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NONLINEAR CONNECTIONS AND DESCRIPTION OF PHOTON-LIKE OBJECTS

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Abstract. The notion of photon-like objects is introduced and briefly discussed. The nonlinear connection view on the Frobenius integrability theory on manifolds is considered as a frame in which appropriate description of photon-like objects to be developed.

1. The Notion Of Photon-Like Objects

We begin with giving the notion of **photon-like object(s)** (PhLO) which notion will be considered further from the point of view of theoretical modeling under the assuming that PhLO are *free*, i.e., interaction of any form of individual PhLO with any other physical object(s) is excluded. The notion we are going to consider reads as follows:

PhLO are real massless time-stable physical objects with
a consistent translational-rotational dynamical structure.

We give now some explanations concerning the above formulated notion of photon-like objects. The feature "real" means:

- PhLO *necessarily* carry energy-momentum
- PhLO can be *created* and *destroyed*
- PhLO are spatially *finite* and they carry finite integral values of physical quantities
- PhLO *propagate* and they do NOT move.

The feature "massless" means:

- their integral energy E and momentum p satisfy $E = cp$, where c is the velocity of light in vacuum

- there exists an isotropic geodesic vector field $\bar{\zeta} = (0, 0, -\varepsilon, 1)$, $\varepsilon = \pm 1$, in Minkowski space-time determining the straight-line direction of translational propagation
- the stress-energy-momentum tensor field $T_{\mu\nu}$ satisfies $T_{\mu\nu}T^{\mu\nu} = 0$.

The feature "time-stable" means:

- after their creation in appropriate conditions PhLO can be destroyed only by external influence. The feature "translational-rotational" means:
  - the propagation has two components: *translational* and *rotational*
  - these both components are of *local* nature
  - these both components exist *simultaneously* and *consistently* and each of them shows definite constancy properties.

The feature "dynamical structure" means:

- some permanent local internal energy-momentum redistribution takes place with time
- PhLO may have interacting, i.e., energy-momentum exchanging, subsystems.

Our purpose now is to find corresponding to this notion appropriate mathematical objects and equations these objects satisfy.

2. Non-Linear Connections

2.1. Projections

These are linear maps P in a linear space W^n satisfying: $P.P = P$ [2]. If P is a projection then $n = \dim(\operatorname{Ker} P) + \dim(\operatorname{Im} P)$. If $(e_1, \dots, e_n)$ and $(\varepsilon_1, \dots, \varepsilon_n)$ are two dual bases in W and $\dim(\operatorname{Ker} P) = p$, $\dim(\operatorname{Im} P) = n - p$, then the bases may be chosen in such a way that P is represented by [3] (summation over the repeated indexes is assumed)

$$P = \varepsilon^a \otimes e_a + (N_i)^a \varepsilon^i \times e_a, \qquad i = 1, \dots, p, \quad a = p + 1, \dots, n.$$

Such bases are usually called P*-adapted*.

2.2. Nonlinear Connections

Let M^n be a smooth (real) manifold with $(x^1, \dots, x^n)$ be local coordinate system. We have the corresponding local frames $\{\mathrm{d}x^1, \dots, \mathrm{d}x^n\}$ and $\{\partial_{x^1}, \dots, \partial_{x^n}\}$. Let for each $x \in M$ we are given a projection P_x of constant rank $p = \dim(\operatorname{Ker} P_x)$, i.e., p does not depend on x, in the tangent space $T_x(M)$. Under this condition we say that a nonlinear connection is given on M [3]. The space $\operatorname{Ker}(P_x) \subset T_x(M)$ is called P*-horizontal*, and the space $\operatorname{Im}(P_x) \subset T_x(M)$ is called P*-vertical*. Thus, we have two distributions on M. The corresponding integrabilities can be defined in terms of P by means of the Nijenhuis bracket $[P, P]$ given by

$$[P, P](X, Y) = 2\{[P(X), P(Y)] + P[X, Y] - P[X, P(Y)] - P[P(X), Y]\}$$

where (X, Y) are two vector fields. Now we add and respectively subtract the term $P[P(X), P(Y)]$, so, the right hand side expression can be represented by

$$[P, P](X, Y) = \mathcal{R}(X, Y) + \bar{\mathcal{R}}(X, Y)$$

where

$$\mathcal{R}(X, Y) = P([(\mathrm{Id} - P)X, (\mathrm{Id} - P)Y]) = P([P_H X, P_H Y])$$

and

$$\bar{\mathcal{R}}(X, Y) = [PX, PY] - P([PX, PY]) = P_H[PX, PY].$$

Since P projects on the vertical subspace $\operatorname{Im} P$, then $(\mathrm{Id} - P) = P_H$ projects on the horizontal subspace. Hence, $\mathcal{R}(X, Y) \neq 0$ measures the nonintegrability of the corresponding horizontal distribution, and $\bar{\mathcal{R}}(X, Y) \neq 0$ measures the nonintegrability of the vertical distribution.

If the vertical distribution is given before-hand and is integrable, then $\mathcal{R}(X, Y) = P([P_H X, P_H Y])$ is called **curvature** of the nonlinear connection P if there exist at least one couple of vector fields (X, Y) such that $\mathcal{R}(X, Y) \neq 0$.

3. Physics + Mathematics

Any physical system with a dynamical structure is characterized with some internal energy-momentum redistributions, i.e., energy-momentum fluxes, during evolution. Any system of energy-momentum fluxes (as well as fluxes of other interesting for the case physical quantities subject to change during evolution, but we limit ourselves just to energy-momentum fluxes here) can be considered mathematically as generated by some system of vector fields. A *consistent* and *interrelated time-stable* system of energy-momentum fluxes can be considered to correspond to an integrable distribution Δ of vector fields according to the principle: *local object generates integral object.*

An integrable distribution Δ may contain various *nonintegrable* subdistributions $\Delta_1, \Delta_2, \ldots$ which subdistributions may be interpreted physically as interacting subsytems. Any physical interaction between two subsystems is necessarily accompanied with available energy-momentum exchange between them, this could be understood mathematically as nonintegrability of each of the two subdistributions of Δ and could be naturally measured by the corresponding curvatures. For example, if Δ is an integrable three-dimensional distribution spent by the vector fields (X_1, X_2, X_3) then we may have, in general, three non-integrable two-dimensional subdistributions (X_1, X_2), (X_1, X_3), (X_2, X_3). Finally, some interaction with the outside world can be described by curvatures of nonintegrable distributions in which elements from Δ and vector fields outside Δ are involved (such processes will not be considered in this paper).

4. Back to PhLO

Our base manifold will be the Minkowski space-time $M = (\mathbb{R}^4, \eta)$, where η is the pseudometric with $\operatorname{sign}\eta = (-,-,-,+)$, canonical coordinates $(x, y, z, \xi = ct)$, and canonical volume form $\omega_o = \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z \wedge \mathrm{d}\xi$. We have the corresponding vector field

$$\bar{\zeta} = -\varepsilon\frac{\partial}{\partial z} + \frac{\partial}{\partial \xi}, \qquad \varepsilon = \pm 1$$

determining that the straight-line of translational propagation of our PhLO is along the spatial coordinate z.

The vector field $\bar{\zeta}$ determines a set of completely integrable three-dimensional Pfaff systems, denoted by $\Delta^*(\bar{\zeta})$. Thus, any element of $\Delta^*(\bar{\zeta})$ is generated by three linearly independent one-forms $(\alpha_1, \alpha_2, \alpha_3)$ which annihilate $\bar{\zeta}$, i.e.,

$$\alpha_1(\bar{\zeta}) = \alpha_2(\bar{\zeta}) = \alpha_3(\bar{\zeta}) = 0, \qquad \alpha_1 \wedge \alpha_2 \wedge \alpha_3 \neq 0.$$

Instead of $(\alpha_1, \alpha_2, \alpha_3)$ we introduce the notation (A, A^*, ζ) and define ζ by

$$\zeta = \varepsilon \mathrm{d}z + \mathrm{d}\xi.$$

Now, since ζ defines one-dimensional completely integrable Pfaff system we have the corresponding completely integrable distribution $(\bar{A}, \bar{A}^*, \bar{\zeta})$. We specify further these objects according to the following

Definition 1. *We shall call these dual systems* ***electromagnetic*** *if they satisfy the following conditions* ($\langle\ ,\ \rangle$ *is the coupling between forms and vectors*):

1. $\langle A, \bar{A}^*\rangle = 0,\ \langle A^*, \bar{A}\rangle = 0$
2. *the vector fields* $(\bar{A}, \bar{A}^*)$ *have no components along* $\bar{\zeta}$
3. *the one-forms* (A, A^*) *have no components along* ζ
4. *the vector fields* $(\bar{A}, \bar{A}^*)$ *are* η*-corresponding to* (A, A^*)*, respectively* .

Further we shall consider only PhLO of electromagnetic nature.

From Conditions 2, 3 and 4 it follows that

$$A = u\,\mathrm{d}x + p\,\mathrm{d}y, \qquad A^* = v\,\mathrm{d}x + w\,\mathrm{d}y$$

$$\bar{A} = -u\,\frac{\partial}{\partial x} - p\frac{\partial}{\partial y}, \qquad \bar{A}^* = -v\,\frac{\partial}{\partial x} - w\frac{\partial}{\partial y}$$

and from Condition 1 it follows $v = -\varepsilon u$, $w = \varepsilon p$, where $\varepsilon = \pm 1$, and (u, p) are two smooth functions on M. Thus, we have

$$A = u\,\mathrm{d}x + p\,\mathrm{d}y, \qquad A^* = -\varepsilon\,p\,\mathrm{d}x + \varepsilon\,u\,\mathrm{d}y$$
$$\bar{A} = -u\,\frac{\partial}{\partial x} - p\frac{\partial}{\partial y}, \qquad \bar{A}^* = \varepsilon\,p\,\frac{\partial}{\partial x} - \varepsilon\,u\,\frac{\partial}{\partial y}\cdot$$

The completely integrable three-dimensional Pfaff system (A, A^*, ζ) contains three two-dimensional subsystems: (A, A^*), (A, ζ) and (A^*, ζ). We have the following

Proposition 1. *The following relations hold*

$$\mathrm{d}A \wedge A \wedge A^* = 0, \qquad \mathrm{d}A^* \wedge A^* \wedge A = 0$$
$$\mathrm{d}A \wedge A \wedge \zeta = \varepsilon[u(p_\xi - \varepsilon p_z) - p(u_\xi - \varepsilon u_z)]\omega_o$$
$$\mathrm{d}A^* \wedge A^* \wedge \zeta = \varepsilon[u(p_\xi - \varepsilon p_z) - p(u_\xi - \varepsilon u_z)]\omega_o.$$

Proof: Immediately checked. □

These relations say that the two-dimensional Pfaff system (A, A^*) is completely integrable for any choice of the two functions (u, p), while the two two-dimensional Pfaff systems (A, ζ) and (A^*, ζ) are NOT completely integrable in general, and the same curvature factor

$$\mathbf{R} = u(p_\xi - \varepsilon p_z) - p(u_\xi - \varepsilon u_z)$$

determines their nonintegrability.

Correspondingly, the three-dimensional completely integrable distribution (or differential system) $\Delta(\zeta)$ contains three two-dimensional subsystems $(\bar{A}, \bar{A}^*)$, $(\bar{A}, \bar{\zeta})$ and $(\bar{A}^*, \bar{\zeta})$. We have the following proposition.

Proposition 2. *The following relations hold* $([X, Y]$ *denotes the Lie bracket*)

$$[\bar{A}, \bar{A}^*] \wedge \bar{A} \wedge \bar{A}^* = 0$$
$$[\bar{A}, \bar{\zeta}] = (u_\xi - \varepsilon u_z)\frac{\partial}{\partial x} + (p_\xi - \varepsilon p_z)\frac{\partial}{\partial y}$$
$$[\bar{A}^*, \bar{\zeta}] = -\varepsilon(p_\xi - \varepsilon p_z)\frac{\partial}{\partial x} + \varepsilon(u_\xi - \varepsilon u_z)\frac{\partial}{\partial y}\cdot$$

Proof: Immediately checked. □

From these last relations and in accordance with Proposition 1 it follows that the distribution $(\bar{A}, \bar{A}^*)$ is integrable, and it can be easily shown that the two distributions $(\bar{A}, \bar{\zeta})$ and $(\bar{A}^*, \bar{\zeta})$ would be completely integrable only if the same curvature factor

$$\mathbf{R} = u(p_\xi - \varepsilon p_z) - p(u_\xi - \varepsilon u_z)$$

is zero.

We mention also that the projections

$$\langle A, [\bar{A}^*, \bar{\zeta}]\rangle = -\langle A^*, [\bar{A}, \bar{\zeta}]\rangle = \varepsilon u(p_\xi - \varepsilon p_z) - \varepsilon p(u_\xi - \varepsilon u_z) = \varepsilon \mathbf{R}$$

give the same factor $\mathbf{R}$. The same curvature factor appears, of course, as coefficient in the exterior products $[\bar{A}^*, \bar{\zeta}] \wedge \bar{A}^* \wedge \bar{\zeta}$ and $[\bar{A}, \bar{\zeta}] \wedge \bar{A} \wedge \bar{\zeta}$. In fact, we obtain

$$[\bar{A}^*, \bar{\zeta}] \wedge \bar{A}^* \wedge \bar{\zeta} = -[\bar{A}, \bar{\zeta}] \wedge \bar{A} \wedge \bar{\zeta} = -\varepsilon \mathbf{R}\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z} + \mathbf{R}\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial \xi} \cdot$$

On the other hand, for the other two projections we obtain

$$\langle A, [\bar{A}, \bar{\zeta}]\rangle = \langle A^*, [\bar{A}^*, \bar{\zeta}]\rangle = \frac{1}{2}\left[(u^2 + p^2)_\xi - \varepsilon(u^2 + p^2)_z\right].$$

Clearly, the last relation may be put in terms of the Lie derivative $L_{\bar{\zeta}}$ as

$$\frac{1}{2}L_{\bar{\zeta}}(u^2 + p^2) = -\frac{1}{2}L_{\bar{\zeta}}\langle A, \bar{A}\rangle = -\langle A, L_{\bar{\zeta}}\bar{A}\rangle = -\langle A^*, L_{\bar{\zeta}}\bar{A}^*\rangle.$$

Remark. Further in the paper we shall denote $\sqrt{u^2 + p^2} \equiv \phi$, and shall assume that ϕ is a *spatially finite* function, so, u and p must also be spatially finite.

Proposition 3. *There is a function* $\psi(u, p)$ *such, that*

$$L_{\bar{\zeta}}\psi = \frac{u(p_\xi - \varepsilon p_z) - p(u_\xi - \varepsilon u_z)}{\phi^2} = \frac{\mathbf{R}}{\phi^2} \cdot$$

Proof: It is immediately checked that $\psi = \arctan \frac{p}{u}$ is such one. □

We note that the function ψ has a natural interpretation of *phase* because of the easily verified now relations $u = \phi \cos \psi$, $p = \phi \sin \psi$, and ϕ acquires the status of *amplitude*. Since the transformation $(u, p) \to (\phi, \psi)$ is non-degenerate this allows to work with the two functions (ϕ, ψ) instead of (u, p).

From Proposition 3 we have

$$\mathbf{R} = \phi^2 L_{\bar{\zeta}}\psi = \phi^2(\psi_\xi - \varepsilon\psi_z).$$

5. Back to Non-Linear Connections

The above relations show that we can introduce two nonlinear connections P and $\tilde{P}$. In fact, since the integrable distribution $(\bar{A}, \bar{A}^*)$ lives in the (x, y)-plane we present the coordinates in order (z, ξ, x, y) and the bases $(\mathrm{d}z, \mathrm{d}\xi, \mathrm{d}x, \mathrm{d}y)$, $(\partial_z, \partial_\xi, \partial_x, \partial_y)$. We choose the vertical distribution to be generated by (∂_x, ∂_y). The corresponding projections look like

$$P_V = \mathrm{d}x \otimes \frac{\partial}{\partial x} + \mathrm{d}y \otimes \frac{\partial}{\partial y} - \varepsilon\, u\, \mathrm{d}z \otimes \frac{\partial}{\partial x} - u\, \mathrm{d}z \otimes \frac{\partial}{\partial y} - \varepsilon\, p\, \mathrm{d}\xi \otimes \frac{\partial}{\partial x} - p\, \mathrm{d}z \otimes \frac{\partial}{\partial y}$$

$$\tilde{P}_V = \mathrm{d}x \otimes \frac{\partial}{\partial x} + \mathrm{d}y \otimes \frac{\partial}{\partial y} + p\, \mathrm{d}z \otimes \frac{\partial}{\partial x} + \varepsilon\, p\, \mathrm{d}z \otimes \frac{\partial}{\partial y} - u\, \mathrm{d}\xi \otimes \frac{\partial}{\partial x} - \varepsilon\, u\, \mathrm{d}\xi \otimes \frac{\partial}{\partial y} \cdot$$

The corresponding matrices look like

$$P_V=\begin{Vmatrix}0&0&0&0\\0&0&0&0\\-\varepsilon u&-u&1&0\\-\varepsilon p&-p&0&1\end{Vmatrix},\qquad P_H=\begin{Vmatrix}1&0&0&0\\0&1&0&0\\\varepsilon u&u&0&0\\\varepsilon p&p&0&0\end{Vmatrix}$$

$$(P_V)^*=\begin{Vmatrix}0&0&-\varepsilon u&-\varepsilon p\\0&0&-u&-p\\0&0&1&0\\0&0&0&1\end{Vmatrix},\qquad (P_H)^*=\begin{Vmatrix}1&0&\varepsilon u&\varepsilon p\\0&1&u&p\\0&0&0&0\\0&0&0&0\end{Vmatrix}$$

$$\tilde P_V=\begin{Vmatrix}0&0&0&0\\0&0&0&0\\p&\varepsilon p&1&0\\-u&-\varepsilon u&0&1\end{Vmatrix},\qquad \tilde P_H=\begin{Vmatrix}1&0&0&0\\0&1&0&0\\-p&-\varepsilon p&0&0\\u&\varepsilon u&0&0\end{Vmatrix}$$

$$(\tilde P_V)^*=\begin{Vmatrix}0&0&p&-u\\0&0&\varepsilon p&-\varepsilon u\\0&0&1&0\\0&0&0&1\end{Vmatrix},\qquad (\tilde P_H)^*=\begin{Vmatrix}1&0&-p&u\\0&1&-\varepsilon p&\varepsilon u\\0&0&0&0\\0&0&0&0\end{Vmatrix}.$$

The projections of the coordinate bases are:

$$\Big(\frac{\partial}{\partial z},\frac{\partial}{\partial \xi},\frac{\partial}{\partial x},\frac{\partial}{\partial y}\Big).P_V=\Big(-\varepsilon u\frac{\partial}{\partial x}-\varepsilon p\frac{\partial}{\partial y},-u\frac{\partial}{\partial x}-p\frac{\partial}{\partial y},\frac{\partial}{\partial x},\frac{\partial}{\partial y}\Big)$$

$$\Big(\frac{\partial}{\partial z},\frac{\partial}{\partial \xi},\frac{\partial}{\partial x},\frac{\partial}{\partial y}\Big).P_H=\Big(\frac{\partial}{\partial z}+\varepsilon u\frac{\partial}{\partial x}+\varepsilon\, p\frac{\partial}{\partial y},\frac{\partial}{\partial \xi}+u\frac{\partial}{\partial x}+p\frac{\partial}{\partial y},0,0\Big)$$

$$(\mathrm{d}z,\mathrm{d}\xi,\mathrm{d}x,\mathrm{d}y).(P_V)^*=(0,0,-\varepsilon u\,\mathrm{d}z-u\,\mathrm{d}\xi+\mathrm{d}x,-\varepsilon p\,\mathrm{d}z-p\,\mathrm{d}\xi+\mathrm{d}y)$$

$$(\mathrm{d}z,\mathrm{d}\xi,\mathrm{d}x,\mathrm{d}y).(P_H)^*=(\mathrm{d}z,\mathrm{d}\xi,\varepsilon u\,\mathrm{d}z+u\,\mathrm{d}\xi,\varepsilon p\,\mathrm{d}z+p\,\mathrm{d}\xi)\,.$$

Consider now the two-forms

$$\begin{aligned}G&=(P_V)^*\mathrm{d}x\wedge(P_H)^*\mathrm{d}x+(P_V)^*\mathrm{d}y\wedge(P_H)^*\mathrm{d}y\\&=\varepsilon u\,\mathrm{d}x\wedge\mathrm{d}z+\varepsilon p\,\mathrm{d}y\wedge\mathrm{d}z+u\,\mathrm{d}x\wedge\mathrm{d}\xi+p\,\mathrm{d}y\wedge\mathrm{d}\xi\\\tilde G&=(\tilde P_V)^*\mathrm{d}x\wedge(\tilde P_H)^*\mathrm{d}x+(\tilde P_V)^*\mathrm{d}y\wedge(\tilde P_H)^*\mathrm{d}y\\&=-p\,\mathrm{d}x\wedge\mathrm{d}z+u\,\mathrm{d}y\wedge\mathrm{d}z-\varepsilon p\,\mathrm{d}x\wedge\mathrm{d}\xi+\varepsilon u\,\mathrm{d}y\wedge\mathrm{d}\xi.\end{aligned}$$

It follows that $G=A\wedge\zeta$, $\tilde G=A^*\wedge\zeta$ and $\tilde G=*G$, where $*$ is the Hodge star operator defined by η. Clearly, the two two-forms $(G,*G)$ represent the two nonintegrable Pfaff systems (A,ζ) and (A^*,ζ).

The corresponding curvatures are

$$\mathcal{R} = \varepsilon(u_\xi - \varepsilon u_z)\,\mathrm{d}z \wedge \mathrm{d}\xi \otimes \frac{\partial}{\partial x} + \varepsilon(p_\xi - \varepsilon p_z)\,\mathrm{d}z \wedge \mathrm{d}\xi \otimes \frac{\partial}{\partial y}$$

$$\tilde{\mathcal{R}} = -(p_\xi - \varepsilon p_z)\,\mathrm{d}z \wedge \mathrm{d}\xi \otimes \frac{\partial}{\partial x} + (u_\xi - \varepsilon u_z)\,\mathrm{d}z \wedge \mathrm{d}\xi \otimes \frac{\partial}{\partial y}\cdot$$

We obtain

$$\mathcal{R}\left(P_H\frac{\partial}{\partial z}, P_H\frac{\partial}{\partial \xi}\right) = [\bar{A}, \bar{\zeta}], \qquad \tilde{\mathcal{R}}\left(\tilde{P}_H\frac{\partial}{\partial z}, \tilde{P}_H\frac{\partial}{\partial \xi}\right) = [\varepsilon\bar{A}^*, \bar{\zeta}].$$

6. Again Physics + Mathematics

The two two-forms obtained $(G, *G)$ suggest to test them as basic constituents of classical electrodynamics, i.e., if they satisfy Maxwell equations. However, it turns out that $\mathrm{d}G \neq 0$ and $\mathrm{d} * G \neq 0$ in general. As for the energy-momentum part of Maxwell theory, determined by the corresponding energy-momentum tensor

$$T_\mu{}^\nu = \frac{1}{2}[G_{\mu\sigma}G^{\nu\sigma} + (*G)_{\mu\sigma}(*G^{\nu\sigma}] \qquad \text{and} \qquad T_{44} = u^2 + p^2 = \phi^2$$

we obtain the following relations

$$\nabla_\nu T_\mu^\nu = \frac{1}{2}\Big[G^{\alpha\beta}(\mathrm{d}G)_{\alpha\beta\mu} + (*G)^{\alpha\beta}(\mathrm{d} * G)_{\alpha\beta\mu}\Big]$$

$$G^{\alpha\beta}(\mathrm{d}G)_{\alpha\beta\mu}\mathrm{d}x^\mu = (*G)^{\alpha\beta}(\mathrm{d} * G)_{\alpha\beta\mu}\mathrm{d}x^\mu = \frac{1}{2}L_{\bar{\zeta}}(u^2 + p^2).\zeta = \frac{1}{2}L_{\bar{\zeta}}\phi^2.\zeta$$

and

$$\left\langle A, \mathcal{R}\left(P_H\frac{\partial}{\partial z}, P_H\frac{\partial}{\partial \xi}\right)\right\rangle = \left\langle \varepsilon A^*, \tilde{\mathcal{R}}\left(\tilde{P}_H\frac{\partial}{\partial z}, \tilde{P}_H\frac{\partial}{\partial \xi}\right)\right\rangle$$
$$= \frac{1}{2}L_{\bar{\zeta}}(u^2 + p^2) = \frac{1}{2}L_{\bar{\zeta}}\phi^2.$$

On the other hand

$$(*G)^{\alpha\beta}(\mathrm{d}G)_{\alpha\beta\mu}\mathrm{d}x^\mu = -G^{\alpha\beta}(\mathrm{d} * G)_{\alpha\beta\mu}\mathrm{d}x^\mu$$
$$= \big[u(p_\xi - \varepsilon p_z) - p(u_\xi - \varepsilon u_z)\big]\zeta = \mathbf{R}.\zeta.$$

Also, we find

$$\left\langle A, \tilde{\mathcal{R}}\left(\tilde{P}_H\frac{\partial}{\partial z}, \tilde{P}_H\frac{\partial}{\partial \xi}\right)\right\rangle = -\left\langle \varepsilon A^*, \mathcal{R}\left(P_H\frac{\partial}{\partial z}, P_H\frac{\partial}{\partial \xi}\right)\right\rangle = -\mathbf{R}.$$

So, if $L_{\bar{\zeta}}\phi = 0$ we can say that our two two-forms $G = A \wedge \zeta$ and $*G = A^* \wedge \zeta$, having zero invariants, are nonlinear solutions to the nonlinear equations

$$G^{\alpha\beta}(\mathrm{d}G)_{\alpha\beta\mu} = 0, \qquad (*G)^{\alpha\beta}(\mathrm{d} * G)_{\alpha\beta\mu} = 0$$

$$G^{\alpha\beta}(\mathrm{d} * G)_{\alpha\beta\mu} + (*G)^{\alpha\beta}(\mathrm{d}G)_{\alpha\beta\mu} = 0.$$

From physical point of view these three equations say that the two subsystems of our PhLO, mathematically represented by the two two-forms G and $*G$ keep the energy-momentum they carry, and are in permanent energy-momentum exchange with each other in equal quantities, i.e., in permanent dynamical equilibrium [1]. The mathematical quantity that guarantees the dynamical nature of this equilibrium is the nonzero curvature $\mathbf{R}$ or $\mathcal{R}$. The permanent nature of this dynamical equilibrium suggests to look for corresponding quantities/parameter(s), which should represent relation(s), charavterizing the state at a given moment of PhLO and its intrinsical capability to overcome the destroying tendencies of the existing nonintegrabilities by means of appropriate propagation properties.

We note the relations

$$\begin{aligned}\left\langle A, P_H\frac{\partial}{\partial\xi}\right\rangle &= \left\langle A^*, \tilde{P}_H\frac{\partial}{\partial z}\right\rangle = -\left\langle A, P_V\frac{\partial}{\partial\xi}\right\rangle = \varepsilon\left\langle A, P_H\frac{\partial}{\partial z}\right\rangle \\ &= -\varepsilon\left\langle A, P_V\frac{\partial}{\partial z}\right\rangle = \varepsilon\left\langle A^*, \tilde{P}_H\frac{\partial}{\partial\xi}\right\rangle = -\left\langle A^*, \tilde{P}_V\frac{\partial}{\partial}\right\rangle \\ &= -\varepsilon\left\langle A^*, \tilde{P}_V\frac{\partial}{\partial\xi}\right\rangle = u^2+p^2 = \phi^2 = -\eta(A,A) = -\eta(A^*,A^*) \\ &\equiv S^2.\end{aligned}$$

On the other hand,

$$\begin{aligned}&\left\langle (P_V)^*(\mathrm{d}x)\wedge(P_V)^*(\mathrm{d}y), \mathcal{R}\left(P_H\frac{\partial}{\partial z}, P_H\frac{\partial}{\partial\xi}\right)\wedge\tilde{\mathcal{R}}\left(\tilde{P}_H\frac{\partial}{\partial z}, \tilde{P}_H\frac{\partial}{\partial\xi}\right)\right\rangle \\ &= \left\langle (\tilde{P}_V)^*(\mathrm{d}x)\wedge(\tilde{P}_V)^*(\mathrm{d}y), \mathcal{R}\left(P_H\frac{\partial}{\partial z}, P_H\frac{\partial}{\partial\xi}\right)\wedge\tilde{\mathcal{R}}\left(\tilde{P}_H\frac{\partial}{\partial z}, \tilde{P}_H\frac{\partial}{\partial\xi}\right)\right\rangle \\ &= \varepsilon[(u_\xi-\varepsilon u_z)^2+(p_\xi-\varepsilon p_z)^2] = \varepsilon\,(\mathcal{R})^2 \equiv \varepsilon Z^2.\end{aligned}$$

Hence, the relation

$$\frac{S^2}{Z^2} = \frac{u^2+p^2}{[(u_\xi-\varepsilon u_z)^2+(p_\xi-\varepsilon p_z)^2]} = \frac{\phi^2}{\phi^2(\psi_\xi-\varepsilon\psi_z)^2} = \frac{1}{(L_{\bar{\zeta}}\psi)^2} \equiv (l_o)^2$$

defines the quantity κl_o, $\kappa=\pm 1$ as an appropriate such parameter.

7. Translational-Rotational Consistency and Equations of Motion

In order to introduce mathematically the translational-rotational consistency we recall the relations

$$\bar{A}\wedge\bar{A}^* = \varepsilon\phi^2\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y}\neq 0, \qquad [\bar{A},\zeta]\wedge[\bar{A}^*,\zeta] = \varepsilon\phi^2(L_{\bar{\zeta}}\psi)^2\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y}\neq 0.$$

Thus, we have two frames $(\bar{A}, \bar{A}^*, \partial_z, \partial_\xi)$ and $([\bar{A},\bar{\zeta}], [\bar{A}^*,\bar{\zeta}], \partial_z, \partial_\xi)$. The internal energy-momentum redistribution during propagation is strongly connected with the existence of linear map transforming the first frame into the second one since both are defined by the dynamical nature of our PhLO. Taking into account that only the first two vectors of these two frames change during propagation we write down this relation in the form

$$([\bar{A},\zeta],[\bar{A}^*,\zeta])=(\bar{A},\bar{A}^*)\begin{Vmatrix}\alpha & \beta\\ \gamma & \delta\end{Vmatrix}.$$

Solving this system with respect to the real numbers $(\alpha,\beta,\gamma,\delta)$ we obtain

$$\begin{Vmatrix}\alpha & \beta\\ \gamma & \delta\end{Vmatrix}=\frac{1}{\phi^2}\begin{Vmatrix}-\frac{1}{2}L_{\bar\zeta}\phi^2 & \varepsilon\mathbf{R}\\ -\varepsilon\mathbf{R} & -\frac{1}{2}L_{\bar\zeta}\phi^2\end{Vmatrix}=-\frac{1}{2}\frac{L_{\bar\zeta}\phi^2}{\phi^2}\begin{Vmatrix}1 & 0\\ 0 & 1\end{Vmatrix}+\varepsilon L_{\bar\zeta}\psi\begin{Vmatrix}0 & 1\\ -1 & 0\end{Vmatrix}.$$

Assuming the conservation law $L_{\bar\zeta}\phi^2=0$, we obtain that the rotational component of propagation is governed by the matrix $\varepsilon L_{\bar\zeta}\psi J$, where J denotes the canonical complex structure in $\mathbb{R}^2$, and since $\phi^2 L_{\bar\zeta}\psi=\mathbf{R}$ we conclude that the rotational component of propagation is available if and only if the Frobenius curvature is NOT zero: $\mathbf{R}\neq 0$. We may also say that a consistent translational-rotational dynamical structure is available if the amplitude $\phi^2=u^2+p^2$ is a running wave along $\bar\zeta$ and the phase $\psi=\mathrm{arctg}\frac{p}{u}$ is NOT a running wave along $\bar\zeta$.

As we have noted before the local conservation law $L_{\bar\zeta}\phi^2=0$, being equivalent to $L_{\bar\zeta}\phi=0$, gives one dynamical linear first order equation. This equation pays due respect to the assumption that our spatially finite PhLO, together with its energy density, propagates translationally with the constant velocity c. We need one more equation in order to specify the phase function ψ. If we pay corresponding respect also to the rotational aspect of the PhLO nature it is desirable this equation *to introduce and guarantee the conservative and constant character of this aspect of PhLO nature*. Since rotation is available only if $L_{\bar\zeta}\psi\neq 0$, the simplest such assumption respecting the constant character of the rotational component of propagation seems to be $L_{\bar\zeta}\psi=\mathrm{const}$, i.e., $l_o=\mathrm{const}$. Thus, the equation $L_{\bar\zeta}\phi=0$ and the frame rotation $(\bar{A},\bar{A}^*,\partial_z,\partial_\xi)\to([\bar{A},\bar\zeta],[\bar{A}^*,\bar\zeta],\partial_z,\partial_\xi)$, i.e., $[\bar{A},\bar\zeta]=-\varepsilon\bar{A}^*L_{\bar\zeta}\psi$ and $[\bar{A}^*,\bar\zeta]=\varepsilon\bar{A}L_{\bar\zeta}\psi$, give the following equations for the two functions (u,p)

$$u_\xi-\varepsilon u_z=-\frac{\kappa}{l_o}p,\qquad p_\xi-\varepsilon p_z=\frac{\kappa}{l_o}u.$$

If we now introduce the complex valued function $\Psi=uI+pJ$, where I is the identity map in $\mathbb{R}^2$, the above two equations are equivalent to

$$L_{\bar\zeta}\Psi=\frac{\kappa}{l_o}J(\Psi)$$

which clearly confirms once again the translational-rotational consistency in the form that *no translation is possible without rotation, and no rotation is possible without translation*, where the rotation is represented by the complex structure J. Since the operator J rotates to angle $\alpha=\pi/2$, the parameter l_o determines the corresponding translational advancement, and $\kappa = \pm 1$ takes care of the left/right orientation of the rotation. Clearly, a full rotation (i.e., 2π-rotation) will require a $4l_o$-translation, so, the natural time-period is $T = 4l_o/c = 1/\nu$, and $4l_o$ is naturally interpreted as the PhLO size along the spatial direction of translational propagation.

In order to find an integral characteristic of the PhLO rotational nature in *action units* we correspondingly modify (i.e., multiply by $\kappa l_o/c$) and consider any of the two equal Frobenius four-forms

$$\frac{\kappa l_o}{c}\mathrm{d}A \wedge A \wedge \zeta = \frac{l_o}{c}\mathrm{d}A^* \wedge A^* \wedge \zeta = \frac{\kappa l_o}{c}\varepsilon \mathbf{R}\omega_o.$$

Integrating this four-form over the four-volume $\mathbb{R}^3 \times 4l_o$ we obtain the quantity $\mathcal{H} = \varepsilon\kappa ET = \pm ET$, where E is the integral energy of the PhLO, which clearly is the analog of the Planck formula $E = h\nu$, i.e., $h = ET$.

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