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LORENTZIAN SUBMANIFOLDS IN SEMI-EUCLIDEAN SPACES WITH POINTWISE 1-TYPE GAUSS MAP

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Abstract. In this work first, we survey the most recent classification results for submanifolds with pointwise 1-type Gauss map. Then, we study a class of hypersurfaces with vanishing Gauss-Kronecker curvature in terms of type of their Gauss map.

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1. Introduction

After the problem "*To what extent does the type of the Gauss map of a submanifold of $\mathbb{E}^m_r$ determine the submanifold?*" was introduced by Chen and Piccinni in [10], submanifolds with pointwise 1-type Gauss map have been worked in many articles, [8–10]. We present a survey of recent results on this topic in Section 3.

Consider an oriented (semi-)Riemannian submanifold M of a (semi-)Euclidean space and its Gauss map G. By the definition, M is said to have pointwise 1-type Gauss map if the Laplacian of its Gauss map take the form

$$\Delta G = f(G + C) \tag{1}$$

for a smooth function f and constant vector C. More precisely, a pointwise 1-type Gauss map is called *of the first kind* if (1) is satisfied for $C = 0$, and *of the second kind* if $C \neq 0$. Moreover, if (1) is satisfied for a non-constant function f, then M is said to have *proper* pointwise 1-type Gauss map. Otherwise, G is said to be (global) 1-type, [8, 13].

2. Preliminaries

In this section, we would like to give a brief summary of basic results on Lorentzian surfaces.

Let $\mathbb{E}^m_t$ denote the semi-Euclidean m-space with the canonical semi-Euclidean metric tensor of index t given by

$$g = -\sum_{i=1}^{t} \mathrm{d}x_i^2 + \sum_{j=t+1}^{m} \mathrm{d}x_j^2$$

where $x_1, x_2, \ldots, x_m$ are rectangular coordinates of the points of $\mathbb{E}^m_t$.

Let $\mathbb{S}^n_1(r^2)$ and $\mathbb{H}^n(-r^2)$ denote the de Sitter space-time and the hyperbolic space of dimension $n > 2$ defined by

$$\begin{aligned} \mathbb{S}^n_1(r^2) &= \{x \in \mathbb{E}^{n+1}_1;\ \langle x, x\rangle = r^{-2}\} \\ \mathbb{H}^n(-r^2) &= \{x \in \mathbb{E}^{n+1}_1;\ \langle x, x\rangle = -r^{-2}\}. \end{aligned}$$

We also put $S^1_1(r)$ and $H^1(-r)$ for the curves in $\mathbb{E}^2_1$ given by $s \mapsto (r \sinh s, r \cosh s)$ and $(r \cosh s, r \sinh s)$, respectively.

We would like to note that all further notations, basic definitions and basic facts that we will use in this paper are described by the author in [35, Section 2] and [37, Section 2]. We also would like to refer to [10, 13, 19] for detailed information of definition and geometrical interpretation of the Gauss map of the submanifolds.

3. Recent Results on Submanifolds with Finite Type Gauss Map

In this section, we would like to present a survey of classification results recently obtained.

3.1. Rotational Surfaces in $\mathbb{E}^4_r$

Over the last few years, rotational surfaces in (semi-)Euclidean spaces of dimension four are studied in some papers in terms of type of their Gauss map. We may note that the study of rotational surfaces in four-dimensional (semi-)Riemannian space forms was initiated in [39] by Yoon. In fact, two different types of rotational surfaces are considered in the Euclidean space $\mathbb{E}^4$, Minkowski space $\mathbb{E}^4_1$ and semi-Euclidean space $\mathbb{E}^4_2$. One of them is called simple rotational surface while the other is general rotational surface. Note that some authors used the term 'Moore-type' rotational surface for the general rotational surfaces after [31].

General (respectively simple) rotational surface is defined as follows. Let P be a plane (respectively hyperplane) in $\mathbb{E}^4_r$ and $P^\perp$ the plane absolutely perpendicular to P. Consider a curve α which is called the profile curve of rotational surface and a one parameter group G_P of orthonormal transformation which leave P and $P^\perp$ invariant as sets (respectively which leave P invariant pointwise). The orbit M of α under the action of G_P is called a general (respectively simple) rotation surface generated by G_P and α (see [7, 20]). Note that the curve α is called the profile curve of the rotational surface M.

In [20], Dursun and the author studied general rotational surfaces in the Euclidean space $\mathbb{E}^4$. Let M be a rotational surface in the Euclidean space $\mathbb{E}^4$ defined by

$$F(s,t) = (x(s)\cos at, x(s)\sin at, z(s)\cos bt, z(s)\sin bt) \tag{2}$$

where a, b are some constants, called rates of rotation of M, and x, z are some smooth functions. The following classification results were obtained in that paper.

Theorem 1 ([20]). *Let M be a non-minimal general rotation surface given by* (2). *Then, M has pointwise* 1-type *Gauss map of the first kind if and only if it is an open part of the surface*

$$F(s,t) = \big(r_0 \cos\frac{s}{r_0}\cos at, r_0\cos\frac{s}{r_0}\sin at, r_0\sin\frac{s}{r_0}\cos bt, r_0\sin\frac{s}{r_0}\sin bt\big).$$

Theorem 2 ([20]). *Let M be a non-planar general rotational surface in $\mathbb{E}^4$ defined by* (2) *for the rates of rotation a and b. Then,*

1. *If $a^2 = b^2$ and M is minimal, then it has proper pointwise* 1-type *Gauss map of the second kind.*
2. *if $a^2 \neq b^2$, then M is minimal and its Gauss map is of pointwise* 1-type *of the second kind if and only if the meridian curve of M is given by*

$$z = cx^{\mp b/a}, \qquad x > 0 \tag{3}$$

for some real number $c \neq 0$.

Most recently, Dursun has moved this study to the Minkowski four-space by considering rotational surfaces to general rotational surfaces who position vector is given by

$$F(s,t) = \big(x(s)\sinh bt, x(s)\cosh bt, z(s)\cos at, z(s)\sin at\big) \tag{4}$$

where a, b are some constants in [16]. Note that the profile curve α of the rotational surface given by (4) is $s \mapsto (x(s), y(s))$. In [16] following theorems were obtained.

Theorem 3 ([16]). *There exists no non-planar maximal space-like rotational surface in E^4_1 defined by* (4) *with pointwise* 1-type *Gauss map of the first kind.*

Theorem 4 ([16]). *A non-maximal space-like rotational surface in E^4_1 defined by* (4) *has pointwise* 1-type *Gauss map of the first kind if and only if its profile curve α is a circle.*

Theorem 5 ([16]). *A space-like rotational surface M in E^4_1 defined by* (4) *with flat normal bundle has pointwise* 1-type *Gauss map of the second kind if and only if M is an open part of a space-like plane in E^4_1.*

Further, in [2], Aksoyak and Yayli studied general rotational surfaces in $\mathbb{E}^4_2$ given by

$$\varphi(s,t) = \big(y(s)\sinh t, x(s)\cosh t, x(s)\sinh t, y(s)\cosh t\big) \tag{5}$$

and

$$\varphi(s,t) = \big(x(s)\cos t, x(s)\sin t, y(s)\cos t, y(s)\sin t\big). \tag{6}$$

In both cases, they proved that a flat general rotational surfaces has pointwise 1-type Gauss map if and only if either M is totally geodesic or its generator curve is given by $x(s) = c_1\mathrm{e}^{-b_0 s+d} + c_2\mathrm{e}^{b_0 s-d}$ and $y(s) = c_1\mathrm{e}^{-b_0 s+d} - c_2\mathrm{e}^{b_0 s-d}$ for some constants b_0, c_1, c_2, d.

On the other hand, in [1] flat simple rotational surfaces in the Minkowski four-space $\mathbb{E}^4_1$ were considered by Aksoyak and Yayli in terms of being pointwise 1-type. A family of simple rotational surfaces in $\mathbb{E}^4_1$ is defined by

$$\varphi(s,t) = \big((x(s)\cosh t, x(s)\sinh t, y(s), z(s)\big) \tag{7}$$

where x, y, z are some smooth functions. Authors obtained that such a flat simple rotational surface has pointwise 1-type Gauss map if and only if its profile curve $s \mapsto (x(s), y(s), z(s))$ is either a circle lying on a space-like plane or a specially chosen helix (see [1, Theorem 1]).

Most recently, Dursun and Bektaş considered Riemannian simple rotational surfaces in the Minkowski space $\mathbb{E}^4_1$ in [18]. Note that such a rotational surface is called elliptic, hyperbolic or parabolic type subject to being invariant under a group of space-like, hyperbolic or screw rotations, respectively. More precisely, an elliptic Riemannian rotational surface is given by

$$F_1(s,t) = \big(w(s), z(s), x(s)\cos t, x(s)\sin t\big) \tag{8}$$

while hyperbolic and parabolic rotational surfaces are

$$F_2(s,t) = (w(s)\sinh t, w(s)\cosh t, x(s), y(s)) \tag{9}$$

and

$$F_3(s,t) = x(s)\eta_1 + \sqrt{2}tw(s)\eta_2 + (z(s) + t^2 w(s))\xi_3 + w(s)\xi_4 \tag{10}$$

respectively, where x, y, w, z are smooth functions and $\{\eta_1, \eta_2, \xi_3, \xi_4\}$ is a pseudo-orthonormal basis of $\mathbb{E}^4_1$ such that $\xi_3 = \frac{1}{\sqrt{2}}(1,1,0,0)$ and $\xi_3 = \frac{1}{\sqrt{2}}(1,-1,0,0)$. Amongs complete classification of flat elliptic and hyperbolic type rotational surfaces with pointwise 1-type Gauss map similar to obtained in [1], they have obtained

Theorem 6 ([18]). *There exists no flat space-like rotational surface of the parabolic type defined by* (10) *in the Minkowski space* $\mathbb{E}^4_1$ *with pointwise* 1-type *Gauss map of the second kind.*

Remark 7. *We would like to announce that most recently, Bektaş, Canfes and Dursun have obtained complete classification of Moore-like rotational surfaces in the Minkowski space* $\mathbb{E}^4_1$ *with pointwise* 1-type *Gauss map of the first and second kind. They have also obtained the classifications of rotational surfaces in* $\mathbb{E}^4_2$ *with zero mean curvature and pointwise* 1-type *Gauss map of second kind.*

3.2. Surfaces in Euclidean Four-Space

A general theory of surfaces with poinwtise 1-type Gauss map is obtained in [17] where Dursun and Arsan give the following results

Theorem 8 ([17]). *An oriented minimal surface* M *in the Euclidean space* $\mathbb{E}^4$ *has pointwise* 1*-type Gauss map of the first kind if and only if* M *has flat normal bundle.*

Theorem 9 ([17]). *An oriented non-minimal surface* M *in the Euclidean space* $\mathbb{E}^4$ *has pointwise* 1*-type Gauss map of the first kind if and only if* M *has parallel mean curvature vector in* $\mathbb{E}^4$.

Theorem 10 ([17]). *A non-planar minimal oriented surface* M *in the Euclidean space* $\mathbb{E}^4$ *has pointwise* 1-type *Gauss map of the second kind if and only if, with respect to some suitable local orthonormal frame* $\{e_1, e_2, e_3, e_4\}$ *on* M*, the shape operators of* M *are given by*

$$A_3 = \begin{pmatrix} \rho & 0 \\ 0 & -\rho \end{pmatrix} \quad \text{and} \quad A_4 = \begin{pmatrix} 0 & \varepsilon\rho \\ \varepsilon\rho & 0 \end{pmatrix} \tag{11}$$

where $\varepsilon = \pm 1$ *and* ρ *is a smooth non-zero function on* M.

On the other hand, in [4], meridian surfaces, one of further generalizations of rotational surfaces, are studied in terms of type of their Gauss map by Arslan, Bulca and Milousheva in [4].

Let $\{e_1, e_2, e_3, e_4\}$ be the standard orthonormal base for $\mathbb{R}$, S^2 denote a unit sphere lying on $\operatorname{span}\{e_1, e_2, e_3\}$ Consider a unit speed curve c lying on S^2 with the parametrization $r = r(v)$. Then, the position vector of a meridian surface in the Euclidean four-space $\mathbb{E}^4$ takes the form

$$z(u, v) = f(u)r(v) + g(u)e_4 \tag{12}$$

where f, g are some smooth functions and where the curve $\alpha : u \mapsto (f(u), g(u))$ is called the meridian curve of M. The following classification theorem appeared in [4].

Theorem 11 ([4]). *Let M be a meridian surface given with parametrization* (12) *and g = 0. Then M has pointwise* 1-type *Gauss map of the second kind if and only if one of the following holds*

i) *the curve c is a circle on $\mathbb{S}^2(1)$ and the meridian curve α is determined by $f(u) = au + a_1$, $g(u) = bu + b_1$, where a, a_1, b, b_1 are constants. In this case, M is a developable ruled surface lying in a three-dimensional space.*

ii) *the curve c is a great circle on $\mathbb{S}^2(1)$ and the meridian curve α is determined by the solutions of a third order differential equation* (*see* [4, p.921]).

Most recently, Arslan and Milousheva have moved this study to Minkowski four-space in [5] by obtaining meridian surfaces of elliptic or hyperbolic type in $\mathbb{E}^4_1$ (see also [24]).

3.3. Submanifolds with Constant Mean Curvature

The Laplacian of Gauss map G of an oriented hypersurface M in $\mathbb{E}^{n+1}_1$ takes the form

$$\Delta G = \varepsilon \|S\|^2 G + n\nabla\alpha \tag{13}$$

where S is the shape operator (or the Weingarten map) and α is the mean curvature of M. In [15], Dursun studied hypersurfaces in Minkowski space $\mathbb{E}^{n+1}_1$ of arbitrary dimension and obtained the following results

Theorem 12 ([15]). *Let M be an oriented hypersurface in the Minkowski space $\mathbb{E}^{n+1}$. Then M has proper pointwise* 1-type *Gauss map of the first kind if and only if M has constant mean curvature and $\|S\|^2$ is non-constant.*

Theorem 13 ([15]). *If an oriented hypersurface M in the Minkowski space $\mathbb{E}^{n+1}$ has proper pointwise* 1-type *Gauss map of the second kind, then the mean curvature α of M is non-constant.*

When the co-dimension is more than one, the study of Gauss map of constant mean curvature surfaces in the Minkowski spaces was initiated in [19], where the author and Dursun obtained

Theorem 14 ([19]). *Let M be an oriented minimal surface in the Minkowski space $\mathbb{E}^4_1$. Then M has pointwise* 1-type *Gauss map of the second kind if and only if it is an open portion of a space-like plane.*

Theorem 15 ([19]). *Let M be an oriented space-like surface in the Minkowski space $\mathbb{E}^4_1$ with flat normal bundle and non-zero constant mean curvature. Then, M has pointwise* 1-type *Gauss map of the second kind if and only if it is congruent to one of the helicial cylinders given by*

$$x_1(s,t) = (a_1 s, b_1 \cos s, b_1 \sin s, t) \tag{14}$$

$$x_2(s,t) = (b_2 \cosh s, b_2 \sinh s, a_2 s, t) \tag{15}$$

and

$$x_3(s,t) = (b_3 \sinh s, b_3 \cosh s, a_3 s, t) \tag{16}$$

for some constants $a_1, b_1, \dots, b_3$ *with* $b_1^2 - a_1^2 > 0$ *and* $a_3^2 - b_3^2 > 0$.

Most recently, the author considered the time-like surfaces and obtain the following classification theorem in [37].

Theorem 16 ([37])**.** *Let M be a non-minimal Lorentzian surface in $\mathbb{E}^4_1$ with normal flat bundle and constant mean curvature. Then, M has pointwise* 1-type *Gauss map of the second kind if and only if it is congruent to one of the following surfaces*

i) *A surface given by* $x(s,t) = \left(s, \frac{a}{\lambda}\cos\lambda t, \frac{a}{\lambda}\sin\lambda t, \sqrt{1-a^2}t\right), 0 < a < 1$

ii) *A surface given by* $x(s,t) = \left(\frac{a^2}{3}t^3 + t, \sqrt{2}at, \frac{a^2}{3}t^3, s\right)$

iii) *A surface given by* $x(s,t) = \left(\frac{\sqrt{a^2-1}}{\lambda}\cosh\lambda t, \frac{\sqrt{a^2-1}}{\lambda}\sinh\lambda t, at, s\right)$, $a > 1$

iv) *A surface given by* $x(s,t) = \left(\frac{\sqrt{a^2+1}}{\lambda}\sinh\lambda t, \frac{\sqrt{a^2+1}}{\lambda}\cosh\lambda t, at, s\right)$

v) *A surface given by* $x(s,t) = \left(\sqrt{1+a^2}t, \frac{a}{\lambda}\cos\lambda t, \frac{a}{\lambda}\sin\lambda t, s\right)$

for a non-zero constant λ.

In the same paper, Lorentzian minimal surfaces in the Minkowski space $\mathbb{E}^4_1$ are classified in term of their Gauss map. Note that the Gauss map of a Lorentzian surface M in the semi-Euclidean space $\mathbb{E}^4_r$, $r = 1, 2$ satisfies

$$\Delta G = 2KG + 2h(f_1, f_1) \wedge h(f_2, f_2) \tag{17}$$

where f_1, f_2 forms a pseudo-orthonormal frame field for the tangent space of M. By using this equation the author obtained following theorems.

Theorem 17 ([37])**.** *Let M be a Lorentzian minimal surface in $\mathbb{E}^4_1$. Also suppose that no open part of M is contained in a hyperplane of $\mathbb{E}^4_1$. Then, the following conditions are equivalent*:

i) *M has pointwise* 1-type *Gauss map.*

ii) *M has pointwise* 1-type *Gauss map of the first kind.*

iii) *M is congruent to the surface given by*

$$x(s,t) = s\eta_0 + \beta(t), \qquad \langle \eta_0, \beta(t)\rangle \neq 0 \tag{18}$$

for a constant light-like vector $\eta_0 \in \mathbb{E}^4_1$ *and a null curve* β *in* $\mathbb{E}^4_1$ *satisfying* $\langle \eta_0, \beta(t) \rangle \neq 0$.

On the other hand, classification of Lorentzian minimal surfaces the semi-Euclidean space $\mathbb{E}^4_2$ is somehow different. In this direction, the following theorems have been obtained by the author and Canfes [21].

Proposition 18 ([21]). *There exist four families of minimal Lorentzian surfaces in the semi-Euclidean space* $\mathbb{E}^4_2$ *with pointwise* 1-type *Gauss map of the first kind.*

i) *A minimal Lorentzian surface lying in a hyperplane of* $\mathbb{E}^4_2$.
ii) *A surface with degenerated relative null space given by* (18).
iii) *A surface lying on a degenerated hyperplane given by*

$$x(s,t) = \big(\phi_1(s) + \phi_2(t), \frac{\sqrt{2}}{2}(s+t), \frac{\sqrt{2}}{2}(s-t), \phi_1(s) + \phi_2(t)\big) \qquad (19)$$

where $\phi_i : I_i \to \mathbb{R}$ *are some smooth, non-vanishing functions and* I_i *are some open intervals for* $i = 1, 2$.

Conversely, every minimal Lorentzian surface with pointwise 1-type *Gauss map of the first kind in the semi-Euclidean space* $\mathbb{E}^4_2$ *is congruent to an open portion of a surface obtained from these type of surfaces.*

Theorem 19 ([21]). *Let* M *be a minimal Lorentzian surface properly contained by the semi-Euclidean space* $\mathbb{E}^4_2$ *with non-harmonic Gauss map. Then* M *has pointwise* 1-type *Gauss map of the second kind if and only if it is locally congruent to the surface given by*

$$x(s,t) = \big(\phi_1(s) + \phi_2(t), s+t, s + \cos c\, t + \sin c\, \phi_2(t), \phi_1(s) - \sin c\, t + \cos c\, \phi_2(t)\big) \qquad (20)$$

for some smooth non-linear functions ϕ_1, ϕ_2 *and a constant* $c \in (0, 2\pi)$, *where* $\varepsilon = \pm 1$. *In this case,* (1) *is satisfied for* $f = 4K$.

3.4. Marginally Trapped Surfaces

Among the constant mean curvature surfaces in Minkowski spaces, marginally trapped surfaces have cought special interest of geometers in terms of type their Gauss map. Note that a space-like surface M in $\mathbb{E}^4_1$ is said to be marginally trapped (or quasi-minimal) if its mean curvature vector H is light-like on M. In this case, if M is free of flat points, then the Laplacian of Gauss map of M becomes

$$\Delta G = -2Kx \wedge y - 2\beta_2 x \wedge n_1 + 2\beta_2 y \wedge n_1 - 4\nu\mu n_1 \wedge n_2$$

where $\{x, y, n_1, n_2\}$ is the geometric frame field and $\beta_1, \beta_2, \nu, \mu$ are the corresponding invariants of M (see [23]). In [30], Milousheva studied marginally trapped surfaces free of flat points with pointwise 1-type Gauss map and obtained

Theorem 20 ([30]). *Let M be a marginally trapped surface free of flat points. Then M is of pointwise* 1-type *Gauss map if and only if M has parallel mean curvature vector field.*

However, if the hypothesis of being free of flat points is removed, then one can obtain marginally trapped surfaces with pointwise 1-type Gauss map and non-parallel mean curvature vector field. Let Ω be an open, bounded subset of $\mathbb{R}^2$, $\psi, \phi : \Omega \to \mathbb{R}$ some smooth functions satisfying

$$\Delta\psi = F(\psi), \qquad \phi(u, v) = \psi(u, v) + c_1 u + c_2 v \tag{21}$$

for some constants $c_1,\ c_2$ and a differentiable non-constant function $F : \psi(\Omega) \to \mathbb{R}$ such that the function $f : \Omega \to \mathbb{R}$ defined by

$$f(u, v) \ = \ F'(\psi(u, v)) \tag{22}$$

is smooth. Consider the partly marginally trapped surface M in the Minkowski space-time $\mathbb{E}^4_1$ given by

$$x(u, v) = \big(\phi(u, v), u, v, \phi(u, v)\big).$$

Then, the Gauss map of M is pointwise 1-type of the second kind.

In [35], the author obtained

Theorem 21 ([35]). *Let M be a marginally trapped surface in the Minkowski space-time $\mathbb{E}^4_1$. Then, M has pointwise* 1-type *Gauss map of the second kind if and only if it is congruent to the surface described above.*

Further in [36], the author moved this study to the de Sitter space $\mathbb{S}^4_1(1)$ and obtain following theorems

Theorem 22 ([36]). *Let M be a quasi-minimal surface lying in $\mathbb{S}^4_1(1)$. Then M has* 1-type *Gauss map if and only if it is congruent to a surface congruent to one of the following six type of surfaces*

i) *A surface given by*

$$x(u, v) = (1, \sin u, \cos u \cos v, \cos u \sin v, 1). \tag{23}$$

ii) *A surface given by*

$$x(u, v) = \frac{1}{2}(2u^2 - 1, 2u^2 - 2, 2u, \sin 2v, \cos 2v). \tag{24}$$

iii) *A surface given by*

$$x(u,v) = \left(\frac{b}{cd}, \frac{\cos cu}{c}, \frac{\sin cu}{c}, \frac{\cos dv}{d}, \frac{\sin dv}{d}\right) \tag{25}$$

where $c = \sqrt{2-b}$ *and* $d = \sqrt{2+b}$ *with* $|b| < 2$.

iv) *A surface given by*

$$x(u,v) = \left(\frac{\cosh cu}{c}, \frac{\sinh cu}{c}, \frac{\cos dv}{d}, \frac{\sin dv}{d}, \frac{b}{cd}\right) \tag{26}$$

where $c = \sqrt{b-2}$ *and* $d = \sqrt{b+2}$ *with* $|b| > 2$.

v) *A surface of curvature one with constant light-like mean curvature vector, lying in* $K_a = \{(t, x_2, x_3, x_4, t+a) | t, x_2, x_3, x_4 \in \mathbb{R}\}$.

vi) *A surface of curvature one lying in* $\mathcal{LC}_1 = \{(y,1) | \langle y, y\rangle = 0, y \in \mathbb{E}^4_1\}$.

Remark 23. *The classification of quasi-minimal surfaces in* $\mathbb{S}^4_1(1)$ *with parallel mean curvature which is obtained by Chen and van der Veken is used in the proof of* Theorem 22 (*see* [11]).

Theorem 24 ([36])**.** *Let* M *be a quasi-minimal surface lying in* $\mathbb{S}^4_1(1)$. *Then* M *has proper pointwise* 1-type *Gauss map if and only if it is congruent to a surface congruent to one of the following two type of surfaces*

i) *A surface lying in* $\mathbb{S}^4_1(1) \cap \mathbb{S}^4(c_0, r^2)$, *where* $c_0 \neq 0$ *and* $r > 0$.

ii) *A surface lying in* $\mathbb{S}^4_1(1) \cap \mathbb{H}^4(c_0, -r^2)$, *where* $c_0 \neq 0$ *and* $r > 0$.

Remark 25. *We also would like to announce that the complete classification of marginally trapped surfaces in the pseudo-Euclidean space* $\mathbb{E}^4_2$ *with neutral metric has been obtained by the author and Milousheva.*

3.5. Some Generalizations of the Notion of Pointwise 1-type Gauss Map

In [27] and [28], Kim and the author presented a generalization of the notion of pointwise 1-type Gauss map in the following sense. Let $L_0, L_1, L_2, \dots, L_k$ denote the linearized operators of the first variation of the $(k+1)$th mean curvature arising from normal variations of an hypersurface M of the Euclidean space $\mathbb{E}^{n+1}$ (see reference [3]). Note that we have $L_0 = -\Delta$ and Then M has said to have L_k pointwise 1-type Gauss map if its Gauss map G satisfies

$$L_k G = f(G + C).$$

In particular, the operator $L_1 = \square$ is called the Cheng-Yau operator, [12]. If M is a surface in $\mathbb{E}^3$, then its Gauss map G satisfies

$$\square G = -\nabla K - 2HKG. \tag{27}$$

We would like to present some of results obtained in [27].

Theorem 26 ([27]). *An oriented surface M in $\mathbb{E}^3$ has $\Box$-pointwise* 1-type *Gauss map of the first kind if and only if it has constant Gaussian curvature.*

Theorem 27 ([27]). *An oriented minimal surface M in $\mathbb{E}^3$ has $\Box$-pointwise* 1-type *Gauss map if and only if it is an open part of a plane.*

Theorem 28 ([27]). *Let M be a surface in $\mathbb{E}^3$ with a constant principal curvature. Then, M has $\Box$-pointwise* 1-type *Gauss map of the first kind if and only if it is either a flat surface or an open part of a sphere.*

On the other hand, if the ambient space is Minkowski 3-space, then the similar results have been very recently obtained by Kim and Turgay when the shape operator of the surface is diagonalizable. On the other hand, if the shape operator is non-diagonalizable then the following results obtained.

Theorem 29 ([29]). *Let M be a surface in $\mathbb{E}^3_1$ with non-diagonalizable shape operator whose characteristic polynomial is of the form of $\mathcal{Q}(\lambda) = (\lambda - k)^2$ for a function k. Then, the followings are equivalent*

i) *M has $\Box$-pointwise* 1-type *Gauss map.*
ii) *M has constant Gaussian curvature, i.e., k is constant.*
iii) *M is a B-scroll.*

Theorem 30 ([29]). *Let M be a surface in $\mathbb{E}^3_1$ with constant mean curvature and non-diagonalizable shape operator whose characteristic polynomial has complex roots. Then, M has $\Box$-pointwise* 1-type *Gauss map if and only if it has proper $\Box$-pointwise* 1-type *Gauss map of the second kind.*

Recently, in [33], Qian and Kim obtain complete classifications of canal surfaces with L_1-pointwise 1-type Gauss map. Up to isometries, the position vector of a canal surface in $\mathbb{E}^3$ is given by

$$x(s,\theta) = c(s) + r(s)\sin\psi(s)\cos\theta N + \sin\psi(s)\sin\theta B + \cos\psi(s)T \qquad (28)$$

where $\alpha(s)$ is a arc-length parametrized curve in $\mathbb{E}^3$ with the Frenet frame $\{T, S, B\}$ and r is a smooth function with $-r'(s) = \cos\psi(s)$. In [33] the authors proved the following theorem.

Theorem 31 ([33]). *An oriented canal surface given by* (28) *has L_1-pointwise* 1-type *Gauss map of the second kind if it is a surface of revolution.*

Note that in [28] the classification of all surface of revolutions with L_1-pointwise 1-type Gauss map was obtained. Further, in [38] Yoon *et al* move this study to pseudo-Galilean space G^3_1.

On the other hand, another generalization of notion of pointwise 1-type Gauss map has been appeared in [6] on which Baba-Hamed and Bekkar studied surface of revolutions whose Gauss map satisfying

$$\Delta^{\mathrm{II}} G = f(G + C) \tag{29}$$

where Δ^{II} denotes the Laplace operator with respect to the second fundamental form. They obtained

Theorem 32 ([6]). *A surface of revolution without parabolic points in a Euclidean 3-space has non-zero constant Gaussian curvature if and only if* (29) *is satisfied for* $C = 0$ *and some non-zero smooth function f.*

4. Lorentzian Hypersurfaces and Their Gauss Maps

In general, a hypersurface in a (semi-)Riemannian space form of dimension $n + 1$ is said to have vanishing Gauss-Kronecker curvature if its second fundamental form h is degenerated at every point or, equivalently, its shape operator satisfies $\det S = 0$. In this section, we deal with a family of Lorentzian hypersurfaces with vanishing Gauss-Kronecker curvature in the Minkowski four-space.

Let

$$\alpha : (a, b) \to \mathbb{S}^3_1(1), \qquad t \mapsto \alpha(t)$$

be a smooth unit speed curve and A, B are vector fields defined on α such that

$$\begin{aligned} &\langle A, \alpha\rangle = \langle A, \alpha'\rangle = \langle B, \alpha\rangle = \langle B, \alpha'\rangle = 0 \\ &\langle A, B\rangle = 0, \qquad \langle A, A\rangle = -\langle B, B\rangle = 1. \end{aligned} \tag{30}$$

We consider the hypersurface M given by

$$x(s, t, u) = s\alpha(t) + c\left(\cosh\frac{u}{c}A(t) + \sinh\frac{u}{c}B(t)\right) \tag{31}$$

Next, we define smooth functions ξ_1, ξ_2, ξ_3 by

$$\langle A', B'\rangle = -\xi_1(t), \qquad \langle A', \alpha'\rangle = \xi_2(t), \qquad \langle B', \alpha'\rangle = \xi_3(t).$$

Then, the induced metric of M becomes

$$g = \mathrm{d}s^2 - \mathrm{d}u^2 + \left(a^{-2} - c^2\xi_1(t)^2\right)\mathrm{d}t^2 - c\xi_1(t)(\mathrm{d}t \otimes \mathrm{d}u + \mathrm{d}u \otimes \mathrm{d}t) \tag{32}$$

and principal directions of M are

$$e_1 = \partial_s, \qquad e_2 = \partial_u, \qquad e_3 = a\left(\partial_t - c\xi_1(t)\partial_u\right)$$

with corresponding principal curvatures

$$k_1 = 0, \qquad k_2 = -\frac{1}{c}, \qquad k_3 = -a\left(\xi_2(t)\cosh\frac{u}{c} + \xi_3(t)\sinh\frac{u}{c}\right)$$

where $a = a(s, t, u)$ is the no-where vanishing function given by

$$a = \left(s + c\xi_2(t)\cosh\frac{u}{c} + c\xi_3(t)\sinh\frac{u}{c}\right)^{-1}. \tag{33}$$

Note that the unit normal vector field of M is

$$N = \cosh\frac{u}{c}A(t) + \sinh\frac{u}{c}B(t).$$

Therefore, the Gauss map of M is

$$G : M \to \mathbb{E}^4_1, \qquad x(s,t,u) \mapsto \cosh\frac{u}{c}A(t) + \sinh\frac{u}{c}B(t).$$

By a direct computation using (32), one can obtain that Levi-Civita connection of M satisfies

$$\begin{aligned} \nabla_{e_1}e_1 &= \nabla_{e_1}e_2 = \nabla_{e_1}e_3 = 0, \qquad \nabla_{e_2}e_1 = \nabla_{e_2}e_2 = \nabla_{e_2}e_3 = 0 \\ \nabla_{e_3}e_1 &= ae_3, \qquad \nabla_{e_3}e_2 = be_3, \qquad \nabla_{e_3}e_3 = -ae_1 + be_2 \end{aligned} \tag{34}$$

for the function $b = b(s,t,u)$ given by $b = a\left(\xi_2(t)\sinh\frac{u}{c} + \xi_3(t)\cosh\frac{u}{c}\right)$. By a further computation, one can see that Codazzi and Gauss equations imply

$$\begin{aligned} e_1(k_3) &= -ak_3, \qquad e_2(k_3) = -b\left(k_3 + \frac{1}{c}\right) \\ e_1(a) &= -a^2, \qquad e_2(a) = -ab \\ e_1(b) &= -ab, \qquad e_2(b) = -b^2 + \frac{1}{c}k_3. \end{aligned} \tag{35}$$

On the other hand, by using (13), we obtain

$$\Delta G = -ak_3e_1 + b\left(k_3 + \frac{1}{c}\right) + e_3(k_3)e_3 + \left(\frac{1}{c^2} + k_3^2\right)N. \tag{36}$$

Now, assume that M has pointwise 1-type Gauss map. Then, (1) is satisfied for a smooth function f and constant vector C. Therefore, we have

$$fC_1 = -ak_3e_1, \qquad fC_2 = b\left(k_3 + \frac{1}{c}\right) \tag{37}$$

where we put $C_i = \langle C, e_i\rangle,\ \ i = 1,2$. Since C is a constant vector, we have $\nabla_{e_1}C = \nabla_{e_2}C = 0$. Therefore, (34) implies

$$e_1(C_1) = e_1(C_2) = 0. \tag{38}$$

By applying e_1 to (37) and using (35) and (38), we get $e_1(f) = -2af$. By using (33) on this equation and then integrating the equation obtained, we have

$$f = \alpha a^2 \tag{39}$$

for a smooth function $\alpha = \alpha(t,u)$. By combining (37), (38) and (39), we get

$$a^2\alpha(t,u)C_2(t,u) \ = \ b\left(k_3 + \frac{1}{c}\right). \tag{40}$$

By applying e_1 to this equation and using (35), we obtain

$$-2a^3\alpha(t,u)C_2(t,u) \;=\; -ab\left(2k_3 + \frac{1}{c}\right). \tag{41}$$

However, by a simple computation using this equation and (40) we obtain $k_3 = 0$ which yields that the shape operator of M is $S = \operatorname{diag}(0, 0, -\frac{1}{c})$. Hence we obtain the following theorem.

Theorem 33. *Let M be a hypersurface in $\mathbb{E}^4_1$ given by* (31) *which has vanishing Gauss-Kronecker curvature. Then M has pointwise* 1-type *Gauss map if and only if it is an open part of either $\mathbb{S}^1_1(r) \times \mathbb{E}^2$ or $\mathbb{H}^1(-r) \times \mathbb{E}^2_1$.*

Finally, we may note that hypersurfaces in space forms with vanishing Gauss-Kronecker curvature have caught interest of many geometers (cf. [14, 25, 26, 34]). Therefore, we would like to present the following problem.

Problem 34. *Classify all hypersurfaces in semi-Euclidean spaces with vanishing Gauss-Kronecker curvature and pointwise* 1-type *Gauss map.*

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