Sixteenth International Conference on
Geometry, Integrability and Quantization
June 6–11, 2014, Varna, Bulgaria
Ivaïlo M. Mladenov, Andrei Ludu
and Akira Yoshioka, Editors
Avangard Prima, Sofia 2015, pp 207–218
doi: 10.7546/giq-16-2015-207-218

AFFINE MODELS OF INTERNAL DEGREES OF FREEDOM AND THEIR QUANTIZATION

AGNIESZKA MARTENS

Institute of Fundamental Technological Research, Polish Academy of Sciences
5^B, Pawińskiego Str., 02-106 Warsaw, Poland

Abstract. We discuss some classical and quantization problems of infinitesimal affinely-rigid bodies moving in two-dimensional manifolds. Considered are highly symmetric models for which the variables can be separated. We follow the standard procedure of quantization in Riemannian manifolds, i.e., we use the L^2-Hilbert space of wave functions in the sense of the usual Riemannian measure (volume element).

MSC: 22E70, 37J15, 81R05
Keywords: affine models, internal degrees of freedom, quantization problems, highly symmetric models, separability problem, Riemannian manifolds, Hilbert space

1. Introduction

Discussed is an affine generalization of the test rigid body model [8]. The general formulae (concerning the kinetic energy, etc.) are presented and later on we concentrate on potential models, which are in some sense isotropic and admit analytical calculations based on the separation of variables method. In particular, we consider a special case, when the translational part of the potential energy has the Bertrand structure [8]. Our results may be physically applicable in mechanics of media with microstructure. We mean micromorphic media which are continua of infinitesimal affinely-rigid bodies. Namely, surfaces of such bodies will behave as two-dimensional continua with the effective microstructure induced by the usual three-dimensional microstructure. There are also other possibilities like continua with the layered molecular structure or surface defects.

2. Classical Description

In Riemannian manifold (M, g) there are no finite affine transformations (with an exception of the trivial one), and therefore, there is no concept of extended affinely-rigid body [11,14]. But we can consider some models of infinitesimal affinely-rigid body.

The treatment consists in replacing extended bodies by structured material points, i.e., by material points with attached linear frames. These bases describe internal degrees of freedom. This means that degrees of freedom are analytically described by the spatial coordinates x^i $(i = 1, \ldots, n)$ and the components $e^i{}_A$ of the attached co-moving bases e_A $(A = 1, \ldots, n)$. The metric tensor g_{ij} is always taken at the point $x \in M$, where the body is instantaneously placed, and the basis $(\ldots, e_A, \ldots)$ is attached, so $e_A \in T_x M$. The quantities $e^i{}_A$ are not generalized coordinates. So, they are not very suitable for analytical calculations. To obtain an effective analytical description, one fixes some, usually non-holonomic field of frames E_A $(A = 1, \ldots, n)$ usually somehow distinguished by the geometry of (M, g). In a general case of affine motion the expression for the total kinetic energy has the form

$$T = T_{\rm tr} + T_{\rm int} = \frac{m}{2} g_{ij} \frac{\mathrm{d}x^i}{\mathrm{d}t} \frac{\mathrm{d}x^j}{\mathrm{d}t} + \frac{1}{2} g_{ij} \frac{\mathrm{De^i{}_A}}{\mathrm{Dt}} \frac{\mathrm{De^j{}_B}}{\mathrm{Dt}} J^{AB}.$$

In this formula the descriptors "tr" and "int" refer obviously to the translational and internal parts, m denotes the mass, and

$$J^{AB} = J^{BA}$$

are co-moving components of the tensor of internal inertia. If we take the following expansion

$$e_A(t) = E_B(x(t)) \phi^B{}_A(t)$$

where $\phi(t) \in \mathrm{GL}(n, \mathbb{R})$ is a general nonsingular matrix then we obtain for the internal part of kinetic energy the expression below

$$T_{\rm int} = \frac{1}{2} \delta_{MN} \phi^M{}_K \phi^N{}_L \widehat{\Omega}^K{}_A \widehat{\Omega}^L{}_B J^{AB}.$$

The affine velocity $\widehat{\Omega}$ in the co-moving representation is defined by

$$\frac{\mathrm{D}e_B}{\mathrm{Dt}} := e_A \widehat{\Omega}^A{}_B$$

then

$$\widehat{\Omega}^A{}_B = (\phi^{-1})^A{}_F \Gamma^F{}_{DC} \phi^D{}_B \phi^C{}_G v^G + (\phi^{-1})^A{}_C \frac{\mathrm{d}\phi^C{}_B}{\mathrm{d}t}$$

where $\Gamma^{F}{}_{DC}$ are the anholonomic components of the Levi-Civita affine connection with respect to E_A and the symbols

$$v^G = e^G{}_i \frac{\mathrm{d}x^i}{\mathrm{d}t}$$

are the co-moving components of the translational velocity.

3. Some Two-Dimensional Problems

Considered is a two-dimensional infinitesimal affinely-rigid body moving in constant-curvature spaces like the spherical space $\mathrm{S}^2(0, R)$ and pseudospherical Lobachevsky space $\mathrm{H}^{2,2,+}(0, R)$ [1]. If no gyroscopic constraints are imposed and the internal motion is affine, then of course there are four internal degrees of freedom; together with translational motion one obtains six degrees of freedom. We use the same, just as in [8], parametrization of these worlds, i.e., (r, φ) coordinates. One of analytical advantages following from the prescribed reference frame E is the possibility of using the polar and two-polar decompositions [5, 12]. We consider highly symmetric systems, when the internal inertia is isotropic, so we will use the two-polar decomposition

$$\phi = LDR^{-1} \in \mathrm{GL}(2, \mathbb{R})$$

where L, R are orthogonal and D is diagonal. It leads to the natural parametrization of the problem

$$L(\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}, \qquad R(\beta) = \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix}$$

$$D(\lambda, \mu) = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}.$$

The two-polar decomposition becomes analytically useful in doubly-isotropic dynamical problems, i.e., ones isotropic both in the physical space M and the micromaterial space. This double isotropy imposes certain restrictions both on the kinetic and potential energies. What concerns the very kinetic energy, the inertial tensor must be isotropic

$$J = I \cdot \mathrm{Id}_n$$

where Id_n denotes the identity matrix, I is a scalar constant. Then we obtain

$$T_{\mathrm{int}} = -\frac{I}{2}\mathrm{Tr}\left(D^2\widehat{\chi}^2\right) - \frac{I}{2}\mathrm{Tr}\left(D^2\widehat{\vartheta}^2\right) + I\,\mathrm{Tr}\left(D\widehat{\chi}D\widehat{\vartheta}\right) + \frac{I}{2}\mathrm{Tr}\left(\left(\frac{\mathrm{d}D}{\mathrm{d}t}\right)^2\right)$$

where corresponding "co-moving" angular velocities are given by the expressions

$$\begin{aligned}
\widehat{\vartheta} &= R^{-1}\frac{\mathrm{d}R}{\mathrm{d}t} \\
\widehat{\chi} &= \widehat{\chi}_{\mathrm{dr}} + \widehat{\chi}_{\mathrm{rl}} = \widehat{\chi}_{\mathrm{dr}} + L^{-1}\frac{\mathrm{d}L}{\mathrm{d}t} \\
\widehat{\chi}_{\mathrm{dr}}{}^{A}{}_{B} &= (L^{-1})^{A}{}_{F}\Gamma^{F}{}_{DC}L^{D}{}_{B}L^{C}{}_{E}v^{E}.
\end{aligned}$$

The labels "dr" and "rl" refer respectively to "drift" (or "drive") and "relative". Let us assume that all angular velocities become one-dimensional objects, denoted by scalar factors χ, ϑ, more precisely, they are equal to $\chi\epsilon$ and $\vartheta\epsilon$, where

$$\epsilon := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

i.e.,

$$\widehat{\chi} = L^{-1}\frac{\mathrm{d}L}{\mathrm{d}t} = \chi\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \qquad \widehat{\vartheta} = R^{-1}\frac{\mathrm{d}R}{\mathrm{d}t} = \vartheta\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

and χ is given by

i) sphere

$$\chi = \chi_{\mathrm{rl}} + \chi_{\mathrm{dr}} = \frac{\mathrm{d}\alpha}{\mathrm{d}t} + \cos\frac{r}{R}\frac{\mathrm{d}\varphi}{\mathrm{d}t}$$

ii) pseudosphere

$$\chi = \chi_{\mathrm{rl}} + \chi_{\mathrm{dr}} = \frac{\mathrm{d}\alpha}{\mathrm{d}t} + \cosh\frac{r}{R}\frac{\mathrm{d}\varphi}{\mathrm{d}t}$$

but ϑ has no "drive" term, i.e.,

$$\vartheta = \frac{\mathrm{d}\beta}{\mathrm{d}t}.$$

Now we can write the internal kinetic energy in the following form

$$T_{\mathrm{int}} = \frac{I}{2}\left(\left(\frac{\mathrm{d}\lambda}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}\mu}{\mathrm{d}t}\right)^2\right) + \frac{I\left(\lambda^2+\mu^2\right)}{2}\chi^2 + \frac{I\left(\lambda^2+\mu^2\right)}{2}\vartheta^2 - 2I\lambda\mu\chi\vartheta$$

and the translational part of the kinetic energy T_{tr} is as follows [8]

i) sphere

$$T_{\mathrm{tr}} = \frac{m}{2}\left(\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 + R^2\sin^2\frac{r}{R}\left(\frac{\mathrm{d}\varphi}{\mathrm{d}t}\right)^2\right)$$

ii) pseudosphere

$$T_{\mathrm{tr}} = \frac{m}{2}\left(\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 + R^2\sinh^2\frac{r}{R}\left(\frac{\mathrm{d}\varphi}{\mathrm{d}t}\right)^2\right).$$

It is convenient to introduce new coordinates for our calculations

$$x := \frac{1}{\sqrt{2}}(\lambda - \mu), \qquad y := \frac{1}{\sqrt{2}}(\lambda + \mu), \qquad \gamma := \alpha + \beta, \qquad \delta := \alpha - \beta.$$

The inverse rules read that

$$\lambda = \frac{1}{\sqrt{2}}(x + y), \qquad \mu = \frac{1}{\sqrt{2}}(y - x), \qquad \alpha = \frac{1}{2}(\gamma + \delta), \qquad \beta = \frac{1}{2}(\gamma - \delta).$$

The canonical momenta satisfy the contragradient rules

$$p_x = \frac{1}{\sqrt{2}}\left(p_\lambda - p_\mu\right), \qquad p_y = \frac{1}{\sqrt{2}}\left(p_\lambda + p_\mu\right)$$
$$p_\gamma = \frac{1}{2}\left(p_\alpha + p_\beta\right), \qquad p_\delta = \frac{1}{2}\left(p_\alpha - p_\beta\right)$$

and conversely

$$p_\lambda = \frac{1}{\sqrt{2}}\left(p_x + p_y\right), \qquad p_\mu = \frac{1}{\sqrt{2}}\left(p_y - p_x\right)$$
$$p_\alpha = p_\gamma + p_\delta, \qquad p_\alpha = p_\gamma - p_\delta.$$

3.1. Sperical Case

Let us order our generalized coordinates q^i, $i = \overline{1, ..., 6}$, as follows

$$r, \varphi, \gamma, \delta, x, y.$$

Then the kinetic energy will be written as follows

$$T = \frac{m}{2} G_{ij}(q) \frac{\mathrm{d}q^i}{\mathrm{d}t} \frac{\mathrm{d}q^j}{\mathrm{d}t}$$

where for the above ordering of variables the matrix $[G_{ij}]$ of the metric tensor G consists of three blocks subsequently placed along the diagonal (looking from the top to bottom):

- the 1×1 block M_1, i.e.,

$$M_1 = [1]$$

- the 3×3 block M_2 given as follows

$$M_2 = \begin{bmatrix} R^2 \sin^2 \frac{r}{R} + \frac{I}{m}\left(x^2 + y^2\right) \cos^2 \frac{r}{R} & \frac{I}{m} x^2 \cos \frac{r}{R} & \frac{I}{m} y^2 \cos \frac{r}{R} \\ \frac{I}{m} x^2 \cos \frac{r}{R} & \frac{I}{m} x^2 & 0 \\ \frac{I}{m} y^2 \cos \frac{r}{R} & 0 & \frac{I}{m} y^2 \end{bmatrix}$$

– the 2×2 isotropic block M_3, i.e.,

$$M_3 = \frac{I}{m} I_2 = \begin{bmatrix} \frac{I}{m} & 0 \\ 0 & \frac{I}{m} \end{bmatrix}$$

where I_2 denotes the 2×2 identity matrix.

One can easily show that

$$\det\left[G_{ij}\right] = R^2 \left(\frac{I}{m}\right)^4 x^2 y^2 \sin^2 \frac{r}{R}.$$

Explicitly, the block matrix $[G_{ij}]$ is given as follows

$$[G_{ij}] = \begin{bmatrix} M_1 & & 0 \\ & M_2 & \\ 0 & & M_3 \end{bmatrix}.$$

The inverse contravariant tensor matrix $\left[G^{ij}\right]$ is obviously given by

$$\left[G^{ij}\right] = \begin{bmatrix} M_1^{-1} & & 0 \\ & M_2^{-1} & \\ 0 & & M_3^{-1} \end{bmatrix}$$

where the inverse blocks have the forms

– $M_1^{-1} = [1]$

– $$M_2^{-1} = \begin{bmatrix} \frac{1}{R^2 \sin^2 \frac{r}{R}} & -\frac{\cos \frac{r}{R}}{R^2 \sin^2 \frac{r}{R}} & -\frac{\cos \frac{r}{R}}{R^2 \sin^2 \frac{r}{R}} \\ -\frac{\cos \frac{r}{R}}{R^2 \sin^2 \frac{r}{R}} & \frac{m}{I}\frac{1}{x^2} + \frac{1}{R^2}\mathrm{ctg}^2 \frac{r}{R} & \frac{1}{R^2}\mathrm{ctg}^2 \frac{r}{R} \\ -\frac{\cos \frac{r}{R}}{R^2 \sin^2 \frac{r}{R}} & \frac{1}{R^2}\mathrm{ctg}^2 \frac{r}{R} & \frac{m}{I}\frac{1}{y^2} + \frac{1}{R^2}\mathrm{ctg}^2 \frac{r}{R} \end{bmatrix}$$

– $M_3^{-1} = \frac{m}{I} I_2 = \begin{bmatrix} \frac{m}{I} & 0 \\ 0 & \frac{m}{I} \end{bmatrix}$.

For potential systems with Lagrangians of the form

$$L = T - V(q)$$

the corresponding kinetic (geodetic) Hamiltonian equals

$$\mathcal{T} = \frac{1}{2m} G^{ij}(q) p_i p_j$$

and the full Hamiltonian is as follows

$$H = \mathcal{T} + V(q).$$

According to our convention of ordering coordinates q^i, $i = \overline{1, ..., 6}$, i.e.,

$$r, \varphi, \gamma, \delta, x, y$$

the corresponding conjugate momenta p_i, $i = \overline{1, ..., 6}$, are denoted and ordered as follows

$$p_r, p_\varphi, p_\gamma, p_\delta, p_x, p_y.$$

In certain expressions it is convenient to use the original momenta p_α, p_β, p_λ, p_μ. First of all this concerns p_α, p_β because of their geometrical interpretation respectively as spin and vorticity [12].

Just as in the gyroscopic case [8], we are dealing here with non-orthogonal coordinates (the 3×3 block M_2) and it is not clear for us whether in some hypothetic orthonormal coordinates (they exist, of course) the system is separable. It is perhaps a little surprising that our kinetic Hamiltonian $\mathcal{T}$ has the separable structure. Of course, for the system with deformative degrees of freedom as above, the geodetic model is not physical because it admits unlimited expansion and contraction. Therefore, some potential must be assumed and this is just the problem, i.e., we could not determine a wide class of potentials compatible with the separability in our non-orthogonal, but nevertheless natural, coordinates. Just as in the gyroscopic case we restrict ourselves to some special class of potentials, assuming in particular that all angles φ, α, β (equivalently φ, γ, δ) are cyclic variables. Let us assume that the potential energy separates explicitly with respect to a cyclic variables, i.e.,

$$V(r, x, y) = V_r(r) + V_x(x) + V_y(y).$$

We consider a special case, when the translational part of the potential energy $V(r)$ has the Bertrand structure [8]

a) oscillatory potentials

$$V_r(r) = \frac{\xi}{2} R^2 \tan^2 \frac{r}{R}$$

b) Kepler-Coulomb potentials

$$V_r(r) = -\frac{\alpha}{R} \cot \frac{r}{R}$$

and the internal part of the potential energy is as follows

$$V(x, y) = \frac{\varkappa}{y^2} + \frac{\varkappa}{2}(x^2 + y^2) \tag{1}$$

where $\varkappa$ is a constant. The first term in (1) prevents any kind of collapse of the two-dimensional body: to the point or to the straight line. The second term of the "harmonic oscillator" type prevents the unlimited expansion.

The above mentioned Bertrand models lead to completely integrable and maximally degenerate (hyperintegrable) problems. But even for the simplest, i.e., geodetic, models with the internal degrees of freedom the situation drastically changes.

There exist interesting and practically applicable integrable models, but as a rule interaction with internal degrees of freedom reduces or completely removes degeneracy [3, 13].

3.2. Pseudospherical Case

Here all symbols concerning internal degrees of freedom are just those used in spherical geometry. The metric tensor G underlying the kinetic energy expression has the form analogous to the spherical case with the trigonometric functions simply replaced by the hyperbolic ones without any change of sign. Thus we have

– $M_1 = [1]$

– $$M_2 = \begin{bmatrix} R^2\sinh^2\frac{r}{R} + \frac{I}{m}\left(x^2+y^2\right)\cosh^2\frac{r}{R} & \frac{I}{m}x^2\cosh\frac{r}{R} & \frac{I}{m}y^2\cosh\frac{r}{R} \\ \frac{I}{m}x^2\cosh\frac{r}{R} & \frac{I}{m}x^2 & 0 \\ \frac{I}{m}y^2\cosh\frac{r}{R} & 0 & \frac{I}{m}y^2 \end{bmatrix}$$

– $M_3 = \frac{I}{m}I_2 = \begin{bmatrix} \frac{I}{m} & 0 \\ 0 & \frac{I}{m} \end{bmatrix}$.

For the inverse contravariant metric $\left[G^{ij}\right]$ underlying the geodetic Hamiltonian, i.e.,

$$\mathcal{T} = \frac{1}{2m}G^{ij}(q)p_ip_j$$

we have the block structure also quite analogous to the spherical formulas

– $M_1^{-1} = [1]$

– $$M_2^{-1} = \begin{bmatrix} \frac{1}{R^2\sinh^2\frac{r}{R}} & -\frac{\cosh\frac{r}{R}}{R^2\sinh^2\frac{r}{R}} & -\frac{\cosh\frac{r}{R}}{R^2\sinh^2\frac{r}{R}} \\ -\frac{\cosh\frac{r}{R}}{R^2\sinh^2\frac{r}{R}} & \frac{m}{I}\frac{1}{x^2} + \frac{1}{R^2}\coth^2\frac{r}{R} & \frac{1}{R^2}\coth^2\frac{r}{R} \\ -\frac{\cosh\frac{r}{R}}{R^2\sinh^2\frac{r}{R}} & \frac{1}{R^2}\coth^2\frac{r}{R} & \frac{m}{I}\frac{1}{y^2} + \frac{1}{R^2}\coth^2\frac{r}{R} \end{bmatrix}$$

– $M_3^{-1} = \frac{m}{I}I_2 = \begin{bmatrix} \frac{m}{I} & 0 \\ 0 & \frac{m}{I} \end{bmatrix}$.

One can easily show that

$$G = \det\left[G_{ij}\right] \; = \; R^2\left(\frac{I}{m}\right)^4 x^2y^2\sinh^2\frac{r}{R}.$$

Again the potential energy does not depend on the angles (φ, α, β), i.e., they are cyclic variables for the total Hamiltonian. Then the potentials have the explicitly separated form

$$V(r, x, y) = V_r(r) + V_x(x) + V_y(y)$$

where $V(r)$ is a Bertrand-type potential [8], i.e.,

a) the "harmonic oscillator"-type potential

$$V_r(r) = \frac{\xi}{2} R^2 \tanh^2 \frac{r}{R}, \qquad \xi > 0$$

b) the "attractive Kepler-Coulomb"-type one

$$V_r(r) = -\frac{\alpha}{R} \coth \frac{r}{R}, \qquad \alpha > 0$$

and $V(x, y)$ is given by (1).

4. The Quantized Problems

Let us formulate the rigorous quantum-mechanical version of the model investigated above. We use the Hilbert space $\mathrm{L}^2(Q, \mu)$ with the usual scalar product

$$\langle \Psi_1 | \Psi_2 \rangle = \int \overline{\Psi}_1(q) \Psi_2(q) \mathrm{d}\mu(q)$$

where

$$\mathrm{d}\mu(q) = \sqrt{|\det[G_{ij}]|} \mathrm{d}q^1 \cdots \mathrm{d}q^f$$

and Ψ_1, Ψ_2 are wave functions, μ is the usual Riemannian measure [2,4,6,7,9,10] and f denotes the number of degrees of freedom, i.e.,

$$f = \dim Q.$$

The Hamiltonian operator $\hat{H}$ is given by the expression

$$\hat{H} = \hat{T} + V(q) = -\frac{\hbar^2}{2} \Delta + V(q)$$

where $\hat{T}$ is the kinetic energy operator and Δ denotes the Laplace-Beltrami operator corresponding to G

$$\Delta = \frac{1}{\sqrt{|G|}} \sum_{i,j} \partial_i \sqrt{|G|} G^{ij} \partial_j = G^{ij} \nabla_i \nabla_j$$

where ∇ is the Levi-Civita covariant differentiation in the G-sense [15–18].

A basis of solutions of the stationary Schrödinger equation

$$\hat{H}\Psi = E\Psi$$

has the form

$$\Psi(r, \varphi, \gamma, \delta, x, y) = f_r(r) f_\varphi(\varphi) f_\gamma(\gamma) f_\delta(\delta) f_x(x) f_y(y).$$

For certain reasons it is convenient to use the new variable

$$\theta = \frac{r}{R}$$

then we put

$$\Psi(\theta, \varphi, \gamma, \delta, x, y) = f_\theta(\theta) f_x(x) f_y(y) \mathrm{e}^{\mathrm{i}s\varphi} \mathrm{e}^{\mathrm{i}j\gamma} \mathrm{e}^{\mathrm{i}u\delta}$$

where s, j, u are integers.

Hence, the stationary Schrödinger equation with an arbitrary potential V leads after the standard separation procedure to the following system of one-dimensional eigenequations. Depending on the considered manifold we have

i) sphere

$$\begin{aligned}
\frac{\mathrm{d}^2 f_x(x)}{\mathrm{d}x^2} + \frac{1}{x}\frac{\mathrm{d}f_x(x)}{\mathrm{d}x} - \left(\frac{(k+l)^2}{4x^2} - \frac{2I}{\hbar^2}\left(E_x(x) - V_x(x)\right)\right) f_x(x) &= 0 \\
\frac{\mathrm{d}^2 f_y(y)}{\mathrm{d}y^2} + \frac{1}{y}\frac{\mathrm{d}f_y(y)}{\mathrm{d}y} - \left(\frac{(k-l)^2}{4y^2} - \frac{2I}{\hbar^2}\left(E_y(y) - V_y(y)\right)\right) f_y(y) &= 0 \\
\frac{\mathrm{d}^2 f_\theta(\theta)}{\mathrm{d}\theta^2} + \frac{\cot\theta}{R}\frac{\mathrm{d}f_\theta(\theta)}{\mathrm{d}\theta} & \\
- \left(\frac{(s - k\cos\theta)^2}{R^2\sin^2\theta} - \frac{2m}{\hbar^2}\left(E - E_x(x) - E_y(y) - V_\theta(\theta)\right)\right) f_\theta(\theta) &= 0
\end{aligned}$$

where E, $E_x(x)$, $E_y(y)$ are fixed values of energies. The relationship between (γ, δ) and (α, β) implies that $k = j + u$ and $l = j - u$.

ii) pseudosphere

$$\begin{aligned}
\frac{\mathrm{d}^2 f_x(x)}{\mathrm{d}x^2} + \frac{1}{x}\frac{\mathrm{d}f_x(x)}{\mathrm{d}x} - \left(\frac{(k+l)^2}{4x^2} - \frac{2I}{\hbar^2}\left(E_x(x) - V_x(x)\right)\right) f_x(x) &= 0 \\
\frac{\mathrm{d}^2 f_y(y)}{\mathrm{d}y^2} + \frac{1}{y}\frac{\mathrm{d}f_y(y)}{\mathrm{d}y} - \left(\frac{(k-l)^2}{4y^2} - \frac{2I}{\hbar^2}\left(E_y(y) - V_y(y)\right)\right) f_y(y) &= 0 \\
\frac{\mathrm{d}^2 f_\theta(\theta)}{\mathrm{d}\theta^2} + \frac{\coth\theta}{R}\frac{\mathrm{d}f_\theta(\theta)}{\mathrm{d}\theta} & \\
- \left(\frac{(s - k\cosh\theta)^2}{R^2\sinh^2\theta} - \frac{2m}{\hbar^2}\left(E - E_x(x) - E_y(y) - V_\theta(\theta)\right)\right) f_\theta(\theta) &= 0.
\end{aligned}$$

In the second case the one-dimensional eigenequations take on the form exactly as in the theory of deformable gyroscope in the spherical space. The only formal difference is that the trigonometric functions of r/R (θ) are replaced by the hyperbolic ones without the change of sign.

Acknowledgements

It is my pleasure to dedicate this paper to Professor Jan Jerzy Sławianowski on the occasion of his 70-th anniversary in 2013. I also would like to thank Professor Ivaïlo M. Mladenov and all Organizers of the Sixteenth International Conference on Geometry, Integrability and Quantization, June 6–11, 2014, Varna, Bulgaria for their cordial hospitality. The paper contains the results obtained within the framework of the research project N N501 049 540. The author is greatly indebted to the Ministry of Science and Higher Education for this financial support.

References

[1] Kobayashi S. and Nomizu K., *Foundations of Differential Geometry*, Interscience Publishers, New York, London 1963.

[2] Mackey G., *The Mathematical Foundations of Quantum Mechanics*, W. A. Benjamin, New York-Amsterdam 1963.

[3] Martens A., *Dynamics of Holonomically Constrained Affinely-Rigid Body*, Rep. Math. Phys. **49** (2002) 295–303.

[4] Martens A., *Quantization of an Affinely-Rigid Body with Constraints*, Rep. Math. Phys. **51** (2003) 287–295.

[5] Martens A., *Hamiltonian Dynamics of Planar Affinely-Rigid Body*, J. Nonlinear Math. Phys. **11** Supplement (2004) 145–150.

[6] Martens A., *Quantization of the Planar Affinely-Rigid Body*, J. Nonlinear Math. Phys. **11** Supplement (2004) 151–156.

[7] Martens A., *Quantization of Two-Dimensional Affine Bodies with Stabilized Dilatations*, Rep. Math. Phys. **62** (2008) 145–155.

[8] Martens A., *Test Rigid Bodies in Riemannian Spaces and Their Quantization*, Rep. Math. Phys. **71** (2013) 381–398.

[9] Martens A. and Sławianowski J., *Affinely-Rigid Body and Oscillatory Dynamical Models on* $\mathrm{GL}(2,\mathbb{R})$, Acta Phys. Pol. B **41** (2010) 1847–1880.

[10] Rubinowicz W., *Quantum Mechanics*, PWN, Warsaw 1968.

[11] Sławianowski J., *The Mechanics of an Affinelly-Rigid Body*, Int. J. Theoret. Phys. **12** (1975) 271–296.

[12] Sławianowski J., *Analytical Mechanics of Deformable Bodies* (in Polish), PWN, Warsaw-Poznań 1982.

[13] Sławianowski J., *Bertrand Systems on Spaces of Constant Sectional Curvature. The Action-Angle Analysis*, Rep. Math. Phys. **46** (2000) 429–460.

[14] Sławianowski J., *Linear Frames in Manifolds, Riemannian Structures and Description of Internal Degrees of Freedom*, Rep. Math. Phys. **51** (2003) 345–369.

[15] Sławianowski J., Kovalchuk V., Martens A., Gołubowska B. and Rożko E., *Quasiclassical and Quantum Systems of Angular Momentum. Part I. Group Algebras as a*

Framework for Quantum-Mechanical Models with Symmetries, J. Geom. Symmetry Phys. **21** (2011) 61–94.

[16] Sławianowski J., Kovalchuk V., Martens A., Gołubowska B. and Rożko E., *Quasi-classical and Quantum Systems of Angular Momentum. Part II. Quantum Mechanics on Lie Groups and Methods of Group Algebras*, J. Geom. Symmetry Phys. **22** (2011) 67–94.

[17] Sławianowski J., Kovalchuk V., Martens A., Gołubowska B. and Rożko E., *Quasi-classical and Quantum Systems of Angular Momentum. Part III. Group Algebra su(2), Quantum Angular Momentum and Quasiclassical Asymptotics*, J. Geom. Symmetry Phys. **23** (2011) 59–95.

[18] Sławianowski J., Kovalchuk V., Martens A., Gołubowska B. and Rożko E., *Quasi-classical and Quantum Dynamics of Systems of Angular Momenta*, Geometry, Integrability & Quantization, **12** (2011) pp 70–155.