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MULTIPARAMETER CONTACT TRANSFORMATIONS

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Abstract. This is a review of multiparameter families of contact transformations and their relationship with the generalized Hamiltonian system. We derive the integrability conditions for the generalized Hamiltonian system and show that when they are satisfied the solutions to this system determine a family of multiparameter contact transformations of the initial conditions. We prove a necessary and sufficient condition for a multiparameter family of contact transformations to be a group and a characterization of the function which describes the group multiplication rule.

1. Introduction

Let us begin by recalling a few facts about one parameter contact transformations. Consider transformations of the (x, y, z, p, q)-space to the (X, Y, Z, P, Q)-space defined by $X = X(x, y, z, p, q)$, $Y = Y(x, y, z, p, q)$, $Z = Z(x, y, z, p, q)$, $P = P(x, y, z, p, q)$, $Q = Q(x, y, z, p, q)$.

Definition 1. *Let T be a one-to-one, onto, continuously differentiable transformation of the (x, y, z, p, q)-space to the (X, Y, Z, P, Q)-space with a nonzero Jacobian. Then T is called a contact transformation if $p\,\mathrm{d}x + q\,\mathrm{d}y - \mathrm{d}z = 0$ implies $P\,\mathrm{d}X + Q\,\mathrm{d}Y - \mathrm{d}Z = 0$.*

Theorem 1. *The one-to-one, onto, continuously differentiable transformation T of the (x, y, z, p, q)-space to the (X, Y, Z, P, Q)-space with a nonzero Jacobian is a contact transformation if and only if there exists a nonzero function $\rho = \rho(x, y, z, p, q)$ such that*

$$P\,\mathrm{d}X + Q\,\mathrm{d}Y - \mathrm{d}Z = \rho(p\,\mathrm{d}x + q\,\mathrm{d}y - \mathrm{d}z). \tag{1}$$

Example 1. The Legendre transformation $X = p,\ Y = q,\ P = x,\ Q = y,$ $Z = p\,x + q\,y - z$ is a contact transformation. For it the function ρ in the previous necessary and sufficient condition is $\rho = -1$.

Let now $x = (x_1, \dots, x_n),\ p = (p_1, \dots, p_n)$ and S_t be a one-parameter family of contact transformations

$$X = X(x, z, p, t)$$
$$Z = Z(x, z, p, t)$$
$$P = P(x, z, p, t)$$

where t is the parameter, $X = (X_1, \dots, X_n)$ stands for the images of $x_1, \dots, x_n$ under S_t, Z is the image of z under S_t and $P = (P_1, \dots, P_n)$ stands for the images of $p_1, \dots, p_n$ under S_t. The summation convention on repeated indices is used for the rest of the paper. For one-parameter families of contact transformations the necessary and sufficient condition (1) for a contact transformation is replaced by

$$P_i\,\mathrm{d}X_i - \mathrm{d}Z = \rho\big(p_i\,\mathrm{d}x_i - \mathrm{d}z\big) + H\mathrm{d}t \tag{2}$$

where

$$H = P_i\,\frac{\mathrm{d}X_i}{\mathrm{d}t} - \frac{\mathrm{d}Z}{\mathrm{d}t}\cdot$$

In the 1930-s Gustav Herglotz proposed a **generalized variational principle** with one independent variable, which generalizes the classical variational principle by defining the functional z, whose extrema are sought, by the differential equation

$$\frac{\mathrm{d}z}{\mathrm{d}t} = L\Big(t, x(t), \frac{\mathrm{d}x(t)}{\mathrm{d}t}, z\Big) \tag{3}$$

where t is the independent variable, and $x(t) \equiv (x_1(t), \dots, x_n(t))$ stands for the argument functions. In order for the equation (3) to define a functional $z = z[x]$ of $x(t)$ equation (3) must be solved with the same fixed initial condition $z(0)$ for all argument functions $x(t)$, and the solution $z(t)$ must be evaluated at the same fixed final time $t = T$ for all argument functions $x(t)$.

The equations whose solutions produce the extrema of this functional are

$$\frac{\partial L}{\partial x_k} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}_k} + \frac{\partial L}{\partial z}\frac{\partial L}{\partial \dot{x}_k} = 0, \qquad k = 1, \dots, n \tag{4}$$

where $\dot{x}_k$ denotes $\mathrm{d}x_k/\mathrm{d}t$. Herglotz called them **generalized Euler-Lagrange equations.**

Remarkably, the solutions of the generalized Euler-Lagrange equations (4), when written in terms of the dependent variables x_k and the associated momenta $p_k = \partial L/\partial \dot{x}_k$, determine a family of **contact transformations** of the initial conditions. In more detail, let's write the defining equation (3) for the functional z and the generalized Euler-Lagrange equations (4) in the following manner

$$\dot{z} = L(x_1, \dots, x_n, \dot{x}_1, \dots, \dot{x}_n, z, t)$$

$$\dot{p}_j = L_j + L_z p_j, \qquad j = 1, \ldots, n \tag{5}$$

where we have denoted

$$\frac{\partial L}{\partial x_j} = L_j, \qquad \frac{\partial L}{\partial \dot{x}_j} = p_j.$$

Let $(x^0, \dot{x}^0, z^0)$ be the initial condition for the system (5) of $n+1$ ordinary differential equations for the functions $x_1(t), \ldots, x_n(t),\ z(t)$. Then the solution of the system (5) with this initial condition is

$$\begin{aligned} x &= x(x^0, \dot{x}^0, z^0, t) \\ \dot{x} &= \dot{x}(x^0, \dot{x}^0, z^0, t) \\ z &= z(x^0, \dot{x}^0, z^0, t). \end{aligned} \tag{6}$$

Theorem 2. *Let $L = L(x, \dot{x}, z, t)$ be such that*

$$\det\left(\frac{\partial^2 L}{\partial \dot{x}_i \partial \dot{x}_j}\right) \neq 0.$$

Then the solution (6) *of the system* (5) *defines a one-parameter family of contact transformations.*

A proof of this theorem can be found in [2].

2. Multiparameter Families of Contact Transformations

Let $x = (x_1, \ldots, x_n)$, $p = (p_1, \ldots, p_n)$ denote points in $\mathbb{R}_n$ so that (x, z, p) is a point in a $(2n+1)$ – dimensional space. $t = (t_1, \ldots, t_r)$ will denote a system of r parameters and $f = f(f_1, \ldots, f_n)$, g and $h = (h_1, \ldots, h_n)$ are functions of (x^0, z^0, p^0, t). We call

$$\begin{aligned} x &= f(x^0, z^0, p^0, t) \\ z &= g(x^0, z^0, p^0, t) \\ p &= h(x^0, z^0, p^0, t) \end{aligned} \tag{7}$$

an r-parameter family of contact transformations if, for each fixed t, the functions f, g and h satisfy the condition

$$p_i \,\mathrm{d}x_i - \mathrm{d}z = \rho\,(p_i^0 \,\mathrm{d}x_i^0 - \mathrm{d}z^0), \qquad \rho \neq 0.$$

It is often convenient to write the transformation (7) in the form

$$(x, z, p) = S_t(x^0, z^0, p^0) \tag{8}$$

to bring out the fact that the point (x^0, z^0, p^0) is carried into the point (x, z, p). We do not demand that the family of transformations $\{S_t\}$ contains the identity, nor that (x^0, z^0, p^0) represent initial values. Rather, (x^0, z^0, p^0) is a generic point in the $(2n+1)$ – dimensional space where the transformations are defined.

Theorem 3. *If* $\{S_t\}$ *is an* r*-parameter family of contact transformations, then there exist functions*

$$H_j = H_j(x, z, p, t), \qquad j = 1, \dots, r \tag{9}$$

such that the (x, z, p) *of* (7) *satisfy the total canonical system*

$$\begin{aligned} \mathrm{d}x_j &= \frac{\partial H_k}{\partial p_j}\mathrm{d}t_k, \qquad j = 1, \dots, n, \qquad k = 1, \dots, r \\ \mathrm{d}z &= \Big(p_j \frac{\partial H_k}{\partial p_j} - H_k\Big)\mathrm{d}t_k \\ \mathrm{d}p_j &= -\Big(\frac{\partial H_k}{\partial x_j} + p_j \frac{\partial H_k}{\partial z}\Big)\mathrm{d}t_k. \end{aligned} \tag{10}$$

Proof: The transformations S_t must satisfy

$$p_i\,\mathrm{d}x_i - \mathrm{d}z = \rho\,(p_i^0\,\mathrm{d}x_i^0 - \mathrm{d}z^0), \qquad \rho \neq 0 \tag{11}$$

which is supposed to hold when the differentials are calculated only with resect to the spatial variables. When x, z, p also depend on $t_1, \dots, t_r$, then

$$\mathrm{d}z = \frac{\partial z}{\partial x_j^0}\mathrm{d}x_j^0 + \frac{\partial z}{\partial z^0}\mathrm{d}z^0 \frac{\partial z}{\partial p_j^0}\mathrm{d}p_j^0 + \frac{\partial z}{\partial t_k}\mathrm{d}t_k, \quad j = 1, \dots, n, \quad k = 1, \dots, r.$$

Similarly for $\mathrm{d}x_i$. Hence the condition (11) is replaced by

$$\big(p_i\,\mathrm{d}x_i - \mathrm{d}z\big) - \Big(p_i \frac{\partial x_i}{\partial t_k} - \frac{\partial z}{\partial t_k}\Big)\,\mathrm{d}t_k = \rho\,(p_i^0\,\mathrm{d}x_i^0 - \mathrm{d}z^0), \qquad \rho \neq 0. \tag{12}$$

From (7) we obtain

$$\frac{\partial x_i}{\partial t_k} = \xi_{ik}(x, z, p, t), \quad \frac{\partial z}{\partial t_k} = \zeta_k(x, z, p, t), \quad \frac{\partial p_i}{\partial t_k} = \pi_{ik}(x, z, p, t), \ \ i = 1, \dots, n.$$

Let us introduce the functions

$$H_k \equiv H_k(x, z, p, t) \equiv p_i\,\xi_{ik}(x, z, p, t) - \zeta_k(x, z, p, t), \qquad k = 1, \dots, r. \tag{13}$$

Then the relation (12) takes the form analogous to (2), namely

$$p_i\,\mathrm{d}x_i - \mathrm{d}z = \rho\,(p_i^0\,\mathrm{d}x_i^0 - \mathrm{d}z^0) + H_k\,\mathrm{d}t_k, \qquad \rho \neq 0. \tag{14}$$

If $\mathrm{d}t_k = 0, k = 1, \dots, r$, equation (14) reduces to (11). Equation (14) represents a system of $(2n + r + 1)$ equations relating the variables (x, z, p, t) with (x^0, z^0, p^0, t), which is obtained by expanding the differentials and comparing coefficients.

Differentiate (14) with respect to t_k and note that the differential operator d commutes with the operator $\partial/\partial t_k$. This leads to

$$\pi_{jk}\,\mathrm{d}x_j + p_j\,\mathrm{d}\xi_{jk} - \mathrm{d}\zeta_k = \rho_{t_k}(p_j^0\,\mathrm{d}x_j^0 - \mathrm{d}z^0) + \frac{\partial H_l}{\partial t_k}\,\mathrm{d}t_l. \tag{15}$$

From (15) and (14) we obtain

$$\pi_{jk}\,\mathrm{d}x_j + p_j\,\mathrm{d}\xi_{jk} - \mathrm{d}\zeta_k - \frac{\partial H_l}{\partial t_k}\,\mathrm{d}t_l = \frac{\rho_{t_k}}{\rho}(p_j\,\mathrm{d}x_j - \mathrm{d}z - H_l\,\mathrm{d}t_l). \tag{16}$$

From (13) we find

$$\mathrm{d}H_k = \xi_{jk}\,\mathrm{d}p_j + p_j\,\mathrm{d}\xi_{jk} - \mathrm{d}\zeta_k, \qquad j = 1,\dots,n$$

so that (16) takes the form

$$\mathrm{d}H_k + \pi_{jk}\,\mathrm{d}x_j - \xi_{jk}\,\mathrm{d}p_j = \frac{\rho_{t_k}}{\rho}(p_j\,\mathrm{d}x_j - \mathrm{d}z) + \Big(\frac{\partial H_l}{\partial t_k} - \frac{\rho_{t_k}}{\rho}\,H_l\Big)\,\mathrm{d}t_l. \tag{17}$$

Expand $\mathrm{d}H_k$ in the form

$$\mathrm{d}H_k = \frac{\partial H_k}{\partial x_j}\,\mathrm{d}x_j + \frac{\partial H_k}{\partial z}\,\mathrm{d}z + \frac{\partial H_k}{\partial p_j}\,\mathrm{d}p_j + \frac{\partial H_k}{\partial t_l}\,\mathrm{d}t_l$$

insert it into (17) and compare coefficients to obtain the following system

$$\begin{aligned} \frac{\partial H_k}{\partial x_j} &= -\,\pi_{jk} + \frac{\rho_{t_k}}{\rho}\,p_j, & \frac{\partial H_k}{\partial p_j} &= \xi_{jk} \\ \frac{\partial H_k}{\partial z} &= -\,\frac{\rho_{t_k}}{\rho}, & \frac{\partial H_k}{\partial t_l} &= \frac{\partial H_l}{\partial t_k} - \frac{\rho_{t_k}}{\rho}\,H_l. \end{aligned} \tag{18}$$

We now obtain expressions for ξ_{jk}, π_{jk} and ζ_k. From (18) by eliminating the quotient ρ_{t_k}/ρ we get

$$\pi_{jk} = -\frac{\partial H_k}{\partial x_j} - p_j\,\frac{\partial H_k}{\partial z}$$

$$\xi_{jk} = \frac{\partial H_k}{\partial p_j}$$

$$\zeta_k = -H_k + p_j\,\xi_{jk} = -H_k + p_j\,\frac{\partial H_k}{\partial p_j}.$$

■

The functions $H_j(x, z, p, t)$ of (9) characterize the particular family of contact transformations and are called characteristic or Hamiltonian functions. Although they may be derived from (7) as indicated, in practical problems one is usually faced with the converse problem of constructing the family (7) or (8) from (10) given (9). In order to carry out the integrations, the H_js must satisfy certain integrability conditions. To obtain them, it is convenient to rewrite the system (10) as

$$\begin{aligned}\frac{\partial x_j}{\partial t_k} &= \frac{\partial H_k}{\partial p_j}, \qquad j = 1, \dots, n, \qquad k = 1, \dots, r \\ \frac{\partial z}{\partial t_k} &= p_j \frac{\partial H_k}{\partial p_j} - H_k \\ \frac{\partial p_j}{\partial t_k} &= -\frac{\partial H_k}{\partial x_j} - p_j \frac{\partial H_k}{\partial z}.\end{aligned} \tag{19}$$

To formulate the integrability conditions, it is advantageous to introduce the bracket symbol

$$[F, G]_{xzp} = \{F, G\}_{xzp} + F\frac{\partial G}{\partial z} - G\frac{\partial F}{\partial z} \tag{20}$$

where $\{F, G\}_{xzp}$ denotes the Mayer bracket of two functions F and G. Recall that the Mayer bracket of the functions f and g is

$$\{f, g\}_{xzp} = \Big(\frac{\partial f}{\partial x_j} + p_j \frac{\partial f}{\partial z}\Big)\frac{\partial g}{\partial p_j} - \Big(\frac{\partial g}{\partial x_j} + p_j \frac{\partial g}{\partial z}\Big)\frac{\partial f}{\partial p_j}.$$

When written out (20) becomes

$$[F, G]_{xzp} = \frac{\partial F}{\partial x_i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial x_i} + \frac{\partial F}{\partial z}\Big(p_i \frac{\partial G}{\partial p_i} - G\Big) - \frac{\partial G}{\partial z}\Big(p_i \frac{\partial F}{\partial p_i} - F\Big).$$

This bracket symbol also satisfies the Jacobi identity

$$[F, [G, H]] + [G, [H, F]] + [H, [F, G]] = 0. \tag{21}$$

Next let us define the symbols

$$H_{kl} \equiv [H_k, H_l]_{xzp} + \frac{\partial H_k}{\partial t_l} - \frac{\partial H_l}{\partial t_k}. \tag{22}$$

The integrability conditions require that the second mixed partials of the functions x_j, z, p_j with respect to the t variables are equal. A calculation making use of (19), (21) and the definition (22) yields the relations

$$\begin{aligned}\frac{\partial^2}{\partial t_l \partial t_k} x_j - \frac{\partial^2}{\partial t_k \partial t_l} x_j &= \frac{\partial}{\partial p_j} H_{kl} \\ \frac{\partial^2}{\partial t_l \partial t_k} p_j - \frac{\partial^2}{\partial t_k \partial t_l} p_j &= -\frac{\partial}{\partial x_j} H_{kl} + p_j \frac{\partial}{\partial z} H_{kl} \\ \frac{\partial^2}{\partial t_l \partial t_k} z - \frac{\partial^2}{\partial t_k \partial t_l} z &= p_j \frac{\partial}{\partial p_j} H_{kl} - H_{kl}.\end{aligned} \tag{23}$$

In order to force the right hand sides of (23) to be zero in these expressions, we see that the H_{kl} must vanish, which in view of (22) says

$$[H_k, H_l] = \frac{\partial H_l}{\partial t_k} - \frac{\partial H_k}{\partial t_l} \tag{24}$$

which are the **integrability conditions**.

To prove the next theorem we need the following

Lemma 4. *Let* $F = F(x, z, p, t)$ *where* (x, z, p) *satisfy* (10) *or equivalently* (19). *Then the differential*

$$\mathrm{d}F = \Big([F, H_i]_{xzp} - F\frac{\partial H_i}{\partial z} + \frac{\partial F}{\partial t_i}\Big)\mathrm{d}t_i \tag{25}$$

or in terms of components

$$\frac{\partial F}{\partial t_i} = [F, H_i]_{xzp} - F\frac{\partial H_i}{\partial z} + \frac{\partial F}{\partial t_i}\cdot \tag{26}$$

Proof: Calculate the differential using (10) to obtain

$$\mathrm{d}F = \frac{\partial F}{\partial x_j}\mathrm{d}x_j + \frac{\partial F}{\partial z}\mathrm{d}z + \frac{\partial F}{\partial p_j}\mathrm{d}p_j + \frac{\partial F}{\partial t_i}\mathrm{d}t_i = \frac{\partial F}{\partial x_j}\frac{\partial H_i}{\partial p_j}\mathrm{d}t_i$$
$$+ \frac{\partial F}{\partial z}\Big(p_j\frac{\partial H_i}{\partial p_j} - H_i\Big)\mathrm{d}t_i - \frac{\partial F}{\partial p_j}\Big(\frac{\partial H_i}{\partial x_j} + p_j\frac{\partial H_i}{\partial z}\Big)\mathrm{d}t_i + \frac{\partial F}{\partial t_i}\mathrm{d}t_i.$$

The formula for the differential follows after some rearranging. ■

If we now use (26) to calculate the second derivatives, we find

$$\frac{\partial^2 F}{\partial t_l \partial t_k} - \frac{\partial^2 F}{\partial t_k \partial t_l} = [F, H_{kl}]_{xzp} - F\frac{\partial H_{kl}}{\partial z}\cdot \tag{27}$$

Next we state and prove the converse of Theorem 3.

Theorem 5. *Suppose the total canonical system* (10) *is given where the characteristic functions satisfy the integrability conditions* (24). *Then the family of transformations* $\{S_t\}$ *obtained by solving* (10) *subject to the initial conditions*

$$(x, z, p)|_{t=0} = (x^0, z^0, p^0)$$

is an r-parameter family of contact transformations.

Proof: We define the linear differential form

$$\omega = p_j\,\mathrm{d}x_j - \mathrm{d}z - H_i\,\mathrm{d}t_i, \qquad j = 1, \dots, n, \quad i = 1, \dots, r \tag{28}$$

when $t = 0$, i.e., $t = (t_1, \dots, t_n) = (0, \dots, 0)$, ω goes over into

$$\omega^0 = p_j^0\,\mathrm{d}x_j^0 - \mathrm{d}z^0, \qquad j = 1, \dots n.$$

Define

$$H_k = p_j\frac{\partial x_j}{\partial t_k} - \frac{\partial z}{\partial t_k}, \qquad k = 1, \dots, r.$$

Then using (19) and (28) we get

$$\mathrm{d}\omega = \frac{\partial \omega}{\partial t_i}\,\mathrm{d}t_i = \Big(\frac{\partial p_j}{\partial t_i}\,\mathrm{d}x_j + p_j\,\mathrm{d}\frac{\partial x_j}{\partial t_i} - \mathrm{d}\frac{\partial z}{\partial t_i} - \frac{\partial H_k}{\partial t_i}\,\mathrm{d}t_k\Big)\mathrm{d}t_i$$

$$
\begin{aligned}
&= -\Big(\frac{\partial H_i}{\partial x_j}\mathrm{d}x_j + \frac{\partial H_i}{\partial p_j}\mathrm{d}p_j + \frac{\partial H_i}{\partial z}\mathrm{d}z + \frac{\partial H_i}{\partial t_k}\,\mathrm{d}t_k\Big)\,\mathrm{d}t_i \\
&\quad +\Big(\frac{\partial H_i}{\partial z}\mathrm{d}z - \frac{\partial H_i}{\partial z}p_j\mathrm{d}x_j + \frac{\partial H_i}{\partial t_k}\mathrm{d}t_k\Big)\mathrm{d}t_i \\
&= -\,\mathrm{d}H_i\,\mathrm{d}t_i - \frac{\partial H_i}{\partial z}\,\omega\,\mathrm{d}t_i + \Big(\frac{\partial H_i}{\partial t_k} - \frac{\partial H_i}{\partial z}\,H_k\Big)\mathrm{d}t_k\,\mathrm{d}t_i.
\end{aligned}
$$

Next we calculate using (19)

$$
\mathrm{d}H_i\,\mathrm{d}t_i = \frac{\partial H_i}{\partial t_k}\mathrm{d}t_k\mathrm{d}t_i = \Big(\frac{\partial H_i}{\partial t_k} - \frac{\partial H_i}{\partial z}H_k\Big)\,\mathrm{d}t_k\,\mathrm{d}t_i.
$$

Thus, we obtain

$$
\mathrm{d}\omega = -\omega\,\frac{\partial H_i}{\partial z}\,\mathrm{d}t_i, \qquad i = 1,\dots,r. \tag{29}
$$

This equation is integrable because it satisfies (27) by hypothesis, i.e.,

$$
\frac{\partial^2\omega}{\partial t_k\partial t_i} - \frac{\partial^2\omega}{\partial t_i\partial t_k} = [\omega, H_{ik}]_{xzp} - \omega\,\frac{\partial}{\partial z}H_{ik} = 0
$$

by the integrability condition.

Now let t be a permissible value for the functions in question. We determine the function

$$
\rho = \rho(x^0, z^0, p^0, t)
$$

from the equation

$$
\ln\rho = -\int_{\Gamma[0,t]}\frac{\partial H_k}{\partial z}\,\mathrm{d}t_k, \qquad k = 1,\dots,r
$$

where the integral is taken over a path Γ, connecting 0 and t. Because of the integrability conditions, the integral is independent of the path. Exponentiate to find for ρ the expression

$$
\rho = \exp\Big(-\int_{\Gamma[0,t]}\frac{\partial H_k}{\partial z}\,\mathrm{d}t_k\Big), \qquad k = 1,\dots,r
$$

and set

$$
\omega = \rho\,\omega^0. \tag{30}
$$

By carrying out the differentiations, it is easy to verify that ω defined by (30) satisfies the total differential equation (29). But (30) is simply

$$
p_j\,\mathrm{d}x_j - \mathrm{d}z = \rho\,(p_j^0\,\mathrm{d}x_j^0 - \mathrm{d}z^0) + H_i\,\mathrm{d}t_i, \qquad j = 1,\dots,n, \quad i = 1,\dots,r
$$

which completes the proof of the assertion.

■

3. Multiparameter Groups of Contact Transformations

In this section the letters t, s, etc., will denote the $r-$tuples $(t_1, \dots, t_r)$, $(s_1, \dots s_r)$, etc. The value $t = 0$ will correspond to the identity transformation.

Definition 2. *An r-parameter group of contact transformations is a family, $\{S_t\}$, of contact transformations which satisfies the following conditions*

1. *The family includes an identity element, S_0, called the identity*
2. *There is an operation called multiplication such that if S_t and S_s are elements of the family, there exists an element, S_σ, of the family, such that*
$$S_\sigma = S_t S_s.$$
This multiplication is determined by a smooth function
$$\phi = (\phi_1, \dots, \phi_r)$$
of the variables (t, s).
3. *$S_t S_0 = S_0 S_t = S_t$, that is*
$$\phi(t, 0) = \phi(0, t) = t$$
and the Jacobi determinant
$$\frac{\partial(\phi_1(t, s), \dots, \phi_r(t, s))}{\partial(t_1, \dots, t_r)} \neq 0$$
for t, s near 0. In particular, $\phi(0, 0) \neq 0$.
4. *The associative law holds, that is*
$$S_t(S_s\, S_\sigma) = (S_t\, S_s)S_\sigma$$
in other words, ϕ satisfies
$$\phi(t, \phi(s, \sigma)) = \phi(\phi(t, s), \sigma).$$

The Condition 3 implies the existence of an inverse, because the equation
$$S_\sigma\, S_t = S_0$$
or more precisely
$$\phi(\sigma, t) = 0$$
is solvable for σ in terms of t. In operator notation, let
$$S_\sigma = S_t^{-1}$$
denote that solution. We must show that also
$$S_t S_\sigma = S_0.$$
For this calculation let S_σ^* be such that $S_\sigma^* S_\sigma = S_0$. Then
$$S_t S_\sigma = S_0(S_t S_\sigma) = (S_\sigma^* S_\sigma)(S_t S_\sigma) = S_\sigma^*(S_\sigma S_t)S_\sigma = S_\sigma^* S_\sigma = S_0$$

so that $S_\sigma = S_t^{-1}$ is both a right and a left inverse and the standard group axioms hold. S_t^{-1} is easily seen to be unique and moreover we find that

$$\frac{\partial(\phi(t,s))}{\partial(s)} \neq 0 \qquad \text{for } \ t,s \ \text{ near } 0.$$

After these preliminaries we state the main theorem of this section.

Theorem 6. *In order that an r-parameter family, $\{S_t\}$, of contact transformations be a group, it is necessary and sufficient that the characteristic functions, H_k, have the form*

$$H_k = H_k(x,z,p,t) = K_i(x,z,p)\,\omega_{ik}(t), \qquad i,k = 1,\dots,r. \tag{31}$$

Here the K_i are independent of t and the ω_{ik} depend only on t. Moreover the functions$K_1,\dots,K_r$ are linearly independent, and the determinant of the $r \times r$ matrix (ω_{ik}) is nonzero.

Before giving a proof of this theorem, we first make a few observations.
Let us set

$$\mathrm{d}\omega_i = \omega_{ik}\,\mathrm{d}t_k, \qquad i,k = 1,\dots,r$$

so that (31) takes the form

$$H_k\,\mathrm{d}t_k = K_i\,\mathrm{d}\omega_i, \qquad i,k = 1,\dots,r.$$

The differential form

$$H_k\,\mathrm{d}t_k, \qquad k = 1,\dots,r$$

is integrable and by the (24) the integrability conditions are

$$[H_k, H_l]_{xzp} = \frac{\partial H_l}{\partial t_k} - \frac{\partial H_k}{\partial t_l}\cdot \tag{32}$$

If the H_k are given by (31) then (32) has the form

$$[K_k, K_l]_{xzp}\,\omega_{k\alpha}\,\omega_{l\beta} = K_k\Big(\frac{\partial \omega_{k\beta}}{\partial t_\alpha} - \frac{\partial \omega_{k\alpha}}{\partial t_\beta}\Big), \qquad k,l = 1\dots r. \tag{33}$$

Since $\det(\omega_{ij}) \neq 0$, the matrix (ω_{ij}) has an inverse which we denote by (η_{ij}). Consequently,

$$\omega_{ik}\,\eta_{kj} = \delta_{ij}\ , \quad \eta_{ik}\,\omega_{kj} = \delta_{ij}, \qquad i,j,k = 1,\dots,r \tag{34}$$

where δ_{ij} is the Kronecker delta. Multiply (33) by $\eta_{\alpha\rho}$ and $\eta_{\beta\sigma}$, sum over α and β, and use (34) to get

$$[K_\rho, K_\sigma] = c_{\rho\sigma j}\,K_j, \qquad \rho,\ \sigma,\ j = 1,\dots,r \tag{35}$$

where

$$c_{\rho\sigma j} = \Big(\frac{\partial \omega_{j\beta}}{\partial t_\alpha} - \frac{\partial \omega_{j\alpha}}{\partial t_\beta}\Big)\eta_{\alpha\rho}\eta_{\beta\sigma}, \qquad \alpha,\ \beta = 1\dots r. \tag{36}$$

Now multiply (36) by $\omega_{\rho k}\,\omega_{\sigma l}$, sum over ρ and σ and use (34) to find

$$\frac{\partial \omega_{jl}}{\partial t_k} - \frac{\partial \omega_{jk}}{\partial t_k} = c_{\rho\sigma j}\,\omega_{\rho k}\,\omega_{\sigma l}, \qquad \rho,\,\sigma = 1,\dots,r. \tag{37}$$

The formulae (35) and (37) are the **Maurer relations**. The $c_{\rho\sigma j}$ are independent of (x, z, p) by their definition, but apparently may depend on t. In fact they are all constant – the structure constants of the group. From their definition, $c_{\rho\sigma j}$ are antisymmetric in the first two indices

$$c_{\rho\sigma j} + c_{\sigma\rho j} = 0.$$

They also satisfy a Jacobi type identity

$$c_{ik\alpha}c_{j\alpha m} + c_{kj\alpha}c_{i\alpha m} + c_{ji\alpha}c_{k\alpha m} = 0, \qquad \alpha = 1,\dots,r. \tag{38}$$

The next theorem characterizes the function $\phi = \phi(t, s)$, which describes the multiplication rule for the multiparameter group of transformations.

Theorem 7. *The function describing the group operation*

$$t' = \phi(t, s)$$

is determined by the Maurer-Cartan system of total differential equations

$$\omega_{ij}(t')\,\mathrm{d}t'_j = \omega_{ij}(t)\,\mathrm{d}t_j, \qquad j = 1,\dots,r, \qquad \textit{briefly} \qquad \mathrm{d}\omega'_i = \mathrm{d}\omega_i \tag{39}$$

which satisfy the initial conditions

$$t' = s \quad \textit{when} \quad t = 0.$$

Proof: Let

$$P(t,s) = \Big(\frac{\partial \phi_i(t,s)}{\partial t_j}\Big) \quad \text{and} \quad Q(t,s) = \Big(\frac{\partial \phi_i(t,s)}{\partial s_j}\Big)$$

denote $r \times r$ matrices and consider the relations

$$\phi_i(\sigma, \phi(t,s)) = \phi_i(\phi(\sigma,t), s), \qquad i = 1,\dots,r. \tag{40}$$

Differentiate (40) successively with respect to $t_1,\dots,t_r$ to obtain the relationship

$$Q(\sigma,\phi(t,s))\,P(t,s) = P(\phi(\sigma,t),s)\,Q(\sigma,t) \tag{41}$$

and then with respect to $\sigma_1,\dots,\sigma_r$ to find

$$P(\sigma,\phi(t,s)) = P(\phi(\sigma,t),s)P(\sigma,t). \tag{42}$$

The matrices P and Q are invertible. Set

$$\Omega(t,s) = P^{-1}(\sigma,t)\,Q(\sigma,t) \tag{43}$$

and in the computation below let

$$t' = \phi(t,s) \quad \text{and} \quad t'' = \phi(\sigma,t).$$

Then by (41) and (42) and the definition (43)

$$\begin{aligned}\Omega(t',\phi)\,P(t,s) &= P^{-1}(\sigma,t')\,Q(\sigma,t')\,P(t,s) = P^{-1}(\sigma,t')\,P(t'',s)\,Q(\sigma,t)\\ &= P^{-1}(\sigma,t')\,P(\sigma,t')\,P^{-1}(\sigma,t)\,Q(\sigma,t) = P^{-1}(\sigma,t)\,Q(\sigma,t) = \Omega(t,\sigma)\end{aligned}$$

so that

$$\Omega(t',\sigma)\,P(t,s) = \Omega(t,\sigma). \tag{44}$$

Set $\sigma = 0$ and let

$$\Omega(t,0) = (\omega_{ij}(t)).$$

Then (44) becomes

$$\omega_{ij}(t')\,\frac{\partial\phi_i}{\partial t_k} = \omega_{ik}(t), \qquad j = 1,\ldots,r. \tag{45}$$

For s fixed

$$\mathrm{d}t'_j = \frac{\partial\phi_j}{\partial t_k}\,\mathrm{d}t_k, \qquad k = 1,\ldots,r$$

so that if we multiply (45) by $\mathrm{d}t_k$ and sum over k, we get

$$\omega_j(t')\,\mathrm{d}t'_j = \omega_j(t)\,\mathrm{d}t_j, \qquad j = 1,\ldots,r$$

which was to be proven. ■

On the other hand, we can derive the associativity of the solution system, $\phi(t,s)$, from these differential equations. To see that, suppose

$$\mathrm{d}\omega'_j = \mathrm{d}\omega_j$$

and let

$$t'' = \phi(t',\sigma), \qquad t' = \phi(t,s).$$

Then from what we have just proven,

$$\mathrm{d}\omega''_j = \mathrm{d}\omega'_j = \mathrm{d}\omega_j\ .$$

In particular, when $t = 0,\ t'' = \phi(s,\sigma)$. By the uniqueness of the solutions

$$t'' = \phi(t,\phi(s,\sigma))$$

and by the definition of t''

$$t'' = \phi(\phi(t,s),\sigma)$$

which proves the associativity of the system of functions $\phi(t,s)$ which appear as solutions to the Maurer-Cartan equations (39).

Proposition 8. *The integrability conditions of the Maurer-Cartan equations* (39) *are the equations* (37).

Proof: Rewrite the conditions $\mathrm{d}\omega_i' = \mathrm{d}\omega_i$, where $t' = \phi(t, s)$, as

$$\omega_{ij}(t')\,\mathrm{d}t_j' = \omega_{ij}(t')\,\frac{\partial \phi_i}{\partial t_k}\,\mathrm{d}t_k = \omega_{ij}(t)\,\mathrm{d}t_k, \qquad j, k = 1, \ldots, r$$

so that the integrability conditions are

$$\frac{\partial}{\partial t_l}\Big(\omega_{ij}(t')\,\frac{\partial \phi_j}{\partial t_k} - \omega_{ik}(t)\Big) = \frac{\partial}{\partial t_k}\Big(\omega_{ij}(t')\,\frac{\partial \phi_j}{\partial t_l} - \omega_{il}(t)\Big), \qquad j = 1, \ldots, r$$

that is

$$\frac{\partial \omega_{ij}(t')}{\partial t_m'}\,\frac{\partial \phi_m}{\partial t_l}\,\frac{\partial \phi_j}{\partial t_k} - \frac{\partial \omega_{ij}(t')}{\partial t_m'}\,\frac{\partial \phi_m}{\partial t_k}\,\frac{\partial \phi_j}{\partial t_l} = \frac{\partial \omega_{ik}}{\partial t_l} - \frac{\partial \omega_{il}}{\partial t_k}, \qquad m, j = 1, \ldots, r$$

by the chain rule. In the first part of the summation, sum first with respect to j and then with respect to m, and in the second, sum first with respect to m and then with respect to j. Rewriting as a single sum, now yields

$$\frac{\partial \omega_{ik}}{\partial t_l} - \frac{\partial \omega_{il}}{\partial t_k} = \Big(\frac{\partial \omega_{ij}(t')}{\partial t_m'} - \frac{\partial \omega_{im}(t')}{\partial t_j'}\Big)\,\frac{\partial \phi_j}{\partial t_k}\,\frac{\partial \phi_m}{\partial t_l}, \qquad m, j = 1, \ldots, r. \quad (46)$$

Now by (46), the $\partial \phi_j / \partial t_k$ are the components of a matrix given by

$$P(t, s) = \Big(\frac{\partial \phi_j(t, s)}{\partial t_k}\Big) = \Omega^{-1}(t', 0)\,\omega(t, 0).$$

The matrix $\Omega^{-1}(t', 0)$ is given by

$$\Omega^{-1}(t', 0) = (\eta_{ij}(t')).$$

Moreover, $\Omega(t, 0) = (\omega_{ij}(t))$ so that after inserting these expressions into (46) and using the definition (36) of the structure constants, we see that (46) is precisely the condition (37). ■

Remark 9. *If the function defining the group operation satisfies*

$$\phi(t, s) = \phi(s, t)$$

then the group is abelian and we can show that

$$\mathrm{d}\omega_i(t) = \omega_{ij}(t)\,\mathrm{d}t_j, \qquad j = 1, \ldots, r$$

is a total differential. The solution to the Maurer-Cartan equations is obtained by a quadrature and one gets

$$\omega_i(t') = \omega_i(t) + \omega_i(s).$$

If we introduce the parameter

$$\tau_i = \omega_i(t)$$

then

$$\tau_i' = \tau_i + \sigma_i$$

where

$$\tau_i' = \omega_i(t'), \qquad \sigma_i = \omega_i(s)$$

which are

$$S_\tau S_\sigma = S_{\tau+\sigma}.$$

In the case $r = 1$, the possibility of introducing an additive parameter follows from the associative law, but if $r \geq 2$, the commutativity condition on the group multiplication must be required in addition to associativity.

We now take up the proof of Theorem 6.

Proof of Theorem 6: Let's first prove that the condition

$$H_k(x, z, p, t) = K_j(x, z, p)\,\omega_{jk}(t), \qquad j = 1, \ldots, r$$

is necessary in order that the H_j generate a group of contact transformations. We assume, therefore, that the family of contact transformations generated by the H_j forms a group and denote the function, describing the group operation, by ϕ so that

$$S_t\,S_s = S_{t'} \quad \text{where} \quad t' = \phi(t, s). \tag{47}$$

Let (x^0, z^0, p^0) and s be fixed but arbitrary, and set

$$(x, z, p) = S_{t'}(x^0, z^0, p^0) = S_t S_s(x^0, z^0, p^0).$$

Then

$$p_\nu\,\mathrm{d}x_\nu - \mathrm{d}z = \Sigma_{j=1}^r\,H_j(x, z, p, t)\,\mathrm{d}t', \qquad \nu = 1, \ldots, n$$

and also

$$p_\nu\,\mathrm{d}x_\nu - \mathrm{d}z = \Sigma_{j=1}^r\,H_j(x, z, p, t)\,\mathrm{d}t, \qquad \nu = 1, \ldots, n$$

hence together with (47)

$$H_j(x, z, p, t')\,\frac{\partial\phi_j(t, s)}{\partial t_l}\,\mathrm{d}t_l = H_l\,\mathrm{d}t_l, \qquad j, l = 1, \ldots, r$$

and consequently

$$\Sigma_{j=1}^r\,H_j(x, z, p, \phi(t, s))\,\frac{\partial\phi(t, s)}{\partial t_l} = H_l(x, z, p, t).$$

Set $t = 0$ to find

$$\Sigma_{j=1}^r\,H_j(x, z, p, s)\,\frac{\partial\phi(0, s)}{\partial t_l} = H_l(x, z, p, 0). \tag{48}$$

Now let

$$K_l(x, z, p) = H_l(x, z, p, 0)$$

and $(\omega_{jk}(s))$ denote the components of the matrix inverse of $(\partial\phi(0, s)/\partial t_l)$. Then (48) becomes with s now replaced by t

$$H_k(x, z, p, t) = K_j(x, z, p)\,\omega_{jk}(t). \tag{49}$$

The $(\omega_{jk}(t))$ obviously has a nonzero determinant. The linear independence of the K_j follows immediately from that of the H_j.

Now let us show that the condition (31) is sufficient. In that case we are assuming that the canonical system generated by the H_j is integrable. We have seen that this implies the validity of the Maurer relations (37), that is the system

$$\mathrm{d}\omega'_j = \mathrm{d}\omega_j \tag{50}$$

is integrable. Let

$$t' = \phi(t, s)$$

be a solution to (50) satisfying

$$\phi(0, s) = s.$$

We must prove that

$$(x, z, p) = S_{t'}(x^0, z^0, p^0)$$

and

$$(x^*, z^*, p^*) = S_t\, S_s(x^0, z^0, p^0)$$

are equal when $t' = \phi(t, s)$. Let s be fixed and arbitrary. We consider $S_{\phi(t,s)}$ and $S_t\, S_s$ as functions of t. For $t = 0$,

$$(x, z, p) = (x^*, z^*, p^*) = S_s(x^0, z^0, p^0).$$

Both the (x, z, p) and (x^*, z^*, p^*) satisfy the same canonical equations, i.e., x satisfies

$$\mathrm{d}x_\nu = \frac{\partial H'_j}{\partial p_\nu}\,\mathrm{d}t'_j = \frac{\partial K_j}{\partial p_\nu}\,\mathrm{d}\omega'_j, \qquad j = 1, \ldots, r$$

$$\mathrm{d}x^*_\nu = \frac{\partial H^*_j}{\partial p_\nu}\,\mathrm{d}t_j = \frac{\partial K_j}{\partial p_\nu}\,\mathrm{d}\omega_j, \qquad j = 1, \ldots, r$$

and by (50) these systems are the same, hence by the uniqueness, x and x^* are equal. The other cases are similar, which proves the theorem. ■

Theorem 6 is Lie's first fundamental theorem proven in the case of contact transformations.

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