\leqslant 0$, (5) then gives $\mathcal{M}\not\models_0\sim\sim\pi\supset\pi$, hence $\mathbf{2}\not\models\sim\sim\pi\supset\pi$.

If we denote by $\mathcal{M}(\alpha)$ the set of points at which α is true in $\mathcal{M}$, i.e. $\mathcal{M}(\alpha)=\{p:\mathcal{M}\models_p\alpha\}$ then the semantic clauses (1), (2) and (3) can be expressed as

(1′) $\mathcal{M}(\pi_i) = V(\pi_i)$

(2′) $\mathcal{M}(\alpha \wedge \beta) = \mathcal{M}(\alpha) \cap \mathcal{M}(\beta)$

(3′) $\mathcal{M}(\alpha \vee \beta) = \mathcal{M}(\alpha) \cup \mathcal{M}(\beta)$.

To re-express (4) and (5) we define, for hereditary S, T,

$$\neg S = \{p\text{: for all } q \text{ such that } p \sqsubseteq q,\ q \notin S\}$$

and

$$S \Rightarrow T = \{p\text{: for all } q \text{ with } p \sqsubseteq q, \text{ if } q \in S \text{ then } q \in T\}.$$

We then have

(4′) $\mathcal{M}(\sim\alpha) = \neg\mathcal{M}(\alpha)$

(5′) $\mathcal{M}(\alpha \supset \beta) = \mathcal{M}(\alpha) \Rightarrow \mathcal{M}(\beta)$.

The notation is of course not accidental. The intersection and union of two hereditary sets are both hereditary, so the poset $\mathbf{P}^+ = (P^+, \subseteq)$ of hereditary sets under the inclusion ordering is a (bounded distributive) lattice with meets and joins given by $\cap$ and $\cup$. $\mathbf{P}^+$ is indeed a Heyting algebra, with $S \Rightarrow T$ being the pseudo-complement of S relative to T. We have

$$U \subseteq S \Rightarrow T \quad \text{iff} \quad S \cap U \subseteq T, \qquad \text{all hereditary } U,$$

and

$$\neg S = S \Rightarrow \emptyset,$$

the pseudo-complement of S (many exercises here for the reader).

Now a $\mathbf{P}$-valuation $V: \Phi_0 \to P^+$ for the frame $\mathbf{P}$ is also by definition a $\mathbf{P}^+$-valuation for the **HA** $\mathbf{P}^+$. This may be extended, using $\cap$, $\cup$, $\neg$, $\Rightarrow$ to obtain elements $V(\alpha)$ of the algebra $\mathbf{P}^+$ in the usual way. But V also yields a model $\mathcal{M} = (\mathbf{P}, V)$ and hence the set $\mathcal{M}(\alpha)$ for each α. By induction, using the two sets of semantic rules above, we find that for any α,

$$\mathcal{M}(\alpha) = V(\alpha),$$

and so

$$\mathcal{M} \models \alpha \quad \text{iff} \quad \mathcal{M}(\alpha) = P \quad \text{iff} \quad V(\alpha) = P.$$

But P is the unit of the lattice $\mathbf{P}^+$, and since this analysis holds for all V, we find for all α that

$$\mathbf{P} \models \alpha \quad \text{iff} \quad \mathbf{P}^+ \models \alpha,$$

i.e. Kripke-validity on the frame **P** is the same as **HA**-validity on the algebra $\mathbf{P}^+$. This contributes to the verification of the basic characterisation theorem for frame validity, which is that for any α.

$$\vdash_{\mathrm{IL}} \alpha \quad \textit{iff} \quad \alpha \textit{ is valid on every frame.}$$

For the soundness part, we note that if $\vdash_{\mathrm{IL}} \alpha$ then α is **HA**-valid, so for any **P**, $\mathbf{P}^+ \models \alpha$, hence $\mathbf{P} \models \alpha$. One way of proving the completeness part would be to use the representation theory of Stone [37] to turn **HA**'s into frames. The original proof of Kripke used a "semantic tableaux" technique. An alternative approach, based on methods first used in classical logic by Leon Henkin [49], has subsequently been developed, and we now describe it briefly.

First, observe that if p is an element of model $\mathcal{M}$, then $\Gamma_p = \{\alpha : \mathcal{M} \models_p \alpha\}$, the set of sentences true in $\mathcal{M}$ at p, satisfies

(i) If $\vdash_{\mathrm{IL}} \alpha$ then $\alpha \in \Gamma_p$ (soundness)
(ii) If $\vdash_{\mathrm{IL}} \alpha \supset \beta$ and $\alpha \in \Gamma_p$, then $\beta \in \Gamma_p$ (closure under detachment)
(iii) there is at least one α such that $\alpha \notin \Gamma_p$ (consistency)
(iv) if $\alpha \vee \beta \in \Gamma_p$ then $\alpha \in \Gamma_p$ or $\beta \in \Gamma_p$ (Γ_p is "prime").

Γ_p could be called a "state-description". It describes the state p by specifying which sentences are true at p. A set $\Gamma \subseteq \Phi$ that satisfies these four conditions will be called *full*. In general a full set can be construed as a state-description, namely the description of that state in which all members of Γ are known to be true and all sentences not in Γ are not known to be true. This introduces us to the *canonical frame for* IL, which is the poset

$$\mathbf{P}_{\mathrm{IL}} = (P_{\mathrm{IL}}, \subseteq),$$

where P_{IL} is the collection of all full sets, and $\subseteq$ as usual is the subset relation. The *canonical model* for IL is $\mathcal{M}_{\mathrm{IL}} = (\mathbf{P}_{\mathrm{IL}}, V_{\mathrm{IL}})$, where

$$V_{\mathrm{IL}}(\pi_i) = \{\Gamma : \pi_i \in \Gamma\},$$

the set of full sets having π_i as a member.
An inductive proof, using facts about IL-derivability and properties of full sets, shows that for any α and Γ,

$$\mathcal{M}_{\mathrm{IL}} \models_{\Gamma} \alpha \quad \textit{iff} \quad \alpha \in \Gamma$$

To derive the completeness theorem we need the further result:

LINDENBAUM'S LEMMA. $\vdash_{\mathrm{IL}} \alpha$ *iff* α *is a member of every full set,* so that we can conclude

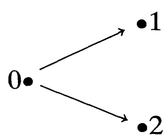
$$\vdash_{\mathrm{IL}} \alpha \quad \textit{iff} \quad \mathcal{M}_{\mathrm{IL}} \models \alpha.$$

From this we get

$$\vdash_{\mathrm{IL}} \alpha \quad \textit{iff} \quad \mathbf{P}_{\mathrm{IL}} \models \alpha$$

and this yields the completeness theorem. (It will also yield, in Chapter 10, a characterisation of the class of topos-valid sentences).

One of the great advantages of the Kripke semantics is that the validity of sentences can be determined by simple conditions on frames. For example, on the poset



if $V(\pi_1)=\{1\}$ and $V(\pi_2)=\{2\}$, then the tautology $(\pi_1 \supset \pi_2) \vee (\pi_2 \supset \pi_1)$ is not true at 0. Notice that this frame is not linearly ordered. In fact it can be shown that:

$\mathbf{P} \models (\alpha \supset \beta) \vee (\beta \supset \alpha)$ iff $\mathbf{P}$ is *weakly linear*, i.e. whenever $p \sqsubseteq q$ and $p \sqsubseteq r$, then $q \sqsubseteq r$ or $r \sqsubseteq q$.

Adjunction of the axiom $(\alpha \supset \beta) \vee (\beta \supset \alpha)$ to IL yields a system, known as LC, first studied by Michael Dummett [59]. The canonical frame method can be adapted to show that the LC-theorems are precisely the sentences valid on all weakly linear frames.

EXERCISE 1. Show $\mathbf{P} \models \alpha \vee \sim\alpha$ iff $\mathbf{P}$ is discrete, i.e. has $p \sqsubseteq q$ iff $p = q$.

EXERCISE 2. $\mathbf{P} \models \sim\alpha \vee \sim\sim\alpha$ iff $\mathbf{P}$ is *directed*, i.e. if $p \sqsubseteq q$ and $p \sqsubseteq r$ then there is an s with $q \sqsubseteq s$ and $r \sqsubseteq s$.

EXERCISE 3. Construct models in which a sentence of the form $\alpha \supset \beta$ has a different truth value to $\sim\alpha \vee \beta$. Similarly for $\alpha \vee \beta$ and $\sim(\sim\alpha \wedge \sim\beta)$.

EXERCISE 4. "$\mathbf{2} \models \alpha$" in Chapter 6 meant "$\alpha$ is valid on the **BA** $2 = \{0, 1\}$". Show this is the same as Kripke-validity on the *discrete* frame $2 = \{0, 1\}$, but different to validity on the non-discrete frame $(2, \leqslant)$ having $0 \leqslant 1$. $\square$

The Kripke semantics is also closely related to the topological interpretation of intuitionism. On any frame $\mathbf{P}$, the collection $\mathbf{P}^+$ of hereditary sets constitutes a topology (a rather special one, as the intersection of any family of open (hereditary) sets is open).

EXERCISE 5. Show that $\mathbf{P}^+$ is the Heyting algebra of open sets for the topology just described, i.e. $\neg S$ is the interior $(-S)^0$ of $-S$, the largest hereditary subset of $-S$, and $S \Rightarrow T$ is $(-S \cup T)^0$, the largest hereditary subset of $-S \cup T$. □

This last section has been a rather rapid survey of what is in fact quite an extensive theory. The full details are readily available in the literature, in the works e.g. of Segerberg [68], Fitting [69], and Thomason [68].

Beth models

Although the Kripke semantics has proven to be the most tractable for many investigations of intuitionistic logic, there is an alternative but related theory due to Evert Beth [56, 59] that is more useful for certain applications (cf. van Dalen [78]). The basic ideas of Beth models can be explained by modifying the semantic rules given in this section for Kripke models.

A *path through* p in a poset $\mathbf{P}$ is a subset A of P that contains p, that is linearly ordered (i.e. $q \sqsubseteq r$ or $r \sqsubseteq q$ for each $q, r \in A$), and that cannot be extended to a larger linearly ordered subset of P. A *bar for* p is a subset B of P with the property that every path through p intersects it. Intuitively, if P represents the possible states of knowledge that can be attained by a mathematician carrying out research, then a path represents a completed course of research. A bar for p is a set of possible states that is unavoidable for any course of research that yields p, i.e. any such course must lead to a state in B.

In a Beth model the connectives $\wedge$, $\sim$, $\supset$ are treated just as in the Kripke theory. The clauses for sentence letters and disjunction however are

$\mathcal{M} \vDash_p \pi_i$ iff there is a bar B for p with $B \subseteq V(\pi_i)$

$\mathcal{M} \vDash_p \alpha \vee \beta$ iff there is a bar B for p with $\mathcal{M} \vDash_q \alpha$ or $\mathcal{M} \vDash_q \beta$ for each $q \in B$.

For further discussion of Beth models in relation to Kripke semantics the reader should consult Kripke's paper and Dummett [77].