MERIDIAN SURFACES OF PARABOLIC TYPE IN THE FOUR-DIMENSIONAL MINKOWSKI SPACE

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Abstract. We construct a special class of spacelike surfaces in the Minkowski 4-space which are one-parameter systems of meridians of the rotational hypersurface with lightlike axis and call these surfaces meridian surfaces of parabolic type. They are analogous to the meridian surfaces of elliptic or hyperbolic type. Using the invariants of these surfaces we give the complete classification of the meridian surfaces of parabolic type with constant Gauss curvature or constant mean curvature. We also classify the Chen meridian surfaces of parabolic type and the meridian surfaces of parabolic type with parallel normal bundle.

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1. Introduction

A fundamental problem of the contemporary differential geometry of surfaces in the Euclidean space $\mathbb{R}^n$ or the pseudo-Euclidean space $\mathbb{R}^n_k$ is the investigation of the basic invariants characterizing the surfaces. Our aim is to investigate various important classes of surfaces in the four-dimensional Minkowski space $\mathbb{R}^4_1$ characterized by conditions on their invariants.

In [6] we developed a local theory of spacelike surfaces in $\mathbb{R}^4_1$ based on the introducing of an invariant linear map γ of Weingarten-type in the tangent plane at any point of the surface. The map γ generates two invariant functions $k = \det \gamma$ and $\varkappa = -\dfrac{1}{2} \operatorname{tr}\gamma$. It turns out that the invariant $\varkappa$ is the curvature of the normal connection of the surface. The existence of principal lines at each point of a spacelike

surface in $\mathbb{R}_1^4$ allows us to introduce a geometrically determined moving frame field at each point of the surface. Writing derivative formulas for this frame field, we obtained eight invariant functions γ_1, γ_2, ν_1, ν_2, λ, μ, β_1, β_2 and proved a fundamental theorem of Bonnet-type, stating that these eight invariants under some natural conditions determine the surface up to a rigid motion in $\mathbb{R}_1^4$.

The basic geometric classes of surfaces in $\mathbb{R}_1^4$ are characterized by conditions on these invariant functions. For example, Chen surfaces are characterized by the condition $\lambda = 0$, minimal surfaces are determined by the equality $\nu_1 + \nu_2 = 0$, surfaces with flat normal connection are described by $\nu_1 = \nu_2$, and surfaces with parallel normal bundle are characterized by $\beta_1 = \beta_2 = 0$.

In the four-dimensional Minkowski space $\mathbb{R}_1^4$ there are three types of rotational hypersurfaces - rotational hypersurfaces with timelike axis, with spacelike axis, and with lightlike axis. In [7] we constructed special families of two-dimensional spacelike surfaces lying on rotational hypersurfaces in $\mathbb{R}_1^4$ with timelike or spacelike axis and called them meridian surfaces of elliptic or hyperbolic type, respectively. These surfaces are analogous to the meridian surfaces in the Euclidean space $\mathbb{R}^4$, which are defined and studied in [5] and [9]. We found all marginally trapped meridian surfaces of elliptic or hyperbolic type. In [10] we found the geometric invariant functions γ_1, γ_2, ν_1, ν_2, λ, μ, β_1, β_2 of the meridian surfaces of elliptic or hyperbolic type and classified those of them with constant Gauss curvature or constant mean curvature. We also gave the complete classification of the Chen meridian surfaces of elliptic or hyperbolic type and the meridian surfaces of elliptic or hyperbolic type with parallel normal bundle.

In [8] we used the idea from the elliptic and hyperbolic case to construct families of two-dimensional spacelike surfaces lying on a rotational hypersurface in $\mathbb{R}_1^4$ with lightlike axis. We called these surfaces *meridian surfaces of parabolic type*. We found all marginally trapped meridian surfaces of parabolic type.

In the present paper we study meridian surfaces of parabolic type in $\mathbb{R}_1^4$ and find the invariant functions γ_1, γ_2, ν_1, ν_2, λ, μ, β_1, β_2 of these surfaces. Using the invariants we classify completely the meridian surfaces of parabolic type with constant Gauss curvature (Theorem 2), with constant mean curvature (Theorem 3), and with constant invariant k (Theorem 4). In Theorem 5 we classify the Chen meridian surfaces of parabolic type and in Theorem 6 we give the classification of the meridian surfaces of parabolic type with parallel normal bundle.

2. Invariants of Meridian Surfaces of Parabolic Type

We consider the four-dimensional Minkowski space $\mathbb{R}_1^4$ endowed with the metric $\langle \cdot , \cdot \rangle$ of signature $(3, 1)$. A surface $M^2 : z = z(u, v)$, $(u, v) \in \mathcal{D}$ $(\mathcal{D} \subset \mathbb{R}^2)$ in $\mathbb{R}_1^4$ is said to be *spacelike* if $\langle \cdot , \cdot \rangle$ induces a Riemannian metric g on M^2. Denote

by ∇' and ∇ the Levi Civita connections on $\mathbb{R}_1^4$ and M^2, respectively. Let x and y be vector fields tangent to M^2 and ξ be a normal vector field. The formulas of Gauss and Weingarten give the decompositions of the vector fields $\nabla'_x y$ and $\nabla'_x \xi$ into tangent and normal components [3]

$$\nabla'_x y = \nabla_x y + \sigma(x, y), \qquad \nabla'_x \xi = -A_\xi x + D_x \xi$$

which define the second fundamental tensor σ, the normal connection D and the shape operator A_ξ with respect to ξ. The mean curvature vector field H of M^2 is defined as $H = \dfrac{1}{2} \operatorname{tr} \sigma$.

Studying spacelike surfaces in $\mathbb{R}_1^4$ whose mean curvature vector at any point is a non-zero spacelike vector or timelike vector, on the base of the principal lines we introduced a geometrically determined orthonormal frame field $\{x, y, b, l\}$ at each point of such a surface [6]. The tangent vector fields x and y are collinear with the principal directions, the normal vector field b is collinear with the mean curvature vector field H. Writing derivative formulas of Frenet-type for this frame field, we obtained eight invariant functions γ_1, γ_2, ν_1, ν_2, λ, μ, β_1, β_2, which determine the surface up to a rigid motion in $\mathbb{R}_1^4$. These invariants are determined by the geometric frame field $\{x, y, b, l\}$ as follows

$$\nu_1 = \langle \nabla'_x x, b \rangle, \qquad \nu_2 = \langle \nabla'_y y, b \rangle, \qquad \lambda = \langle \nabla'_x y, b \rangle, \qquad \mu = \langle \nabla'_x y, l \rangle$$
$$\gamma_1 = \langle \nabla'_x x, y \rangle, \qquad \gamma_2 = \langle \nabla'_y y, x \rangle, \qquad \beta_1 = \langle \nabla'_x b, l \rangle, \qquad \beta_2 = \langle \nabla'_y b, l \rangle.$$

The invariants k, $\varkappa$, and the Gauss curvature K of M^2 are expressed by the functions $\nu_1, \nu_2, \lambda, \mu$ as follows

$$k = -4\nu_1 \nu_2 \mu^2, \qquad \varkappa = (\nu_1 - \nu_2)\mu, \qquad K = \varepsilon(\nu_1 \nu_2 - \lambda^2 + \mu^2)$$

where $\varepsilon = \operatorname{sign}\langle H, H \rangle$.

In the present section we give the construction of meridian surfaces of parabolic type and find their invariant functions γ_1, γ_2, ν_1, ν_2, λ, μ, β_1, β_2.

Let $\{e_1, e_2, e_3, e_4\}$ be the standard orthonormal frame in the Minkowski space $\mathbb{R}_1^4$, i.e., $\langle e_1, e_1 \rangle = \langle e_2, e_2 \rangle = \langle e_3, e_3 \rangle = 1$, $\langle e_4, e_4 \rangle = -1$. We denote $\xi_1 = \dfrac{e_3 + e_4}{\sqrt{2}}$, $\xi_2 = \dfrac{-e_3 + e_4}{\sqrt{2}}$ and consider the pseudo-orthonormal base $\{e_1, e_2, \xi_1, \xi_2\}$ of $\mathbb{R}_1^4$. Note that $\langle \xi_1, \xi_1 \rangle = 0$, $\langle \xi_2, \xi_2 \rangle = 0$, $\langle \xi_1, \xi_2 \rangle = -1$. The rotational hypersurface with lightlike axis in $\mathbb{R}_1^4$ can be parametrized by

$$\mathcal{M}''' : Z(u, w^1, w^2) = f(u) \, w^1 \cos w^2 \, e_1 + f(u) \, w^1 \sin w^2 \, e_2$$
$$+ \left(f(u) \tfrac{(w^1)^2}{2} + g(u) \right) \xi_1 + f(u) \, \xi_2$$

where $f = f(u)$, $g = g(u)$ are smooth functions, defined in an interval $I \subset \mathbb{R}$, such that $-f'(u)g'(u) > 0$, $f(u) > 0$, $u \in I$.

Let $w^1 = w^1(v)$, $w^2 = w^2(v)$, $v \in J$, $J \subset \mathbb{R}$ and assume that $(\dot{w}^1)^2 + (\dot{w}^2)^2 \neq 0$. We consider the surface $\mathcal{M}'''_m$ in $\mathbb{R}^4_1$ defined as follows

$$\mathcal{M}'''_m : z(u,v) = Z(u, w^1(v), w^2(v)) \tag{1}$$

where $u \in I$, $v \in J$. The surface $\mathcal{M}'''_m$, defined by (1), is a one-parameter system of meridians of the rotational hypersurface $\mathcal{M}'''$ with lightlike axis. We call $\mathcal{M}'''_m$ a *meridian surface of parabolic type*.

Without loss of generality we assume that $w^1 = \varphi(v)$, $w^2 = v$. Then the surface $\mathcal{M}'''_m$ is parameterized as follows:

$$\begin{aligned}
\mathcal{M}'''_m : z(u,v) = \ & f(u)\,\varphi(v)\cos v\, e_1 + f(u)\,\varphi(v)\sin v\, e_2 \\
& + \left(f(u)\frac{(\varphi(v))^2}{2} + g(u) \right) \xi_1 + f(u)\,\xi_2.
\end{aligned} \tag{2}$$

The parametric u-lines of $\mathcal{M}'''_m$ are curves congruent in $\mathbb{R}^4_1$ and the curvature of each u-line is $\dfrac{f'g'' - g'f''}{(-2f'g')^{\frac{3}{2}}}$. These curves are the meridians of $\mathcal{M}'''_m$. We denote $\kappa_m(u) = \dfrac{f'g'' - g'f''}{(-2f'g')^{\frac{3}{2}}}$. For each $u = u_0 = $ const the curvature of the corresponding parametric v-line is $\varkappa_{c_v} = \dfrac{\varphi\ddot{\varphi} - 2\dot{\varphi}^2 - \varphi^2}{a(\dot{\varphi}^2 + \varphi^2)^{\frac{3}{2}}}$, where $a = f(u_0)$. Let us denote $\kappa(v) = \dfrac{\varphi\ddot{\varphi} - 2\dot{\varphi}^2 - \varphi^2}{(\dot{\varphi}^2 + \varphi^2)^{\frac{3}{2}}}$. Then, the curvature of the v-line $u = u_0$ is expressed as $\varkappa_{c_v} = \dfrac{1}{a}\kappa(v)$ [8].

The tangent vector fields of $\mathcal{M}'''_m$ are

$$\begin{aligned}
z_u &= f'\varphi\cos v\, e_1 + f'\varphi\sin v\, e_2 + \left(f'\frac{\varphi^2}{2} + g' \right)\xi_1 + f'\,\xi_2 \\
z_v &= f(\dot{\varphi}\cos v - \varphi\sin v)\, e_1 + f(\dot{\varphi}\sin v + \varphi\cos v)\, e_2 + f\varphi\dot{\varphi}\,\xi_1.
\end{aligned}$$

So, the coefficients of the first fundamental form of $\mathcal{M}'''_m$ are

$$E = -2f'(u)g'(u), \qquad F = 0, \qquad G = f^2(u)(\dot{\varphi}^2(v) + \varphi^2(v)).$$

Note that the first fundamental form is positive definite, since $-f'g' > 0$.

Without loss of generality we assume that $-2f'(u)g'(u) = 1$, i.e., the meridians are parameterized by the arc-length. Then $\kappa_m(u) = \dfrac{f''(u)}{f'(u)}$.

Denote $X = z_u$, $Y = \dfrac{z_v}{f\sqrt{\dot\varphi^2 + \varphi^2}}$ and consider the following orthonormal nor-mal frame field

$$n_1 = \frac{1}{\sqrt{\dot\varphi^2 + \varphi^2}} \left((\dot\varphi \sin v + \varphi \cos v)\, e_1 + (-\dot\varphi \cos v + \varphi \sin v)\, e_2 + \varphi^2\, \xi_1 \right)$$

$$n_2 = f' \left(\varphi \cos v\, e_1 + \varphi \sin v\, e_2 + \frac{f'\varphi^2 - 2g'}{f'}\, \xi_1 + \xi_2 \right).$$

Thus we obtain a frame field $\{X, Y, n_1, n_2\}$ of $\mathcal{M}'''_m$, such that $\langle n_1, n_1 \rangle = 1$, $\langle n_2, n_2 \rangle = -1$, $\langle n_1, n_2 \rangle = 0$.

Calculating the second partial derivatives of $z(u,v)$ and using the normal frame $\{n_1, n_2\}$ given above, we get the following derivative formulas

$$
\begin{aligned}
\nabla'_X X &= -\,\kappa_m\, n_2, & \nabla'_X n_1 &= 0 \\[4pt]
\nabla'_X Y &= 0, & \nabla'_Y n_1 &= -\frac{\kappa}{f}\, Y \\[4pt]
\nabla'_Y X &= \frac{f'}{f}\, Y, & \nabla'_X n_2 &= \kappa_m\, X \\[4pt]
\nabla'_Y Y &= -\frac{f'}{f}\, X + \frac{\kappa}{f}\, n_1 - \frac{f'}{f}\, n_2, & \nabla'_Y n_2 &= \frac{f'}{f}\, Y.
\end{aligned}
\tag{3}
$$

The invariants k and $\varkappa$ of the meridian surface of parabolic type are given by the following formulas:

$$k = -\frac{\kappa_m^2(u)\,\kappa^2(v)}{f^2(u)}, \qquad \varkappa = 0.$$

Since $\varkappa$ is the curvature of the normal connection, from the equality $\varkappa = 0$ we get the following result.

Proposition 1. *The meridian surface of parabolic type $\mathcal{M}'''_m$, defined by (2), is a surface with flat normal connection.*

Taking into account (3) and using that $\kappa_m = \dfrac{f''}{f'}$, we find the Gauss curvature K and the mean curvature vector field H of $\mathcal{M}'''_m$

$$K = -\frac{f''(u)}{f(u)} \tag{4}$$

$$H = \frac{\kappa(v)}{2f(u)}\, n_1 - \frac{f(u)f''(u) + f'^2(u)}{2f(u)f'(u)}\, n_2. \tag{5}$$

We distinguish the following three cases (see [8])

   I. $\kappa(v) = 0$. In this case $n_1 = \text{const}$ and $\mathcal{M}_m'''$ lies in the hyperplane $\mathbb{R}_1^3$ of $\mathbb{R}_1^4$ orthogonal to n_1, i.e., $\mathcal{M}_m'''$ lies in $\mathbb{R}_1^3 = \text{span}\{x, y, n_2\}$.

  II. $\kappa_m(u) = 0$. In this case $\mathcal{M}_m'''$ is a developable ruled surface in $\mathbb{R}_1^4$.

 III. $\kappa_m(u)\,\kappa(v) \neq 0$.

In the first two cases the surface $\mathcal{M}_m'''$ consists of flat points, i.e., $k = \varkappa = 0$. It is known that surfaces consisting of flat points either lie in a hyperplane of $\mathbb{R}_1^4$ or are developable ruled surfaces. So, we consider the third (general) case, i.e., we assume that $\kappa_m \neq 0$ and $\kappa \neq 0$.

The mean curvature vector field H is expressed by formula (5). Since $\kappa \neq 0$, the surface $\mathcal{M}_m'''$ is non-minimal, i.e., $H \neq 0$. Recall that a spacelike surface in $\mathbb{R}_1^4$ is called marginally trapped if $H \neq 0$ and $\langle H, H \rangle = 0$. The marginally trapped meridian surfaces of parabolic type are described in [8]. So, here we consider the case $\langle H, H \rangle \neq 0$.

The orthonormal frame field $\{X, Y, n_1, n_2\}$ defined above is not the geometric frame field of the surface, since X and Y are not principal tangents. The principal tangents of $\mathcal{M}_m'''$ are determined by

$$x = \frac{X + Y}{\sqrt{2}}, \qquad y = \frac{-X + Y}{\sqrt{2}}.$$

In the case $\langle H, H \rangle > 0$, i.e., $\kappa^2 f'^2 - (f f'' + f'^2)^2 > 0$, the geometric normal frame field $\{b, l\}$ is given by

$$b = \frac{1}{\sqrt{\kappa^2 f'^2 - (f f'' + f'^2)^2}} \left(\kappa f'\, n_1 - (f f'' + f'^2)\, n_2 \right)$$

$$l = \frac{1}{\sqrt{\kappa^2 f'^2 - (f f'' + f'^2)^2}} \left((f f'' + f'^2)\, n_1 - \kappa f'\, n_2 \right).$$

In this case the normal vector fields b and l satisfy $\langle b, b \rangle = 1$, $\langle b, l \rangle = 0$, $\langle l, l \rangle = -1$.

In the case $\langle H, H \rangle < 0$, i.e., $\kappa^2 f'^2 - (f f'' + f'^2)^2 < 0$, the geometric normal frame field $\{b, l\}$ is given by

$$b = -\frac{1}{\sqrt{(f f'' + f'^2)^2 - \kappa^2 f'^2}} \left(\kappa f'\, n_1 - (f f'' + f'^2)\, n_2 \right)$$

$$l = \frac{1}{\sqrt{(f f'' + f'^2)^2 - \kappa^2 f'^2}} \left(-(f f'' + f'^2)\, n_1 + \kappa f'\, n_2 \right).$$

In this case we have $\langle b, b \rangle = -1$, $\langle b, l \rangle = 0$, $\langle l, l \rangle = 1$.

Using the geometric frame field $\{x, y, b, l\}$ of $\mathcal{M}_m'''$ and derivative formulas (3), we obtain that the geometric invariant functions of $\mathcal{M}_m'''$ are expressed by the formulas:

$$\gamma_1 = -\gamma_2 = \frac{f'}{\sqrt{2}f}, \qquad \nu_1 = \nu_2 = \frac{\sqrt{\varepsilon(\kappa^2 f'^2 - (ff'' + f'^2)^2)}}{2ff'}$$

$$\lambda = \varepsilon \frac{\kappa^2 f'^2 + f^2 f''^2 - f'^4}{2ff'\sqrt{\varepsilon(\kappa^2 f'^2 - (ff'' + f'^2)^2)}}, \qquad \mu = \frac{\kappa f''}{\sqrt{\varepsilon(\kappa^2 f'^2 - (ff'' + f'^2)^2)}}$$

$$\beta_1 = \frac{-\varepsilon f'^2}{\sqrt{2}(\kappa^2 f'^2 - (ff'' + f'^2)^2)}\left(\kappa\left(\frac{ff'' + f'^2}{f'}\right)' - \dot{\kappa}\frac{ff'' + f'^2}{ff'\sqrt{\dot{\varphi}^2 + \varphi^2}}\right)$$

$$\beta_2 = \frac{\varepsilon f'^2}{\sqrt{2}(\kappa^2 f'^2 - (ff'' + f'^2)^2)}\left(\kappa\left(\frac{ff'' + f'^2}{f'}\right)' + \dot{\kappa}\frac{ff'' + f'^2}{ff'\sqrt{\dot{\varphi}^2 + \varphi^2}}\right)$$

$$(6)$$

where $\varepsilon = \text{sign}\langle H, H\rangle$ and $\dot{\kappa} = \frac{\mathrm{d}}{\mathrm{d}v}(\kappa)$.

In the following sections, using the invariants of the meridian surface $\mathcal{M}_m'''$, we shall describe and classify some special classes of meridian surfaces of parabolic type.

3. Meridian Surfaces of Parabolic Type with Constant Gauss Curvature

The study of surfaces with constant Gauss curvature is one of the main topics in differential geometry. Surfaces with constant Gauss curvature in Minkowski space have drawn the interest of many geometers, see for example [4], [13], and the references therein.

The Gauss curvature of a meridian surface of parabolic type $\mathcal{M}_m'''$ depends only on the meridian curve m and is expressed by formula (4). The following theorem describes the meridian surfaces of parabolic type with constant non-zero Gauss curvature.

Theorem 2. *Let $\mathcal{M}_m'''$ be a meridian surface of parabolic type from the general class. Then $\mathcal{M}_m'''$ has constant non-zero Gauss curvature K if and only if the meridian m is given by*

$$f(u) = \alpha \cos\sqrt{K}u + \beta \sin\sqrt{K}u, \qquad if \qquad K > 0$$
$$f(u) = \alpha \cosh\sqrt{-K}u + \beta \sinh\sqrt{-K}u, \quad if \qquad K < 0$$

$$(7)$$

where α and β are constants, $g(u)$ is defined by $g'(u) = -\dfrac{1}{2f'(u)}$.

Proof: It follows from (4) that the Gauss curvature $K = \text{const} \neq 0$ if and only if the function $f(u)$ satisfies the following differential equation

$$f''(u) + Kf(u) = 0.$$

The general solution of the above equation is given by (7), where α and β are constants. The function $g(u)$ is determined by $g'(u) = -\dfrac{1}{2f'(u)}$. ■

4. Meridian Surfaces of Parabolic Type with Constant Mean Curvature

Surfaces with constant mean curvature in arbitrary spacetime are important objects for the special role they play in the theory of general relativity. The study of constant mean curvature surfaces (CMC surfaces) involves not only geometric methods but also PDE and complex analysis, that is why the theory of CMC surfaces is of great interest not only for mathematicians but also for physicists and engineers. Surfaces with constant mean curvature in Minkowski space have been studied intensively in the last years. See for example [1], [2], [11], [12], [14].

Let $\mathcal{M}_m'''$ be a meridian surface of parabolic type. Equality (5) implies that the mean curvature of $\mathcal{M}_m'''$ is given by

$$||H|| = \sqrt{\frac{\varepsilon(\kappa^2 f'^2 - (ff'' + f'^2)^2)}{4f^2 f'^2}}. \tag{8}$$

The following theorem gives the classification of the meridian surfaces of parabolic type with constant mean curvature.

Theorem 3. *Let $\mathcal{M}_m'''$ be a meridian surface of parabolic type from the general class. Then $\mathcal{M}_m'''$ has constant mean curvature $||H|| = a = $ const, $a \neq 0$ if and only if $\kappa = $ const $= b$, $b \neq 0$, and the meridian m is determined by $f' = y(f)$ where*

$$y(t) = \frac{1}{t}\left(C \pm \frac{t}{2}\sqrt{b^2 - 4a^2t^2} \pm \frac{b^2}{4a}\arcsin\frac{2at}{b}\right), \qquad \text{if} \quad \langle H, H\rangle > 0$$

$$y(t) = \frac{1}{t}\left(C \pm \frac{t}{2}\sqrt{b^2 + 4a^2t^2} \pm \frac{b^2}{4a}\ln|2at + \sqrt{b^2 + 4a^2t^2}|\right), \text{ if } \langle H, H\rangle < 0$$

$C = $ const, $g(u)$ is defined by $g'(u) = -\dfrac{1}{2f'(u)}$.

Proof: Using (8) we obtain that $||H|| = a$ if and only if

$$\kappa^2(v) = \frac{(ff'' + f'^2)^2 + \varepsilon 4a^2 f^2 f'^2}{f'^2}$$

which implies

$$\kappa = \text{const} = b, \qquad b \neq 0$$
$$(ff'' + f'^2)^2 + \varepsilon 4a^2 f^2 f'^2 = b^2 f'^2. \tag{9}$$

If we set $f' = y(f)$ in the second equality of (9), we obtain that the function $y = y(t)$ is a solution of the following differential equation

$$tyy' + y^2 = \pm y\sqrt{b^2 - \varepsilon 4a^2t^2}. \tag{10}$$

In the case $\varepsilon = 1$ the general solution of equation (10) is given by the formula

$$y(t) = \frac{1}{t}\left(C \pm \frac{t}{2}\sqrt{b^2 - 4a^2t^2} \pm \frac{b^2}{4a}\arcsin\frac{2at}{b}\right), \qquad C = \text{const}. \tag{11}$$

In the case $\varepsilon = -1$ the general solution of (10) is given by

$$y(t) = \frac{1}{t}\left(C \pm \frac{t}{2}\sqrt{b^2 + 4a^2t^2} \pm \frac{b^2}{4a}\ln|2at + \sqrt{b^2 + 4a^2t^2}|\right) \tag{12}$$

where $C = \text{const}$. The function $f(u)$ is determined by $f' = y(f)$ and (11) or (12), respectively. The function $g(u)$ is defined by $g'(u) = -\dfrac{1}{2f'(u)}.$

■

5. Meridian Surfaces of Parabolic Type with Constant Invariant k

Let $\mathcal{M}_m'''$ be a meridian surface of parabolic type. Then the invariant k is given by the formula

$$k = -\frac{\kappa_m^2(u)\,\kappa^2(v)}{f^2(u)}. \tag{13}$$

In the following theorem we describe the meridian surfaces of parabolic type with constant invariant k.

Theorem 4. *Let $\mathcal{M}_m'''$ be a meridian surface of parabolic type from the general class. Then $\mathcal{M}_m'''$ has constant invariant $k = \text{const} = -a^2$, $a \neq 0$ if and only if $\kappa = \text{const} = b$, $b \neq 0$, and the meridian m is determined by $f' = y(f)$ where*

$$y(t) = c \pm \frac{at^2}{2b}, \qquad c = \text{const}$$

$g(u)$ is defined by $g'(u) = -\dfrac{1}{2f'(u)}.$

Proof: Using that $\kappa_m(u) = \dfrac{f''(u)}{f'(u)}$, from (13) we obtain that $k = \text{const} = -a^2$, $a \neq 0$ if and only if

$$\kappa^2(v) = \frac{a^2 f^2(u) f'^2(u)}{f''^2(u)}.$$

The last equality implies

$$\kappa = \text{const} = b, \qquad b \neq 0$$
$$bf''(u) = \pm a f(u) f'(u). \tag{14}$$

Setting $f' = y(f)$ in the second equality of (14), we obtain that the function $y = y(t)$ is a solution of the following differential equation

$$byy' = \pm aty.$$

The general solution of the above equation is given by

$$y(t) = c \pm \frac{at^2}{2b}, \qquad c = \text{const}. \tag{15}$$

The function $f(u)$ is determined by $f' = y(f)$ and (15). The function $g(u)$ is defined by $g'(u) = -\dfrac{1}{2f'(u)}.$

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6. Chen Meridian Surfaces of Parabolic Type

In [6] we showed that a spacelike surface in $\mathbb{R}^4_1$ is a non-trivial Chen surface if and only if the invariant function λ is zero. In the next theorem we give the classification of all Chen meridian surfaces of parabolic type.

Theorem 5. *Let $\mathcal{M}'''_m$ be a meridian surface of parabolic type from the general class. Then $\mathcal{M}'''_m$ is a Chen surface if and only if $\kappa = \text{const} = b, \ b \neq 0$, and the meridian m is determined by $f' = y(f)$ where*

$$y(t) = \frac{1}{2c\,t^{\pm 1}} \left(c^2 t^{\pm 2} + b^2 \right), \qquad c = \text{const} \neq 0$$

$g(u)$ is defined by $g'(u) = -\dfrac{1}{2f'(u)}.$

Proof: It follows from (6) that $\lambda = 0$ if and only if

$$\kappa^2(v) = \frac{f'^4(u) - f^2(u)f''^2(u)}{f'^2(u)}$$

which implies

$$\kappa = \text{const} = b, \qquad b \neq 0$$
$$f'^4(u) - f^2(u)f''^2(u) = b^2 f'^2(u).$$

Hence, the function $f(u)$ is a solution of the following differential equation

$$ff'' = \pm f'\sqrt{f'^2 - b^2}. \tag{16}$$

Setting $f' = y(f)$ in equation (16), we obtain that the function $y = y(t)$ is a solution of the equation

$$tyy' = \pm y\sqrt{y^2 - b^2}. \tag{17}$$

Since $y \neq 0$ the last equation is equivalent to

$$\frac{y'}{\sqrt{y^2 - b^2}} = \pm\frac{1}{t}. \tag{18}$$

Integrating both sides of (18), we get

$$y + \sqrt{y^2 - b^2} = c\,t^{\pm 1}, \qquad c = \text{const}.$$

Hence, the general solution of differential equation (17) is given by

$$y(t) = \frac{1}{2c\,t^{\pm 1}}\left(c^2 t^{\pm 2} + b^2\right), \qquad c = \text{const} \neq 0.$$

∎

7. Meridian Surfaces with Parallel Normal Bundle

Surfaces with parallel normal bundle are characterized by the conditions $\beta_1 = 0$, $\beta_2 = 0$ (see [10]). In this section we describe the meridian surfaces of parabolic type with parallel normal bundle.

Theorem 6. *Let $\mathcal{M}_m'''$ be a meridian surface of parabolic type from the general class. Then $\mathcal{M}_m'''$ has parallel normal bundle if and only if one of the following cases holds*

a) *the meridian m is defined by*

$$f(u) = \pm(cu + d)^{\frac{1}{2}}, \qquad g(u) = \mp\frac{2}{3c^2}(cu + d)^{\frac{3}{2}} + a$$

where a, c, and d are constants

b) *$\kappa = \text{const} = b$, $b \neq 0$, and the meridian m is determined by $f' = y(f)$ where*

$$y(t) = \frac{c + at}{t}, \qquad a = \text{const} \neq 0, \qquad c = \text{const}$$

$g(u)$ is defined by $g'(u) = -\dfrac{1}{2f'(u)}.$

Proof: Using formulas (6) we get that $\beta_1 = \beta_2 = 0$ if and only if

$$\begin{aligned}
\kappa\frac{\mathrm{d}}{\mathrm{d}u}\left(\frac{ff'' + f'^2}{f'}\right) - \frac{\mathrm{d}}{\mathrm{d}v}(\kappa)\frac{ff'' + f'^2}{ff'\sqrt{\dot\varphi^2 + \varphi^2}} &= 0 \\
\kappa\frac{\mathrm{d}}{\mathrm{d}u}\left(\frac{ff'' + f'^2}{f'}\right) + \frac{\mathrm{d}}{\mathrm{d}v}(\kappa)\frac{ff'' + f'^2}{ff'\sqrt{\dot\varphi^2 + \varphi^2}} &= 0.
\end{aligned} \tag{19}$$

It follows from (19) that there are two possible cases

Case a): $ff'' + f'^2 = 0$. The general solution of this differential equation is $f(u) = \pm\sqrt{cu + d}$, $c = \text{const}$, $d = \text{const}$. Using that $g'(u) = -\dfrac{1}{2f'(u)}$, we get $g' = \mp\dfrac{\sqrt{cu + d}}{c}$. Integrating both sides of the last equation we obtain $g(u) = \mp\dfrac{2}{3c^2}(cu + d)^{\frac{3}{2}} + a$, $a = \text{const}$. Consequently, the meridian m is defined as described in a).

Case b): $\dfrac{ff'' + f'^2}{f'} = a = \text{const}$, $a \neq 0$ and $\kappa = b = \text{const}$, $b \neq 0$. In this case the meridian m is determined by the following differential equation

$$ff'' + f'^2 = af', \qquad a = \text{const} \neq 0. \tag{20}$$

The solutions of differential equation (20) can be found in the following way. Setting $f' = y(f)$ in equation (20), we obtain that the function $y = y(t)$ is a solution of the equation

$$tyy' + y^2 = ay.$$

Since $y \neq 0$ the last equation is equivalent to the equation

$$y' + \frac{1}{t}y = \frac{a}{t}$$

whose general solution is given by the formula

$$y(t) = \frac{c + at}{t}, \qquad a = \text{const} \neq 0, \qquad c = \text{const}.$$

■

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