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GREEN'S FUNCTION FOR 5D $SU(2)$ MIC-KEPLER PROBLEM

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Abstract. The Green's function for 5-dimensional counterpart of the MIC-Kepler problem (Kepler potential plus $SU(2)$ Yang–Mills instanton plus Zwanziger-like $1/R^2$ centrifugal term) is constructed on the basis of the Green's function for the 8-dimensional harmonic oscillator.

1. Introduction

Coulomb Green's functions in a n-dimensional Euclidean space have been constructed in [1]. The results for the cases $n = 2, 3, 5$ can be deduced from the oscillator Green's functions in $N = 2, 4, 8$ dimensions due to Levi-Civita, Kustaanheimo–Stiefel [2] and Hurwitz transformations [3], respectively.

Moreover [4], the $N = 4$ oscillator representation allows to obtain Green's function for 3-dimensional MIC-Kepler problem [5] (Kepler–Coulomb potential plus $U(1)$ Dirac monopole plus Zwanziger's [6] $1/R^2$ centrifugal term).

In this paper we construct the Green's function for 5-dimensional counterpart of the MIC-Kepler problem [7] (Kepler potential plus $SU(2)$ Yang–Mills instanton plus Zwanziger-like $1/R^2$ centrifugal term). We avoid a tedious procedure of path integration and deduce our result from the well-known expression for the 8-dimensional oscillator Green's function by exploiting the Hurwitz correspondence between these 5- and 8-dimensional problems [7–9].

2. Correspondence Between 5- and 8-Dimensional Problems

Under the certain known conditions [7–9] there appears the correspondence between the 8-dimensional harmonic oscillator problem

$$H\psi^{(8)} = E\psi^{(8)}, \quad H = -\frac{1}{2}\Delta_8 + \frac{\omega^2}{2}\left(|u|^2 + |v|^2\right) \tag{1}$$

and 5-dimensional $SU(2)$ MIC-Kepler problem

$$\mathcal{H}^l\phi^l = \mathcal{E}^l\phi^l, \quad \mathcal{H}^l = \frac{\pi_\mu^2}{2} + \frac{l(l+1)}{2R^2} - \frac{a}{R}, \tag{2}$$

where the covariant derivative $\pi_\mu = -i\partial_\mu - A_\mu^a\Lambda_a^{2l+1}$ contains $SU(2)$ Yang–Mills instanton [10] as the gauge potential defined due to

$$A_\mu^a dr_\mu = \frac{1}{R(R+r_0)}\left(-r_4 dr_a + r_a dr_4 - \varepsilon_{abc} r_b dr_c\right), \tag{3}$$
$$\mu = 0, \ldots, 4, \quad a, b, c = 1, 2, 3,$$

and Λ_a^{2l+1} are the generators of the $(2l+1)$-dimensional representation of $SU(2)$.

These conditions are the following.

1. The coordinates of $5D$ Euclidean space are expressed through that of $8D$ one by means of the Hurwitz transformation

$$r_0 = |u|^2 - |v|^2, \tag{4}$$
$$r = 2u\bar{v}, \tag{5}$$

where $u = u_0 + u_a e_a$, $v = v_0 + v_a e_a$, $r = r_4 + r_a e_a$ $(a = 1, 2, 3)$ are the real quaternions.

We recall that quaternion's algebra

$$e_a e_b = -\delta_{ab} + \varepsilon_{abc} e_c, \qquad e_0 e_a = e_a e_0 = e_a$$

has the involution — quaternionic conjugation — which is an antiautomorphism of the algebra: $\overline{(u\overline{v})} = v\overline{u}$. One can define the norm $|u| = \sqrt{u\bar{u}}$, scalar $(u)_S = 1/2\,(u + \bar{u}) = u_0$ and vector $(u)_V = 1/2\,(u - \bar{u}) = u_a e_a = \mathbf{u}$ parts.

The Hurwitz transformation possesses the property

$$R \equiv \sqrt{r_0^2 + |r|^2} = |u|^2 + |v|^2. \tag{6}$$

To make the change of coordinates (4)–(5) complete, we represent $u = |u|g$ (and, therefore, $v = |v|\bar{r}g/|r|$) where g is unimodular quaternion. It is relevant to note that there is the isomorphism between the unimodular quaternions and the group $SU\,(2)$. We can introduce parameters (following [11] we shall call them vector parameters)

$$g = \pm\frac{1+\mathbf{z}}{\sqrt{1+\mathbf{z}^2}}, \qquad \mathbf{z} = \frac{\mathbf{u}}{u_0}\,, \tag{7}$$

and choose $z_a = u_a/u_0$ as an additional coordinates.

2. The eigenvalues of one problem are expressed through the parameters of another one and vice versa:

$$E = 4a, \qquad \omega^2 = -8\mathcal{E}^l\,; \tag{8}$$

3. The equivariance condition

$$\mathbf{K}^2\psi^{(8)} = l(l+1)\psi^{(8)} \tag{9}$$

is supposed to hold. It allows to establish the correspondence between the respective Hilbert spaces

$$\psi^{(8)}\,(u,v) = \mathrm{trace}(\Psi^l\,(\bar{g})\,\phi^l\,(r_\mu)), \quad \Psi^l\,(\bar{g}) = \left[\Psi^l\,(g)\right]^\dagger\,. \tag{10}$$

Here $\Psi^l\,(g)$ is the matrix of the $(2l+1)$-dimensional representation of $SU\,(2)$ which components are the eigenfunctions of the mutually commuting operators $\mathbf{K}^2$, K_3, T_3:

$$\begin{aligned} \mathbf{K}^2\Psi^l_{mm'} &= l(l+1)\Psi^l_{mm'}\,, \qquad & -K_3\Psi^l_{mm'} &= m\Psi^l_{mm'} \\ T_3\Psi^l_{mm'} &= m'\Psi^l_{mm'}\,, \qquad & -l \le m, m' &\le l\,. \end{aligned} \tag{11}$$

When written in the vector parametrization, the operators K_a and T_a read [11]

$$K_a = -\frac{\mathrm{i}}{2}\left(z_a z_b\frac{\partial}{\partial z_b} + \frac{\partial}{\partial z_a} + \varepsilon_{abc} z_b\frac{\partial}{\partial z_c}\right), \tag{12}$$

$$T_a = \frac{\mathrm{i}}{2}\left(z_a z_b\frac{\partial}{\partial z_b} + \frac{\partial}{\partial z_a} - \varepsilon_{abc} z_b\frac{\partial}{\partial z_c}\right). \tag{13}$$

The well-known formula for the $SU(2)$ matrix elements [12]

$$\begin{aligned} \Psi^l_{mm'}\,(g) = {} & \sqrt{\frac{(l-m)!\,(l-m')!}{(l+m)!\,(l+m')!}}\,\frac{\delta^{m+m'}}{\beta^m\gamma^{m'}} \\ & \times \sum_{j=\max(m,m')}^{l} \frac{(l+j)!\,(\beta\gamma)^j}{(l-j)!\,(j-m)!\,(j-m')!}\,, \end{aligned} \tag{14}$$

where $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ with $\{\alpha, \beta, \gamma, \delta \in \mathbb{C};\ \alpha\delta - \beta\gamma = 1\}$ can be expressed in terms of vector parameters if we choose

$$g = \pm \frac{1}{\sqrt{1+\mathbf{z}^2}} \begin{pmatrix} 1 - \mathrm{i} z_3 & -\mathrm{i}\,(z_1 - \mathrm{i} z_2) \\ -\mathrm{i}\,(z_1 + \mathrm{i} z_2) & 1 + \mathrm{i} z_3 \end{pmatrix} = \pm \frac{1 - \mathrm{i}\sigma_a z_a}{\sqrt{1+\mathbf{z}^2}} \tag{15}$$

(compare with (7). Note that there is the representation for quaternion's basis $e_a = -\mathrm{i}\sigma_a$).

In the spherical coordinates

$$\begin{aligned} z_1 &= n_1 \tan\chi = \tan\chi \sin\theta \cos\varphi\,, \\ z_2 &= n_2 \tan\chi = \tan\chi \sin\theta \sin\varphi\,, \\ z_3 &= n_3 \tan\chi = \tan\chi \cos\theta\,, \\ 0 \le \chi < \pi\,, &\quad 0 \le \theta < \pi\,, \quad 0 \le \varphi < 2\pi\,, \end{aligned} \tag{16}$$

the group element g and its representation $\Psi^l\,(g)$ are parametrized

$$\begin{aligned} g &= \exp\,(\mathbf{n}\chi) = \cos\chi - \mathrm{i}\sigma_a n_a \sin\chi \\ &= \begin{pmatrix} \cos\chi - \mathrm{i}\sin\chi\cos\theta & -\mathrm{i}\sin\chi\sin\theta\exp\,(-\mathrm{i}\varphi) \\ -\mathrm{i}\sin\chi\sin\theta\exp\,(\mathrm{i}\varphi) & \cos\chi + \mathrm{i}\sin\chi\cos\theta \end{pmatrix} \end{aligned} \tag{17}$$

and

$$\begin{aligned} \Psi^l_{mm'}\,(g) = & \sqrt{\frac{(l-m)!\,(l-m')!}{(l+m)!\,(l+m')!}} \left(\frac{\cos\chi + \mathrm{i}\sin\chi\cos\theta}{-\mathrm{i}\sin\chi\sin\theta} \right)^{m+m'} \mathrm{e}^{\mathrm{i}\left(m-m'\right)\varphi} \\ & \times \sum_{j=\max(m,m')}^{l} \frac{(l+j)!\,[-\mathrm{i}\sin\chi\sin\theta]^{2j}}{(l-j)!\,(j-m)!\,(j-m')!}\,, \end{aligned} \tag{18}$$

respectively.

Representation $\Psi^l\,(g)$ coincides with that used in [7] up to the complex conjugation.

3. Green's Function

The equation defining the Green's function of the 8-dimensional harmonic oscillator is

$$(H - E)\,G(u, v, u', v'; E) = -\mathrm{i}\delta^{(4)}\,(u - u')\,\delta^{(4)}\,(v - v')\;. \tag{19}$$

Its solution is well-known [3]

$$G = \int_0^\infty \mathrm{d}t \exp\left(\mathrm{i}4at\right)\left(\frac{\omega}{2\pi\sin\omega t}\right)^4 \times \exp\left[\frac{\mathrm{i}\omega}{2\sin\omega t}\left(\left(|u|^2+|v|^2+|u'|^2+|v'|^2\right)\cos\omega t - 2\left(u\bar{u}'+v\bar{v}'\right)_S\right)\right]. \tag{20}$$

Let us express it in $(r_\mu, \mathbf{z})$-coordinates. In this section we now assume $u = |u|h$ and $u' = |u|h'$. The notation g we shall reserve for $g = h\bar{h}'$.
First of all, note that

$$\begin{aligned} 2\left(u\bar{u}'+v\bar{v}'\right)_S &= 2\left(|u|\,|u'|h\bar{h}' + |v|\,|v'|\frac{\bar{r}}{|r|}h\bar{h}'\frac{r'}{|r'|}\right)_S \\ &= 2\left(\left(|u|\,|u'| + |v|\,|v'|\frac{r'\bar{r}}{|r'|\,|r|}\right)h\bar{h}'\right)_S = \left(\bar{F}g\right)_S \end{aligned} \tag{21}$$

where

$$\begin{aligned} F &= 2\left(|u|\,|u'| + |v|\,|v'|\frac{r\bar{r}'}{|r'|\,|r|}\right) = 2|u|\,|u'|\left(1+\frac{r\bar{r}'}{4|u|^2|u'|^2}\right) \\ &= \frac{RR' + Rr_0' + r_0R' + r_\mu r_\mu' + \left(r\bar{r}'\right)_V}{\sqrt{(R+r_0)(R'+r_0')}}\,. \end{aligned} \tag{22}$$

The norm of the quaternion F is

$$\begin{aligned} |F| &= \sqrt{2\left(RR' + r_\mu r_\mu'\right)} = 2\sqrt{RR'}\cos\frac{\Theta}{2}\,, \\ \cos\Theta &= r_\mu r_\mu'/RR', \end{aligned} \tag{23}$$

and then we can introduce the unimodular quaternion f which is

$$f \equiv \frac{F}{|F|} = \frac{RR' + Rr_0' + r_0R' + r_\mu r_\mu' + \left(r\bar{r}'\right)_V}{\sqrt{2\left(RR'+r_\mu r_\mu'\right)(R+r_0)(R'+r_0')}}\,. \tag{24}$$

Then

$$G\left(r_\mu, r_\mu', g; E\right) = \int_0^\infty \mathrm{d}t\left(\frac{\omega}{2\pi\sin\omega t}\right)^4 \exp\left[\mathrm{i}4at + \frac{\mathrm{i}\omega}{2}\left(R+R'\right)\cot\omega t\right] \times\exp\left(-\frac{\mathrm{i}\omega|F|}{2\sin\omega t}\left(\bar{f}g\right)_S\right). \tag{25}$$

To obtain the expression for the 5-dimensional Green's function we make the following simple manipulations on Eq. (19):

$$4R\Psi^{l}\left(\bar{h}\right)\left(\mathcal{H}^{l}-\mathcal{E}^{l}\right)\Psi^{l}\left(h\right)G=-\mathrm{i}\delta^{(4)}\left(u-u'\right)\delta^{(4)}\left(v-v'\right)\,, \tag{26}$$

then

$$\left(\mathcal{H}^{l}-\mathcal{E}^{l}\right)\Psi^{l}\left(h\bar{h}'\right)G=-\frac{1}{4R}\mathrm{i}\delta^{(4)}\left(u-u'\right)\delta^{(4)}\left(v-v'\right)\Psi^{l}\left(h\bar{h}'\right)\,. \tag{27}$$

On the analogy to the symbolic identity $\delta(x)f(x)=\delta(x)f(0)$ we can write

$$\delta^{(4)}\left(u-u'\right)\Psi^{l}\left(\frac{u\bar{u}'}{|u|\,|u'|}\right)=\delta^{(4)}\left(u-u'\right)\Psi^{l}(1)=\delta^{(4)}\left(u-u'\right)\,. \tag{28}$$

Integrating (27) over the group we obtain

$$\left(\mathcal{H}^{l}-\mathcal{E}^{l}\right)\int\mathrm{d}\tau\left(g\right)\Psi^{l}\left(g\right)G=-\frac{1}{4R}\mathrm{i}\int\mathrm{d}\tau\left(g\right)\delta^{(4)}\left(u-u'\right)\delta^{(4)}\left(v-v'\right). \tag{29}$$

Because the identity proven in [3]

$$\int\mathrm{d}\tau\left(g\right)\delta^{(4)}\left(u-u'\right)\delta^{(4)}\left(v-v'\right)=\frac{16R}{\pi^{2}}\delta^{(5)}\left(r_{\mu}-r'_{\mu}\right) \tag{30}$$

we are led to the equation defining the Green's function for the 5-dimensional problem

$$\left(\mathcal{H}^{l}-\mathcal{E}^{l}\right)\mathcal{G}^{l}\left(r_{\mu},r'_{\mu};\mathcal{E}^{l}\right)=-\mathrm{i}\delta^{(5)}\left(r_{\mu}-r'_{\mu}\right). \tag{31}$$

It can be solved easily by evaluation of the integral

$$\mathcal{G}^{l}\left(r_{\mu},r'_{\mu};\mathcal{E}^{l}\right)=\frac{\pi^{2}}{4}\int\mathrm{d}\tau\left(g\right)\Psi^{l}\left(g\right)G\left(r_{\mu},r'_{\mu},g;E\right)\,. \tag{32}$$

Due to the properties of the invariant measure $d\tau\left(g\right)$ the next expression is valid

$$\mathcal{G}^{l}\left(r_{\mu},r'_{\mu};\mathcal{E}^{l}\right)=\frac{\pi^{2}}{4}\Psi^{l}\left(f\right)\int\mathrm{d}\tau\left(g\right)\Psi^{l}\left(g\right)G\left(r_{\mu},r'_{\mu},fg;E\right)\,. \tag{33}$$

To achieve the final result we have to perform the integration over the group volume in the expression

$$\begin{aligned}\mathcal{G}^{l}\left(r_{\mu},r'_{\mu};\mathcal{E}^{l}\right)=\frac{\pi^{2}}{4}\Psi^{l}\left(f\right)\int_{0}^{\infty}\mathrm{d}t\int\mathrm{d}\tau\left(g\right)\Psi^{l}\left(g\right)\exp\left(\mathrm{i}x\left(g\right)_{S}\right)\\ \times\left(\frac{\omega}{2\pi\sin\omega t}\right)^{4}\exp\left[\mathrm{i}4at+\frac{\mathrm{i}\omega}{2}\left(R+R'\right)\cot\omega t\right].\end{aligned} \tag{34}$$

where it is introduced

$$x = -\frac{\omega |F|}{2\sin \omega t}\,. \tag{35}$$

Due to the identity

$$\int \mathrm{d}\tau\,(g)\,\Psi^l\,(g)\exp\left(\mathrm{i}x\,(g)_S\right) = \mathrm{i}^{2l}\,\frac{2}{x}\,J_{2l+1}(x)\,,$$

where $J_{2l+1}(x)$ is the Bessel function, we obtain

$$\mathcal{G}^l\left(r_\mu, r'_\mu; \mathcal{E}^l\right) = \Psi^l\,(f)\,\frac{(-\mathrm{i})^{2l}\omega^3}{16\pi^2|F|}\int\limits_0^\infty \mathrm{d}t J_{2l+1}\left(\frac{\omega|F|}{2\sin\omega t}\right) \times \frac{\exp\left[\mathrm{i}4at + \frac{\mathrm{i}\omega}{2}\left(R+R'\right)\cot\omega t\right]}{\sin^3\omega t}. \tag{36}$$

To bring our result to the notations of [1] we introduce $q = -\mathrm{i}\omega t,\ \omega = 2\mathrm{i}k,\ p' = -\mathrm{i}a/k$ and finally have

$$\mathcal{G}^l\left(r_\mu, r'_\mu; \mathcal{E}^l\right) = \Psi^l(f)\frac{(-\mathrm{i})^{2l}k^2}{8\pi^2\sqrt{RR'}\cos\frac{\Theta}{2}}\int\limits_0^\infty \mathrm{d}q J_{2l+1}\left(\frac{2k\sqrt{RR'}\cos\frac{\Theta}{2}}{\sinh q}\right) \times \frac{\exp\left[-2p'q + \mathrm{i}k\left(R+R'\right)\coth q\right]}{\sinh^3 q}\,. \tag{37}$$

For the case of the trivial constraints $l = 0$ the expression

$$\mathcal{G}^0\left(r_\mu, r'_\mu; \mathcal{E}^0\right) = \frac{k^2}{8\pi^2\sqrt{RR'}\cos\frac{\Theta}{2}}\int\limits_0^\infty \mathrm{d}q J_1\left(\frac{2k\left(RR'\right)^{1/2}}{\sinh q}\cos\frac{\Theta}{2}\right) \times \frac{\exp\left[-2p'q + \mathrm{i}k\left(R+R'\right)\coth q\right]}{\sinh^3 q} \tag{38}$$

appears to be the same as the respective result in [1] for $n = 5$.

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