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ON NEW IDEAS OF NONLINEARITY IN QUANTUM MECHANICS

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Abstract. Our main idea is to suggest a new model of non-perturbative and geometrically motivated nonlinearity in quantum mechanics. The Schrödinger equation and corresponding relativistic linear wave equations derivable from variational principles are analyzed as usual self-adjoint equations of mathematical physics. It turns out that introducing the second-order time derivatives to dynamical equations, even as small corrections, can help to obtain the regular Legendre transformation. Following the conceptual transition from the special to general theory of relativity, where the metric tensor loses its status of the absolute geometric object and becomes included into degrees of freedom (gravitational field), in our treatment the Hilbert-space scalar product becomes a dynamical quantity which satisfies together with the state vector the system of differential equations. The structure of obtained Lagrangian and equations of motion is very beautiful, as usually in high-symmetry problems.

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Introduction

It is well known that quantum mechanics is still plagued with some paradoxes concerning

- decoherence
- measurement process
- reduction of the state vector.

There is an opinion that the main concern is the linearity of the Schrödinger equation which seems to be drastically incompatible with the above-mentioned problems. But at the same time the linearity works well when

- describing the unobserved unitary quantum evolution
- finding the energy levels
- in all statistical predictions.

It seems that we deal here with a very sophisticated and delicate nonlinearity which becomes active and remarkable just in the process of interaction between quantum systems and "large" classical objects.

Therefore, the main idea is to analyze the Schrödinger equation and corresponding relativistic linear wave equations as usual self-adjoint equations of mathematical physics which are derivable from variational principles. It is easy to construct their Lagrangians, but some problems appear when trying to formulate Hamiltonian formalism, because Lagrangians for the Schrödinger or Dirac equations are highly degenerate and the corresponding Legendre transformation is non-invertible and leads to constraints in the phase space. Nevertheless, using the Dirac formalism for such Lagrangians, one can find the corresponding Hamiltonian formalism.

Incidentally, it turns out that introducing the second-order time derivatives to dynamical equations, even as small corrections, one can obtain the regular Legendre transformation. In non-relativistic quantum mechanics there are certain hints suggesting just such a modification in the nano-scale physics [3, 4]

1. As the first step a formal analogy between the quantum Fourier equation which describes the heat (or mass) diffusion on the atomic level and the free Schrödinger equation is constructed with the help of the following substitution
$$\frac{\partial T}{\partial t} = \frac{\hbar}{m}\nabla^2 T$$
where $t \to \mathrm{i}t/2$ and $T \to \psi$, then we obtain the free Schrödinger equation as it was expected
$$\mathrm{i}\hbar\frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi.$$
2. Then the complete Schrödinger equation with the potential term V
$$\mathrm{i}\hbar\frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + V\psi$$
after the reverse substitutions $t \to -2\mathrm{i}t$ and $\psi \to T$ gives us the parabolic quantum Fokker-Planck equation
$$\frac{\partial T}{\partial t} = \frac{\hbar}{m}\nabla^2 T - \frac{2V}{\hbar}T$$

which describes the quantum heat transport for time intervals $\Delta t > \tau$, where $\tau = \hbar/m\alpha^2c^2 \sim 10^{-17}$ sec, and distances $c\tau \sim 1$ nm.

3. From the other side for ultrashort time processes when $\Delta t < \tau$ one can obtain the generalized quantum hyperbolic heat transport equation

$$\tau\frac{\partial^2 T}{\partial t^2} + \frac{\partial T}{\partial t} = \frac{\hbar}{m}\nabla^2 T - \frac{2V}{\hbar}T$$

which leads us to the second-order modified Schrödinger equation

$$2\tau\hbar\frac{\partial^2\psi}{\partial t^2} + \mathrm{i}\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + V\psi$$

in which the additional term describes the interaction of electrons with surrounding space-time filled with virtual positron-electron pairs.

This situation also reminds us the other problem of Klein-Gordon-Dirac equation studied in a bit different context motivated by the idea of conformal invariance [1, 5, 6]. There the wave equation with the superposition of Dirac and d'Alembert operators were studied.

The main idea of our approach is to follow the conceptual transition from special to general theory of relativity [2]

- in specially-relativistic theories the metric tensor is fixed once for all as an absolute geometric object, whereas all physical fields are "flexible" and satisfy differential equations as a rule derivable from the variational principle
- in generally-relativistic theories the metric tensor becomes "flexible" as well, it is included into degrees of freedom and satisfies differential equations together with the other "physical" fields; moreover it becomes itself the physical field, in this case the gravitational one.

Similarly, in our treatment the Hilbert-space scalar product becomes a dynamical quantity which satisfies (together with the state vector) the system of differential equations.

So, the main idea is that there is no fixed scalar product metric and the dynamical term of Lagrangian describing the self-interaction of the metric is invariant under the total group $\mathrm{GL}(n, \mathbb{C})$. But this invariance is possible only for models non-quadratic in the metric, just like in certain problems of the dynamics of affinely-rigid bodies [9–11].

There is a natural metric of this kind and it introduces to the theory a very strong nonlinearity which induces also the effective nonlinearity of the wave equation even if there is no "direct nonlinearity" in it. The structure of Lagrangian and equations of motion is very beautiful as usually in high-symmetry problems. Nevertheless, the very strong nonlinearity prevents us from finding a rigorous solution.

But some partial results are possible if we fix the behaviour of the wave function to some simple form, then for the scalar product behaviour there are rigorous exponential solutions, including ones which are infinitely growing and ones which are exponentially decaying in the future. This makes some hope for describing, e.g., some decay/reduction phenomena.

So, we are dealing with two kinds of degrees of freedom, i.e., the dynamical variables: the wave function and the scalar product, which are mutually interacting.

1. N-Level Quantum System in the N-Dimensional "Hilbert" Space

Let us interpret the unitary evolution of a quantum system described by the Schrödinger equation as a Hamiltonian system on Hilbert space [2, 7, 8]. The most convenient way to visualize this is to start from finite-dimensional, i.e., "n-level", quantum system ($n < \infty$). Then we can define the "wave function" as a following n-vector

$$\psi = \begin{bmatrix} \psi^1 \\ \vdots \\ \psi^n \end{bmatrix}, \qquad \psi^a = \psi(a) \in \mathbb{C}.$$

Let H be a unitary space (n-dimensional "Hilbert space" $\mathbb{C}^n$) with the scalar product (a sesquilinear hermitian form) defined as follows

$$G : H \times H \to \mathbb{C}.$$

Let us consider the following most general form of the Lagrangian admitting the second time derivatives of both the wave function ψ and scalar product G

$$\begin{aligned} L \;=\; & \alpha_1 \mathrm{i} G_{\bar{a}b} \left(\overline{\psi}^{\bar{a}} \dot{\psi}^b - \dot{\overline{\psi}}^{\bar{a}} \psi^b \right) + \alpha_2 G_{\bar{a}b} \dot{\overline{\psi}}^{\bar{a}} \dot{\psi}^b \\ & + \alpha_3 \left[G^{b\bar{a}} + \alpha_9 \overline{\psi}^{\bar{a}} \psi^b \right] \dot{G}_{\bar{a}b} + \Omega[\psi, G]^{d\bar{c}b\bar{a}} \dot{G}_{\bar{a}b} \dot{G}_{\bar{c}d} \\ & + \left[\alpha_4 G_{\bar{a}b} + \alpha_5 H_{\bar{a}b} \right] \overline{\psi}^{\bar{a}} \psi^b - \mathcal{V}(\psi, G) \end{aligned} \tag{1}$$

where the introduced auxiliary object $\Omega[\psi, G]$ is defined as follows

$$\begin{aligned} \Omega[\psi, G]^{d\bar{c}b\bar{a}} \;=\; & \alpha_6 \left[G^{d\bar{a}} + \alpha_9 \overline{\psi}^{\bar{a}} \psi^d \right] \left[G^{b\bar{c}} + \alpha_9 \overline{\psi}^{\bar{c}} \psi^b \right] \\ & + \alpha_7 \left[G^{b\bar{a}} + \alpha_9 \overline{\psi}^{\bar{a}} \psi^b \right] \left[G^{d\bar{c}} + \alpha_9 \overline{\psi}^{\bar{c}} \psi^d \right] \\ & + \alpha_8 \, \overline{\psi}^{\bar{a}} \psi^b \overline{\psi}^{\bar{c}} \psi^d \end{aligned}$$

and it satisfies also the following symmetry condition

$$\Omega[\psi, G]^{d\bar{c}b\bar{a}} = \Omega[\psi, G]^{b\bar{a}d\bar{c}}.$$

In the above formulae $\overline{\psi}^{\bar{a}} = \overline{\psi^a}$ denotes the usual complex conjugation and α_i, $i = \overline{1, ..., 9}$, $\varkappa$ are some constants.

The interpretation of the consecutive terms in Lagrangian (1) is as follows

- the first and second terms (those with α_1 and α_2) describe the free evolution of wave function ψ while G is fixed
- the third term (that with α_4) describes the trivial part of the linear dynamics and it can be also taken in the more general form
$$f\left(G_{\bar{a}b}\overline{\psi}^{\bar{a}}\psi^{b}\right)$$
where $f : \mathbb{R} \to \mathbb{R}$
- the forth term (that with α_5) corresponds to the Schrödinger dynamics while G is fixed and then
$$H^{a}{}_{b} = G^{a\bar{c}}H_{\bar{c}b}$$
is the usual Hamilton operator (if we properly choose the constants α_1 and α_5, then we obtain precisely the Schrödinger equation — see Section 5.2)
- the next two terms (those linear and quadratic in the time derivatives of G) describe the dynamics of the scalar product G
- and the last term is the potential $\mathcal{V}(\psi, G)$ which can be taken, for instance, in the following quartic form
$$\mathcal{V}(\psi, G) = \varkappa\left(G_{\bar{a}b}\overline{\psi}^{\bar{a}}\psi^{b}\right)^{2}.$$

2. $\mathrm{GL}(n, \mathbb{C})$-Invariance and Conservation Laws

So, if we investigate the invariance of our general Lagrangian (1) under the group $\mathrm{GL}(n, \mathbb{C})$ and consider some one-parameter group of transformations
$$\{\exp(A\tau)\ ;\ \tau \in \mathbb{R}\}, \qquad A \in \mathrm{L}(n, \mathbb{C})$$
then the infinitesimal transformations rules for ψ and G are as follows
$$\psi^{a} \mapsto L^{a}{}_{b}\psi^{b}, \qquad G^{a\bar{c}} \mapsto L^{a}{}_{b}\overline{L}^{\bar{c}}{}_{\bar{e}}G^{b\bar{e}}, \qquad G_{\bar{a}b} \mapsto G_{\bar{c}d}\overline{L^{-1}}^{\bar{c}}{}_{\bar{a}}L^{-1d}{}_{b}$$
where
$$L^{a}{}_{b} = \delta^{a}{}_{b} + \epsilon A^{a}{}_{b}, \qquad L^{-1a}{}_{b} \approx \delta^{a}{}_{b} - \epsilon A^{a}{}_{b}, \qquad \epsilon \approx 0.$$
So leaving only the first-order terms with respect to ϵ we obtain that the variations of ψ and G are as follows
$$\delta\psi^{a} = \epsilon A^{a}{}_{b}\psi^{b}, \qquad \delta\overline{\psi}^{\bar{a}} = \epsilon\overline{A}^{\bar{a}}{}_{\bar{c}}\overline{\psi}^{\bar{c}}$$
$$\delta G^{a\bar{c}} = \epsilon\left(A^{a}{}_{b}G^{b\bar{c}} + \overline{A}^{\bar{c}}{}_{\bar{e}}G^{a\bar{e}}\right), \qquad \delta G_{\bar{a}b} = -\epsilon\left(G_{\bar{c}b}\overline{A}^{\bar{c}}{}_{\bar{a}} + G_{\bar{a}d}A^{d}{}_{b}\right).$$

Then

$$\frac{1}{\epsilon}\left(\frac{\partial L}{\partial\dot{\overline{\psi}}^{\bar{a}}}\delta\overline{\psi}^{\bar{a}}+\frac{\partial L}{\partial\dot{\psi}^{b}}\delta\psi^{b}\right)=G_{\bar{a}b}\left(\alpha_{2}\dot{\overline{\psi}}^{\bar{a}}+\alpha_{1}\mathrm{i}\overline{\psi}^{\bar{a}}\right)A^{b}{}_{d}\psi^{d} + G_{\bar{a}b}\left(\alpha_{2}\dot{\psi}^{b}-\alpha_{1}\mathrm{i}\psi^{b}\right)\overline{A}^{\bar{a}}{}_{\bar{c}}\overline{\psi}^{\bar{c}} \quad (2)$$

and

$$\begin{aligned}\frac{1}{\epsilon}\frac{\partial L}{\partial\dot{G}_{\bar{a}b}}\delta G_{\bar{a}b} &= -\left[\alpha_{3}\left(\delta^{b}{}_{f}+\alpha_{9}G_{\bar{a}f}\overline{\psi}^{\bar{a}}\psi^{b}\right)+2\Omega[\psi,G]^{b\bar{a}d\bar{c}}G_{\bar{a}f}\dot{G}_{\bar{c}d}\right]A^{f}{}_{b} \\ &\quad -\left[\alpha_{3}\left(\delta^{\bar{a}}{}_{\bar{e}}+\alpha_{9}G_{\bar{e}b}\overline{\psi}^{\bar{a}}\psi^{b}\right)+2\Omega[\psi,G]^{b\bar{a}d\bar{c}}G_{\bar{e}b}\dot{G}_{\bar{c}d}\right]\overline{A}^{\bar{e}}{}_{\bar{a}}.\end{aligned} \quad (3)$$

If we consider some fixed scalar product G_0 and take the G_0-hermitian A's, then

$$A^{a}{}_{b}={G_{0}}^{a\bar{c}}\widetilde{A}_{\bar{c}b},\qquad \overline{A}^{\bar{a}}{}_{\bar{c}}=\widetilde{A}_{\bar{c}b}G_{0}^{b\bar{a}},\qquad \widetilde{A}^{\dagger}=\widetilde{A}$$

and therefore the expressions (2) and (3) are written together in the matrix form as follows

$$\mathcal{J}\left(A\right)=\mathrm{Tr}\left(V\widetilde{A}\right)$$

where the hermitian tensor V describing the system of conserved physical quantities is given as follows

$$\begin{aligned}V &= \alpha_{2}\left(\psi\dot{\psi}^{\dagger}GG_{0}^{-1}+G_{0}^{-1}G\dot{\psi}\psi^{\dagger}\right) \\ &\quad +\left(\alpha_{1}\mathrm{i}-\alpha_{3}\alpha_{9}\right)\psi\psi^{\dagger}GG_{0}^{-1}-\left(\alpha_{1}\mathrm{i}+\alpha_{3}\alpha_{9}\right)G_{0}^{-1}G\psi\psi^{\dagger} \\ &\quad -2\alpha_{3}G_{0}^{-1}-2\left(G_{0}^{-1}G\omega[\psi,G]+\omega[\psi,G]GG_{0}^{-1}\right)\end{aligned}$$

where

$$\omega\left[\psi,G\right]^{b\bar{a}}=\Omega\left[\psi,G\right]^{b\bar{a}d\bar{c}}\dot{G}_{\bar{c}d}.$$

Similarly for the G_0-antihermitian A's, i.e., when $\widetilde{A}^{\dagger}=-\widetilde{A}$, we obtain another hermitian tensor W as a conserved value

$$\mathcal{J}\left(A\right)=\mathrm{Tr}\left(iW\widetilde{A}\right)$$

where

$$\begin{aligned}\mathrm{i}W &= \alpha_{2}\left(\psi\dot{\psi}^{\dagger}GG_{0}^{-1}-G_{0}^{-1}G\dot{\psi}\psi^{\dagger}\right) \\ &\quad +(\alpha_{1}i-\alpha_{3}\alpha_{9})\psi\psi^{\dagger}GG_{0}^{-1}+(\alpha_{1}i+\alpha_{3}\alpha_{9})G_{0}^{-1}G\psi\psi^{\dagger} \\ &\quad +2\left(G_{0}^{-1}G\omega[\psi,G]-\omega[\psi,G]GG_{0}^{-1}\right).\end{aligned}$$

3. Euler-Lagrange Equations of Motion

Applying the variational procedure we obtain the following Euler-Lagrange equations of motion for the system "wave function + flexible scalar products" which are essentially nonlinear (this is the non-perturbative nonlinearity, i.e., it cannot be interpreted as an artificial extra correction to some basic linear background)

$$\begin{aligned}\frac{\delta L}{\delta\overline{\psi}^{\bar{a}}} \;=\;& \alpha_2 G_{\bar{a}b}\ddot{\psi}^b + \left(\alpha_2\dot{G}_{\bar{a}b} - 2\alpha_1 \mathrm{i} G_{\bar{a}b}\right)\dot{\psi}^b - 2\alpha_8 \dot{G}_{\bar{a}b}\psi^b \dot{G}_{\bar{c}d}\overline{\psi}^{\bar{c}}\psi^d \\ & -2\alpha_9\left(\alpha_6 \dot{G}_{\bar{a}d}\dot{G}_{\bar{c}b} + \alpha_7 \dot{G}_{\bar{a}b}\dot{G}_{\bar{c}d}\right)\psi^b\left(G^{d\bar{c}} + \alpha_9\overline{\psi}^{\bar{c}}\psi^d\right) \\ & + \left[\left(2\varkappa G_{\bar{c}d}\overline{\psi}^{\bar{c}}\psi^d - \alpha_4\right)G_{\bar{a}b} - \alpha_5 H_{\bar{a}b} - \left[\alpha_3\alpha_9 + \alpha_1\mathrm{i}\right]\dot{G}_{\bar{a}b}\right]\psi^b = 0\end{aligned}$$

and

$$\begin{aligned}\frac{\delta L}{\delta G_{\bar{a}b}} \;=\;& 2\Omega[\psi,G]^{b\bar{a}d\bar{c}}\ddot{G}_{\bar{c}d} + 2\dot{\Omega}[\psi,G]^{b\bar{a}d\bar{c}}\dot{G}_{\bar{c}d} + \left(2\varkappa G_{\bar{c}d}\overline{\psi}^{\bar{c}}\psi^d - \alpha_4\right)\overline{\psi}^{\bar{a}}\psi^b \\ & +2G^{d\bar{a}}\left[\alpha_6 G^{b\bar{e}}\left(G^{f\bar{c}} + \alpha_9\overline{\psi}^{\bar{c}}\psi^f\right) + \alpha_7 G^{b\bar{c}}\left(G^{f\bar{e}} + \alpha_9\overline{\psi}^{\bar{e}}\psi^f\right)\right]\dot{G}_{\bar{c}d}\dot{G}_{\bar{e}f} \\ & -\alpha_2\dot{\overline{\psi}}^{\bar{a}}\dot{\psi}^b + \left[\alpha_3\alpha_9 + \alpha_1\mathrm{i}\right]\dot{\overline{\psi}}^{\bar{a}}\psi^b + \left[\alpha_3\alpha_9 - \alpha_1\mathrm{i}\right]\overline{\psi}^{\bar{a}}\dot{\psi}^b = 0.\end{aligned}$$

The explicit form of the time derivative of the auxiliary object $\Omega[\psi, G]$ can be written as follows

$$\begin{aligned}\dot{\Omega}[\psi,G]^{b\bar{a}d\bar{c}} \;=\;& \alpha_8\left(\dot{\overline{\psi}}^{\bar{a}}\psi^b\overline{\psi}^{\bar{c}}\psi^d + \overline{\psi}^{\bar{a}}\dot{\psi}^b\overline{\psi}^{\bar{c}}\psi^d + \overline{\psi}^{\bar{a}}\psi^b\dot{\overline{\psi}}^{\bar{c}}\psi^d + \overline{\psi}^{\bar{a}}\psi^b\overline{\psi}^{\bar{c}}\dot{\psi}^d\right) \\ & +\alpha_6\alpha_9\left(\left[\dot{\overline{\psi}}^{\bar{a}}\psi^d + \overline{\psi}^{\bar{a}}\dot{\psi}^d\right]\left[G^{b\bar{c}} + \alpha_9\overline{\psi}^{\bar{c}}\psi^b\right]\right. \\ & \qquad\left. + \left[\dot{\overline{\psi}}^{\bar{c}}\psi^b + \overline{\psi}^{\bar{c}}\dot{\psi}^b\right]\left[G^{d\bar{a}} + \alpha_9\overline{\psi}^{\bar{a}}\psi^d\right]\right) \\ & +\alpha_7\alpha_9\left(\left[\dot{\overline{\psi}}^{\bar{a}}\psi^b + \overline{\psi}^{\bar{a}}\dot{\psi}^b\right]\left[G^{d\bar{c}} + \alpha_9\overline{\psi}^{\bar{c}}\psi^d\right]\right. \\ & \qquad\left. + \left[\dot{\overline{\psi}}^{\bar{c}}\psi^d + \overline{\psi}^{\bar{c}}\dot{\psi}^d\right]\left[G^{b\bar{a}} + \alpha_9\overline{\psi}^{\bar{a}}\psi^b\right]\right) \\ & -\alpha_6\left[G^{d\bar{e}}G^{f\bar{a}}\left(G^{b\bar{c}} + \alpha_9\overline{\psi}^{\bar{c}}\psi^b\right)\right. \\ & \qquad\left. + G^{b\bar{e}}G^{f\bar{c}}\left(G^{d\bar{a}} + \alpha_9\overline{\psi}^{\bar{a}}\psi^d\right)\right]\dot{G}_{\bar{e}f} \\ & -\alpha_7\left[G^{b\bar{e}}G^{f\bar{a}}\left(G^{d\bar{c}} + \alpha_9\overline{\psi}^{\bar{c}}\psi^d\right)\right. \\ & \qquad\left. + G^{d\bar{e}}G^{f\bar{c}}\left(G^{b\bar{a}} + \alpha_9\overline{\psi}^{\bar{a}}\psi^b\right)\right]\dot{G}_{\bar{e}f}.\end{aligned}$$

4. Canonical Formalism and Hamiltonian

The above obtained Euler-Lagrange equations of motion may be expressed in Hamiltonian terms. Therefore, the Legendre transformations leads us to the following canonical variables

$$\pi_b = \frac{\partial L}{\partial \dot{\psi}^b} = \alpha_2 G_{\bar{a}b}\dot{\overline{\psi}}^{\bar{a}} + \alpha_1 \mathrm{i} G_{\bar{a}b}\overline{\psi}^{\bar{a}} \tag{4}$$

$$\overline{\pi}_{\bar{a}} = \frac{\partial L}{\partial \dot{\overline{\psi}}^{\bar{a}}} = \alpha_2 G_{\bar{a}b}\dot{\psi}^b - \alpha_1 \mathrm{i} G_{\bar{a}b}\psi^b \tag{5}$$

$$\pi^{\bar{a}b} = \frac{\partial L}{\partial \dot{G}_{\bar{a}b}} = \alpha_3\left[G^{b\bar{a}} + \alpha_9 \overline{\psi}^{\bar{a}}\psi^b\right] + 2\Omega[\psi, G]^{b\bar{a}d\bar{c}}\dot{G}_{\bar{c}d}. \tag{6}$$

The energy of our n-level Hamiltonian system is given as follows

$$E = \dot{\overline{\psi}}^{\bar{a}}\frac{\partial L}{\partial \dot{\overline{\psi}}^{\bar{a}}} + \dot{\psi}^b \frac{\partial L}{\partial \dot{\psi}^b} + \dot{G}_{\bar{a}b}\frac{\partial L}{\partial \dot{G}_{\bar{a}b}} - L$$

$$= \alpha_2 G_{\bar{a}b}\dot{\overline{\psi}}^{\bar{a}}\dot{\psi}^b - (\alpha_4 G_{\bar{a}b} + \alpha_5 H_{\bar{a}b})\,\overline{\psi}^{\bar{a}}\psi^b + \Omega[\psi, G]^{\bar{a}b\bar{c}d}\dot{G}_{\bar{a}b}\dot{G}_{\bar{c}d} + \varkappa\left(G_{\bar{a}b}\overline{\psi}^{\bar{a}}\psi^b\right)^2.$$

Inverting the expressions (4)–(6) we obtain that

$$\begin{aligned}
\dot{\overline{\psi}}^{\bar{a}} &= \frac{1}{\alpha_2}G^{b\bar{a}}\pi_b - \frac{\alpha_1}{\alpha_2}\mathrm{i}\overline{\psi}^{\bar{a}}, \qquad \dot{\psi}^b = \frac{1}{\alpha_2}G^{b\bar{a}}\overline{\pi}_{\bar{a}} + \frac{\alpha_1}{\alpha_2}\mathrm{i}\psi^b \\
\dot{G}_{\bar{a}b} &= \frac{1}{2}\Omega[\psi, G]^{-1}_{\bar{a}b\bar{c}d}\left(\pi^{\bar{c}d} - \alpha_3\left[G^{d\bar{c}} + \alpha_9\overline{\psi}^{\bar{c}}\psi^d\right]\right)
\end{aligned} \tag{7}$$

where

$$\begin{aligned}
\Omega[\psi, G]^{-1}_{\bar{a}b\bar{c}d} &= \Lambda[\psi, G]^{-1}_{\bar{a}b\bar{c}d} \\
&\quad - \frac{\alpha_8}{1 + \alpha_8\theta[\psi, G]}\Lambda[\psi, G]^{-1}_{\bar{a}b\bar{e}f}\overline{\psi}^{\bar{e}}\psi^f\Lambda[\psi, G]^{-1}_{\bar{c}d\bar{g}h}\overline{\psi}^{\bar{g}}\psi^h \\
\Lambda[\psi, G]^{-1}_{\bar{a}b\bar{c}d} &= \frac{1}{\alpha_6}\lambda[\psi, G]^{-1}_{\bar{a}d}\lambda[\psi, G]^{-1}_{\bar{c}b} \\
&\quad - \frac{\alpha_7}{\alpha_6(\alpha_6 + n\alpha_7)}\lambda[\psi, G]^{-1}_{\bar{a}b}\lambda[\psi, G]^{-1}_{\bar{c}d} \\
\lambda[\psi, G]^{-1}_{\bar{a}b} &= G_{\bar{a}b} - \frac{\alpha_9}{1 + \alpha_9 G_{\bar{e}f}\overline{\psi}^{\bar{e}}\psi^f}G_{\bar{a}d}G_{\bar{c}b}\overline{\psi}^{\bar{c}}\psi^d
\end{aligned}$$

and

$$\theta[\psi, G] = \Lambda[\psi, G]^{-1}_{\bar{a}b\bar{c}d}\overline{\psi}^{\bar{a}}\psi^b\overline{\psi}^{\bar{c}}\psi^d = \frac{\alpha_6 + (n-1)\,\alpha_7}{\alpha_6(\alpha_6 + n\alpha_7)}\left(\frac{G_{\bar{a}b}\overline{\psi}^{\bar{a}}\psi^b}{1 + \alpha_9 G_{\bar{c}d}\overline{\psi}^{\bar{c}}\psi^d}\right)^2.$$

So, finally the Hamiltonian has the following form

$$\begin{aligned} H \;=\; & \frac{1}{\alpha_2} G^{b\bar{a}} \overline{\pi}_{\bar{a}} \pi_b + \frac{1}{4} \Omega[\psi, G]^{-1}_{\bar{a}b\bar{c}d} \pi^{\bar{a}b} \pi^{\bar{c}d} \\ & + \frac{\alpha_1}{\alpha_2} \mathrm{i} \left(\psi^b \pi_b - \overline{\psi}^{\bar{a}} \overline{\pi}_{\bar{a}} \right) - \frac{\alpha_3}{2} \Omega[\psi, G]^{-1}_{\bar{a}b\bar{c}d} \left[G^{b\bar{a}} + \alpha_9 \overline{\psi}^{\bar{a}} \psi^b \right] \pi^{\bar{c}d} \\ & + \frac{\alpha_3^2}{4} \Omega[\psi, G]^{-1}_{\bar{a}b\bar{c}d} \left[G^{b\bar{a}} + \alpha_9 \overline{\psi}^{\bar{a}} \psi^b \right] \left[G^{d\bar{c}} + \alpha_9 \overline{\psi}^{\bar{c}} \psi^d \right] \\ & - \left[\left(\alpha_4 - \frac{\alpha_1^2}{\alpha_2} \right) G_{\bar{a}b} + \alpha_5 H_{\bar{a}b} \right] \overline{\psi}^{\bar{a}} \psi^b + \varkappa \left(G_{\bar{a}b} \overline{\psi}^{\bar{a}} \psi^b \right)^2 . \end{aligned}$$

5. The Special Cases

5.1. The Pure Dynamics for the Scalar Product G

The equations of motion for the pure dynamics of scalar product G while the wave function ψ is fixed are written as follows

$$\begin{aligned} \Omega[\psi, G]^{b\bar{a}d\bar{c}} \ddot{G}_{\bar{c}d} \;=\; & \left(\frac{\alpha_4}{2} - \varkappa G_{\bar{c}d} \overline{\psi}^{\bar{c}} \psi^d \right) \overline{\psi}^{\bar{a}} \psi^b \\ & + \alpha_7 G^{d\bar{e}} G^{f\bar{c}} \dot{G}_{\bar{c}d} \dot{G}_{\bar{e}f} \left(G^{b\bar{a}} + \alpha_9 \overline{\psi}^{\bar{a}} \psi^b \right) \\ & + \alpha_6 \dot{G}_{\bar{c}d} \dot{G}_{\bar{e}f} \left(\gamma[\psi, G]^{b\bar{e}f\bar{c}d\bar{a}} + \gamma[\psi, G]^{f\bar{a}d\bar{e}b\bar{c}} - \gamma[\psi, G]^{b\bar{e}d\bar{a}f\bar{c}} \right) \end{aligned}$$

where

$$\gamma[\psi, G]^{f\bar{e}d\bar{c}b\bar{a}} \;=\; G^{f\bar{e}} G^{d\bar{c}} \left(G^{b\bar{a}} + \alpha_9 \overline{\psi}^{\bar{a}} \psi^b \right).$$

If we additionally suppose that $\alpha_4 = \alpha_8 = \alpha_9 = \varkappa = 0$, then the above expression simplifies significantly

$$\left(\alpha_6 G^{b\bar{c}} G^{d\bar{a}} + \alpha_7 G^{b\bar{a}} G^{d\bar{c}} \right) \left(\ddot{G}_{\bar{c}d} - \dot{G}_{\bar{c}f} G^{f\bar{e}} \dot{G}_{\bar{e}d} \right) = 0.$$

Hence, the pure dynamics of the scalar product is described by the following equations

$$\ddot{G}_{\bar{a}b} - \dot{G}_{\bar{a}d} G^{d\bar{c}} \dot{G}_{\bar{c}b} = 0.$$

Let us now demand that $\dot{G}G^{-1}$ is equal to some constant value E, i.e., $\dot{G} = EG$, then

$$\ddot{G} = E\dot{G} = E^2 G$$

and

$$\dot{G} G^{-1} \dot{G} = EGG^{-1}EG = E^2 G.$$

Therefore our equations of motion are fulfilled automatically and the solution is as follows

$$G(t)_{\bar{a}b} = \left[\exp(Et)\right]^{\bar{c}}{}_{\bar{a}} G_{0\bar{c}b}.$$

Similarly if we demand that $G^{-1}\dot{G}$ is equal to some other constant E', then

$$\dot{G} = GE', \qquad \ddot{G} = \dot{G}E'^2 = GE'^2$$
$$\dot{G}G^{-1}\dot{G} = GE'G^{-1}GE' = GE'^2.$$

The equations of motion are also fulfilled and the solution is as follows

$$G(t)_{\bar{a}b} = G_{0\bar{a}d}\left[\exp(E't)\right]^{d}{}_{b}.$$

The connection between these two different constants E and E' is written below

$$\dot{G}(0) = \dot{G}_0 = G_0E' = EG_0.$$

5.2. Usual and First-Order Modified Schrödinger Equations

The second interesting special case is obtained when we suppose that the scalar product G is fixed, i.e., the equations of motion are as follows

$$\alpha_2\ddot{\psi}^a - 2\alpha_1\mathrm{i}\dot{\psi}^a + \left(2\varkappa G_{\bar{c}d}\overline{\psi}^{\bar{c}}\psi^d - \alpha_4\right)\psi^a - \alpha_5 H^a{}_b\psi^b = 0.$$

Then if all constants of model vanish except of the following ones

$$\alpha_1 = \frac{\hbar}{2}, \qquad \alpha_5 = -1$$

we end up with the well-known usual Schrödinger equation

$$\mathrm{i}\hbar\dot{\psi}^a = H^a{}_b\psi^b.$$

Its first-order modified version is obtained when we suppose that G is a dynamical variable and α_2 is equal to zero, i.e.,

$$\begin{aligned}
\mathrm{i}\hbar\dot{\psi}^a \;=\; & H^a{}_b\psi^b - \left[\frac{\mathrm{i}\hbar}{2} + \alpha_3\alpha_9\right] G^{a\bar{c}}\dot{G}_{\bar{c}b}\psi^b \\
& + \left(2\varkappa G_{\bar{c}d}\overline{\psi}^{\bar{c}}\psi^d - \alpha_4\right)\psi^a - 2\alpha_8 G^{a\bar{c}}\dot{G}_{\bar{c}b}\psi^b\dot{G}_{\bar{e}d}\overline{\psi}^{\bar{e}}\psi^d \\
& - 2\alpha_9 G^{a\bar{c}}\left(\alpha_6\dot{G}_{\bar{c}d}\dot{G}_{\bar{e}b} + \alpha_7\dot{G}_{\bar{c}b}\dot{G}_{\bar{e}d}\right)\psi^b\left(G^{d\bar{e}} + \alpha_9\overline{\psi}^{\bar{e}}\psi^d\right)
\end{aligned}$$

and

$$\begin{aligned}
2\Omega[\psi,G]^{b\bar{a}d\bar{c}}\ddot{G}_{\bar{c}d} \;=\; & \left[\frac{\mathrm{i}\hbar}{2} - \alpha_3\alpha_9\right]\overline{\psi}^{\bar{a}}\dot{\psi}^b - \left[\frac{\mathrm{i}\hbar}{2} + \alpha_3\alpha_9\right]\dot{\overline{\psi}}^{\bar{a}}\psi^b \\
& - 2G^{d\bar{a}}\Big[\alpha_6 G^{b\bar{e}}\left(G^{f\bar{c}} + \alpha_9\overline{\psi}^{\bar{c}}\psi^f\right) \\
& \qquad + \alpha_7 G^{b\bar{c}}\left(G^{f\bar{e}} + \alpha_9\overline{\psi}^{\bar{e}}\psi^f\right)\Big]\dot{G}_{\bar{c}d}\dot{G}_{\bar{e}f} \\
& - \left(2\varkappa G_{\bar{c}d}\overline{\psi}^{\bar{c}}\psi^d - \alpha_4\right)\overline{\psi}^{\bar{a}}\psi^b - 2\dot{\Omega}[\psi,G]^{b\bar{a}d\bar{c}}\dot{G}_{\bar{c}d}.
\end{aligned}$$

We can rewrite the above equation of motion for ψ in the following form

$$\mathrm{i}\hbar\dot{\psi}^a = {H_{\mathrm{eff}}}^a{}_b\psi^b$$

where the effective Hamilton operator is given as follows

$$\begin{aligned}{H_{\mathrm{eff}}}^a{}_b \;=\; & H^a{}_b - \left[\frac{\mathrm{i}\hbar}{2} + \alpha_3\alpha_9\right] G^{a\bar{c}}\dot{G}_{\bar{c}b} \\ & + \left(2\varkappa G_{\bar{c}d}\overline{\psi}^{\bar{c}}\psi^d - \alpha_4\right)\delta^a{}_b - 2\alpha_8 G^{a\bar{c}}\dot{G}_{\bar{c}b}\dot{G}_{\bar{e}d}\overline{\psi}^{\bar{e}}\psi^d \\ & - 2\alpha_9 G^{a\bar{c}}\left(\alpha_6\dot{G}_{\bar{c}d}\dot{G}_{\bar{e}b} + \alpha_7\dot{G}_{\bar{c}b}\dot{G}_{\bar{e}d}\right)\left(G^{d\bar{e}} + \alpha_9\overline{\psi}^{\bar{e}}\psi^d\right).\end{aligned}$$

Final Remarks

Let us summarize the presented ideas. We started from introducing the first- and second-order in time modified Schrödinger equation for a finite-level quantum system. We also constructed the "direct nonlinearity" as a non-quadratic term in the Lagrangian and included the scalar product metric into the dynamical variables for which we obtained the separate equations of motion. As usually in high-symmetry problems the structure of Lagrangian and resulting equations of motion is very beautiful but strongly nonlinear which prevented us from finding a rigorous analytical solution of the problem. Nevertheless, there is the special case when we fixed the behaviour of the wave function and the resulting behaviour of the scalar product allowed the rigorous exponential solutions, including ones infinitely growing and exponentially decaying in the future. This makes some hope for describing some quantum decay/reduction phenomena, although the full answer will be possible only when we find a rigorous solution for the total system.

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