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A MODULI SPACE OF MINIMAL ANNULI

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Abstract. In this paper, we will report a recent study about moduli spaces of branched and complete minimal surfaces in Euclidean space of genus one with two ends and total curvature -4π.

1. Introduction

The purpose of this paper is to report recent results on moduli spaces of certain minimal surfaces and their geometric properties [5].

A complete conformal minimal immersion X from an open Riemann surface M to the Euclidean space $\mathbb{R}^n$ of finite total curvature is an immersion such that M can be compactified conformally. Moduli spaces of these minimal immersions have been studied from several viewpoints (cf. Pérez and Ros [7, 8], Ros [11], Yang [12], Kusner and Schmitt [2], Pirola [9, 10], and Moriya [3]).

The Riemann surface M can also be compactified conformally in the case where $X\colon M \to \mathbb{R}^3/T(v)$ is a branched complete conformal minimal immersion of finite total curvature, where $T(v)$ is the discrete group of isometries generated by a translation by $v \in \mathbb{R}^3$ (Lemma 2.1). We will call a branched complete conformal minimal immersion $X\colon M \to \mathbb{R}^3/T(v)$ of finite total curvature a minimal surface of algebraic type, or simply, an algebraic minimal surface.

Xiaokang Mo studied a moduli space of Weierstrass data for algebraic minimal surfaces in $\mathbb{R}^3$ in terms of **divisor spaces** and Kichoon Yang introduced it in his book [12].

We will call an algebraic minimal surface of genus 0 with two puncture points an algebraic minimal annulus. In Moriya [4], an example of a moduli space of Weierstrass data for algebraic minimal annuli is investigated in terms of divisor spaces and the defining equations of the moduli space is obtained.

In the paper [5], we discussed concrete examples of moduli spaces of Weierstrass data for algebraic minimal annuli in terms of linear systems.

Let us denote by P a pair of integers (P_0, P_∞) which belongs to the set $I := \{(3,3),(2,3),(3,2),(1,3),(2,2),(3,1)\}$. We will denote by $\mathcal{W}(P,v)$ the set of Weierstrass data for algebraic minimal annuli from $\mathbb{C}\setminus\{0\}$ to $\mathbb{R}^3/T(v)$ satisfying the following conditions:

1. The Gauss map $g(z) = z$.
2. The order of the puncture point 0 and that of the puncture point ∞ are equal to P_0 and P_∞ respectively.

We will define the sets of Weierstrass data $\mathcal{W}(v)$ and $\mathcal{W}$ by

$$\mathcal{W}(v) := \bigcup_{P\in I} \mathcal{W}(P,v)\,, \qquad \mathcal{W} := \bigcup_{P\in I}\bigcup_{v\in\mathbb{R}^3} \mathcal{W}(P,v).$$

Our main result is that $\mathcal{W}$ has a structure of a 3-dimensional smooth manifold with a Hermitian metric satisfying the following conditions:

1. The scalar curvature is a positive constant.
2. Each $\mathcal{W}(v)$ $(v\in\mathbb{R}^3)$ is a totally real submanifold in $\mathcal{W}$.
3. A point in $\mathcal{W}$ is a helicoid if and only if it is a point in $\mathcal{W}((0,0,v_3))$ $(v_3 \neq 0)$ which attains the maximal value of the scalar curvature of $\mathcal{W}((0,0,v_3))$ with respect to the metric induced from the Hermitian metric on $\mathcal{W}$.
4. The curve in $\mathcal{W}$ which corresponds to an associated family of a minimal annulus in $\mathcal{W}$ is a geodesic.

2. A Classification of Weierstrass Data

In this section, we will overview the minimal surface theory briefly and classify Weierstrass data for algebraic minimal annuli by the orders of the branch points and those of the puncture points. For more details, see Hoffman and Karcher [1], Moriya [3,4], and Yang [12].

For a divisor D on a compact Riemann surface $\bar{M}$, we will denote by $\operatorname{mult}_p D$ the multiplicity of D at p and by $\operatorname{supp} D$ the support of D:

$$\operatorname{supp} D := \{p\in\bar{M};\ \operatorname{mult}_p D \neq 0\}\,.$$

Let us define two nonnegative divisors D_+ and D_- on $\bar{M}$ by

$$D_+ := \sum_{p\in\bar{M}} \max\{\operatorname{mult}_p D, 0\}\cdot p\,,$$
$$D_- := \sum_{p\in\bar{M}} \max\{-\operatorname{mult}_p D, 0\}\cdot p\,.$$

Then, we can see that $D = D_+ - D_-$.

For a triplet $\zeta = (\zeta_1, \zeta_2, \zeta_3)$ of meromorphic one-forms on $\bar{M}$, we will define the divisor (ζ) of ζ by

$$(\zeta) := \sum_{p \in \bar{M}} \left(\min_{i=1,2,3} \mathrm{ord}_p \, \zeta_i \right) \cdot p \,,$$

where $\mathrm{ord}_p \, \zeta_i$ is the order of ζ_i at p $(i = 1, 2, 3)$.

Let $X \colon M \to \mathbb{R}^3/T(v)$ be a branched complete conformal minimal immersion. When $v \neq (0, 0, 0)$, we will assume that there exists a branched complete conformal minimal immersion $\tilde{X} \colon \tilde{M} \to \mathbb{R}^3$, where $\pi \colon \tilde{M} \to M$ is a holomorphic covering, $\tilde{M}$ connected, and $\Pi \colon \mathbb{R}^3 \to \mathbb{R}^3/T(v)$ the natural covering such that $\tilde{X} \circ \Pi = \pi \circ X$. We will identify X with $\tilde{X}$.

We will denote by Ψ_i the holomorphic one-form on M defined by $\Psi_i := (\partial X_i / \partial z) \, \mathrm{d}z$, $(i = 1, 2, 3)$, where z is a holomorphic coordinate on M. We will modify the Chern-Osserman theorem as follows:

Lemma 2.1. *Let $X \colon M \to \mathbb{R}^3/T(v)$ be a branched conformal minimal immersion. Then the following two conditions are equivalent:*

i) *The Riemann surface M is a certain compact Riemann surface $\bar{M}$ with finitely many points $\{p_1, \ldots, p_r\}$ removed, M is complete, and X is of finite total curvature.*
ii) *There exist a compact Riemann surface $\bar{M}$ and a triplet $\bar{\Psi} = (\bar{\Psi}_1, \bar{\Psi}_2, \bar{\Psi}_3)$ of meromorphic one-forms on $\bar{M}$ such that $\mathrm{supp}(\bar{\Psi})_- = \{p_1, \ldots, p_r\}$ and that $\bar{\Psi}_i|_M = \Psi_i$ $(i = 1, 2, 3)$.*

Definition 2.1.

1) *We will call a point p_i a **puncture point** of X, $i = 1, \ldots, r$. Assume that $(\bar{\Psi}) = \sum_{j=1}^{l} B_j \cdot b_j - \sum_{i=1}^{r} P_i \cdot p_i$ ($B_j > 0, P_i > 0$).*
2) *We will call B_j the order of a branch point b_j, $j = 1, \ldots, l$ and P_i the order of a puncture point p_i, $(i = 1, \ldots, r)$.*
3) *We will call $\tilde{B} := \sum_{j=1}^{l} B_j$ the **total order** of branch points of X and $\tilde{P} := \sum_{i=1}^{r} P_i$ the total order of puncture points of X.*

We can see that the set of points $\{b_1, \ldots, b_l\}$ coincides with the set of **branch points** of X.

Assume that $\bar{M} = \mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$ and the set of puncture points is $\{0, \infty\}$. Then we can show a representation formula as follows:

Lemma 2.2. *Let* $X\colon \mathbb{C}\setminus\{0\} \to \mathbb{R}^3/T(v)$ *be an algebraic minimal surface. Then the triplet* $\bar{\Psi} = (\bar{\Psi}_1, \bar{\Psi}_2, \bar{\Psi}_3)$ *of meromorphic one-forms satisfy the following conditions:*

$$(\bar{\Psi}_1 \otimes \bar{\Psi}_1) + (\bar{\Psi}_2 \otimes \bar{\Psi}_2) + (\bar{\Psi}_3 \otimes \bar{\Psi}_3) = 0\,, \tag{2.1}$$

$$(\bar{\Psi}) = \sum_{j=1}^{l} B_j \cdot b_j - P_0 \cdot 0 - P_\infty \cdot \infty\,, \tag{2.2}$$

$$-\,2\pi \operatorname{Im}\operatorname{Res}(\bar{\Psi}_i; 0) = v_i\,, \tag{2.3}$$

where B_j $(j = 1, \dots, l)$, $P_0 > 0$, $P_\infty > 0$ *and* $\operatorname{Res}(\bar{\Psi}_i; 0)$ *is the residue of* $\bar{\Psi}_i$ *at* 0.

Conversely, if there exists a triplet $\bar{\Psi} = (\bar{\Psi}_1, \bar{\Psi}_2, \bar{\Psi}_3)$ *of meromorphic one-forms on* $\mathbb{CP}^1$ *satisfying the conditions* (2.1), (2.2), *and* (2.3), *then we can obtain an algebraic minimal annulus* $X\colon \mathbb{C}\setminus\{0\} \to \mathbb{R}^3/T(v)$ *by integration*

$$X(z) := \operatorname{Re}\int_{z_0}^{z} \bar{\Psi}\,,$$

where $z_0 \in \mathbb{C}\setminus\{0\}$. *If we choose another base point in the integral, the image of the map shifts by a translation in* $\mathbb{R}^3$.

Definition 2.2. *We call the condition* (2.1), (2.2) *and* (2.3) *the* ***conformality condition****, the* ***divisor condition*** *and the* ***period condition*** *of* $\bar{\Psi}$ *respectively.*

A triplet $\bar{\Psi} = (\bar{\Psi}_1, \bar{\Psi}_2, \bar{\Psi}_3)$ of meromorphic one-forms on $\mathbb{CP}^1$ satisfying the conformality condition (2.1) and the divisor condition (2.2) is equivalent to a pair (g, η) consisting of a meromorphic function g on $\mathbb{CP}^1$ and a meromorphic one-form η on $\mathbb{CP}^1$ by the relation

$$(g, \eta) = \left(\frac{\bar{\Psi}_3}{\bar{\Psi}_1 - \mathrm{i}\bar{\Psi}_2}, \bar{\Psi}_3 \right),$$

$$(\bar{\Psi}_1, \bar{\Psi}_2, \bar{\Psi}_3) = \left(\frac{1}{g} - g, \mathrm{i}\left(\frac{1}{g} + g\right), 2 \right) \frac{\eta}{2}\,.$$

Hence, the following relation holds among $(\bar{\Psi})$, (g), and (η):

$$(\bar{\Psi}) = -(g)_+ - (g)_- + (\eta)\,.$$

Then the divisor condition (2.2) becomes

$$-\,(g)_+ - (g)_- + (\eta) = \sum_{j=1}^{l} B_j \cdot b_j - P_0 \cdot 0 - P_\infty \cdot \infty \tag{2.4}$$

where (g) is the divisor of g. Thus, a pair (g, η) corresponds to an algebraic minimal annuli if and only if (g, η) satisfies the condition (2.4) and the triplet $\bar{\Psi}$ of meromorphic one-forms on $\mathbb{CP}^1$ equivalent to (g, η) satisfies the period condition (2.3). We will call the pair (g, η) associated to an algebraic minimal annulus $X\colon \mathbb{C} \setminus \{0\} \to \mathbb{R}^3/T(v)$ **Weierstrass data** of X.

Let $\mathcal{W}(P, v)$, $\mathcal{W}(v)$, and $\mathcal{W}$ be the sets of Weierstrass data defined in Section 1. We will denote by $\mathcal{A}(P, v)$ the set of algebraic minimal annuli whose Weierstrass data belong to $\mathcal{W}(P, v)$. The notation $X \sim Y$ for X and Y in $\mathcal{A}(P, v)$ means that $X + x = Y$ for some $x \in \mathbb{R}^3$. Summarizing the above discussion, we can prove the following lemmas:

Lemma 2.3. *There exists a bijective correspondence between the set $\mathcal{A}(P, v)/{\sim}$ and the set $\mathcal{W}(P, v)$.*

Lemma 2.4. *The set $\mathcal{W}$ is equal to the set*

$$\left\{ \left(z, \frac{c_0 z^2 + (c_1/\sqrt{2}) z + c_2}{z^2} \, \mathrm{d}z \right) ; \ (c_0, c_1, c_2) \in \mathbb{C}^3 \setminus \{0\} \right\} .$$

Lemma 2.5. *The set $\mathcal{W}(v)$ consists of each element (z, η) of $\mathcal{W}$ satisfying the following conditions:*

$$\begin{aligned} &-2\pi \operatorname{Im} \operatorname{Res} \left(\left(\frac{1}{z} - z \right) \frac{\eta}{2}, 0 \right) = v_1 \, , \\ &-2\pi \operatorname{Im} \operatorname{Res} \left(\mathrm{i} \left(\frac{1}{z} + z \right) \frac{\eta}{2}, 0 \right) = v_2 \, , \\ &-2\pi \operatorname{Im} \operatorname{Res}(\eta, 0) = v_3 \, . \end{aligned} \tag{2.5}$$

Lemma 2.6. *The relations among the total order of branch points $\tilde{B}$, the order of each puncture point P_0 and P_∞, and the values of c_0, c_1, and c_2 become as Table 1, where $*$ means any complex number.*

Table 1. The relation among (P_0, P_∞), $\tilde{B}$, and (c_0, c_1, c_2)

P_0	P_∞	$\tilde{B}$	c_0	c_1	c_2
3	3	2	$\neq 0$	$*$	$\neq 0$
2	3	1	$\neq 0$	$\neq 0$	0
3	2	1	0	$\neq 0$	$\neq 0$
1	3	0	$\neq 0$	0	0
2	2	0	0	$\neq 0$	0
3	1	0	0	0	$\neq 0$

We will consider the set $\mathbb{S}^1 := \{\,\mathrm{e}^{\mathrm{i}t}\,;\, t \in \mathbb{R}\}$ as a Lie group. Let $\mu\colon \mathcal{W}{\times}\mathbb{S}^1 \to \mathcal{W}$ be the map defined by

$$\mu((z,\eta),\, \mathrm{e}^{\mathrm{i}t}) = (z, (\,\mathrm{e}^{\mathrm{i}t})\eta)\,.$$

Then, μ is an action of $\mathbb{S}^1$ on $\mathcal{W}$. This action is transitive and effective. The orbit of an element $(z,\eta) \in \mathcal{W}$ is the set of Weierstrass data for the associated family of a minimal annulus produced by η.

Definition 2.3. *We will call the orbit of* $(z,\eta) \in \mathcal{W}$ *by* μ *the* ***associated orbit*** *of* (z,η).

We will recall that an unbranched complete minimal annulus in $\mathbb{R}^3$ with total curvature -4π is a catenoid (Osserman [6]) and a helicoid is the conjugate surface of a catenoid. Then we can prove the following lemmas:

Lemma 2.7. *A point* $(z,\eta) \in \mathcal{W}((0,0,0))$ *corresponds to a catenoid if and only if* $\eta = r\,\mathrm{d}z/z$ *(*$r \in \mathbb{R}, r \neq 0$*).*

Lemma 2.8. *A point* $(z,\eta) \in \mathcal{W}((0,0,v_3))$ $(v_3 \neq 0)$ *corresponds to a helicoid if and only if* $\eta = -\mathrm{i}v_3\,\mathrm{d}z/z$.

3. Geometry of Moduli Spaces

In this section, we will describe geometric properties of moduli spaces $\mathcal{W}$, $\mathcal{W}(v)$, and $\mathcal{W}(P)$.

We can consider $\mathcal{W}$ as $\mathbb{C}^3 \setminus \{0\}$ with holomorphic coordinates (c_0, c_1, c_2). Let us denote by J_0 the complex structure. We will define real coordinates $(u_1,\ldots,u_6)$ on $\mathcal{W}$ by $u_1 := \operatorname{Re} c_0$, $u_2 := \operatorname{Im} c_0$, $u_3 := \operatorname{Re} c_1$, $u_4 := \operatorname{Im} c_1$, $u_5 := \operatorname{Re} c_2$, and $u_6 := \operatorname{Im} c_2$.

Theorem 3.1. *The set* $\mathcal{W}(v)$ *is a 3-dimensional connected real algebraic smooth submanifold of* $\mathcal{W}$ *defined by*

$$\mathcal{W}(v) = \{(u_1, u_2, u_3, -\sqrt{2}v_3, -u_1 - 2v_2, u_2 + 2v_1) \in \mathcal{W}\}\,. \tag{3.1}$$

Let $\mathcal{W}(P)$ be the subset of $\mathcal{W}$ defined by $\mathcal{W}(P) := \bigcup_{v\in\mathbb{R}^3} \mathcal{W}(P,v)$, where $P = (P_0, P_\infty) \in I$.

Theorem 3.2. *The set* $\mathcal{W}(P)$ *is a* $(\tilde{B}+1)$*-dimensional algebraic complex submanifold of* $(\mathcal{W}, J_0)$ *where* $\tilde{B}$ *is the total order of branch points corresponding to* P *as in Lemma 2.8.*

Let $\rho = (\rho_1, \dots, \rho_7)\colon \mathcal{W} \to \mathbb{R}^7$ be an immersion such that

$$\rho_\alpha(u_1, \dots, u_6) = \frac{u_\alpha}{u} \qquad (\alpha = 1, \dots, 6),\ \rho_7(u_1, \dots, u_6) = \log u,$$

where $u = (\sum_{\alpha=1}^{6} u_\alpha^2)^{1/2}$. Then, $\rho(\mathcal{W})$ is a high-dimensional cylinder, that is $\rho(\mathcal{W}) = \mathbb{S}^5 \times \mathbb{R}$, where $\mathbb{S}^5$ is a 5-dimensional sphere.

Let g_0 be the metric on $\mathcal{W}$ induced from the standard Riemannian metric on $\mathbb{R}^7$ by ρ. Then,

$$g_0 = \frac{1}{u^2}\left(\sum_{\alpha=1}^{6} \mathrm{d}u_\alpha^2\right).$$

We can calculate geometric quantities of $\mathcal{W}$ and obtain the following:

Proposition 3.1. *The scalar curvature of $(\mathcal{W}, g_0)$ is equal to 20.*

Proposition 3.2. *An associated orbit of an element of $\mathcal{W}$ is a geodesic in $(\mathcal{W}, g_0)$.*

Next, we will describe the geometry of $\mathcal{W}(v)$. We will denote by g_1 the metric on $\mathcal{W}(v)$ induced from g_0 by the inclusion map. We can calculate geometric quantities of $\mathcal{W}(v)$, too. Then, we obtain

Proposition 3.3. *The scalar curvature σ_1 of $(\mathcal{W}(v), g_1)$ is 2 if and only if $v = (0,0,0)$.*

The moduli space $\mathcal{W}((0,0,0))$ and a helicoid are characterized as follows:

Proposition 3.4. *The submanifold $\mathcal{W}(v)$ of $(\mathcal{W}, g_0)$ is totally geodesic if and only if $v = (0,0,0)$.*

Theorem 3.3. *A point in $\mathcal{W}((0,0,v_3))$ ($v_3 \neq 0$) corresponds to a helicoid if and only if the point attains the maximum value of the scalar curvature σ_1 of $(\mathcal{W}((0,0,v_3)), g_1)$.*

Let J be the tensor field on $\mathcal{W}$ which is an endomorphism of the tangent space $T_u\mathcal{W}$ at every point u of $\mathcal{W}$ such that

$$\begin{aligned} J\tilde{e}_1 &= \tilde{e}_5\,, & J\tilde{e}_2 &= \tilde{e}_4\,, & J\tilde{e}_3 &= \tilde{e}_6\,, \\ J\tilde{e}_4 &= -\tilde{e}_2\,, & J\tilde{e}_5 &= -\tilde{e}_1\,, & J\tilde{e}_6 &= -\tilde{e}_3\,, \end{aligned}$$

where $(\tilde{e}_1, \dots, \tilde{e}_6)$ is the dual frame of $(\tilde{\theta}^1, \dots, \tilde{\theta}^6)$.

Lemma 3.1. *The tensor field J is an orthogonal* ***integrable complex structure*** *of $\mathcal{W}$.*

Remark 3.1. *The metric g_0 is a Hermitian metric on $(\mathcal{W}, J)$. On the other hand, g_0 is not a Kähler metric.*

By Theorem 3.1 and Lemma 3.1, $\mathcal{W}(v)$ is a 3-dimensional real submanifold of a 6-dimensional Hermitian manifold $(\mathcal{W}, g_0, J)$. Moreover, we can see the following theorem holds:

Theorem 3.4. *The submanifold $\mathcal{W}(v)$ is a totally real submanifold of $(\mathcal{W}, g_0, J)$.*

A real submanifold in a Kähler manifold is called **Lagrangian** if it satisfies the following conditions:

1. The real dimension of the submanifold is equal to half of the real dimension of the ambient Kähler manifold.
2. The submanifold is a totally real submanifold of the ambient Kähler manifold.

It is an interesting problem to find a Kähler metric on $\mathcal{W}$ which makes $\mathcal{W}(v)$ Lagrangian. We will mention a sufficient condition for a Kähler metric on $\mathcal{W}$ to make $\mathcal{W}(v)$ Lagrangian.

Theorem 3.5. *Let g' be a Kähler metric on $(\mathcal{W}, J) \cong \mathbb{R}^6 \setminus \{0\}$ conformal to the standard metric g of $\mathbb{R}^6$. If $(\mathcal{W}, g', J)$ makes $\mathcal{W}(v)$ Lagrangian, then g' is homothetic to g, that is $g' = cg$ where c is a positive constant.*

Remark 3.2. *The metric cg gives a good geometric property to $\mathcal{W}(v)$. However, this metric does not distinguish moduli spaces $\{\mathcal{W}(v)\}$ by their curvature. Hence, it is impossible to characterize a helicoid and a catenoid by the curvature of the moduli space. Thus, we can find that this metric is not so useful to obtain geometric properties of each individual minimal annulus.*

Finally, we will mention the geometry of $\mathcal{W}(P)$. We will denote by g_2 the metrics on $\mathcal{W}(P)$ induced from g_0 by the inclusion map.

Proposition 3.5. *The submanifold $\mathcal{W}(P)$ is a totally geodesic submanifold of $(\mathcal{W}, g_0)$.*

References

[1] Hoffman D. and Karcher H., *Complete Embedded Minimal Surfaces of Finite Total Curvature*, Vol. V, Geometry, Springer, Berlin 1997, pp 5–93, 267–272.

[2] Kusner R. and Schmitt N., *The Spinor Representation of Surfaces in Space*, (dg-ga/9610005).

[3] Moriya K., *On a Variety of Algebraic Minimal Surfaces in Euclidean 4-space*, Tokyo J. Math. **21**(1) (1998) 121–134.

[4] Moriya K., *On a Variety of Weierstrass Data for Branched Minimal Surfaces in Euclidean Space*, In: *Differential Geometry and Applications* (Brno, 1998), Masaryk Univ., Brno 1999, pp 207–213.

[5] Moriya K., *On a Moduli Space of Minimal Annuli*, Osaka J. Math. **38**(2) (2001) 271–285.

[6] Osserman R., *Global Properties of Minimal Surfaces in E^3 and E^n*, Ann. of Math. **80** (1964) 340–364.

[7] Pérez J. and Ros A., *The Space of Properly Embedded Minimal Surfaces with Finite Total Curvature*, Indiana Univ. Math. J. **45**(1) (1996) 177–204.

[8] Pérez J. and Ros A., *The Space of Complete Minimal Surfaces with Finite Total Curvature as Lagrangian Submanifold*, Trans. Amer. Math. Soc. **351**(10) (1999) 3935–3952.

[9] Pirola G., *Algebraic Curves and Non-rigid Minimal Surfaces in the Euclidean Space*, Pacific J. Math. **183**(2) (1998) 333–357.

[10] Pirola G., *The Infinitesimal Variation of the Spin Abelian Differentials and Periodic Minimal Surfaces*, Comm. Anal. Geom. **6**(3) (1998) 393–426.

[11] Ros A., *Compactness of Spaces of Properly Embedded Minimal Surfaces with Finite Total Curvature*, Indiana Univ. Math. J. **44**(1) (1995) 139–152.

[12] Yang K., *Complete Minimal Surfaces of Finite Total Curvature*, Kluwer Academic Publishers Group, Dordrecht 1994.