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A NEW CHARACTERIZATION OF EULERIAN ELASTICAS

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Abstract. Here we present a new characterization of Euler elastica via the Weierstrassian functions.

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1. Euler Elastica and the Problem of Pure Bending

The problem of pure bending of elastic rods and laminae in its general form was stated for the first time in a clear mathematical formulation by Daniel Bernoulli. In 1742, in a letter to Leonhard Euler, Daniel Bernoulli suggested to him to find all the possible equilibrium shapes assumed by an elastic lamina (or rod) under two bending terminal loads (Fig. 1). On the base of the ideas of his uncle – James Bernoulli, Daniel Bernoulli suggested also an expression for the total potential energy of the lamina – an integral of the squared curvature over the profile curve of the bent lamina (see the integral in (1)), and raised himself the conjecture of the minimum potential energy of the lamina in equilibrium.

Euler presented the solution two years later in the appendix to his historical treatise on the calculus of variations, published in 1744. Following Bernoulli's suggestions, Euler made use of the "isoperimetric method", as it was named at that days the calculus of variations technique. The equation that he obtained – the *Euler-Lagrange equation*, was a fourth order ordinary differential equation. Euler succeeded in integrating it to a first order equation and obtained an integral-form solution. Actually it was an elliptic integral, which Euler went on analyzing from graphical-qualitative perspectives. The elliptic functions machinery had not been invented yet and Euler made a series of exceptional graphical observations that finally led him to the classification (graphical) of all the possible species of the *elastica*, as it was customary then to call the profile curve of a bent elastic lamina or rod. Since the times of Bernoulli and Euler such a curve is known as *Euler elastica*. In more precise settings and modern terminology, *Euler elastica* is a curve (elastic in nature) that is found as a solution to a variational problem of minimizing the integral of the squared curvature (the strain potential energy) for curves of a fixed length satisfying certain conditions about tangent directions at their ends.

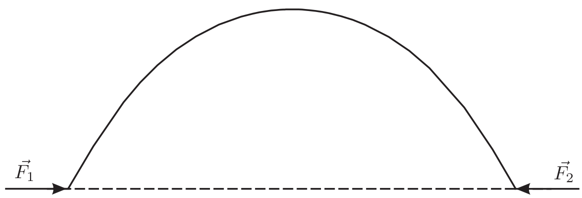

Figure 1. A thin elastic rod subjected to pure bending deformation under the two equal and opposite forces $\vec{F}_1$ and $\vec{F}_2$ applied at its ends.

According to the Bernoulli-Euler theory [6], the shape of the rod (or elastica) can be found by minimizing the integral of the energy of bending

$$U = EI \int\limits_{\text{(rod)}} \kappa^2(s)\mathrm{d}s \quad \rightarrow \quad \min \tag{1}$$

where $\kappa(s)$ and s denote respectively the curvature and the arclength along the bent rod, provided that the total length L of the deformed rod is kept fixed

$$\int\limits_{\text{(rod)}} \mathrm{d}s = L = \text{const.} \tag{2}$$

The physical (structural) characteristics of the rod are the Young's modulus of elasticity E and the second moment of inertia of the cross-section I about the neutral axis (central line) of the rod.

Let us recall, in the Bernoulli-Euler theory the rod experiences a bending stress, which is normal (perpendicular) to the cross section of the rod and increases linearly on moving away from the neutral axis. A neutral axis is an axial line of the rod that in the process of bending does not extend or contract. It means that the neutral axis is a zero-stress axial line. The axial lines that are on the "top" side of the neutral line undergo a positive (tensile) stress and stretch, while the axial lines at the "bottom" side of the neutral line are in a negative (compressive) state of stress and their length diminishes. The second moment of inertia (or the second area moment) of the cross-section I is an important structural parameter characterizing the stiffness of the rod in bending through the profile of the cross-section of the rod. It is furnished to describe the distribution (squared) of all the "patches" (differential elements) $\mathrm{d}A$ of the cross-section A of the rod with respect to the neutral axis, which is achieved by calculating the integral

$$I = \int\limits_A \rho^2 \mathrm{d}A$$

where ρ is the distance measured from each $\mathrm{d}A$ to the aforementioned axis.

2. Euler Elastica and the Weierstrassian Functions

The curvature function $\kappa(s)$ that satisfies the integral condition (1) under the constraint (2) can be obtained as a solution of a second order differential equation, the so called intrinsic equation of the Euler elastica [7, p. 119]

$$\ddot{\kappa}(s) + \frac{1}{2}\kappa^3(s) + \sigma\kappa(s) = 0$$

where σ is a tension parameter and the dots denote the derivatives with respect to the arc length parameter s. As it was shown in [3] (based on the theorem about the uniqueness of a plane curve determined up to a rigid motion in the plane by its own curvature function $\kappa(s)$ [9]) the elastica curve is equivalently described by the trajectory of a particle moving in the XOZ-plane (the plane of the elastica) with a parametric equation

$$\mathbf{x}(s) = (x(s), z(s))$$

governed by a fictitious system of two coupled ordinary differential equations

$$\begin{aligned} \ddot{x} - \lambda z \dot{z} &= 0 \\ \ddot{z} + \lambda z \dot{x} &= 0 \end{aligned} \tag{3}$$

where λ is an arbitrary positive constant. By integrating once the fictitious dynamical system (3) takes the form

$$\begin{aligned} \dot{x} &= \frac{\lambda z^2}{2} + \mu \\ \dot{z}^2 &= -\frac{\lambda^2 z^4}{4} - \lambda\mu z^2 - \mu^2 + 1 \end{aligned} \tag{4}$$

where $\mu < 1$ is an integration constant. In obtaining the above equations it was assumed that the particle trajectory is traced with unit speed [3]

$$\dot{x}^2 + \dot{z}^2 = 1.$$

Now, we proceed with the integration of the system (4) by making use of the Weierstrassian functions. From the second equation it follows that the solution for $z(s)$ is obtained by inverting the quadrature

$$s + C = \int_a^z \left(-\frac{\lambda^2 t^4}{4} - \lambda\mu t^2 - \mu^2 + 1 \right)^{-1/2} \mathrm{d}t \tag{5}$$

where a is an arbitrary root of the polynomial

$$f(t) = -\frac{\lambda^2 t^4}{4} - \lambda\mu t^2 - \mu^2 + 1$$

and C is an integration constant. After two successive substitutions (for more details, cf [4,10]) the quartic and the quadratic terms in the polynomial are removed and the latter integral is transformed to the Weierstrassian form

$$s + C = \int_\eta^\infty \frac{\mathrm{d}t}{\sqrt{4t^3 - g_2 t - g_3}} \tag{6}$$

where

$$g_2 = \frac{\lambda^2(4\mu^2 - 3)}{12}, \qquad g_3 = \frac{\lambda^3\mu(9 - 8\mu^2)}{216}$$

are the so called invariants of the polynomial $f(t)$. The lower limit of the integral is

$$\eta = \frac{f'(a)}{4(z-a)} + \frac{f''(a)}{24} \tag{7}$$

where $f'(a) \equiv \mathrm{d}f(t)/\mathrm{d}t_{|t=a}$, etc. The equation (6) is equivalent to the relation

$$\eta = \wp(s + C; g_2, g_3) \tag{8}$$

where $\wp(s + C; g_2, g_3)$ is the Weierstrassian $\wp$-function. Combining the equations (7) and (8) we arrive at

$$z(s) = a + \frac{f'(a)}{4}\left(\wp(s + C;\, g_2, g_3) - \frac{f''(a)}{24} \right)^{-1}.$$

As it can be easily seen from the factorization of the polynomial

$$f(t) = \frac{\lambda^2}{4}\left(t^2 + \frac{2(1+\mu)}{\lambda} \right)\left(-t^2 + \frac{2(1-\mu)}{\lambda} \right)$$

there exists, for $\lambda > 0$ and $\mu < 1$, a positive root of $f(t)$ which in our further considerations will be taken as the lower limit of the integral in (5), i.e.,

$$a = \sqrt{\frac{2(1-\mu)}{\lambda}}. \tag{9}$$

Finally, by the substitution of (9) in the expression for $z(s)$ derived above, and choosing $C = 0$, we obtain

$$z(s) = \sqrt{\frac{2(1-\mu)}{\lambda}} \cdot \frac{12\wp(s) - 2\lambda\mu - 3\lambda}{12\wp(s) - 2\lambda\mu + 3\lambda} \tag{10}$$

where $\wp(s) \equiv \wp(s;\, g_2, g_3)$. Now, the first equation in (4) can be directly integrated, producing the equation

$$x(s) = \frac{\lambda^2(1-\mu)}{4} J_2(s) - \lambda(1-\mu)J_1(s) + s$$

where the integrals

$$J_k(s) = \int \frac{\mathrm{d}s}{[\wp(s) - \wp(\mathring{s})]^k}, \qquad k = 1, 2$$

are expressible via the Weierstrassian functions $\wp(s)$, $\sigma(s)$ and $\zeta(s)$ [1,5]

$$J_1(s) = \frac{1}{\wp'(\mathring{s})} \left(2\zeta(\mathring{s})s + \ln \frac{\sigma(s - \mathring{s})}{\sigma(s + \mathring{s})} \right)$$

$$J_2(s) = -\frac{1}{\wp'^2(\mathring{s})} \left(\wp''(\mathring{s})J_1(s) + 2\wp(\mathring{s})s + \zeta(s - \mathring{s}) + \zeta(s + \mathring{s}) \right).$$

Here $\sigma(s) \equiv \sigma(s;\, g_2, g_3)$, $\zeta(s) \equiv \zeta(s;\, g_2, g_3)$, $\mathring{s}$ is the argument of $\wp(s)$ such that $\wp(\mathring{s}) = \lambda(2\mu - 3)/\ 12$, and $\wp'(\mathring{s}) \equiv \mathrm{d}\wp(s)/\mathrm{d}s_{|s=\mathring{s}}$, etc. After some algebraic computations we end up with the solution

$$x(s) = \frac{2}{\lambda} \left(2\zeta(s) + \frac{12\wp'(s)}{12\wp(s) - 2\lambda\mu + 3\lambda} \right) + \frac{2\mu}{3} s. \tag{11}$$

Thus we have derived the explicit expressions (10) and (11) which describe all possible shapes of the elastica. The graphics of these elasticas, produced by the above formulas and illustrating the most typical forms depicted for $\lambda = 4$ and different values of the parameter μ are presented in Fig. 2. Alternative parameterization of the Eulerian elasticas can be found in [2] and [3].

3. Some Concluding Geometrical Remarks

We have considered the problem of finding and classifying the fundamental forms of the Euler elastica by directly availing of the Weierstrassian method for expressing elliptic integrals, discovered by Karl Weierstrass in 1862/1863. As a result we obtained an explicit characterization of the shape of the Euler elastica, presented here in terms of the Weierstrassian functions $\wp(s)$, $\sigma(s)$ and $\zeta(s)$. By utilizing the computer program *Mathematica*® we have plotted (cf. Fig. 2) the graphics of elasticas corresponding to the equations (10) and (11) for different values of the parameters involved, reproducing thereby the Euler's classification picture of the elastica. In this way we have found a unified explicit description of all the shapes of the Euler elastica, respectively all the profile curves of an elastic rod subjected to pure bending deformation, given by the parametric representation of the equations (10) and (11). Note that the graphics in Fig. 2, for the cases a), b), d) and f), represent only fragments of the respective periodic elastica curves. Another interesting fact about elastica is the existence of a single closed type equilibrium state, which is the figure "eight" in case c). The value $\mathring{\mu}$ of the parameter μ responsible for the closure of the elastica is estimated (up to five decimal digits) to be $\mathring{\mu} \approx -0.65223$ (for more details see [3]).

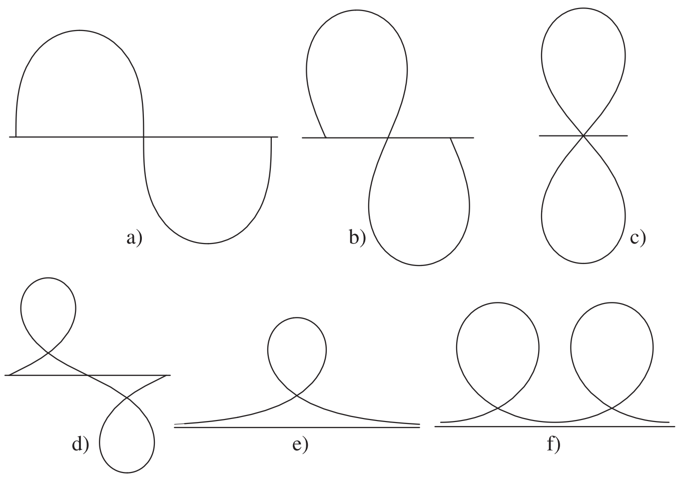

Figure 2. Various graphics of the Euler elasticas, produced by *Mathematica*® via parameterization formulas (10) and (11) in which $\lambda = 4$. The values of the parameter μ and scales of axes are different: a) $\mu = 0$; b) $\mu = -0.4$; c) $\mu \equiv \mathring{\mu}$; d) $\mu = -0.9$; e) $\mu = -1$; f) $\mu = -1.2$.

An alternative parametrization of the Euler elastica in terms of another set of elliptic functions – the so called Jacobian elliptic functions (Carl Jacobi introduced them in 1829), was suggested by Djondjorov *et al* in [3]. The functions used therein are the Jacobian amplitude $\mathrm{am}(u,k)$, Jacobian cosine $\mathrm{cn}(u,k)$ and Jacobian delta $\mathrm{dn}(u,k)$ functions, the incomplete elliptic integral of second order $E(u,k)$, with argument u and modulus k, and the hyperbolic functions $\mathrm{sech}(u)$ and $\tanh(u)$ (more about the Jacobian elliptic functions and the elliptic integrals can be found in [1] and [5]). The elastica curves in [3] are distinguished in three classes:

for $\mu \in (-1,1)$

$$x(s) = \frac{2}{\sqrt{\lambda}} E(\mathrm{am}(\sqrt{\lambda} s, k), k) - s, \qquad z(s) = a\, \mathrm{cn}(\sqrt{\lambda} s, k) \tag{12}$$

for $\mu \equiv -1$

$$x(s) = \frac{2\tanh(\sqrt{\lambda}s)}{\sqrt{\lambda}} - s, \qquad z(s) = \frac{2\,\mathrm{sech}(\sqrt{\lambda}s)}{\sqrt{\lambda}} \tag{13}$$

and for $\mu < -1$

$$x(s) = aE(\mathrm{am}(\sqrt{\frac{\lambda(1-\mu)}{2}}s, \hat{k}), \hat{k}) + \mu s, \quad z(s) = a\,\mathrm{dn}(\sqrt{\frac{\lambda(1-\mu)}{2}}s, \hat{k}). \tag{14}$$

The moduli k and $\hat{k}$ of the Jacobian functions are respectively

$$k = \sqrt{\frac{1-\mu}{2}}, \qquad \hat{k} = \sqrt{\frac{2}{1-\mu}}$$

and a is the same constant as in (9), i.e., a is the positive root of the polynomial $f(t)$. Using *Mathematica*® we have established the full coincidence of the graphics in Fig. 2 with the graphics of the curves described and plotted in [3] for any of the three classes of elasticas.

Finally we will consider one geometrical application of the Euler elastica concerning the area enclosed between the elastica curve and the coordinate axes OX, OZ. This area has shown a particular practical interest in the logboom towing, as it is noted in [3] (more about elastica-like shapes in the logboom industry can be found in [8]). It can be deduced from (4) that the area $\mathcal{A}$ of the region between the elastica and the axes (the area of the regions marked with stripes in Fig. 3) can be obtained by calculating the quadrature (cf. formula (37) in [3])

$$\mathcal{A} = \int_0^a z\frac{\mathrm{d}x}{\mathrm{d}z}\mathrm{d}z = \int_0^a \frac{z^3 + 2\mu z/\lambda}{\sqrt{(a^2-z^2)(z^2+c^2)}}\mathrm{d}z \tag{15}$$

where

$$c = \sqrt{\frac{2(1+\mu)}{\lambda}}. \tag{16}$$

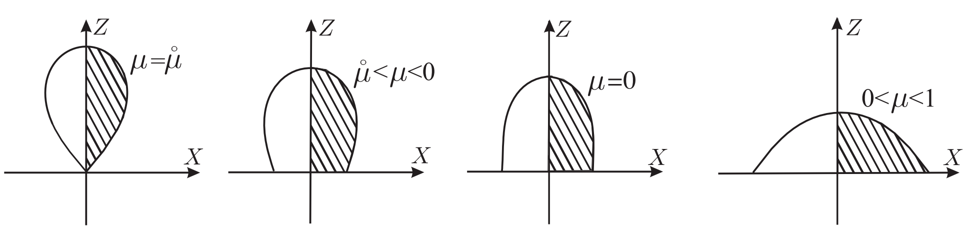

Figure 3. The striped regions present the area $\mathcal{A}$ given by formula (17).

It should be noted that the area defined by (15) corresponds to the elastica that satisfies certain initial conditions, namely the elastica curve crosses the Z axis at the point with the coordinates

$$x = 0, \qquad z = a = \sqrt{\frac{2(1-\mu)}{\lambda}}\cdot$$

By making use of the parameterization (12) in [3] it was found that the sought area is given by the formula

$$\mathcal{A} = \frac{\sqrt{1-\mu^2}}{\lambda}\cdot \tag{17}$$

Yet this amazingly simple expression can be obtained without making any reference to the concrete parameterization of the elastica. Calculating directly the quadrature (15) in the limits of the integration we have

$$\mathcal{A} = F(a) - F(0)$$

where

$$F(z) = -\frac{1}{2}\sqrt{(a^2-z^2)(z^2+c^2)} - \frac{\lambda(a^2-c^2)+4\mu}{4\lambda}\tan^{-1}\frac{a^2-c^2-2z^2}{2\sqrt{(a^2-z^2)(z^2+c^2)}}$$

and replacing in the last expression a and c by their values given in (9) and (16) we end up with the final result (17).

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