

ON SUPERINTEGRABILITY OF THE MANEV PROBLEM AND ITS REAL HAMILTONIAN FORM

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Abstract. We construct Ermanno–Bernoulli type invariants for the Manev model dynamics which may be viewed upon as remnants of the Laplace–Runge–Lenz vector in the Kepler model. If the orbits are bounded these invariants exist only when a certain rationality condition is met and thus we have superintegrability only on a subset of initial values. Manev model's dynamics is demonstrated to be bi-Hamiltonian and a recursion operator is constructed. We analyze real form dynamics of the Manev model and derive that it is always superintegrable. We also discuss the symmetry algebras of the Manev model and its real Hamiltonian form.

1. Preliminaries

1.1. The Manev Problem

By **Manev model** [15] we mean here the dynamics given by the Hamiltonian

$$H = \frac{1}{2}(p_x^2 + p_y^2 + p_z^2) - \frac{A}{r} - \frac{B}{r^2} \quad (1)$$

where $r = \sqrt{x^2 + y^2 + z^2}$; A and B are assumed arbitrary real constants whose positive values correspond to attractive forces. The model itself as proposed by G. Manev involved specific expression for the constant $B = \frac{3G}{2c^2}A$ determined by applying Max Planck's more general action–reaction principle. It offers a surprisingly good practical approximation to Einstein's relativistic dynamics – at least at

a solar system level – capable to describe both the perihelion advance of the inner planets and the Moon’s perigee motion. In the planetary approximation the Manev model is the natural classical analog of the Schwarzschild problem [7]. In the last decade it has enjoyed an increased interest both as a very suitable approximation from astronomers’ point of view and as a toy model for applying different techniques of modern mechanics (see e.g. [17, 18, 5, 2, 7, 12]).

There is close a connection between the solutions of Kepler and Manev problems – the latter always being certain Newton trajectory with added precession. Due to Hamiltonian’s rotational invariance each component of the angular momentum \vec{L}

$$L_j = \varepsilon_{jkm} p_k x_m \quad \text{with} \quad (x_1, x_2, x_3) = (x, y, z) \quad (2)$$

is an obvious first integral $\{H, L_j\} = 0$ and the model is integrable since H , \vec{L}^2 and any component, say L_z , of the angular momentum are in involution. Components themselves are not in involution – L_j form an $\mathfrak{so}(3)$ algebra with respect to the Poisson brackets

$$\{L_j, L_k\} = \varepsilon_{jkm} L_m \quad (3)$$

and if we approach the question of the integrability in their terms we obtain the most simple example of non-commutative integrability [1, 19, 21].

The dynamics is confined on a plane which we assume to be Oxy and is separable in radial coordinates. On the reduced phase space (see e.g. [10] for the generalities of the reduction procedure) obtained by fixing the angular momentum $\pi_\theta = L_z \equiv L$ to a certain value ℓ the motion is governed by

$$H_{\text{eff}} = \frac{1}{2} \left(p_r^2 + \frac{\ell^2 - 2B}{r^2} \right) - \frac{A}{r}. \quad (4)$$

The dynamics behave like radial motion of Kepler dynamics with angular momentum squared $\ell^2 - 2B$; while the case $2B > \ell^2$ corresponds to overall centripetal effect. On the other hand, the angular equation of motion $\dot{\theta} = L/r^2$ is still governed by the ‘authentic’ angular momentum ℓ (and r is as just described). Thus we may have not only purely classical perihelion shifts but also if $2B \geq \ell^2 \neq 0$ we may have collapsing trajectories which are spirals, even though in phase space they are symplectic transformations; while in the Kepler dynamics the only allowed fall down is along straight lines. Spiraling here has nothing to do with non-conservative forces but follows from the fact that in the Manev model collapse is possible for non-vanishing angular momentum as well. For this reason the set of initial data leading to collision has a positive measure and this may offer an explanation why collisions in the solar system are estimated to happen more often than Newton theory predicts [6].

Consequently, the remarkable properties of Kepler dynamics that all negative energy orbits are closed and the frequencies of radial and angular motions coincide

(for any initial conditions) are no more true. In the generic case and when $\ell^2 > 2B$ bounded orbits shall not close except for discrete, rational values of $\sqrt{\ell^2 - 2B}/\ell$ in which case orbits close and form a rosette (see also [20]).

We shall not reproduce here a comprehensive analysis of Manev model's dynamical properties (see e.g. [17, 18, 5, 2, 7]) but we shall rather note the equation of trajectories and concentrate on its symmetry features.

1.2. The Kepler Problem Invariants

In the case of **Kepler problem**, corresponding to $B = 0$, we have more first integrals (for details and historical notes see e.g. [11, 13, 22, 4])

$$J_x = p_y L - \frac{A}{r} x, \quad J_y = -p_x L - \frac{A}{r} y, \quad \{H_K, \vec{J}\} = 0 \quad (5)$$

where H_K is the Kepler Hamiltonian and J_x and J_y are the components of the Laplace–Runge–Lenz vector. They are not independent since

$$J^2 = 2H_K L^2 + A^2. \quad (6)$$

Together with the Hamiltonian and angular momentum they close on an algebra with respect to the Poisson brackets

$$\{H_K, L\} = 0, \quad \{L, J_x\} = J_y, \quad \{L, J_y\} = -J_x, \quad \{J_x, J_y\} = -2H_K L. \quad (7)$$

After redefining $\vec{E} = \vec{J}/\sqrt{|-2H_K|}$ we get

$$\{L, E_x\} = E_y, \quad \{L, E_y\} = -E_x, \quad \{E_x, E_y\} = -\text{sign}(H_K)L \quad (8)$$

which makes obvious the fact that we have an $\mathfrak{so}(3)$ algebra for negative energies and $\mathfrak{so}(2, 1)$ for positive ones. In the case of the 3-dimensional Kepler problem the components of the angular momentum give us another copy of $\mathfrak{so}(3)$, see equation (3), so the full symmetry algebra is $\mathfrak{so}(4)$ or $\mathfrak{so}(3, 1)$ depending on the sign of H_K .

Remark 1. Let us note that instead of \vec{J} one may also choose the components of **Hamilton's vector** $\vec{K} = \vec{L} \times \vec{J}/L^2$ [11, 13]

$$K_x = p_x + \frac{A}{L} \frac{y}{r}, \quad K_y = p_y - \frac{A}{L} \frac{x}{r}, \quad K^2 = 2H_K + \frac{A^2}{L^2} \quad (9)$$

for which we have the Poisson brackets

$$\{L, K_x\} = K_y, \quad \{L, K_y\} = -K_x, \quad \{K_x, K_y\} = \frac{A^2}{L}, \quad \{H_K, \vec{K}\} = 0. \quad (10)$$

Historically (according to [13]) the first use of these first integrals was made by J. Hermann (= J. Ermanno) in 1710 (in order to find *all* possible orbits under an inverse square law force) in the disguise of **Ermanno–Bernoulli constants**

$$J_{\pm} = J_x \pm iJ_y = \left(\frac{L^2}{r} - A \mp iLp_r \right) e^{\pm i\theta} \tag{11}$$

which satisfy

$$\{H_K, J_{\pm}\} = 0, \quad \{L, J_{\pm}\} = \pm iJ_{\pm}, \quad \{J_+, J_-\} = -4iH_KL. \tag{12}$$

The Kepler problem is among the integrable models having some ‘extra’ independent conserved quantities. It is intriguing whether there are some remnants of these first integrals in the Manev model. Here we report that the answer is ‘yes’.

2. The Manev Problem Invariants

In order to obtain the equation for the trajectories of the Manev model in the case of non-vanishing angular momentum we note that due to $\dot{\theta} = L/r^2$ we have $dt = \frac{r^2}{L}d\theta$. As a result the equation for the radial motion takes the form

$$\frac{d^2}{d\theta^2} \frac{L^2}{r} + \frac{\ell^2 - 2B}{\ell^2} \frac{L^2}{r} - A = 0. \tag{13}$$

Denoting

$$\nu^2 = \frac{\ell^2 - 2B}{\ell^2}, \quad w = \frac{L^2}{r} - \frac{\ell^2}{\ell^2 - 2B}A \tag{14}$$

we obtain an oscillator-type equation

$$\frac{d^2}{d\theta^2}w + \nu^2w = 0 \tag{15}$$

from which one could easily read off the properties of the trajectories [5].

In an effort to obtain new invariants we look for invariant (complex) polarizations. Following [3] we seek functions on the phase space F and G such that

$$\{H, F\} = fF, \quad \{H, G\} = -fG \tag{16}$$

and hence

$$\{H, FG\} = 0. \tag{17}$$

In order to apply this procedure we note that when $\nu \neq 0$ and $\ell^2 > 2B$

$$\frac{d}{d\theta} \left(\nu w \pm i \frac{d}{d\theta}w \right) = \mp i\nu \left(\nu w \pm i \frac{d}{d\theta}w \right). \tag{18}$$

Therefore $(\nu w - i \frac{d}{d\theta}w)$ is a candidate for F and then G should be $e^{-i\nu\theta}$. Hence

$$\frac{d}{d\theta} \left[\left(\nu w \pm i \frac{d}{d\theta}w \right) e^{\pm i\nu\theta} \right] = 0 \tag{19}$$

and since $\frac{d}{d\theta} = \frac{r^2}{L} \frac{d}{dt}$ we obtain

$$\frac{d}{dt} \left[\left(\nu \frac{L^2}{r} - \frac{A}{\nu} \mp iLp_r \right) e^{\pm i\nu\theta} \right] = 0. \tag{20}$$

Remark 2. *In the special case of circular Kepler orbits the Laplace–Runge–Lenz vector is not well-defined ($J_{\pm} = 0$) and therefore our procedure for obtaining new invariants seems to degenerate as $\pm i \frac{d}{d\theta} w = 0$. However in this case we have the obvious invariant w , and hence r as well.*

In the case when $\ell \neq 0$, $\ell^2 > 2B$, $H < 0$ and $A > 0$ the motion is on a 2-dimensional torus. In order to have globally defined constants of motion in this case we have to require that ν be rational, i.e.

$$\nu = \sqrt{\ell^2 - 2B} : \ell = m : k \tag{21}$$

with m and k integers. Then due to equation (19)

$$\mathcal{J}_{\pm} = \mathcal{J}_{\mp}^* = \left[\frac{m}{k} \frac{L^2}{r} - \frac{k}{m} A \mp iLp_r \right] e^{\pm im\theta/k} \tag{22}$$

are conserved by the flow of equation (1) on a surface $L = \ell$ satisfying the rationality condition (21). Thus we have conditional constants of motion which exist only for disjoint but infinite set of values ℓ (c.f. invariant relations of [14]).

Obviously, if $B = 0$ then $\nu = 1$ or $m : k = 1 : 1$, $\mathcal{J}_{\pm} = J_{\pm}$ and we recover the ‘Ermanno–Bernoulli’ constants. To make the comparison with the Kepler case more transparent we note that the radial/angular components of corresponding conserved vectors take the form

$$J_r = \frac{L^2}{r} - A, \quad J_{\theta} = -Lp_r \tag{23}$$

while $\mathcal{J}_r + i\mathcal{J}_{\theta} = \left(\nu \frac{L^2}{r} - \frac{A}{\nu} - iLp_r \right) e^{i(\nu-1)\theta}$ and hence

$$\mathcal{J}_r = \left(\nu \frac{L^2}{r} - \frac{A}{\nu} \right) \cos(\nu - 1)\theta + Lp_r \sin(\nu - 1)\theta \tag{24}$$

$$\mathcal{J}_{\theta} = -Lp_r \cos(\nu - 1)\theta + \left(\nu \frac{L^2}{r} - \frac{A}{\nu} \right) \sin(\nu - 1)\theta. \tag{25}$$

Also equation (6) is to be compared with

$$\mathcal{J}^2 = \mathcal{J}_x^2 + \mathcal{J}_y^2 = \mathcal{J}_+ \mathcal{J}_- = L^2 \left(2H - \frac{2B}{r^2} \left(\frac{L^2}{\ell^2} - 1 \right) + \frac{A^2}{\nu^2 L^2} \right). \tag{26}$$

where $H < 0$ and $A > 0$, ν is assumed to be the rational m/k .

Turning to the algebraic properties of the new invariants one finds that the Poisson brackets between the real and imaginary parts of $\mathcal{J}_{\pm} = \mathcal{J}_0 \mp i\mathcal{J}_1 = \mathcal{J}_x \mp i\mathcal{J}_y$ are

$$\begin{aligned} \{H, \mathcal{J}_{0,1}\} &= 0, & \{L, \mathcal{J}_0\} &= \frac{m}{k}\mathcal{J}_1, & \{L, \mathcal{J}_1\} &= -\frac{m}{k}\mathcal{J}_0 \\ \{\mathcal{J}_0, \mathcal{J}_1\} &= -\frac{m}{k}L \left[2H - \frac{2B}{r^2} \left(\frac{2L^2}{\ell^2} - 1 \right) \right]. \end{aligned} \tag{27}$$

Here we have a $1/r^2$ term which seems to obstruct the Poisson brackets to form a closed algebra. Fortunately, redefining $\mathcal{E}_{0,1} = \mathcal{J}_{0,1}/L\sqrt{L^2 - \ell^2}$ we obtain the closed algebra $\mathfrak{g}_{H,L}$

$$\begin{aligned} \{H, \mathcal{E}_{0,1}\} &= 0, & \{L, \mathcal{E}_0\} &= \frac{m}{k}\mathcal{E}_1, & \{L, \mathcal{E}_1\} &= -\frac{m}{k}\mathcal{E}_0 \\ \{\mathcal{E}_0, \mathcal{E}_1\} &= \frac{m}{k} \frac{1}{(L^2 - \ell^2)^2} \left[2HL + \frac{k^2}{m^2} \frac{A^2}{L^3} \frac{2L^2 - \ell^2}{L^2 - \ell^2} \right] \end{aligned} \tag{28}$$

in which H is a central element and L, \mathcal{E}_0 and \mathcal{E}_1 can be viewed as Cartan and root-vector generators. Due to (28) $\mathfrak{g}_{H,L}$ is a deformation of $\mathfrak{gl}(2)$. Of course we have in addition the $\mathfrak{so}(3)$ algebra (3).

Similarly, in the case when $\ell^2 < 2B$ we may denote $\frac{2B-\ell^2}{\ell^2} = v^2$ with v real and

$$\mathcal{E}_{\pm} = \left[v \frac{L^2}{r} + \frac{A}{v} \mp Lp_r \right] \frac{e^{\mp v\theta}}{L\sqrt{L^2 - \ell^2}} \tag{29}$$

are first integrals for any ℓ and they satisfy

$$\begin{aligned} \{H, \mathcal{E}_{\pm}\} &= 0, & \{L, \mathcal{E}_{\pm}\} &= \mp v\mathcal{E}_{\pm} \\ \{\mathcal{E}_+, \mathcal{E}_-\} &= \frac{2v}{(L^2 - \ell^2)^2} \left[2HL - \frac{A^2}{v^2 L^3} \frac{2L^2 - \ell^2}{L^2 - \ell^2} \right]. \end{aligned} \tag{30}$$

The algebra $\mathfrak{g}'_{H,L}$ satisfied by H, L and \mathcal{E}_{\pm} is quite analogous to $\mathfrak{g}_{H,L}$ but with a different function at the right hand side of the bracket $\{\mathcal{E}_+, \mathcal{E}_-\}$.

When $\ell^2 = 2B$ we have the first integral

$$j = Lp_r + A\theta \tag{31}$$

satisfying $\{H, j\} = 0, \{L, j\} = A$.

Recently [16], it was shown that Kepler dynamics admits a second Hamiltonian description and consequently, a recursion operator can be constructed for it. Following the same route one can note that the action variables for the case when $\ell^2 > 2B, H < 0$ and $A > 0$ are $I_{\theta} = L$ and $I_r = -\frac{m}{k}L + \frac{A}{\sqrt{-2H}}$, so the

dynamical vector field Γ is Hamiltonian with respect to

$$H = \frac{-A^2}{2\left(I_r + \frac{m}{k}L\right)^2}, \quad \omega = dI_r \wedge d\theta_r + dI_\theta \wedge d\theta \equiv \sum_j dI_j \wedge d\theta_j \quad (32)$$

where θ_r is the angle coordinate of the Liouville tori conjugate to I_r . Following [16] one can show that Γ is a Hamiltonian vector field also with respect to H_1 and the symplectic form ω_1

$$H_1 = \frac{-A^2}{\left(I_r + \frac{m}{k}L\right)}, \quad \omega_1 = \sum_{j,k} S_{jk} dI_j \wedge d\theta_k \quad (33)$$

with the matrix

$$S = \begin{pmatrix} I_r & \frac{m^2}{k^2}L \\ \frac{2m}{k}L & \frac{2m}{k}I_r \end{pmatrix} \quad (34)$$

which is *not* a transformation jacobian. We can now define an invariant mixed tensor field

$$T = \omega_1 \circ \omega^{-1} = \sum_{j,k} \left(S_{jk} dI_j \otimes \frac{\partial}{\partial I_k} + (S^\dagger)_{jk} d\theta_j \otimes \frac{\partial}{\partial \theta_k} \right) \quad (35)$$

and it is easy to check that its Nijenhuis torsion vanishes and its eigenvalues are doubly degenerated, so it is a recursion operator. The arguments in [16] that due to the Energy–Period Theorem [9] we could not expect a recursion operator to produce new functionally independent constants of motion, are valid here as well.

3. Real Form Dynamics

Here we briefly recall the notion of real form (RF) dynamics referring the reader to [8] for more details and a list of examples.

We start from a standard (real) Hamiltonian system $\mathcal{H} \equiv \{\mathcal{M}, \omega, H\}$ with n degrees of freedom and at the present stage we assume that our phase space is just a vector space $\mathcal{M} = \mathbb{R}^{2n}$.

Let's consider its complexification $\mathcal{H}^{\mathbb{C}} \equiv \{\mathcal{M}^{\mathbb{C}}, H^{\mathbb{C}}, \omega^{\mathbb{C}}\}$ where $\mathcal{M}^{\mathbb{C}}$ can be viewed as a linear space over the field of complex numbers

$$\mathcal{M}^{\mathbb{C}} = \mathcal{M} \oplus i\mathcal{M}.$$

In other words the dynamical variables in $\mathcal{M}^{\mathbb{C}}$ now take complex values. We assume that the Hamiltonian H (as well as all other possible first integrals in involution I_k) are *real analytic functions* on \mathcal{M} which can naturally be extended to $\mathcal{M}^{\mathbb{C}}$. We introduce on the phase space \mathcal{M} an involutive, **symplectic automorphism** $\mathcal{C} : \mathcal{M} \rightarrow \mathcal{M}$:

$$\mathcal{C}^2 = \mathbb{1}, \quad \mathcal{C}(\{F, G\}) = \{\mathcal{C}(F), \mathcal{C}(G)\} \quad (36)$$

where with some abuse of terminology we use the same notation for the action of \mathcal{C} on the dual of the phase space.

Since \mathcal{C} has eigenvalues 1 and -1 , it naturally splits \mathcal{M} into two eigenspaces

$$\mathcal{M} = \mathcal{M}_+ \oplus \mathcal{M}_- \quad (37)$$

whose dimensions need not be equal. Due to the fact that \mathcal{C} is symplectic \mathcal{M}_- and \mathcal{M}_+ are symplectic subspaces of \mathcal{M} and we will write

$$\omega = \omega_+ \oplus \omega_-.$$

Assuming a symplectic frame adapted to \mathcal{C} we have

$$\omega = \sum_{k=1}^{n_+} dp_{k+} \wedge dq_{k+} + \sum_{k=1}^{n_-} dp_{k-} \wedge dq_{k-}.$$

The automorphism \mathcal{C} can naturally be extended to $\mathcal{M}^{\mathbb{C}}$ and it splits it again into a direct sum of two eigenspaces

$$\mathcal{M}^{\mathbb{C}} = \mathcal{M}_-^{\mathbb{C}} \oplus \mathcal{M}_+^{\mathbb{C}}.$$

Similarly, the action of the complex conjugation $*$ produces splitting into real and imaginary parts of the corresponding spaces. By construction \mathcal{C} commutes with $*$ and their composition $\tilde{\mathcal{C}} \equiv \mathcal{C} \circ * = * \circ \mathcal{C}$ is also an involutive symplectic automorphism on $\mathcal{M}^{\mathbb{C}}$; then we define $\mathcal{M}_{\mathbb{R}}$ to be the fixed point set of $\tilde{\mathcal{C}}$ i.e.

$$\mathcal{M}_{\mathbb{R}} = \text{Re } \mathcal{M}_+^{\mathbb{C}} \oplus i \text{Im } \mathcal{M}_-^{\mathbb{C}}$$

and it is again a symplectic subspace. From now on we will be interested in dynamics on $\mathcal{M}_{\mathbb{R}}$ and its connection to the initial real dynamical system.

In order to construct ‘real form dynamics’ we shall assume that the Hamiltonian is \mathcal{C} -invariant, i.e.

$$\mathcal{C}(H) = H. \quad (38)$$

Then the Hamiltonian on the complexified phase space $H^{\mathbb{C}}$ (being the same analytical function of the complexified variables) will share this property.

The **real form dynamics** may be defined either as:

- i) complexified Hamilton equations on $\mathcal{M}^{\mathbb{C}}$ being consistently restricted to $\mathcal{M}_{\mathbb{R}}$. This gives a real vector field tangent to $\mathcal{M}_{\mathbb{R}}$ and satisfying the equations of motion given by the real part of $H^{\mathbb{C}}$.

or

ii) dynamics on $\mathcal{M}_{\mathbb{R}}$ defined by the restricted $H^{\mathbb{C}}$ and $\omega^{\mathbb{C}}$ (whose restrictions are real on $\mathcal{M}_{\mathbb{R}}$)

$$\begin{aligned} H|_{\mathcal{M}_{\mathbb{R}}} &= \frac{H + \tilde{\mathcal{C}}(H)}{2} = \frac{H + \mathcal{C}(H)^*}{2} = \text{Re } H^{\mathbb{C}} \\ \omega^{\mathbb{C}}|_{\mathcal{M}_{\mathbb{R}}} &= d \text{Re } p_+^{\mathbb{C}} \wedge d \text{Re } q_+^{\mathbb{C}} - d \text{Im } p_-^{\mathbb{C}} \wedge d \text{Im } q_-^{\mathbb{C}}. \end{aligned} \tag{39}$$

Now we have a well defined dynamical system $\mathcal{H}_{\mathbb{R}} = \{\mathcal{M}_{\mathbb{R}}, \omega|_{\mathcal{M}_{\mathbb{R}}}, H|_{\mathcal{M}_{\mathbb{R}}}\}$ with *real Hamiltonian* and *real symplectic form* on a subspace of the complexified phase space.

The ‘real form dynamics’ corresponding to a Liouville integrable Hamiltonian system is Liouville integrable again [8]. The complexification provides us with $2n$ integrals of motion \mathcal{I}_k and \mathcal{I}_k^* which are also in involution. One can check that after restricting ourselves to $\mathcal{M}_{\mathbb{R}}$ exactly n of the integrals are preserved and mutually commute

$$I_{k,\mathbb{R}} = \frac{1}{2} \left(\mathcal{I}_k + \tilde{\mathcal{C}}(\mathcal{I}_k) \right) \Big|_{\mathcal{M}_{\mathbb{R}}}$$

and they are those invariant with respect to $\tilde{\mathcal{C}}$. As a result, those initial I_j which have definite \mathcal{C} -parity will produce real first integrals for the real form dynamics and those with minus \mathcal{C} -parity will produce purely imaginary integrals.

Similarly, the ‘real form dynamics’ corresponding to a superintegrable Hamiltonian system is superintegrable again. In such case we have $\kappa \in [n + 1, 2n - 1]$ independent constants of motion which are no more in involution. It could easily be checked that they will again produce κ independent constants of motion of the RF dynamics.

4. Real Form Dynamics of the Manev Problem

The Manev Hamiltonian (and the canonical symplectic form as well) is invariant under the involution \mathcal{C} reflecting the y -degree of freedom:

$$\begin{aligned} \mathcal{C}(x) &= x, & \mathcal{C}(y) &= -y, & \mathcal{C}(z) &= z \\ \mathcal{C}(p_x) &= p_x, & \mathcal{C}(p_y) &= -p_y, & \mathcal{C}(p_z) &= p_z. \end{aligned} \tag{40}$$

Consequently, the ‘real form dynamics’ of Manev model for this choice of involution will be given by

$$\begin{aligned} H_{\mathbb{R}} &= \frac{1}{2}(p_x^2 - p_x^2 + p_z^2) - \frac{A}{\rho} - \frac{B^2}{\rho} \\ \omega_{\mathbb{R}} &= dp_x \wedge dx - dp_y \wedge dy + dp_z \wedge dz \end{aligned} \tag{41}$$

where $\rho = \sqrt{x^2 - y^2 + z^2}$. This is not an ordinary central field dynamics but rather an ‘indefinite metric central field’ as $H_{\mathbb{R}}$ depends on indefinite metric distance ρ . The real form Hamiltonian $H_{\mathbb{R}}$ and the appropriate ‘angular momentum’

\tilde{L}_j are still commuting first integrals and the model is integrable. The involution acts on \tilde{L}_j according to $\mathcal{C}(\tilde{L}_j) = (-1)^j \tilde{L}_j$ and

$$\{\tilde{L}_j, \tilde{L}_k\} = \varepsilon_{jki} (-1)^{j+k+1} \tilde{L}_i \tag{42}$$

instead of equation (3); the corresponding algebra is $\mathfrak{so}(2, 1)$ which is the real form of $\mathfrak{so}(3)$ obtained with a \mathcal{C} -induced Cartan involution.

We shall assume again that the motion is on the Oxy -plane and in order to avoid the question of the behavior of trajectories on the singularities we restrict our attention on the \mathcal{C} -invariant configuration space

$$\{(x, y, z) \in \mathbb{R}^2; z = 0, x^2 > y^2, x > 0\}.$$

Then dynamics is separable in pseudo-radial coordinates $\vartheta = \operatorname{arctanh}(y/x) \in (-\infty, \infty)$ and $\rho \in (0, \infty)$

$$H = \frac{1}{2} \left(p_\rho^2 - \frac{\pi_\vartheta^2}{\rho^2} \right) - \frac{A}{\rho} - \frac{B}{\rho^2} \tag{43}$$

$$\omega = dp_\rho \wedge d\rho + d\pi_\vartheta \wedge d\vartheta$$

with $\tilde{L} \equiv \tilde{L}_z = \pi_\vartheta$, hence $\pi_\vartheta = 0$ and $\dot{\vartheta} = -\tilde{L}/\rho^2$. Due to the different symplectic form \tilde{L} generates now transformations which preserve ρ .

The type of the ρ -trajectories could be easily read off after the observation that the value of the real form Hamiltonian will be

$$h = \frac{1}{2} \left(\dot{\rho}^2 - \frac{\ell^2}{\rho^2} \right) - \frac{A}{\rho} - \frac{B}{\rho^2}$$

due to $\dot{x}^2 - \dot{y}^2 = \dot{\rho}^2 - \ell^2/\rho^2$ and denoting the value of \tilde{L} by ℓ . Introducing $v = \dot{\rho}$ and $u = 1/\rho$ we obtain an equation describing conics in the (u, v) -space

$$u^2(\ell^2 + 2B) + 2Au + (2h - v^2) = 0.$$

Note that due to the ‘real form dynamics’ the sign in front of ℓ^2 is plus – in the standard Manev model there would be minus in the corresponding equation for $1/r$ and \dot{r} , c.f. [18]. Performing the same type of analysis as in [18] we may conclude that we may have three types of qualitatively different dynamical regimes:

- for $\ell^2 + 2B > 0$ we have a family of hyperbolas.
- for $\ell^2 + 2B = 0$ we have a family of parabolas for $A \neq 0$ which degenerate at $A = 0$ into pair of lines parallel to the $1/\rho$ -axis.
- for $\ell^2 + 2B < 0$ (only possible for repulsive Manev term) we have a family of ellipses.

Of course, in all these cases we have to exclude the region $u < 0$.

In order to obtain more specific information about the motion we will need an equation for the trajectories. Let’s note again that in the case of non-vanishing

angular momentum we have $dt = -\frac{\ell^2}{\tilde{L}}d\vartheta$. As a result the equation for the ρ -motion takes the form

$$\frac{d^2}{d\vartheta^2} \frac{\tilde{L}^2}{\rho} - \frac{\ell^2 + 2B}{\ell^2} \frac{\tilde{L}^2}{\rho} - A = 0. \quad (44)$$

Assuming $\ell^2 + 2B \neq 0$ we introduce

$$v^2 = \frac{\ell^2 + 2B}{\ell^2}, \quad w = \frac{\tilde{L}^2}{\rho} + \frac{\ell^2}{\ell^2 + 2B} A \quad (45)$$

and obtain an inverted oscillator-type equation

$$\frac{d^2}{d\vartheta^2} w - v^2 w = 0. \quad (46)$$

Denoting by c_j the integration constants below we conclude that:

- If $\ell^2 + 2B > 0$, the solution $\rho^{-1}(\vartheta)$ is

$$\frac{\tilde{L}^2}{\rho} = c_1 \cosh(v\vartheta) + c_2 \sinh(v\vartheta) - \frac{A}{v^2}. \quad (47)$$

Trajectories may collapse ($\rho \rightarrow 0$) for $\vartheta \rightarrow \pm\infty$ and $c_1 > c_2 > 0$, or ρ may tend to ∞ as ϑ tends to certain values ϑ_{min} and ϑ_{max} .

- If $\ell^2 + 2B = 0$, the solution of equation (44) is

$$\rho^{-1} = \frac{A}{2\tilde{L}^2} \vartheta^2 + c_3 \vartheta + c_4. \quad (48)$$

If $A > 0$, trajectories collapse for $\vartheta \rightarrow \pm\infty$ and if $A < 0$ then $\rho \rightarrow \infty$ as ϑ tends to some ϑ_{min} and ϑ_{max} . The case $A = 0$ leads to linear solution $\rho^{-1}(\vartheta)$ and corresponds either to motion along fixed ρ or to trajectory starting at $\rho = \infty$ and some value of ϑ and collapsing for $\vartheta \rightarrow \infty$ (or its reverse).

- If $\ell^2 + 2B < 0$, we have a solution which oscillates harmonically between some values ρ_{min} and ρ_{max}

$$\frac{\tilde{L}^2}{\rho} = c_5 \cos(v\vartheta) + c_6 \sin(v\vartheta) - \frac{A}{v^2}. \quad (49)$$

The case when $\rho_{min} < 0$ means that ‘acceptable’ motions will be trajectories coming from $\rho = \infty$ at $\vartheta = \vartheta_{min}$ and going to $\rho = \infty$ for some $\vartheta = \vartheta_{ax}$.

In the special case of vanishing angular momentum we have 1-dimensional motion along the ray $\vartheta = \text{const}$. It may be oscillating between some ρ_{min} and ρ_{max} or heading to collapse, or to infinity.

It is worth noting that the only trajectories which are compact in the (x, y) -space are those collapsing at both their ends in the origin tangentially to the boundaries

$x = \pm y$ and those oscillating on a line interval of $\vartheta = \text{const}$. This is to be contrasted to the standard Manev or Kepler problems.

Obviously, when $B = 0$ we will obtain a real form dynamics of the Kepler model. In this case we have even fewer possibilities for compact trajectories as we could not have oscillating along the line of $\vartheta = \text{const}$.

In order to determine the first integrals when $\ell^2 > -2B$ and proceeding as in Section 2 we obtain that

$$\mathcal{J}_{\pm} = \left[v \frac{\tilde{L}^2}{\rho} + \frac{A}{v} \pm \tilde{L} p_{\rho} \right] e^{\mp v \vartheta}. \tag{50}$$

As before we can introduce the renormalized quantities $\mathcal{E}_{\pm} = \mathcal{J}_{\pm} / \tilde{L} \sqrt{\tilde{L}^2 - \ell^2}$ and derive for them the following symmetry algebra $\mathfrak{g}'_{H, \tilde{L}}$

$$\begin{aligned} \{H_{\mathbb{R}}, \mathcal{E}_{\pm}\} &= 0, & \{\tilde{L}, \mathcal{E}_{\pm}\} &= \mp v \mathcal{E}_{\pm} \\ \{\mathcal{E}_{+}, \mathcal{E}_{-}\} &= \frac{2v}{(\tilde{L}^2 - \ell^2)^2} \left[2H_{\mathbb{R}} \tilde{L} - \frac{A^2}{v^2 \tilde{L}^3} \frac{2\tilde{L}^2 - \ell^2}{\tilde{L}^2 - \ell^2} \right]. \end{aligned} \tag{51}$$

Like in (28) above $\mathfrak{g}'_{H, \tilde{L}}$ is a deformation of $\mathfrak{gl}(2)$ having the same H, \tilde{L} dependence in the right hand side of (51), though L and \tilde{L} have different properties.

Note that the algebras $\mathfrak{g}_{H, L}$ and $\mathfrak{g}'_{H, \tilde{L}}$ seem very close, i.e. they do not change effectively when passing from one real Hamiltonian form to the other. The reason for this is the fact, that all its generators are invariant with respect to the involution \mathcal{C} . The situation changes when we consider the algebra satisfied by \tilde{L}_j , see equation (42).

In the case when $\ell^2 < -2B$ let $\nu^2 = \frac{-(\ell^2 + 2B)}{\ell^2}$ and then invariants take the form

$$\mathcal{J}_{\pm} = \mathcal{J}_{\mp}^* = \mathcal{J}_0 \mp i \mathcal{J}_1 = \left[\nu \frac{\tilde{L}^2}{\rho} - \frac{A}{\nu} \mp i \tilde{L} p_{\rho} \right] e^{\mp i \nu \vartheta} \tag{52}$$

for any ℓ . Redefining again $\mathcal{E}_{0,1} = \mathcal{J}_{0,1} / \tilde{L} \sqrt{\tilde{L}^2 - \ell^2}$ we obtain the brackets

$$\begin{aligned} \{H_{\mathbb{R}}, \mathcal{E}_{0,1}\} &= 0, & \{\tilde{L}, \mathcal{E}_0\} &= -\nu \mathcal{E}_1, & \{\tilde{L}, \mathcal{E}_1\} &= \nu \mathcal{E}_0 \\ \{\mathcal{E}_0, \mathcal{E}_1\} &= \frac{-\nu}{(\tilde{L}^2 - \ell^2)^2} \left[2H_{\mathbb{R}} \tilde{L} + \frac{A^2}{\nu^2 \tilde{L}^3} \frac{2\tilde{L}^2 - \ell^2}{\tilde{L}^2 - \ell^2} \right]. \end{aligned} \tag{53}$$

When $\ell^2 = 2B$ we have the first integral

$$j = \tilde{L} p_{\rho} - A \vartheta \tag{54}$$

satisfying $\{H, j\} = 0, \{\tilde{L}, j\} = -A$.

5. Conclusions

The existence of Ermanno–Bernoulli type invariants strengthens our belief that Manev model has an exceptional position among the central field theories. Not only it provides a better description of the real motion of the heavenly bodies than Kepler model but to a large extent it shares its superintegrability, one of its most celebrated mathematical features. As a result it provides also a testground for analysing the intricate interplay between integrability and superintegrability, having the advantage of being realistic.

Also, from the viewpoint of RF dynamics enthusiasts we see here a curious (and encouraging) example when the RF dynamics – exotic as it may be – behaves ‘better’ than the original problem remaining always superintegrable.

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