SYMPLECTIC LEAVES OF W-ALGEBRAS FROM THE REDUCED KAC–MOODY POINT OF VIEW

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Abstract. The symplectic leaves of W-algebras are the intersections of
the symplectic leaves of the Kac–Moody algebras and the hypersur-
face of the second class constraints, which define the W-algebra. This
viewpoint enables us to classify the symplectic leaves and also to give a
representative for each of them. The case of the W_2 (Virasoro) algebra
is investigated in detail, where the positivity of the energy functional is
also analyzed.

1. Introduction

W-algebras have attracted a great interest since their first appearance [1] thanks
to the fact that their quantized versions [2], (the extensions of the Virasoro
algebra with higher spin currents), are relevant not only in the classification
of two dimensional conformal field theories but also in describing various
statistical physical models. For a review on W-algebras and their application
see [3] and references therein. Later it was shown in [4] that the Toda models
(which carry the W-algebras as symmetry algebras) are Hamiltonian reductions
of the **Wess-Zumino-Witten** (WZW) **models**. Under the reduction procedure,
which can be implemented by second class constraints, the symmetry algebra
of the WZW model, namely the **Kac–Moody** (KM) **algebra**, reduces to the
symmetry algebra of the Toda models, the W-algebra.

The quantization of W-algebras started by a free field construction [2], and then
the BRST method [5] was adopted to produce their quantum counterparts. None
of the approaches mentioned however, are relied on the classical geometry of
W-algebras and loose useful information in this way. The aim of this paper
is to reveal the classical geometry of W-algebras, more precisely to analyze

their symplectic leaves, which can be used to quantize them via geometric quantization [6], later. Since this problem has its own mathematical relevance there were some efforts in this direction previously.

In the simplest case, which corresponds to the Virasoro algebra the program mentioned above means the investigation of the coadjoint orbits of the Virasoro group [9]. The classification of these orbits are well-known [7–13]. Although only in [11] the authors give explicit representative for each class. Moreover, they also investigated the positivity of the energy functional on the orbits, which is relevant for finding the highest weight representations at the quantum level. In the next simplest case, namely in the case of W_3 the orbits were classified in [12], while the case of W_n was considered in [10].

In this paper I will show that the symplectic leaves of W-algebras are the intersections of the symplectic leaves of KM algebras and the hypersurface determined by the second class constraints. This approach not only provides a unified viewpoint for the orbit classifications obtained previously but also gives explicit representative for each orbit. Let us note that a Lagrangian realization of this idea was applied in one particular case, namely in the case of some special orbits of the Virasoro algebra in [13].

The paper is organized as follows. In Section 2 I give the definition of the WZW phase space with its Hamiltonian and review how it is connected to the symplectic leaves of its symmetry algebra the KM algebra. By means of these I classify the symplectic leaves and provide a representative for each of them. In Section 3 I show how the W-algebras arise as reductions of the system above and how this fact can be used classifying their symplectic leaves. At each stage the results are demonstrated via $SL(2, \mathbb{R})$.

2. The Phase Space of the WZW Model and the Symplectic Leaves of the KM Algebra

The Hamiltonian formulation of the WZW theories is the following. Take a maximally non-compact Lie group G (this is necessary in order to carry out the reduction) and define the phase space to be

$$M_{WZW} = \{(g, J)\, ;\ g \in LG\, ,\ J \in L\mathfrak{g}\}\, ,$$

where LG denotes the loop group of G, that is the space of the smooth maps from $S^1 = \{\, \mathrm{e}^{\mathrm{i}x}\, ;\ x \in [0, 2\pi)\}$ to G and $L\mathfrak{g}$ is its Lie algebra. It is equipped with the Poisson brackets:

$$\{g(x), g(y)\} = 0\, , \qquad \{J_a(x), g(y)\} = t_a g(y)\delta(x - y)\, ,$$
$$\{J_a(x), J_b(y)\} = f_{ab}^c J_c(y)\delta(x - y) + k\kappa_{ab}\delta'(x - y)\, ,$$

where the current is decomposed as $J(x) = J^a(x)t_a$; $t_a \in \mathfrak{g}$, $[t_a, t_b] = f_{ab}^c t_c$, and the indices are lowered and raised by the Cartan metric $\kappa_{ab} = \mathrm{trace}(t_a t_b)$ and its inverse κ^{ab}, respectively, and $\delta(x - y)$ represents the 2π periodic delta function: $\delta(x) = \dfrac{1}{2\pi} \sum_{n \in \mathbb{Z}} e^{\mathrm{i}nx}$. The second line contains the defining relations of the KM algebra at level k. If we introduce

$$\tilde{J} = -g^{-1}Jg + kg^{-1}g',$$

then the Hamiltonian takes the following simple form:

$$H_{wzw} = \frac{1}{2k} \int\limits_{0}^{2\pi} \mathrm{d}x \, \mathrm{Tr}(J(x)^2 + \tilde{J}(x)^2).$$

In light cone coordinates, $x_\pm = x \pm t$, it generates the flow:

$$\partial_- J(x,t) = 0, \qquad k\partial_+ g(x,t) = J(x,t)g(x,t). \tag{1}$$

The most general solutions are $J(x,t) = J(x_+)$ and $g(x,t) = g(x_+)g(x_-)$. The symmetries of the model consist of local transformations, $h(x_+) \in LG$:

$$g(x_+) \to h(x_+)g(x_+),$$
$$J(x_+) \to h(x_+)J(x_+)h^{-1}(x_+) + kh'(x_+)h^{-1}(x_+)$$

and global transformations, $h \in G$:

$$g(x_+) \to g(x_+)h, \qquad J(x_+) \to J(x_+).$$

Observe that the local transformations are generated by the conserved KM currents $J(x_+)$ and are nothing but the coadjoint action of the centrally extended loop group on its Lie algebra and left translation on itself. Note that if $k \neq 0$ what we will suppose in the sequel, we can absorb k into the definition of J, that is, we change J to J/k and analyze all the orbits in one turn by substituting $k = 1$ in the formulae from now on.

Now the key point in analyzing the coadjoint orbits is the fact, that there is one-to-one correspondence between the currents $J(x_+)$ and the elements $g(x_+)$ with $g(0) = e$ property via the equations of motions (1). This means that instead of analyzing the coadjoint action on $J(x_+)$ we can analyze the much simpler action on $g(x_+)$. Of course we have to modify the action in order to ensure $g(0) = e$. This can be achieved by combining the local and global transformations in the following way:

$$g(x_+) \to h(x_+)g(x_+)h(0)^{-1},$$
$$J(x_+) \to h(x_+)J(x_+)h^{-1}(x_+) + kh'(x_+)h^{-1}(x_+).$$

According to this action we can split any transformation as

$$h(x_+) = \left(h(0), h(x_+)h^{-1}(0)\right),$$

where $h(0)$ is a constant loop, $h(x_+)h^{-1}(0) \in \Omega G$ and ΩG consists of those loops which starts at the identity.

Thus topologically we can write the loop group as $LG = G \times \Omega G$. The group structure is a semi-direct product: $(a, g)(b, h) = (ab, g\,\mathrm{Ad}_a h)$.

Remember that we have split the original periodic $g(x, t)$ as $g(x, t) = g(x_+)g(x_-)$. Consequently $g(x_+)$ is not necessary periodic, but only quasi-periodic

$$g(x_+ + 2\pi) = g(x_+)M, \qquad M = g(2\pi).$$

M is called the monodromy matrix. Under a loop group transformation $g(x_+) \rightarrow h(x_+)g(x_+)h(0)^{-1}$ the monodromy matrix changes as $M \rightarrow h(0)Mh(0)^{-1}$ which means that the conjugacy class of the monodromy matrix is an invariant of the orbit. It is in fact the only invariant, since if M_1 and M_2 are conjugated $M_2 = hM_1h^{-1}$ and g_i corresponds to M_i then $g_1 h^{-1} g_2^{-1}$ is periodic and maps g_2 to g_1. Note however, that an orbit corresponding to a conjugacy class is not necessarily connected. Its connected components are labelled by $\Pi_0(LG) = \Pi_1(G)$ that is by the fundamental group of G.

Concluding, the connected components of the coadjoint orbits of the centrally extended loop groups are characterized by the conjugacy class of the monodromy matrix and the discrete invariant labelled by the fundamental group of G.

In order to analyze the topology of the orbit we need to determine the stabilizer of a given J. Clearly $h(x_+)$ will stabilize a given $J(x_+)$ if and only if $h(x_+)g(x_+)h(0)^{-1} = g(x_+)$ holds. This means that $h(0)$ must stabilize the monodromy matrix, $h(0) \in H_M = \{g \in G;\ gMg^{-1} = M\}$, moreover $h \in G$ completely determines the stabilizing $h(x_+)$ as $h(x_+) = g(x_+)hg(x_+)^{-1}$. The topology of the orbit is

$$LG/H_M = G \times \Omega G/H_M = (G/H_M) \times \Omega G,$$

where in the last equality we used the fact that LG/H_M is a principal ΩG bundle over H/M which turns out to be trivial.

2.1. The Example of the $SL(2, \mathbb{R})$ WZW Model

The coadjoint orbits are labelled by the conjugacy classes of $SL(2, \mathbb{R})$ and by a $\mathbb{Z} = \Pi_1(SL(2, \mathbb{R}))$ valued discrete parameter. For each conjugacy class we present a representative and then give its stabilizer group. Furthermore we determine that particular $g(x_+)$ which gives rise to this monodromy matrix and

compute the corresponding $J(x_+) = g(x_+)'g^{-1}(x_+)$. We proceed from case to case:

1. Elliptic case:

$$M = \begin{pmatrix} \cos(2\pi\omega) & -\sin(2\pi\omega) \\ \sin(2\pi\omega) & \cos(2\pi\omega) \end{pmatrix}$$

$$H_M = \left\{ \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix}; \quad \alpha \in (0, 2\pi] \right\},$$

$$g(x_+) = \begin{pmatrix} \cos(\omega x_+) & -\sin(\omega x_+) \\ \sin(\omega x_+) & \cos(\omega x_+) \end{pmatrix}$$

$$J(x_+) = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix}.$$

(2)

These correspond to the leaf labelled by $n = 0$. In order to move to the n-th leaf we multiply $g(x_+)$ by

$$T_n = \begin{pmatrix} \cos(nx_+) & -\sin(nx_+) \\ \sin(nx_+) & \cos(nx_+) \end{pmatrix},$$

which amounts to shift ω to $\omega + n$ in (2).

2. Exceptional case

$$M = \eta \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \eta = \pm 1, \qquad H_M = SL(2, \mathbb{R}).$$

This can be recovered from the elliptic case by taking $\omega = n \in \mathbb{Z}$ for $\eta = 1$ and $\omega = n + 1/2 \in \mathbb{Z} + 1/2$ for $\eta = -1$.

3. Hyperbolic case

$$M = \eta \begin{pmatrix} e^{2b\pi} & 0 \\ 0 & e^{-2b\pi} \end{pmatrix} \quad \eta = \pm 1, \qquad H_M = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}; \quad a \neq 0 \right\}.$$

The corresponding $g(x_+)$ and currents are

$$g(x_+) = T_n \begin{pmatrix} e^{bx_+} & 0 \\ 0 & e^{-bx_+} \end{pmatrix},$$

$$J(x_+) = n \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + b \begin{pmatrix} \cos(2nx_+) & \sin(2nx_+) \\ \sin(2nx_+) & -\cos(2nx_+) \end{pmatrix},$$

where similarly to the previous case $n \in \mathbb{Z}$ for $\eta = 1$ and $n \in \mathbb{Z} + 1/2$ for $\eta = -1$.

4. Parabolic case

$$M = \eta \begin{pmatrix} 1 & 0 \\ q & 1 \end{pmatrix} \quad \eta = \pm 1, \, q = \pm 1, \qquad H_M = \left\{ \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}; \quad a \in \mathbb{R} \right\},$$

$$g(x_+) = T_n \begin{pmatrix} 1 & 0 \\ \frac{qx}{2\pi} & 1 \end{pmatrix}$$

$$J(x_+) = n \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + \frac{q}{2\pi} \begin{pmatrix} \frac{1}{2}\sin(2nx_+) & -\sin^2(nx_+) \\ \cos^2(nx_+) & -\frac{1}{2}\sin(2nx_+) \end{pmatrix}.$$

Here, as before $n \in \mathbb{Z}$ for $\eta = 1$ and $n \in \mathbb{Z} + 1/2$ for $\eta = -1$.

3. The Definition of W-Algebras and their Symplectic Leaves

In order to give a self-contained description of W-algebras we collect the main results of [4]. For simplicity we restrict ourselves to the case of the $SL(n, \mathbb{R})$ WZW model. The generalization for other groups is straightforward. Impose the following constraints on the KM current:

$$\Phi_\alpha = J_\alpha - \chi(J_\alpha) = 0 \quad \text{where } \chi(J_\alpha) = \begin{cases} 1 & \text{if } \alpha \in \Delta_- \\ 0 & \text{if } \alpha \in \Phi_- \setminus \Delta_- \end{cases}$$

$$J_{\text{const}} = \begin{pmatrix} * & \cdots & \cdots & \cdots & * \\ 1 & * & \cdots & \cdots & * \\ 0 & 1 & * & \cdots & * \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 1 & * \end{pmatrix}$$

where Φ_- (Δ_-) denotes the set of negative (simple) roots of $sl(n, \mathbb{R})$. These constraints are first class so they generate gauge transformations, which are nothing but the KM transformations generated by the currents associated to the positive roots. One possible gauge fixing is the so-called Wronsky gauge:

$$J_{gf} = \begin{pmatrix} 0 & W_2 & W_3 & \cdots & W_n \\ 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix},$$

where the W-s are gauge invariant polynomials of the unconstrained KM currents. The resulting phase space carries a Poisson algebra structure, which is inherited from the WZW phase space and can be computed either by computing the Poisson brackets of the gauge invariant quantities or by using the Dirac bracket. The resulting Poisson algebra, which closes in general only on polynomials of the fields

$$\{W_i, W_j\} = P_{ij}(W)$$

is called the W-algebra associated to the group $SL(n, \mathbb{R})$ and is denoted by W_n. It always contains the Virasoro algebra:

$$\{W_2(x), W_2(y)\} = -\frac{1}{2}\delta'''(x-y) - 2W_2(y)\delta'(x-y) + W_2'(x)\delta(x-y)$$

and there exist such combinations of the remaining generators that are primary fields of weights $3, 4, \ldots, n$ with respect to this Virasoro algebra. Having fixed the gauge, the equations of motions (1) forces g to be of the form

$$g = \begin{pmatrix} \psi_1^{(n-1)} & \psi_2^{(n-1)} & \cdots & \psi_n^{(n-1)} \\ \cdots & \cdots & \cdots & \cdots \\ \psi_1' & \psi_2' & \cdots & \psi_n' \\ \psi_1 & \psi_2 & \cdots & \psi_n \end{pmatrix},$$

where any of the ψ_i-s satisfies the following n-th order differential equation:

$$\psi_i^{(n)} - W_2\psi_i^{(n-2)} - W_3\psi_i^{(n-3)} - \cdots - W_{n-1}\psi_i' - W_n\psi_i = 0.$$

Moreover, any infinitesimal W-transformation generated by $Q_i = \int_0^{2\pi} dx\, \epsilon_i(x)W_i(x)$ can be implemented by appropriately chosen field-dependent KM transformation $J_{imp}(W)$:

$$\delta_i J_{gf} = \begin{pmatrix} 0 & \{Q_i, W_2\} & \{Q_i, W_3\} & \cdots & \{Q_i, W_n\} \\ 0 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 0 & 0 \end{pmatrix} = [J_{imp}, J_{gf}] + J_{imp}'.$$

The left hand side defines the tangent vector to the W-symplectic leaf which, as a consequence of the KM implementation, is also a tangent vector to the coadjoint orbit of the loop group. This shows that the W-symplectic leaves can be obtained by considering the intersection of the coadjoint orbits of the loop group with the gauge fixed surface.

This viewpoint allows one to classify the W-symplectic leaves: We have the conjugacy class of the monodromy matrix as an invariant and the $\Pi_1(G)$ valued discrete topological invariant. This classification is not the finest one however, since we also have to count the number of the connected components of the intersection surface. We may not have any intersection, or we may have one connected component or more than one. This analysis turns out to be very complicated and we cannot cope with the general case, that is why we proceed from case to case. Note however, that instead of analyzing the intersection problem at the level of J we can analyze it in the language of g (since we

know that they are uniquely connected). This amounts to classify the homotopy classes of nondegenerate curves in $\mathbb{R}^n$ or equivalently in $\mathbb{RP}^{n-1}$. This problem was investigated in [10].

3.1. The Example of the Virasoro Algebra

Let us focus on the $SL(2, \mathbb{R})$ case. Parameterizing the current as $J = \begin{pmatrix} J_0 & J_+ \\ J_- & -J_0 \end{pmatrix}$ the constraint reads as $J_- = 1$ and the gauge fixed form is

$$J_{gf} = \begin{pmatrix} 0 & L \\ 1 & 0 \end{pmatrix} , \qquad L = J_+ + J_0^2 - J_0' .$$

The Poisson bracket is simply

$$\{L(x), L(y)\} = -\frac{1}{2}\delta'''(x-y) - 2L(y)\delta'(x-y) + L'(x)\delta(x-y) ,$$

which is just the defining relation of the Virasoro algebra. Thus the coadjoint orbits of the Virasoro group can be obtained by analyzing the intersection of the coadjoint orbits of the loop group (previous section) and the gauge fixing hypersurface. As in the case of the KM algebras it is transparent to work at the level of g. The gauge fixing forces g to be of the form of

$$g = \begin{pmatrix} \psi_1' & \psi_2' \\ \psi_1 & \psi_2 \end{pmatrix} , \qquad \text{where } \psi_i'' - L\psi_i = 0$$

that is all the ψ_i-s satisfies the Hill equation. From the previous section it follows that instead of analyzing the coadjoint orbits of the Virasoro group we can analyze the conformally nonequivalent solutions of the Hill equation or equivalently the homotopy classes of nondegenerate curves in the plane.

It can be shown in the general case, that if the KM orbit has intersection with the gauge fixed surface then the intersection is necessarily connected. In other words we do not have new invariant compared to the KM case, and the only thing we have to check for each connected orbit whether the intersection exists or not.

We can parameterize the representative of the KM currents listed in the previous section for any leaf having $n = 0$ as $J_0 = \begin{pmatrix} j_0 & j_- \\ j_+ & -j_0 \end{pmatrix}$ with constant j-s. The representative, J_n, on the n-th leaf can be obtained by acting with T_n as

$$J_n = T_n J_0 T_n^{-1} + T_n' T_n^{-1}$$

$$= \begin{pmatrix} j_0 \cos(2nx_+) & -n - j_2 + j_0 \sin(2nx_+) \\ - j_1 \sin(2nx_+) & + j_1 \cos(2nx_+) \\ n + j_2 + j_0 \sin(2nx_+) & -j_0 \cos(2nx_+) \\ + j_1 \cos(2nx_+) & + j_1 \sin(2nx_+) \end{pmatrix}.$$

where $j_\pm = j_1 \pm j_2$. Now if $R^{-2} = n + j_2 + j_0 \sin(2nx_+) + j_1 \cos(2nx_+) > 0$ then

$$h = \begin{pmatrix} R^{-1} & R' \left(1 + \frac{1}{n} R^{-2}\right) \\ 0 & R \end{pmatrix}$$

is periodic and is in the trivial homotopy class as a loop. It maps J_n into the gauge fixed form and gives

$$L = C + 2n(n^2 + C)R^2 + 3n(n^2 + C + 2nj_3)R^4 \,;$$

where $C = -\det J = j_0^2 + j_1^2 - j_2^2 \,.$

Now let us proceed form case to case:

1. Elliptic case: the intersection exists for $n \geq 0$ and we have the following representative of the orbit

$$L = -(n + \omega)^2 \,, \qquad \omega \in (0, 1) \,, \qquad \omega \neq \frac{1}{2} \,.$$

The stabilizer subgroup is S^1.

2. Exceptional case: this case can be recovered from the elliptic by putting $\omega = n$ or $\omega = n + 1/2$ where $n \in \mathbb{Z}$. In the first case however, only $n > 0$ is allowed and the stabilizer is the whole $SL(2, \mathbb{R})$.

3. Hyperbolic case: conjugating the current $J = b \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ with the constant $\begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}$ one can arrange that $R^{-2} = (n - 1) + (b^2 + 1) \cos^2(nx_+) + (b \cos nx_+ + \sin nx_+)^2 > 0$ holds. The intersection exists for $n \geq 0$ and we have

$$L = b^2 + 2n(n^2 + b^2)R^2 + 3n^2(b^2 + 2bn - n^2)R^4 \,.$$

The stabilizer subgroup is $\mathbb{R}$.

4. Parabolic case: we have intersection for $q = -1$ if $n > 0$ and for $q = 1$ if $n \geq 0$. Since $R^{-2} = n + \frac{q}{2\pi} \cos^2(nx_+) > 0$ the representative of the orbit is

$$L = n^3 \left[2R^2 - 3\left(n + \frac{q}{2\pi}\right)R^4 \right]$$

and the stabilizer subgroup is $\mathbb{R}$.

The positivity of the energy functional, $E = \int\limits_0^{2\pi} L(x_+)\,\mathrm{d}x_+$, is necessary to obtain highest weight representation for the Virasoro algebra since the energy in a quantum theory is bounded from below.

In the local analysis we demand that $\delta E = 0$ and $\delta\delta E > 0$ in order to have a minimum for the energy. Since

$$\delta_\epsilon E = \int\limits_0^{2\pi} \epsilon(x_+)\{L(x_+), L(y_+)\}\,\mathrm{d}y_+ = -\int\limits_0^{2\pi} \epsilon(y_+)L'(y_+)\,\mathrm{d}y_+\,,$$

(we have dropped the total derivatives!) it is clear that only constant L can give local minimum of the energy. The second variation

$$\delta_\epsilon\delta_\epsilon E = \delta_\epsilon(E + \delta_\epsilon E) = -\int\limits_0^{2\pi} (L(\epsilon')^2 + \frac{1}{4}(\epsilon'')^2)$$

shows that $L \geq -\frac{1}{4}$ is necessary. The global analysis is much more involved. The result is the following. The lowest energy is on the exceptional orbit for $\omega = 1/2$ with stabilizer $SL(2,\mathbb{R})$. So it is a good candidate for the classical vacuum. The energy has a minimum also on the elliptic orbits for $n = 0$ and $\omega < 1/2$. On the hyperbolic orbit for $n = 0$ it has a minimum. Moreover, surprisingly on the orbit corresponding to $q = -1$ and $n = 1$ in the parabolic case the energy is bounded from below, however this lower bound is never reached. This indicates that the representation obtained by quantizing this orbit is not of the highest weight type similarly to the case for quantizing the cone-like coadjoint orbit of the group $SL(2,\mathbb{R})$.

Acknowledgements

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