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QUADRATIC HAMILTON-POISSON SYSTEMS ON $\mathfrak{so}_-^*(3)$: CLASSIFICATION AND INTEGRATION

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Abstract. We classify, under affine equivalence, the quadratic Hamilton-Poisson systems on the Lie-Poisson space $\mathfrak{so}_-^*(3)$. For the simplest strictly inhomogeneous quadratic system, we find explicit expressions for the integral curves in terms of Jacobi elliptic functions.

1. Introduction

Quadratic Hamilton-Poisson systems on Lie-Poisson spaces have received attention from several authors in recent years (see, e.g., [3, 6–8, 14]). Equivalence of quadratic Hamilton-Poisson systems has been considered by Tudoran [12,13]. The use of equivalence (in reducing to normal forms) has proved promising for the analysis of various classes of such systems (see, e.g., [3, 9]).

Homogeneous quadratic Hamilton-Poisson systems on the orthogonal Lie-Poisson space $\mathfrak{so}_-^*(3)$ have been treated in [5,9] were both stability and integration were addressed. In this paper we classify, under affine equivalence, the homogeneous and *inhomogeneous* quadratic Hamilton-Poisson systems on $\mathfrak{so}_-^*(3)$ and an exhaustive list of normal forms are exhibited. (Two systems are said to be affinely equivalent if their associated vector fields are compatible with an affine isomorphism.) Among the inhomogeneous systems obtained as normal forms, we integrate the simplest one. Three qualitatively different cases are identified for this system. In each case explicit expressions for the integral curves are found in terms of Jacobi elliptic functions.

1.1. The Lie-Poisson Structure

Let $\mathfrak{g}$ be a (real) Lie algebra; its dual space $\mathfrak{g}^*$ admits a natural Poisson structure

$$\{F, G\}(p) = -p([\mathrm{d}F(p), \mathrm{d}G(p)])$$

called the (*minus*) *Lie-Poisson structure*. Here $p \in \mathfrak{g}^*$, $F, G \in C^\infty(\mathfrak{g}^*)$, and $\mathrm{d}F(p), \mathrm{d}G(p) \in g^{**}$ are identified with elements of $\mathfrak{g}$. To each function $H \in C^\infty(\mathfrak{g}^*)$ we associate a *Hamiltonian vector field* $\vec{H}$ on $\mathfrak{g}^*$ specified by $\vec{H}[F] = \{F, H\}$. A function $C \in C^\infty(\mathfrak{g}^*)$ is a *Casimir function* provided $\{C, F\} = 0$ for all $F \in C^\infty(\mathfrak{g}^*)$. A linear isomorphism $\psi : \mathfrak{g}^* \to \mathfrak{g}^*$ is called a *linear Poisson automorphism* if $\{F, G\} \circ \psi = \{F \circ \psi, G \circ \psi\}$ for all $F, G \in C^\infty(\mathfrak{g}^*)$. Linear Poisson automorphisms are the dual maps of Lie algebra automorphisms.

Given a Lie-Poisson space, a *quadratic Hamilton-Poisson system* is specified by

$$H_{A,\mathcal{Q}} : \mathfrak{g}^*_- \to \mathbb{R}, \qquad p \mapsto p(A) + \mathcal{Q}(p).$$

Here $A \in \mathfrak{g}$ and $\mathcal{Q}$ is a quadratic form on $\mathfrak{g}^*$. If $A = 0$, then the system is called *homogeneous* and otherwise, it is called *inhomogeneous*.

1.2. The Lie-Poisson Space $\mathfrak{so}^*_-(3)$

The three-dimensional orthogonal Lie algebra

$$\mathfrak{so}(3) = \left\{A \in \mathbb{R}^{3\times 3};\ A^\top + A = \mathbf{0}\right\}$$

has standard (ordered) basis

$$E_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \qquad E_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \qquad E_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The commutator relations are

$$[E_2, E_3] = E_1, \qquad [E_3, E_1] = E_2, \qquad [E_1, E_2] = E_3.$$

Let (E_1^*, E_2^*, E_3^*) denote the dual of the standard basis. We shall write an element $p = p_1 E_1^* + p_2 E_2^* + p_3 E_3^*$ as $p = \begin{bmatrix} p_1 & p_2 & p_3 \end{bmatrix}$. The group of linear Poisson automorphisms is given by

$$\{p \mapsto p\Psi\,;\ \Psi \in \mathbb{R}^{3\times 3},\ \Psi\Psi^\top = \mathbf{1},\ \det\Psi = 1\} \cong \mathrm{SO}(3).$$

The Poisson structure on $\mathfrak{so}^*_-(3)$ is given by

$$\Pi = \begin{bmatrix} 0 & -p_3 & p_2 \\ p_3 & 0 & -p_1 \\ -p_2 & p_1 & 0 \end{bmatrix}.$$

(The Hamiltonian vector field associated to a function H is specified by $\vec{H} = \Pi \cdot \nabla H$.) Note that $C(p) = p_1^2 + p_2^2 + p_3^2$ is a Casimir function.

2. Classification

We say that two quadratic Hamilton-Poisson systems G and H on $\mathfrak{g}_-^*$ are (affinely) *equivalent* if their associated vector fields $\vec{G}$ and $\vec{H}$ are compatible with an affine isomorphism, i.e., there exists an affine isomorphism $\psi : \mathfrak{g}^* \to \mathfrak{g}^*$ such that $T\psi \cdot \vec{G} = \vec{H} \circ \psi$. (The map ψ establishes a one-to-one correspondence between the integral curves of $\vec{G}$ and $\vec{H}$.) The following Hamilton-Poisson systems are equivalent to $H_{A,\mathcal{Q}}$

$\mathfrak{E}1)$ $H_{A,\mathcal{Q}} \circ \psi$, where ψ is a linear Poisson automorphism

$\mathfrak{E}2)$ $H_{A,r\mathcal{Q}}$, where $r \neq 0$

$\mathfrak{E}3)$ $H_{A,\mathcal{Q}} + C$, where C is a Casimir function.

We now proceed to classify the quadratic Hamilton-Poisson systems on the Lie-Poisson space $\mathfrak{so}_-^*(3)$. The above types of equivalence $\mathfrak{E}1)$-$\mathfrak{E}3)$ are not always sufficient to reduce a system to its normal form. In such cases, we find an explicit affine isomorphism with respect to which the vector fields are compatible.

Theorem 1. *Let H be a quadratic Hamilton-Poisson system on $\mathfrak{so}_-^*(3)$. If H is homogeneous, then it is equivalent to exactly one of the systems*

$$H^0(p) = 0, \qquad H^1(p) = \tfrac{1}{2}p_1^2, \qquad H^2(p) = p_1^2 + \tfrac{1}{2}p_2^2.$$

If H is inhomogeneous, then it is equivalent to exactly one of the systems

$$H^0_{1,\alpha}(p) = \alpha p_1$$
$$H^1(p) = \tfrac{1}{2}p_1^2, \qquad H^1_1(p) = p_2 + \tfrac{1}{2}p_1^2, \qquad H^1_{2,\alpha}(p) = p_1 + \alpha p_2 + \tfrac{1}{2}p_1^2$$
$$H^2_{\boldsymbol{\beta}}(p) = \beta_1 p_1 + \beta_2 p_2 + \beta_3 p_3 + p_1^2 + \tfrac{1}{2}p_2^2.$$

Here $\alpha > 0$ and $\boldsymbol{\beta} \in \{(\beta_1,0,0),(0,\beta_2,0),(\beta_1,\beta_2,0),(\beta_1,0,\beta_3),(\beta_1,\beta_2,\beta_3)\,;$ $\beta_1 \geq |\beta_3| > 0,\ \beta_2 > 0\}$ parametrize families of class representatives, each different value corresponding to a distinct non-equivalent representative.

Proof: Let $H(p) = pA + pQp^\top$, where Q is a symmetric 3×3 matrix. Here $A = a_1E_1 + a_2E_2 + a_3E_3 \in \mathfrak{so}(3)$ is identified with $\begin{bmatrix} a_1, & a_2, & a_3 \end{bmatrix}^\top$. We may assume that Q is positive definite. If Q is not positive definite, then H is equivalent to a system $H + \mu C$ for which the quadratic form is positive definite (for some sufficiently large μ).

Given a linear Poisson automorphism $\psi : p \mapsto p\Psi$, we have

$$(H \circ \psi)(p) = p\Psi A + p\Psi Q \Psi^\top p^\top.$$

As any real symmetric matrix can be diagonalized by an orthogonal matrix (see, e.g., [11]), it follows that there exists a linear Poisson automorphism ψ such that

$(H \circ \psi)(p) = p\Psi A + p\ \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)\, p^\top$ with $\lambda_1 \geq \lambda_2 \geq \lambda_3 > 0$. Thus

$$(H \circ \psi)(p) - \lambda_3 C(p) = p\Psi A + p\ \mathrm{diag}(\lambda_1 - \lambda_3, \lambda_2 - \lambda_3, 0)\, p^\top$$

with $\lambda_1 - \lambda_3 \geq \lambda_2 - \lambda_3 \geq 0$. If $\lambda_1 - \lambda_3 = 0$, then (by $\mathfrak{E}1$) and $\mathfrak{E}3$)) H is equivalent to an intermediate system $G^0_B(p) = pB$, where $B = \Psi A$. On the other hand, if $\lambda_1 - \lambda_3 > 0$, then

$$(H \circ \psi)(p) - \lambda_3 C(p) = p\Psi A + (\lambda_1 - \lambda_3)\, p\ \mathrm{diag}\left(1, \frac{\lambda_2 - \lambda_3}{\lambda_1 - \lambda_3}, 0\right)\, p^\top$$

and so H is equivalent to

$$H'(p) = p\Psi A + p_1^2 + \alpha p_2^2, \qquad \alpha = \frac{\lambda_2 - \lambda_3}{\lambda_1 - \lambda_3}\cdot$$

If $\alpha = 0$, then $H'(p) = p\Psi A + p_1^2$ and so H' is equivalent to (an intermediate system) $G^1_B(p) = pB + \frac{1}{2}p_1^2$ with $B = \Psi A$. Suppose $\alpha = 1$. Then

$$\psi' = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

is a linear Poisson automorphism such that $(H' \circ \psi' - C)(p) = p\Psi'\Psi A - p_1^2$. So H' is equivalent to $G^1_B(p) = pB + \frac{1}{2}p_1^2$, where $B = \Psi'\Psi A$. On the other hand, suppose $0 < \alpha < 1$. Then the vector fields associated to

$$H'(p) = a_1' p_1 + a_2' p_2 + a_3' p_3 + p_1^2 + \alpha p_2^2, \qquad A' = \Psi A$$

and

$$G^2_B(p) = b_1 p_1 + b_2 p_2 + b_3 p_3 + p_1^2 + \tfrac{1}{2} p_2^2$$

are compatible with the affine isomorphism

$$p \mapsto p \begin{bmatrix} -\sqrt{2(1-\alpha)} & 0 & 0 \\ 0 & 2\sqrt{\alpha(1-\alpha)} & 0 \\ 0 & 0 & -\sqrt{2\alpha} \end{bmatrix} + \begin{bmatrix} -\frac{1-2\alpha}{\sqrt{2(1-\alpha)}} a_1' \\ \frac{1-2\alpha}{2\sqrt{\alpha(1-\alpha)}} a_2' \\ -\frac{1-2\alpha}{\sqrt{2\alpha}} a_3' \end{bmatrix}$$

provided $b_1 = -\frac{\alpha\sqrt{2(1-\alpha)}}{1-\alpha} a_1'$, $b_2 = \frac{1}{2\sqrt{\alpha(1-\alpha)}} a_2'$, and $b_3 = -\frac{\sqrt{2}(1-\alpha)}{\sqrt{\alpha}} a_3'$.

Suppose that H is homogeneous, i.e., $A = 0$. Then, by the above argument, H is equivalent to $G^0_{\mathbf{0}} = H^0$, $G^1_{\mathbf{0}} = H^1$ or $G^2_{\mathbf{0}} = H^2$. No two of the systems H^0, H^1 and H^2 are equivalent. Indeed, if two systems are equivalent, then there exists an affine bijection between their equilibria. However, the set of equilibria for H^1 is the union of a plane and a line, whereas the set of equilibria for H^2 is the union of three lines.

On the other hand, suppose that H is inhomogeneous. Then, by the above argument, H is equivalent to one of the intermediate systems

$$G^0_B(p) = pB, \qquad G^1_B(p) = pB + \tfrac{1}{2}p_1^2, \qquad G^2_B(p) = pB + p_1^2 + \tfrac{1}{2}p_2^2$$

for some $B \in \mathfrak{so}(3)$. (Here B is the image of A under some linear isomorphism and so $B \neq \mathbf{0}$). We note that G^1_B and $G^2_{B'}$ are not equivalent for any $B, B' \in \mathfrak{so}(3)$. Indeed, if they were equivalent, then a simple calculation shows that their homogeneous parts H^1 and H^2 would be equivalent, a contradiction. Likewise, G^0_B cannot be equivalent to $G^1_{B'}$ or $G^2_{B'}$ (for any $B, B' \in \mathfrak{so}(3)$).

Suppose H is equivalent to G^0_B. As $\mathrm{SO}(3)$ acts transitively on any sphere, there exists a linear Poisson automorphism ψ such that $(G^0_B \circ \psi)(p) = \alpha p_1$ for some $\alpha > 0$. Thus H is equivalent to $H^0_{1,\alpha}$. We claim that $H^0_{1,\alpha}$ and $H^0_{1,\beta}$ are equivalent only if $\alpha = \beta$ (i.e., no further reduction is possible). Indeed, if they are equivalent, then there exists an affine isomorphism $\psi : p \mapsto p\Psi + q$ such that $T\psi \cdot \vec{H}^0_{1,\alpha} = \vec{H}^0_{1,\beta} \circ \psi$, i.e.,

$$\begin{aligned} -\alpha\Psi_{31}p_2 + \alpha\Psi_{21}p_3 &= 0 \\ -\alpha\Psi_{32}p_2 + \alpha\Psi_{22}p_3 - \beta(\Psi_{31}p_1 + \Psi_{32}p_2 + \Psi_{33}p_3 + q_3) &= 0 \\ -\alpha\Psi_{33}p_2 + \alpha\Psi_{23}p_3 - \beta(\Psi_{21}p_1 + \Psi_{22}p_2 + \Psi_{23}p_3 + q_2) &= 0 \end{aligned}$$

for all $p \in \mathfrak{so}(3)^*$. However this implies that $\alpha = \beta$. (Here $\Psi = [\Psi_{ij}]$.)

Next, suppose H is equivalent to G^1_B. Given a linear Poisson automorphism $\psi : p \mapsto p\Psi$, we have that $(G^1_B \circ \psi)(p) = p\Psi B + p\Psi \operatorname{diag}(\frac{1}{2}, 0, 0)\Psi^\top p^\top$. Now Ψ leaves $\operatorname{diag}(\frac{1}{2}, 0, 0)$ invariant, i.e., $\Psi \operatorname{diag}(\frac{1}{2}, 0, 0)\Psi^\top = \operatorname{diag}(\frac{1}{2}, 0, 0)$ if and only if $\Psi = \begin{bmatrix} \det(S) & 0 \\ 0 & S \end{bmatrix}$, $S \in \mathrm{O}(2)$. Thus, there exists a linear Poisson automorphism ψ such that

$$H'(p) = (G^1_B \circ \psi)(p) = \gamma_1 p_1 + \gamma_2 p_2 + \tfrac{1}{2}p_1^2$$

for some $\gamma_1, \gamma_2 \geq 0$, $(\gamma_1, \gamma_2) \neq (0, 0)$. Assume $\gamma_1 = 0$. Then the vector fields associated to $H'(p) = \gamma_2 p_2 + \frac{1}{2}p_1^2$ and $H^1_1(p) = p_2 + \frac{1}{2}p_1^2$ are compatible with the affine isomorphism

$$p \mapsto p \begin{bmatrix} 1 & 0 & 0 \\ 0 & \gamma_2 & 0 \\ 0 & 0 & \gamma_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 - \gamma_2^2 \\ 0 \end{bmatrix}.$$

Assume $\gamma_2 = 0$. Then the vector fields associated to $H'(p) = \gamma_1 p_1 + \frac{1}{2}p_1^2$ and $H^1(p) = \frac{1}{2}p_1^2$ are compatible with the affine isomorphism

$$p \mapsto p \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} -\gamma_1 \\ 0 \\ 0 \end{bmatrix}.$$

Assume $\gamma_1, \gamma_2 > 0$. Then the vector fields associated to $H'(p) = \gamma_1 p_1 + \gamma_2 p_2 + \frac{1}{2}p_1^2$ and $H^1_{2,\alpha}(p) = p_1 + \alpha p_2 + \frac{1}{2}p_1^2$, $\alpha = \gamma_2\sqrt{\gamma_1}$ are compatible with the affine isomorphism

$$p \mapsto p \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{\gamma_1}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{\gamma_1}} \end{bmatrix} + \begin{bmatrix} \gamma_1 - 1 \\ \frac{(\gamma_1 - 1)\gamma_2}{\sqrt{\gamma_1}} \\ 0 \end{bmatrix}.$$

By direct computation, it is straightforward to show that $T\psi \cdot \vec{H}^1_{2,\alpha} = \vec{H}^1_{2,\beta} \circ \psi$ for some affine isomorphism ψ only if $\alpha = \beta$. Thus $H^1_{2,\alpha}$ is equivalent to $H^1_{2,\beta}$ only if $\alpha = \beta$.

Lastly, suppose H is equivalent to $G^2_B(p) = b_1p_1 + b_2p_2 + b_3p_3 + p_1^2 + \frac{1}{2}p_2^2$. Assume b_1, b_2, b_3 are all nonzero. Note that

$$\psi : p \mapsto p\Psi, \qquad \Psi = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

is a linear Poisson automorphism such that $(G^2_B \circ \psi - C)(p) = b_3p_1 + b_2p_2 - b_1p_3 - p_1^2 - \frac{1}{2}p_2^2$. Thus G^2_B is equivalent to a system $H'(p) = b'_1p_1 + b'_2p_2 + b'_3p_3 + p_1^2 + \frac{1}{2}p_2^2$, where $|b'_1| \geq |b'_3| > 0$. The linear Poisson automorphisms $\mathrm{diag}(-1, 1, -1)$, $\mathrm{diag}(1, -1, -1)$, and $\mathrm{diag}(-1, -1, 1)$ allow us to change the signs of b_1 and b_2. Thus H' is equivalent to $H^2_{\boldsymbol{\beta}}(p) = \beta_1p_1 + \beta_2p_2 + \beta_3p_3 + p_1^2 + \frac{1}{2}p_2^2$ for some $\boldsymbol{\beta}$, where $\beta_1 \geq |\beta_3| > 0$ and $\beta_2 > 0$. Likewise, if $b_1 = 0$, $b_2 = 0$, $b_3 = 0$, $b_1 = b_2 = 0$, $b_1 = b_3 = 0$, or $b_2 = b_3 = 0$, then H is equivalent to $H^2_{\boldsymbol{\beta}}$ for some $\boldsymbol{\beta} \in \{(\beta_1, 0, 0), (0, \beta_2, 0), (\beta_1, \beta_2, 0), (\beta_1, 0, \beta_3), (\beta_1, \beta_2, \beta_3)\,;\ \beta_1 \geq |\beta_3| > 0,\ \beta_2 > 0\}$. Direct computations shows that $H^2_{\boldsymbol{\beta}}$ is equivalent to $H^2_{\boldsymbol{\beta}'}$ only if $\boldsymbol{\beta} = \boldsymbol{\beta}'$. ■

3. Integration

Among the inhomogeneous quadratic systems on $\mathfrak{so}^*_-(3)$, as listed in theorem 1, the system $H^1_1(p) = p_2 + \frac{1}{2}p_1^2$ is the simplest. (We discount $H^0_{1,\alpha}$ as it has a linear Hamiltonian.) In this paper we shall investigate only H^1_1 and the remaining systems will be treated elsewhere.

The equations of motion for H^1_1 are given by

$$\dot{p}_1 = -p_3, \qquad \dot{p}_2 = p_1p_3, \qquad \dot{p}_3 = p_1(1 - p_2). \tag{1}$$

Remark 2. *H^1_1 has equilibria $e^\mu_1 = (0, \mu, 0)$ and $e^\nu_2 = (\nu, 1, 0)$. The states e^μ_1, $\mu \leq 1$ are stable, the states e^μ_1, $\mu > 1$ are (spectrally) unstable, and the states e^ν_2, $\nu \neq 0$ are stable. (A proof for these claims will appear elsewhere.)*

Before finding explicit expressions for the integral curves of H_1^1, we determine the conditions which lead to qualitatively different behaviour. Clearly, the integral curves of this system evolve along the intersections of the level sets of the Hamiltonian H_1^1 and the Casimir function $C(p) = p_1^2 + p_2^2 + p_3^2$. The qualitative behaviour of the system changes when these two level sets are tangent (cf [2, 3]). This happens exactly when

$$\nabla H_1^1(p) = \lambda \nabla C(p) \qquad \Longleftrightarrow \qquad \begin{bmatrix} p_1 & 1 & 0 \end{bmatrix} = \lambda \begin{bmatrix} 2p_1, & 2p_2, & 2p_3 \end{bmatrix}$$

for some nonzero $\lambda \in \mathbb{R}$ and $p \in \mathfrak{so}(3)^*$. Let $h_0 = H_1^1(p)$ and $c_0 = C(p)$. If $\lambda \neq \frac{1}{2}$, then $h_0 = p_2$ and $c_0 = p_2^2$ which implies that $c_0 = h_0^2$. This motivates us to distinguish between the cases $c_0 < h_0^2$, $c_0 = h_0^2$, and $c_0 > h_0^2$. On the other hand, if $\lambda = \frac{1}{2}$, then $p_3 = 0$ and $p_2 = 1$; hence $h_0 = 1 + \frac{1}{2}p_1^2$ and $c_0 = p_1^2 + 1$ which implies that $c_0 = 2h_0 - 1$. However, the case $c_0 < 2h_0 - 1$ is impossible (in this case the intersection $(H_1^1)^{-1}(h_0) \cap C^{-1}(c_0)$ is empty), whereas $c_0 = 2h_0 - 1$ corresponds to constant solutions (in this case the intersection is one or two distinct points). In figure 1 we graph the level sets of H_1^1 and C and their intersection for some typical values of h_0 and c_0. (The stable and unstable equilibria are plotted in blue and red, respectively.)

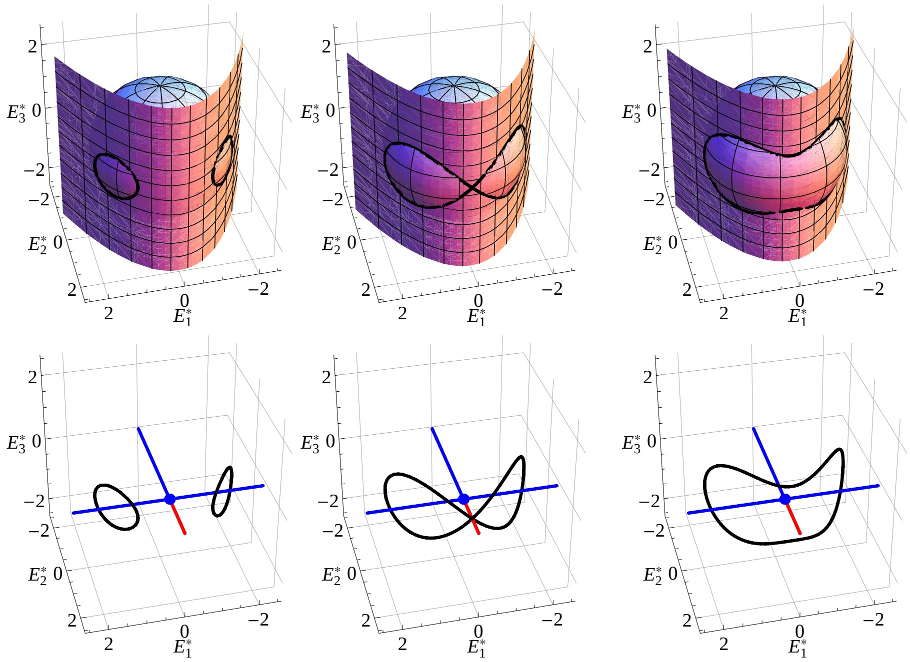
Note 1. *Every Hamiltonian vector field on $\mathfrak{so}^*_-(3)$ is complete as the integral curves evolve on the compact subsets $C^{-1}(c_0)$, $c_0 \geq 0$* (cf [1]).

It turns out that the expressions for the integral curves are expressible in terms of Jacobi elliptic functions. Given the modulus $k \in [0, 1]$, the basic *Jacobi elliptic functions* $\mathrm{sn}(\cdot, k)$, $\mathrm{cn}(\cdot, k)$, and $\mathrm{dn}(\cdot, k)$ can be defined as (see, e.g., [4, 10])

$$\begin{aligned} \mathrm{sn}(x, k) &= \sin \mathrm{am}(x, k) \\ \mathrm{cn}(x, k) &= \cos \mathrm{am}(x, k) \\ \mathrm{dn}(x, k) &= \sqrt{1 - k^2 \sin^2 \mathrm{am}(x, k)} \end{aligned}$$

where $\mathrm{am}(\cdot, k) = F(\cdot, k)^{-1}$ is the amplitude and $F(\varphi, k) = \int_0^\varphi \frac{\mathrm{d}t}{\sqrt{1 - k^2 \sin^2 t}}$. The number K is given by the formula $K = F(\frac{\pi}{2}, k)$. (The functions $\mathrm{sn}(\cdot, k)$ and $\mathrm{cn}(\cdot, k)$ are $4K$ periodic, whereas $\mathrm{dn}(\cdot, k)$ is $2K$ periodic.)

Theorem 3. *Suppose $p(\cdot)$ is a (nonconstant) integral curve of H_1^1. Let $h_0 = H_1^1(p(0))$ and $c_0 = C(p(0))$.*



(a) $c_0 < h_0^2$ **(b)** $c_0 = h_0^2$ **(c)** $c_0 > h_0^2$

Figure 1. Typical integral curves of H_1^1

(a) *If* $c_0 < h_0^2$*, then there exist* $t_0 \in \mathbb{R}$ *and* $\sigma \in \{-1, 1\}$ *such that* $p(t) = \bar{p}(t + t_0)$*, where*

$$\bar{p}_1(t) = \sigma\sqrt{2\delta}\,\frac{1 + k\ \operatorname{sn}(\Omega\,t,\,k)}{\operatorname{dn}(\Omega\,t,\,k)}$$

$$\bar{p}_2(t) = h_0 + \delta - \frac{2\delta}{1 - k\ \operatorname{sn}(\Omega\,t,\,k)}$$

$$\bar{p}_3(t) = -\sigma k\Omega\sqrt{2\delta}\,\frac{\operatorname{cn}(\Omega\,t,\,k)}{1 - k\ \operatorname{sn}(\Omega\,t,\,k)}.$$

Here $\Omega = \sqrt{h_0 - 1 + \delta}$, $k = \frac{\sqrt{h_0 - 1 - \delta}}{\sqrt{h_0 - 1 + \delta}}$, *and* $\delta = \sqrt{h_0^2 - c_0}$.

(b) *If* $c_0 = h_0^2$*, then there exist* $t_0 \in \mathbb{R}$ *and* $\sigma \in \{-1, 1\}$ *such that* $p(t) = \bar{p}(t + t_0)$*, where*

$$\bar{p}_1(t) = 2\sigma\sqrt{h_0 - 1}\ \operatorname{sech}\left(\sqrt{h_0 - 1}\, t\right)$$
$$\bar{p}_2(t) = h_0 - 2(h_0 - 1)\ \operatorname{sech}\left(\sqrt{h_0 - 1}\, t\right)^2$$
$$\bar{p}_3(t) = 2\sigma(h_0 - 1)\ \operatorname{sech}\left(\sqrt{h_0 - 1}\, t\right)\ \tanh\left(\sqrt{h_0 - 1}\, t\right).$$

(c) *If* $c_0 > h_0^2$*, then there exists* $t_0 \in \mathbb{R}$ *such that* $p(t) = \bar{p}(t + t_0)$*, where*

$$\bar{p}_1(t) = \sqrt{2}\sqrt{h_0 + \delta - 1}\ \operatorname{cn}(\Omega\, t,\, k)$$
$$\bar{p}_2(t) = h_0 - (h_0 + \delta - 1)\ \operatorname{cn}(\Omega\, t,\, k)^2$$
$$\bar{p}_3(t) = \sqrt{2}\sqrt{h_0 + \delta - 1}\,\Omega\ \operatorname{dn}(\Omega\, t,\, k)\ \operatorname{sn}(\Omega\, t,\, k)\,.$$

Here $\Omega = \sqrt{\delta}$*,* $k = \sqrt{\frac{h_0+\delta-1}{2\delta}}$*, and* $\delta = \sqrt{1 + c_0 - 2h_0}$*.*

Proof: a) We start by explaining how the expression for $\bar{p}(\cdot)$ was found. Suppose $\bar{p}(\cdot)$ is an integral curve of H_1^1 such that $c_0 < h_0^2$, where $h_0 = H_1^1(\bar{p}(0))$ and $c_0 = C(\bar{p}(0))$. As $\bar{p}(\cdot)$ satisfies equation (1), $H_1^1(\bar{p}(\cdot)) = h_0$, and $C(\bar{p}(\cdot)) = c_0$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\bar{p}_2 = \sqrt{2(h_0 - \bar{p}_2)(c_0 - 2(h_0 - \bar{p}_2) - \bar{p}_2^2)}. \tag{2}$$

We transform (2) into standard form (see, e.g., [4, 10]). First, we can rewrite (2) as

$$\frac{\mathrm{d}\bar{p}_2}{\mathrm{d}t} = \sqrt{2}\sqrt{(A_1(\bar{p}_2 - r_1)^2 + B_1(\bar{p}_2 - r_2)^2)\,(A_2(\bar{p}_2 - r_1)^2 + B_2(\bar{p}_2 - r_2)^2)}$$

where $r_1 = h_0 + \delta$, $r_2 = h_0 - \delta$, $\delta = \sqrt{h_0^2 - c_0}$, and

$$A_1 = \frac{1}{4\delta} > 0, \qquad A_2 = \frac{h_0 - 1 - \delta}{2\delta} > 0$$
$$B_1 = -\frac{1}{4\delta} < 0, \qquad B_2 = -\frac{h_0 - 1 + \delta}{2\delta} < 0.$$

Hence

$$\int \sqrt{2}\,\mathrm{d}t = \int \frac{\mathrm{d}\bar{p}_2}{\sqrt{(A_1(\bar{p}_2 - r_1)^2 + B_1(\bar{p}_2 - r_2)^2)\,(A_2(\bar{p}_2 - r_1)^2 + B_2(\bar{p}_2 - r_2)^2)}}\,.$$

Making the change of variable $s = \frac{\bar{p}_2 - r_1}{\bar{p}_2 - r_2}$ yields

$$\sqrt{2}\,t = \frac{1}{(r_1 - r_2)\sqrt{A_1 A_2}} \int_0^{\frac{\bar{p}_2(t) - r_1}{\bar{p}_2(t) - r_2}} \frac{\mathrm{d}s}{\left(s^2 - \frac{B_2}{A_2}\right)(s^2 - 1)}\,.$$

Applying the elliptic integral formula (see, e.g., [4, 10])

$$\int_a^x \frac{\mathrm{d}t}{\sqrt{(t^2-a^2)(t^2-b^2)}} = \tfrac{1}{a}\,\mathrm{dc}^{-1}\left(\tfrac{1}{a}\,x,\ \tfrac{b}{a}\right), \qquad b < a \leq x$$

and rearranging, we obtain

$$\bar{p}_2(t) = \frac{r_2\sqrt{-\frac{B_2}{A_2}}\ \mathrm{dc}\left((r_1-r_2)\sqrt{2A_1A_2}\sqrt{-\frac{B_2}{A_2}},\ \frac{1}{\sqrt{-\frac{B_2}{A_2}}}\right) - r_1}{\sqrt{-\frac{B_2}{A_2}}\ \mathrm{dc}\left((r_1-r_2)\sqrt{2A_1A_2}\sqrt{-\frac{B_2}{A_2}},\ \frac{1}{\sqrt{-\frac{B_2}{A_2}}}\right) - 1}.$$

Here $\mathrm{dc}(x,k) = \frac{\mathrm{dn}(x,k)}{\mathrm{cn}(x,k)}$. Substituting the values for $A_1, A_2, B_1, B_2, r_1, r_2$ and simplifying then yields

$$\bar{p}_2(t) = h_0 + \frac{\delta\,(k + \mathrm{dc}(\Omega\,t,\ k))}{k - \mathrm{dc}(\Omega\,t,\ k)}$$

where $k = \frac{\sqrt{h_0-1-\delta}}{\sqrt{h_0-1+\delta}}$. Now $\mathrm{dc}(x,k) = \frac{1}{\mathrm{sn}(x+K,\,k)}$. Thus, by making a suitable translation in t, we obtain the following (prospective) expression

$$\bar{p}_2(t) = h_0 + \delta - \frac{2\delta}{1 - k\ \mathrm{sn}(\Omega\,t,\ k)}.$$

As $\bar{p}_2(t) + \frac{1}{2}\bar{p}_1(t)^2 = h_0$, we get

$$\bar{p}_1(t)^2 = 2(h_0 - \bar{p}_2(t)) = -2\delta + \frac{2\delta}{1 - k\ \mathrm{sn}(\Omega\,t,\ k)}.$$

By multiplying by $\frac{1+k\ \mathrm{sn}(\Omega\,t,\,k)}{1+k\ \mathrm{sn}(\Omega\,t,\,k)}$ and simplifying, we find that

$$\bar{p}_1(t) = \sigma\sqrt{2\delta}\ \frac{1 + k\ \mathrm{sn}(\Omega\,t,\ k)}{\mathrm{dn}(\Omega\,t,\ k)}$$

for some $\sigma \in \{-1, 1\}$. Note that $1-k\ \mathrm{sn}(\Omega\,t,\ k) > 0$ and $\mathrm{dn}(\Omega\,t, k) > 0$ provided $0 < k < 1$. By (1) it follows that

$$\bar{p}_3(t) = -\frac{\mathrm{d}}{\mathrm{d}t}\bar{p}_1(t) = -\sigma\ \frac{\sqrt{2\delta}\,k\,\Omega\ \mathrm{cn}(\Omega\,t,\ k)}{1 - k\ \mathrm{sn}(\Omega\,t,\ k)}.$$

We now verify that $\bar{p}(\cdot)$ is an integral curve for $\sigma \in \{-1, 1\}$. After some simplification, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\bar{p}_2(t) - \bar{p}_1(t)\bar{p}_3(t) = \frac{2k\delta(\sigma^2-1)\Omega\ \mathrm{cn}(\Omega\,t,\ k)\ \mathrm{dn}(\Omega\,t,\ k)}{(1 - k\ \mathrm{sn}(\Omega\,t,\ k))^2}.$$

Therefore, as $\sigma \in \{-1, 1\}$, it follows that $\frac{\mathrm{d}}{\mathrm{d}t}\bar{p}_2(t) = \bar{p}_1(t)\bar{p}_3(t)$. Similarly, we have that $\frac{\mathrm{d}}{\mathrm{d}t}\bar{p}_3(t) = \bar{p}_1(t)(1 - \bar{p}_2(t))$ and $\frac{\mathrm{d}}{\mathrm{d}t}\bar{p}_1(t) = -\bar{p}_3(t)$. This motivates $\bar{p}(\cdot)$

as a prospective integral curve. It is not difficult to verify that the constants δ, k, and Ω are real and $0 < k < 1$.

Suppose $p(\cdot)$ is an integral curve such that $c_0 < h_0^2$, where $h_0 = H_1^1(p(0))$ and $c_0 = C(p(0))$. We claim that $p(\cdot)$ must be of the form $p(t) = \bar{p}(t + t_0)$ for some $\sigma \in \{-1, 1\}$ and $t_0 \in \mathbb{R}$. As $p_2(0) + \frac{1}{2}p_1(0)^2 = h_0$ and $p_1(0)^2 + p_2(0)^2 + p_3(0)^2 = c_0$, we have $(2h_0 - c_0) - 2p_2(0) + p_2^2(0) \leq 0$. Thus

$$1 - \sqrt{1 + c_0 - 2h_0} \leq p_2(0) \leq 1 + \sqrt{1 + c_0 - 2h_0}.$$

Now $\bar{p}_2(\frac{K}{\Omega}) = 1 - \sqrt{1 + c_0 - 2h_0}$ and $\bar{p}_2(\frac{3K}{\Omega}) = 1 + \sqrt{1 + c_0 - 2h_0}$. Thus there exists $t_1 \in [\frac{K}{\Omega}, \frac{3K}{\Omega}]$ such that $\bar{p}_2(t_1) = p_2(0)$. As

$$\tfrac{1}{2}p_1(0)^2 = h_0 - p_2(0) \geq h_0 - 1 - \sqrt{1 + c_0 - 2h_0} > 0$$

it follows that $p_1(0) \neq 0$. Let $\sigma = \operatorname{sign}(p_1(0))$. We have

$$\tfrac{1}{2}p_1(0)^2 = h_0 - p_2(0) = h_0 - \bar{p}_2(t_1) = \tfrac{1}{2}\bar{p}_1(t_1)^2.$$

Hence, as $\operatorname{sign}(\bar{p}_1(t_1)) = \sigma$, we get $p_1(0) = \bar{p}_1(t_1)$. Therefore

$$p_3(0)^2 = c_0 - p_1(0)^2 - p_2(0)^2 = c_0 - \bar{p}_1(t_1)^2 - \bar{p}_2(t_1)^2 = \bar{p}_3(t_1)^2$$

and so $p_3(0) = \pm\bar{p}_3(t_1)$. Now

$$\begin{aligned}
\bar{p}_1(-t_1 + \tfrac{2K}{\Omega}) &= \bar{p}_1(t_1) \\
\bar{p}_2(-t_1 + \tfrac{2K}{\Omega}) &= \bar{p}_2(t_1) \\
\bar{p}_3(-t_1 + \tfrac{2K}{\Omega}) &= -\bar{p}_3(t_1).
\end{aligned}$$

Thus there exists $t_0 \in \mathbb{R}$ ($t_0 = t_1$ or $t_0 = -t_1 + \frac{2K}{\Omega}$) such that $p(0) = \bar{p}(t_0)$. Consequently, the integral curves $t \mapsto p(t)$ and $t \mapsto \bar{p}(t + t_0)$ solve the same Cauchy problem and therefore are identical.

b) The expression for $\bar{p}(\cdot)$ is found by limiting $c_0 \to h_0^2$ from the right (i.e., limiting the expression for $\bar{p}(\cdot)$ in case c)) and allowing for possible changes in sign. Likewise, for any integral curve $p(\cdot)$ satisfying the conditions of case b), there exist $t_0 \in \mathbb{R}$ and $\sigma \in \{-1, 1\}$ such that $p(t) = \bar{p}(t + t_0)$.

c) The argument is very similar to that for case a). However, for this case we found it more convenient to transform the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\bar{p}_1 = -\sqrt{(1 - \delta - h_0 + \tfrac{1}{2}\bar{p}_1^2)(-1 - \delta + h_0 - \tfrac{1}{2}\bar{p}_1^2)}$$

into standard form. ■

4. Final Remark

A comprehensive treatment of the remaining inhomogeneous systems, addressing both stability and integration, will appear elsewhere.

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