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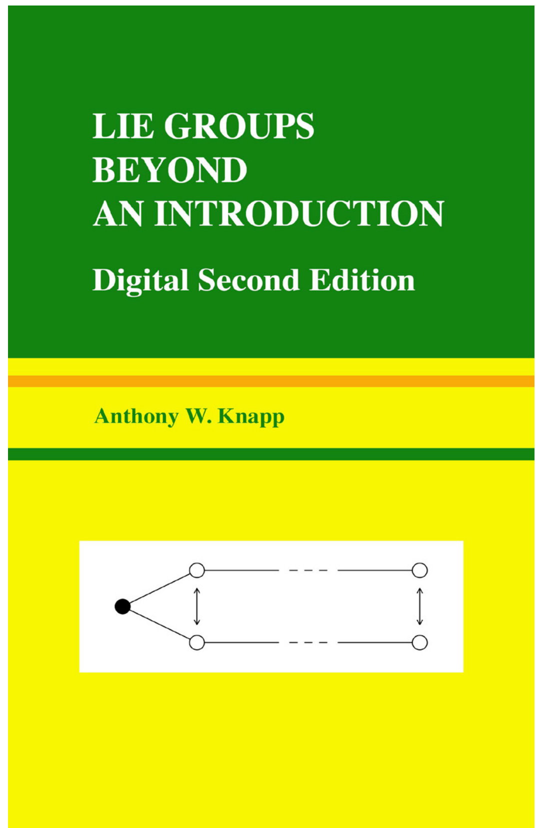

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Pages vii–xviii and 1–812 are the same in the digital and printed second editions. A list of corrections as of June 2023 has been included as pages 813–820 of the digital second edition. The corrections have not been implemented in the text.

Cover: Vogan diagram of $\mathfrak{sl}(2n, \mathbb{R})$. See page 399.

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CHAPTER VI

Structure Theory of Semisimple Groups

Abstract. Every complex semisimple Lie algebra has a compact real form, as a consequence of a particular normalization of root vectors whose construction uses the Isomorphism Theorem of Chapter II. If $\mathfrak{g}_0$ is a real semisimple Lie algebra, then the use of a compact real form of $(\mathfrak{g}_0)^{\mathbb{C}}$ leads to the construction of a "Cartan involution" θ of $\mathfrak{g}_0$. This involution has the property that if $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ is the corresponding eigenspace decomposition or "Cartan decomposition," then $\mathfrak{k}_0 \oplus i\mathfrak{p}_0$ is a compact real form of $(\mathfrak{g}_0)^{\mathbb{C}}$. Any two Cartan involutions of $\mathfrak{g}_0$ are conjugate by an inner automorphism. The Cartan decomposition generalizes the decomposition of a classical matrix Lie algebra into its skew-Hermitian and Hermitian parts.

If G is a semisimple Lie group, then a Cartan decomposition $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ of its Lie algebra leads to a global decomposition $G = K \exp \mathfrak{p}_0$, where K is the analytic subgroup of G with Lie algebra $\mathfrak{k}_0$. This global decomposition generalizes the polar decomposition of matrices. The group K contains the center of G and, if the center of G is finite, is a maximal compact subgroup of G.

The Iwasawa decomposition $G = KAN$ exhibits closed subgroups A and N of G such that A is simply connected abelian, N is simply connected nilpotent, A normalizes N, and multiplication from $K \times A \times N$ to G is a diffeomorphism onto. This decomposition generalizes the Gram–Schmidt orthogonalization process. Any two Iwasawa decompositions of G are conjugate. The Lie algebra $\mathfrak{a}_0$ of A may be taken to be any maximal abelian subspace of $\mathfrak{p}_0$, and the Lie algebra of N is defined from a kind of root-space decomposition of $\mathfrak{g}_0$ with respect to $\mathfrak{a}_0$. The simultaneous eigenspaces are called "restricted roots," and the restricted roots form an abstract root system. The Weyl group of this system coincides with the quotient of normalizer by centralizer of $\mathfrak{a}_0$ in K.

A Cartan subalgebra of $\mathfrak{g}_0$ is a subalgebra whose complexification is a Cartan subalgebra of $(\mathfrak{g}_0)^{\mathbb{C}}$. One Cartan subalgebra of $\mathfrak{g}_0$ is obtained by adjoining to the above $\mathfrak{a}_0$ a maximal abelian subspace of the centralizer of $\mathfrak{a}_0$ in $\mathfrak{k}_0$. This Cartan subalgebra is θ stable. Any Cartan subalgebra of $\mathfrak{g}_0$ is conjugate by an inner automorphism to a θ stable one, and the subalgebra built from $\mathfrak{a}_0$ as above is maximally noncompact among all θ stable Cartan subalgebras. Any two maximally noncompact Cartan subalgebras are conjugate, and so are any two maximally compact ones. Cayley transforms allow one to pass between any two θ stable Cartan subalgebras, up to conjugacy.

A Vogan diagram of $\mathfrak{g}_0$ superimposes certain information about the real form $\mathfrak{g}_0$ on the Dynkin diagram of $(\mathfrak{g}_0)^{\mathbb{C}}$. The extra information involves a maximally compact θ stable Cartan subalgebra and an allowable choice of a positive system of roots. The effect of θ on simple roots is labeled, and imaginary simple roots are painted if they are "noncompact," left unpainted if they are "compact." Such a diagram is not unique for $\mathfrak{g}_0$, but it determines

$\mathfrak{g}_0$ up to isomorphism. Every diagram that looks formally like a Vogan diagram arises from some $\mathfrak{g}_0$.

Vogan diagrams lead quickly to a classification of all simple real Lie algebras, the only difficulty being eliminating the redundancy in the choice of positive system of roots. This difficulty is resolved by the Borel and de Siebenthal Theorem. Using a succession of Cayley transforms to pass from a maximally compact Cartan subalgebra to a maximally noncompact Cartan subalgebra, one readily identifies the restricted roots for each simple real Lie algebra.

1. Existence of a Compact Real Form

An important clue to the structure of semisimple Lie groups comes from the examples of the classical semisimple groups in §§I.8 and I.17. In each case the Lie algebra $\mathfrak{g}_0$ is a real Lie algebra of matrices over $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$ closed under conjugate transpose $(\,\cdot\,)^*$. This fact is the key ingredient used in Proposition 1.59 to detect semisimplicity of $\mathfrak{g}_0$.

Using the techniques at the end of §I.8, we can regard $\mathfrak{g}_0$ as a Lie algebra of matrices over $\mathbb{R}$ closed under transpose $(\,\cdot\,)^*$. Then $\mathfrak{g}_0$ is the direct sum of the set $\mathfrak{k}_0$ of its skew-symmetric members and the set $\mathfrak{p}_0$ of its symmetric members. The real vector space $\mathfrak{u}_0 = \mathfrak{k}_0 \oplus i\mathfrak{p}_0$ of complex matrices is closed under brackets and is a Lie subalgebra of skew-Hermitian matrices.

Meanwhile we can regard the complexification $\mathfrak{g}$ of $\mathfrak{g}_0$ as the Lie algebra of complex matrices $\mathfrak{g} = \mathfrak{g}_0 + i\mathfrak{g}_0$. Putting $\mathfrak{k} = (\mathfrak{k}_0)^{\mathbb{C}}$ and $\mathfrak{p} = (\mathfrak{p}_0)^{\mathbb{C}}$, we write $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ as vector spaces. The complexification of $\mathfrak{u}_0$ is the same set of matrices: $(\mathfrak{u}_0)^{\mathbb{C}} = \mathfrak{k} \oplus \mathfrak{p}$.

Since $\mathfrak{g}_0$ has been assumed semisimple, $\mathfrak{g}$ is semisimple by Corollary 1.53, and $\mathfrak{u}_0$ is semisimple by the same corollary. The claim is that $\mathfrak{u}_0$ is a compact Lie algebra in the sense of §IV.4. In fact, let us introduce the inner product $\langle X, Y\rangle = \operatorname{Re}\operatorname{Tr}(XY^*)$ on $\mathfrak{u}_0$. The proof of Proposition 1.59 shows that

$$\langle (\operatorname{ad} Y)X, Z\rangle = \langle X, (\operatorname{ad}(Y^*))Z\rangle$$

and hence

$$(\operatorname{ad} Y)^* = \operatorname{ad}(Y^*). \tag{6.1}$$

Since $Y^* = -Y$, $\operatorname{ad} Y$ is skew Hermitian. Thus $(\operatorname{ad} Y)^2$ has eigenvalues ≤ 0, and the Killing form $B_{\mathfrak{u}_0}$ of $\mathfrak{u}_0$ satisfies

$$B_{\mathfrak{u}_0}(Y, Y) = \operatorname{Tr}((\operatorname{ad} Y)^2) \le 0.$$

Since $\mathfrak{u}_0$ is semisimple, $B_{\mathfrak{u}_0}$ is nondegenerate (Theorem 1.45) and must be negative definite. By Proposition 4.27, $\mathfrak{u}_0$ is a compact Lie algebra.

In the terminology of real forms as in §I.3, the splitting of any of the classical semisimple Lie algebras $\mathfrak{g}_0$ in §I.8 is equivalent with associating to $\mathfrak{g}_0$ the compact Lie algebra $\mathfrak{u}_0$ that is a real form of the complexification of $\mathfrak{g}_0$. Once we have this splitting of $\mathfrak{g}_0$, the arguments in §I.17 allowed us to obtain a polar-like decomposition of the analytic group of matrices G with Lie algebra $\mathfrak{g}_0$. This polar-like decomposition was a first structure theorem for the classical groups, giving insight into the topology of G and underlining the importance of a certain compact subgroup K of G.

The idea for beginning an investigation of the structure of a general semisimple Lie group G, not necessarily classical, is to look for this same kind of structure. We start with the Lie algebra $\mathfrak{g}_0$ and seek a decomposition into skew-symmetric and symmetric parts. To get this decomposition, we look for the occurrence of a compact Lie algebra $\mathfrak{u}_0$ as a real form of the complexification $\mathfrak{g}$ of $\mathfrak{g}_0$.

Actually not just any $\mathfrak{u}_0$ of this kind will do. The real forms $\mathfrak{u}_0$ and $\mathfrak{g}_0$ must be aligned so that the skew-symmetric part $\mathfrak{k}_0$ and the symmetric part $\mathfrak{p}_0$ can be recovered as $\mathfrak{k}_0 = \mathfrak{g}_0 \cap \mathfrak{u}_0$ and $\mathfrak{p}_0 = \mathfrak{g}_0 \cap i\mathfrak{u}_0$. The condition of proper alignment for $\mathfrak{u}_0$ is that the conjugations of $\mathfrak{g}$ with respect to $\mathfrak{g}_0$ and to $\mathfrak{u}_0$ must commute with each other.

The first step will be to associate to a complex semisimple Lie algebra $\mathfrak{g}$ a real form $\mathfrak{u}_0$ that is compact. This construction will occupy us for the remainder of this section. In §2 we shall address the alignment question when $\mathfrak{g}$ is the complexification of a real semisimple Lie algebra $\mathfrak{g}_0$. The result will yield the desired Lie algebra decomposition $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$, known as the "Cartan decomposition" of the Lie algebra. Then in §3 we shall pass from the Cartan decomposition of the Lie algebra to a "Cartan decomposition" of the Lie group that generalizes the polar-like decomposition in Proposition 1.143.

The argument in the present section for constructing a compact real form from a complex semisimple $\mathfrak{g}$ will be somewhat roundabout. We shall use the Isomorphism Theorem (Theorem 2.108) to show that root vectors can be selected so that the constants arising in the bracket products of root vectors are all real. More precisely this result gives us a real form of $\mathfrak{g}$ known as a "split real form." It is not a compact Lie algebra but in a certain sense is as noncompact as possible. When $\mathfrak{g}$ is $\mathfrak{sl}(2, \mathbb{C})$, the real subalgebra $\mathfrak{sl}(2, \mathbb{R})$ is a split form, and the desired real form that is compact is $\mathfrak{su}(2)$. In general we obtain the real form that is compact by taking suitable linear combinations of the root vectors that define the split real form.

For the remainder of this section, let $\mathfrak{g}$ be a complex semisimple Lie algebra, let $\mathfrak{h}$ be a Cartan subalgebra, let $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$ be the set of roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$, and let B be the Killing form. (The Killing form has the property that it is invariant under all automorphisms of $\mathfrak{g}$, according to Proposition 1.119, and this property is not always shared by other forms. To take advantage of this property, we shall insist that B is the Killing form in §§1–3. After that, we shall allow more general forms in place of B.)

For each pair $\{\alpha, -\alpha\}$ in Δ, we fix $E_\alpha \in \mathfrak{g}_\alpha$ and $E_{-\alpha} \in \mathfrak{g}_{-\alpha}$ so that $B(E_\alpha, E_{-\alpha}) = 1$. Then $[E_\alpha, E_{-\alpha}] = H_\alpha$ by Lemma 2.18a. Let α and β be roots. If $\alpha + \beta$ is in Δ, define $C_{\alpha,\beta}$ by

$$[E_\alpha, E_\beta] = C_{\alpha,\beta} E_{\alpha+\beta}.$$

If $\alpha + \beta$ is not in Δ, put $C_{\alpha,\beta} = 0$.

Lemma 6.2. $C_{\alpha,\beta} = -C_{\beta,\alpha}$.

PROOF. This follows from the skew symmetry of the bracket.

Lemma 6.3. If α, β, and γ are in Δ and $\alpha + \beta + \gamma = 0$, then

$$C_{\alpha,\beta} = C_{\beta,\gamma} = C_{\gamma,\alpha}.$$

PROOF. By the Jacobi identity,

$$[[E_\alpha, E_\beta], E_\gamma] + [[E_\beta, E_\gamma], E_\alpha] + [[E_\gamma, E_\alpha], E_\beta] = 0.$$

Thus $$C_{\alpha,\beta}[E_{-\gamma}, E_\gamma] + C_{\beta,\gamma}[E_{-\alpha}, E_\alpha] + C_{\gamma,\alpha}[E_{-\beta}, E_\beta] = 0$$

and $$C_{\alpha,\beta} H_\gamma + C_{\beta,\gamma} H_\alpha + C_{\gamma,\alpha} H_\beta = 0.$$

Substituting $H_\gamma = -H_\alpha - H_\beta$ and using the linear independence of $\{H_\alpha, H_\beta\}$, we obtain the result.

Lemma 6.4. Let α, β, and $\alpha + \beta$ be in Δ, and let $\beta + n\alpha$, with $-p \le n \le q$, be the α string containing β. Then

$$C_{\alpha,\beta}, C_{-\alpha,-\beta} = -\tfrac{1}{2} q(1 + p)|\alpha|^2.$$

PROOF. By Corollary 2.37,

$$[E_{-\alpha}, [E_\alpha, E_\beta]] = \tfrac{1}{2} q(1 + p)|\alpha|^2 B(E_\alpha, E_{-\alpha}) E_\beta.$$

The left side is $C_{-\alpha,\alpha+\beta}C_{\alpha,\beta}E_\beta$, and $B(E_\alpha, E_{-\alpha}) = 1$ on the right side. Therefore

$$C_{-\alpha,\alpha+\beta}C_{\alpha,\beta} = \tfrac{1}{2}q(1+p)|\alpha|^2. \tag{6.5}$$

Since $(-\alpha)+(\alpha+\beta)+(-\beta) = 0$, Lemmas 6.3 and 6.2 give

$$C_{-\alpha,\alpha+\beta} = C_{-\beta,-\alpha} = -C_{-\alpha,-\beta},$$

and the result follows by substituting this formula into (6.5).

Theorem 6.6. Let $\mathfrak{g}$ be a complex semisimple Lie algebra, let $\mathfrak{h}$ be a Cartan subalgebra, and let Δ be the set of roots. For each $\alpha \in \Delta$, it is possible to choose root vectors $X_\alpha \in \mathfrak{g}_\alpha$ such that, for all α and β in Δ,

$$\begin{aligned}
[X_\alpha, X_{-\alpha}] &= H_\alpha \\
[X_\alpha, X_\beta] &= N_{\alpha,\beta}X_{\alpha+\beta} && \text{if } \alpha+\beta \in \Delta \\
[X_\alpha, X_\beta] &= 0 && \text{if } \alpha+\beta \neq 0 \text{ and } \alpha+\beta \notin \Delta
\end{aligned}$$

with constants $N_{\alpha,\beta}$ that satisfy

$$N_{\alpha,\beta} = -N_{-\alpha,-\beta}.$$

For any such choice of the system $\{X_\alpha\}$ of root vectors, the constants $N_{\alpha,\beta}$ satisfy

$$N_{\alpha,\beta}^2 = \tfrac{1}{2}q(1+p)|\alpha|^2,$$

where $\beta + n\alpha$, with $-p \leq n \leq q$, is the α string containing β.

PROOF. The transpose of the linear map $\varphi : \mathfrak{h} \to \mathfrak{h}$ given by $\varphi(h) = -h$ carries Δ to Δ, and thus φ extends to an automorphism $\widetilde{\varphi}$ of $\mathfrak{g}$, by the Isomorphism Theorem (Theorem 2.108). (See Example 3 at the end of §II.10.) Since $\widetilde{\varphi}(E_\alpha)$ is in $\mathfrak{g}_{-\alpha}$, there exists a constant $c_{-\alpha}$ such that $\widetilde{\varphi}(E_\alpha) = c_{-\alpha}E_{-\alpha}$. By Proposition 1.119,

$$B(\widetilde{\varphi}X, \widetilde{\varphi}Y) = B(X, Y) \qquad \text{for all } X \text{ and } Y \text{ in } \mathfrak{g}.$$

Applying this formula with $X = E_\alpha$ and $Y = E_{-\alpha}$, we obtain

$$c_{-\alpha}c_\alpha = c_{-\alpha}c_\alpha B(E_{-\alpha}, E_\alpha) = B(\widetilde{\varphi}E_\alpha, \widetilde{\varphi}E_{-\alpha}) = B(E_\alpha, E_{-\alpha}) = 1.$$

Thus $c_{-\alpha}c_\alpha = 1$. Because of this relation we can choose a_α for each $\alpha \in \Delta$ such that

$$a_\alpha a_{-\alpha} = +1 \tag{6.7a}$$

$$a_\alpha^2 = -c_\alpha. \tag{6.7b}$$

For example, fix a pair $\{\alpha, -\alpha\}$, and write $c_\alpha = re^{i\theta}$ and $c_{-\alpha} = r^{-1}e^{-i\theta}$; then we can take $a_\alpha = r^{1/2}ie^{i\theta/2}$ and $a_{-\alpha} = -r^{-1/2}ie^{-i\theta/2}$.

With the choices of the a_α's in place so that (6.7) holds, define $X_\alpha = a_\alpha E_\alpha$. The root vectors X_α satisfy

$$[X_\alpha, X_{-\alpha}] = a_\alpha a_{-\alpha}[E_\alpha, E_{-\alpha}] = H_\alpha \qquad \text{by (6.7a)}$$

and

$$\begin{aligned} \widetilde{\varphi}(X_\alpha) = a_\alpha \widetilde{\varphi}(E_\alpha) &= a_\alpha c_{-\alpha} E_{-\alpha} \\ &= a_{-\alpha}^{-1} c_{-\alpha} E_{-\alpha} && \text{by (6.7a)} \\ &= -a_{-\alpha} E_{-\alpha} && \text{by (6.7b)} \\ &= -X_{-\alpha}. \end{aligned} \tag{6.8}$$

Define constants $N_{\alpha,\beta}$ relative to the root vectors X_γ in the same way that the constants $C_{\alpha,\beta}$ are defined relative to the root vectors E_γ. Then (6.8) gives

$$\begin{aligned} -N_{\alpha,\beta}X_{-\alpha-\beta} &= \widetilde{\varphi}(N_{\alpha,\beta}X_{\alpha+\beta}) = \widetilde{\varphi}[X_\alpha, X_\beta] \\ &= [\widetilde{\varphi}X_\alpha, \widetilde{\varphi}X_\beta] = [-X_{-\alpha}, -X_{-\beta}] = N_{-\alpha,-\beta}X_{-\alpha-\beta}, \end{aligned}$$

and we find that $N_{\alpha,\beta} = -N_{-\alpha,-\beta}$. The formula for $N_{\alpha,\beta}^2$ follows by substituting into Lemma 6.4, and the proof is complete.

Theorem 6.6 has an interpretation in terms of real forms of the complex Lie algebra $\mathfrak{g}$. With notation as in Theorem 6.6, define

$$\mathfrak{h}_0 = \{H \in \mathfrak{h} \mid \alpha(H) \in \mathbb{R} \text{ for all } \alpha \in \Delta\}, \tag{6.9}$$

and put

$$\mathfrak{g}_0 = \mathfrak{h}_0 \oplus \bigoplus_{\alpha \in \Delta} \mathbb{R}X_\alpha.$$

The formula $N_{\alpha,\beta}^2 = \frac{1}{2}q(1+p)|\alpha|^2$ shows that $N_{\alpha,\beta}$ is real. Therefore $\mathfrak{g}_0$ is a subalgebra of $\mathfrak{g}^{\mathbb{R}}$. Since it is clear that $\mathfrak{g}^{\mathbb{R}} = \mathfrak{g}_0 \oplus i\mathfrak{g}_0$ as real vector spaces, $\mathfrak{g}_0$ is a real form of $\mathfrak{g}$. A real form of $\mathfrak{g}$ that contains $\mathfrak{h}_0$ as in (6.9) for some Cartan subalgebra $\mathfrak{h}$ is called a **split real form** of $\mathfrak{g}$. We summarize the above remarks as follows.

Corollary 6.10. Any complex semisimple Lie algebra contains a split real form.

EXAMPLES. It is clear from the computations in §II.1 that $\mathfrak{sl}(n, \mathbb{R})$ and $\mathfrak{sp}(n, \mathbb{R})$ are split real forms of $\mathfrak{sl}(n, \mathbb{C})$ and $\mathfrak{sp}(n, \mathbb{C})$, respectively. We shall see in §4 that $\mathfrak{so}(n+1, n)$ and $\mathfrak{so}(n, n)$ are isomorphic to split real forms of $\mathfrak{so}(2n+1, \mathbb{C})$ and $\mathfrak{so}(2n, \mathbb{C})$, respectively.

As we indicated at the beginning of this section, we shall study real semisimple Lie algebras by relating them to other real forms that are compact Lie algebras. A real form of the complex semisimple Lie algebra $\mathfrak{g}$ that is a compact Lie algebra is called a **compact real form** of $\mathfrak{g}$.

Theorem 6.11. If $\mathfrak{g}$ is a complex semisimple Lie algebra, then $\mathfrak{g}$ has a compact real form $\mathfrak{u}_0$.

REMARKS.

1) The compact real forms of the classical complex semisimple Lie algebras are already familiar. For $\mathfrak{sl}(n, \mathbb{C})$, $\mathfrak{so}(n, \mathbb{C})$, and $\mathfrak{sp}(n, \mathbb{C})$, they are $\mathfrak{su}(n)$, $\mathfrak{so}(n)$, and $\mathfrak{sp}(n)$, respectively. In the case of $\mathfrak{sp}(n, \mathbb{C})$, this fact uses the isomorphism $\mathfrak{sp}(n) \cong \mathfrak{sp}(n, \mathbb{C}) \cap \mathfrak{u}(2n)$ proved in §I.8.

2) We denote the compact real forms of the complex Lie algebras of types E_6, E_7, E_8, F_4, and G_2 by $\mathfrak{e}_6$, $\mathfrak{e}_7$, $\mathfrak{e}_8$, $\mathfrak{f}_4$, and $\mathfrak{g}_2$, respectively. Corollary 6.20 will show that these compact real forms are well defined up to isomorphism.

PROOF. Let $\mathfrak{h}$ be a Cartan subalgebra, and define root vectors X_α as in Theorem 6.6. Let

$$\mathfrak{u}_0 = \sum_{\alpha \in \Delta} \mathbb{R}(iH_\alpha) + \sum_{\alpha \in \Delta} \mathbb{R}(X_\alpha - X_{-\alpha}) + \sum_{\alpha \in \Delta} \mathbb{R}i(X_\alpha + X_{-\alpha}). \tag{6.12}$$

It is clear that $\mathfrak{g}^{\mathbb{R}} = \mathfrak{u}_0 \oplus i\mathfrak{u}_0$ as real vector spaces. Let us see that $\mathfrak{u}_0$ is closed under brackets. The term $\sum \mathbb{R}(iH_\alpha)$ on the right side of (6.12) is abelian, and we have

$$[iH_\alpha, (X_\alpha - X_{-\alpha})] = |\alpha|^2 i(X_\alpha + X_{-\alpha})$$
$$[iH_\alpha, i(X_\alpha + X_{-\alpha})] = -|\alpha|^2 (X_\alpha - X_{-\alpha}).$$

Therefore the term $\sum \mathbb{R}(iH_\alpha)$ brackets $\mathfrak{u}_0$ into $\mathfrak{u}_0$. For the other brackets of elements of $\mathfrak{u}_0$, we recall from Theorem 6.6 that $N_{\alpha,\beta} = -N_{-\alpha,-\beta}$, and we compute for $\beta \neq \pm\alpha$ that

$$\begin{aligned}
&[(X_\alpha - X_{-\alpha}), (X_\beta - X_{-\beta})] \\
&\qquad = N_{\alpha,\beta} X_{\alpha+\beta} + N_{-\alpha,-\beta} X_{-\alpha-\beta} - N_{-\alpha,\beta} X_{-\alpha+\beta} - N_{\alpha,-\beta} X_{\alpha-\beta} \\
&\qquad = N_{\alpha,\beta}(X_{\alpha+\beta} - X_{-(\alpha+\beta)}) - N_{-\alpha,\beta}(X_{-\alpha+\beta} - X_{-(-\alpha+\beta)})
\end{aligned}$$

and similarly that

$$\begin{aligned}&[(X_\alpha - X_{-\alpha}), i(X_\beta + X_{-\beta})]\\ &\qquad = N_{\alpha,\beta} i(X_{\alpha+\beta} + X_{-(\alpha+\beta)}) - N_{-\alpha,\beta} i(X_{-\alpha+\beta} + X_{-(-\alpha+\beta)})\end{aligned}$$

and

$$\begin{aligned}&[i(X_\alpha + X_{-\alpha}), i(X_\beta + X_{-\beta})]\\ &\qquad = -N_{\alpha,\beta}(X_{\alpha+\beta} - X_{-(\alpha+\beta)}) - N_{-\alpha,\beta}(X_{-\alpha+\beta} - X_{-(-\alpha+\beta)}).\end{aligned}$$

Finally

$$[(X_\alpha - X_{-\alpha}), i(X_\alpha + X_{-\alpha})] = 2iH_\alpha,$$

and therefore $\mathfrak{u}_0$ is closed under brackets. Consequently $\mathfrak{u}_0$ is a real form.

To show that $\mathfrak{u}_0$ is a compact Lie algebra, it is enough, by Proposition 4.27, to show that the Killing form of $\mathfrak{u}_0$ is negative definite. The Killing forms $B_{\mathfrak{u}_0}$ of $\mathfrak{u}_0$ and B of $\mathfrak{g}$ are related by $B_{\mathfrak{u}_0} = B|_{\mathfrak{u}_0\times\mathfrak{u}_0}$, according to (1.20). The first term on the right side of (6.12) is orthogonal to the other two terms by Proposition 2.17a, and B is positive on $\sum \mathbb{R}H_\alpha$ by Corollary 2.38. Hence B is negative on $\sum \mathbb{R}iH_\alpha$. Next we use Proposition 2.17a to observe for $\beta \neq \pm\alpha$ that

$$\begin{aligned}B((X_\alpha - X_{-\alpha}), (X_\beta - X_{-\beta})) &= 0\\ B((X_\alpha - X_{-\alpha}), i(X_\beta + X_{-\beta})) &= 0\\ B(i(X_\alpha + X_{-\alpha}), i(X_\beta + X_{-\beta})) &= 0.\end{aligned}$$

Finally we have

$$\begin{aligned}B((X_\alpha - X_{-\alpha}), (X_\alpha - X_{-\alpha})) &= -2B(X_\alpha, X_{-\alpha}) = -2\\ B(i(X_\alpha + X_{-\alpha}), i(X_\alpha + X_{-\alpha})) &= -2B(X_\alpha, X_{-\alpha}) = -2,\end{aligned}$$

and therefore $B|_{\mathfrak{u}_0\times\mathfrak{u}_0}$ is negative definite.

2. Cartan Decomposition on the Lie Algebra Level

To detect semisimplicity of some specific Lie algebras of matrices in §I.8, we made critical use of the conjugate transpose mapping $X \mapsto X^*$. Slightly better is the map $\theta(X) = -X^*$, which is actually an **involution**, i.e., an automorphism of the Lie algebra with square equal to the identity. To see that θ respects brackets, we just write

$$\theta[X,Y] = -[X,Y]^* = -[Y^*,X^*] = [-X^*,-Y^*] = [\theta(X),\theta(Y)].$$

Let B be the Killing form. The involution θ has the property that $B_\theta(X,Y) = -B(X,\theta Y)$ is symmetric and positive definite because Proposition 1.119 gives

$$\begin{aligned} B_\theta(X,Y) &= -B(X,\theta Y) = -B(\theta X,\theta^2 Y) \\ &= -B(\theta X, Y) = -B(Y,\theta X) = B_\theta(Y,X) \end{aligned}$$

and (6.1) gives

$$\begin{aligned} B_\theta(X,X) &= -B(X,\theta X) = -\operatorname{Tr}((\operatorname{ad} X)(\operatorname{ad}\theta X)) \\ &= \operatorname{Tr}((\operatorname{ad} X)(\operatorname{ad} X^*)) = \operatorname{Tr}((\operatorname{ad} X)(\operatorname{ad} X)^*) \geq 0. \end{aligned}$$

An involution θ of a real semisimple Lie algebra $\mathfrak{g}_0$ such that the symmetric bilinear form

$$B_\theta(X,Y) = -B(X,\theta Y) \tag{6.13}$$

is positive definite is called a **Cartan involution**. We shall see that any real semisimple Lie algebra has a Cartan involution and that the Cartan involution is unique up to inner automorphism. As a consequence of the proof, we shall obtain a converse to the arguments of §I.8: Every real semisimple Lie algebra can be realized as a Lie algebra of real matrices closed under transpose.

Theorem 6.11 says that any complex semisimple Lie algebra $\mathfrak{g}$ has a compact real form. According to the next proposition, it follows that $\mathfrak{g}^{\mathbb{R}}$ has a Cartan involution.

Proposition 6.14. Let $\mathfrak{g}$ be a complex semisimple Lie algebra, let $\mathfrak{u}_0$ be a compact real form of $\mathfrak{g}$, and let τ be the conjugation of $\mathfrak{g}$ with respect to $\mathfrak{u}_0$. If $\mathfrak{g}$ is regarded as a real Lie algebra $\mathfrak{g}^{\mathbb{R}}$, then τ is a Cartan involution of $\mathfrak{g}^{\mathbb{R}}$.

REMARK. The real Lie algebra $\mathfrak{g}^{\mathbb{R}}$ is semisimple by (1.61).

PROOF. It is clear that τ is an involution. The Killing forms $B_{\mathfrak{g}}$ of $\mathfrak{g}$ and $B_{\mathfrak{g}^{\mathbb{R}}}$ of $\mathfrak{g}^{\mathbb{R}}$ are related by

$$B_{\mathfrak{g}^{\mathbb{R}}}(Z_1, Z_2) = 2\mathrm{Re}\, B_{\mathfrak{g}}(Z_1, Z_2),$$

according to (1.60). Write $Z \in \mathfrak{g}$ as $Z = X + iY$ with X and Y in $\mathfrak{u}_0$. Then

$$\begin{aligned} B_{\mathfrak{g}}(Z, \tau Z) &= B_{\mathfrak{g}}(X + iY, X - iY) \\ &= B_{\mathfrak{g}}(X, X) + B_{\mathfrak{g}}(Y, Y) \\ &= B_{\mathfrak{u}_0}(X, X) + B_{\mathfrak{u}_0}(Y, Y), \end{aligned}$$

and the right side is < 0 unless $Z = 0$. In the notation of (6.13), it follows that

$$(B_{\mathfrak{g}^{\mathbb{R}}})_\tau(Z_1, Z_2) = -B_{\mathfrak{g}^{\mathbb{R}}}(Z_1, \tau Z_2) = -2\mathrm{Re}\, B_{\mathfrak{g}}(Z_1, \tau Z_2)$$

is positive definite on $\mathfrak{g}^{\mathbb{R}}$, and therefore τ is a Cartan involution of $\mathfrak{g}^{\mathbb{R}}$.

Now we address the problem of aligning a compact real form properly when we start with a real semisimple Lie algebra $\mathfrak{g}_0$ and obtain $\mathfrak{g}$ by complexification. Corollaries give the existence and uniqueness (up to conjugacy) of Cartan involutions.

Lemma 6.15. Let $\mathfrak{g}_0$ be a real finite-dimensional Lie algebra, and let ρ be an automorphism of $\mathfrak{g}_0$ that is diagonable with positive eigenvalues $d_1, \ldots, d_m$ and corresponding eigenspaces $(\mathfrak{g}_0)_{d_j}$. For $-\infty < r < \infty$, define ρ^r to be the linear transformation on $\mathfrak{g}_0$ that is d_j^r on $(\mathfrak{g}_0)_{d_j}$. Then $\{\rho^r\}$ is a one-parameter group in $\mathrm{Aut}\,\mathfrak{g}_0$. If $\mathfrak{g}_0$ is semisimple, then ρ^r lies in $\mathrm{Int}\,\mathfrak{g}_0$.

PROOF. If X is in $(\mathfrak{g}_0)_{d_i}$ and Y is in $(\mathfrak{g}_0)_{d_j}$, then

$$\rho[X, Y] = [\rho X, \rho Y] = d_i d_j [X, Y]$$

since ρ is an automorphism. Hence $[X, Y]$ is in $(\mathfrak{g}_0)_{d_i d_j}$, and we obtain

$$\rho^r[X, Y] = (d_i d_j)^r [X, Y] = [d_i^r X, d_j^r Y] = [\rho^r X, \rho^r Y].$$

Consequently ρ^r is an automorphism. Therefore $\{\rho^r\}$ is a one-parameter group in $\mathrm{Aut}\,\mathfrak{g}_0$, hence in the identity component $(\mathrm{Aut}\,\mathfrak{g}_0)_0$. If $\mathfrak{g}_0$ is semisimple, then Propositions 1.120 and 1.121 show that $(\mathrm{Aut}\,\mathfrak{g}_0)_0 = \mathrm{Int}\,\mathfrak{g}_0$, and the lemma follows.

Theorem 6.16. Let $\mathfrak{g}_0$ be a real semisimple Lie algebra, let θ be a Cartan involution, and let σ be any involution. Then there exists $\varphi \in \operatorname{Int} \mathfrak{g}_0$ such that $\varphi\theta\varphi^{-1}$ commutes with σ.

PROOF. Since θ is given as a Cartan involution, B_θ is an inner product for $\mathfrak{g}_0$. Put $\omega = \sigma\theta$. This is an automorphism of $\mathfrak{g}_0$, and Proposition 1.119 shows that it leaves B invariant. From $\sigma^2 = \theta^2 = 1$, we therefore have

$$B(\omega X, \theta Y) = B(X, \omega^{-1}\theta Y) = B(X, \theta\omega Y)$$

and hence $$B_\theta(\omega X, Y) = B_\theta(X, \omega Y).$$

Thus ω is symmetric, and its square $\rho = \omega^2$ is positive definite. Write ρ^r for the positive-definite r^{th} power of ρ, $-\infty < r < \infty$. Lemma 6.15 shows that ρ^r is a one-parameter group in $\operatorname{Int} \mathfrak{g}_0$. Consideration of ω as a diagonal matrix shows that ρ^r commutes with ω. Now

$$\rho\theta = \omega^2\theta = \sigma\theta\sigma\theta\theta = \sigma\theta\sigma = \theta\theta\sigma\theta\sigma = \theta\omega^{-2} = \theta\rho^{-1}.$$

In terms of a basis of $\mathfrak{g}_0$ that diagonalizes ρ, the matrix form of this equation is

$$\rho_{ii}\theta_{ij} = \theta_{ij}\rho_{jj}^{-1} \qquad \text{for all } i \text{ and } j.$$

Considering separately the cases $\theta_{ij} = 0$ and $\theta_{ij} \neq 0$, we see that

$$\rho_{ii}^r\theta_{ij} = \theta_{ij}\rho_{jj}^{-r}$$

and therefore that

$$\rho^r\theta = \theta\rho^{-r}. \tag{6.17}$$

Put $\varphi = \rho^{1/4}$. Then two applications of (6.17) give

$$\begin{aligned}(\varphi\theta\varphi^{-1})\sigma &= \rho^{1/4}\theta\rho^{-1/4}\sigma = \rho^{1/2}\theta\sigma \\ &= \rho^{1/2}\omega^{-1} = \rho^{-1/2}\rho\omega^{-1} \\ &= \rho^{-1/2}\omega = \omega\rho^{-1/2} \\ &= \sigma\theta\rho^{-1/2} = \sigma\rho^{1/4}\theta\rho^{-1/4} = \sigma(\varphi\theta\varphi^{-1}),\end{aligned}$$

as required.

Corollary 6.18. If $\mathfrak{g}_0$ is a real semisimple Lie algebra, then $\mathfrak{g}_0$ has a Cartan involution.

PROOF. Let $\mathfrak{g}$ be the complexification of $\mathfrak{g}_0$, and choose by Theorem 6.11 a compact real form $\mathfrak{u}_0$ of $\mathfrak{g}$. Let σ and τ be the conjugations of $\mathfrak{g}$ with respect to $\mathfrak{g}_0$ and $\mathfrak{u}_0$. If we regard $\mathfrak{g}$ as a real Lie algebra $\mathfrak{g}^{\mathbb{R}}$, then σ and τ are involutions of $\mathfrak{g}^{\mathbb{R}}$, and Proposition 6.14 shows that τ is a Cartan involution. By Theorem 6.16 we can find $\varphi \in \operatorname{Int}(\mathfrak{g}^{\mathbb{R}}) = \operatorname{Int}\mathfrak{g}$ such that $\varphi\tau\varphi^{-1}$ commutes with σ.

Here $\varphi\tau\varphi^{-1}$ is the conjugation of $\mathfrak{g}$ with respect to $\varphi(\mathfrak{u}_0)$, which is another compact real form of $\mathfrak{g}$. Thus

$$(B_{\mathfrak{g}^{\mathbb{R}}})_{\varphi\tau\varphi^{-1}}(Z_1, Z_2) = -2\operatorname{Re} B_{\mathfrak{g}}(Z_1, \varphi\tau\varphi^{-1}Z_2)$$

is positive definite on $\mathfrak{g}^{\mathbb{R}}$.

The Lie algebra $\mathfrak{g}_0$ is characterized as the fixed set of σ. If $\sigma X = X$, then

$$\sigma(\varphi\tau\varphi^{-1}X) = \varphi\tau\varphi^{-1}\sigma X = \varphi\tau\varphi^{-1}X.$$

Hence $\varphi\tau\varphi^{-1}$ restricts to an involution θ of $\mathfrak{g}_0$. We have

$$B_\theta(X, Y) = -B_{\mathfrak{g}_0}(X, \theta Y) = -B_{\mathfrak{g}}(X, \varphi\tau\varphi^{-1}Y) = \tfrac{1}{2}(B_{\mathfrak{g}^{\mathbb{R}}})_{\varphi\tau\varphi^{-1}}(X, Y).$$

Thus B_θ is positive definite on $\mathfrak{g}_0$, and θ is a Cartan involution.

Corollary 6.19. If $\mathfrak{g}_0$ is a real semisimple Lie algebra, then any two Cartan involutions of $\mathfrak{g}_0$ are conjugate via $\operatorname{Int}\mathfrak{g}_0$.

PROOF. Let θ and θ' be two Cartan involutions. Taking $\sigma = \theta'$ in Theorem 6.16, we can find $\varphi \in \operatorname{Int}\mathfrak{g}_0$ such that $\varphi\theta\varphi^{-1}$ commutes with θ'. Here $\varphi\theta\varphi^{-1}$ is another Cartan involution of $\mathfrak{g}_0$. So we may as well assume that θ and θ' commute from the outset. We shall prove that $\theta = \theta'$.

Since θ and θ' commute, they have compatible eigenspace decompositions into $+1$ and -1 eigenspaces. By symmetry it is enough to show that no nonzero $X \in \mathfrak{g}_0$ is in the $+1$ eigenspace for θ and the -1 eigenspace for θ'. Assuming the contrary, suppose that $\theta X = X$ and $\theta' X = -X$. Then we have

$$0 < B_\theta(X, X) = -B(X, \theta X) = -B(X, X)$$

$$0 < B_{\theta'}(X, X) = -B(X, \theta' X) = +B(X, X),$$

contradiction. We conclude that $\theta = \theta'$, and the proof is complete.

Corollary 6.20. If $\mathfrak{g}$ is a complex semisimple Lie algebra, then any two compact real forms of $\mathfrak{g}$ are conjugate via $\operatorname{Int}\mathfrak{g}$.

PROOF. Each compact real form has an associated conjugation of $\mathfrak{g}$ that determines it, and this conjugation is a Cartan involution of $\mathfrak{g}^{\mathbb{R}}$, by Proposition 6.14. Applying Corollary 6.19 to $\mathfrak{g}^{\mathbb{R}}$, we see that the two conjugations are conjugate by a member of $\operatorname{Int}(\mathfrak{g}^{\mathbb{R}})$. Since $\operatorname{Int}(\mathfrak{g}^{\mathbb{R}}) = \operatorname{Int}\mathfrak{g}$, the corollary follows.

Corollary 6.21. If $A = (A_{ij})_{i,j=1}^{l}$ is an abstract Cartan matrix, then there exists, up to isomorphism, one and only one compact semisimple Lie algebra $\mathfrak{g}_0$ whose complexification $\mathfrak{g}$ has a root system with A as Cartan matrix.

PROOF. Existence of $\mathfrak{g}$ is given in Theorem 2.111, and uniqueness of $\mathfrak{g}$ is given in Example 1 of §II.10. The passage from $\mathfrak{g}$ to $\mathfrak{g}_0$ is accomplished by Theorem 6.11 and Corollary 6.20.

Corollary 6.22. If $\mathfrak{g}$ is a complex semisimple Lie algebra, then the only Cartan involutions of $\mathfrak{g}^{\mathbb{R}}$ are the conjugations with respect to the compact real forms of $\mathfrak{g}$.

PROOF. Theorem 6.11 and Proposition 6.14 produce a Cartan involution of $\mathfrak{g}^{\mathbb{R}}$ that is conjugation with respect to some compact real form of $\mathfrak{g}$. Any other Cartan involution is conjugate to this one, according to Corollary 6.19, and hence is also the conjugation with respect to a compact real form of $\mathfrak{g}$.

A Cartan involution θ of $\mathfrak{g}_0$ yields an eigenspace decomposition

$$\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0 \tag{6.23}$$

of $\mathfrak{g}_0$ into $+1$ and -1 eigenspaces, and these must bracket according to the rules

$$[\mathfrak{k}_0, \mathfrak{k}_0] \subseteq \mathfrak{k}_0, \quad [\mathfrak{k}_0, \mathfrak{p}_0] \subseteq \mathfrak{p}_0, \quad [\mathfrak{p}_0, \mathfrak{p}_0] \subseteq \mathfrak{k}_0 \tag{6.24}$$

since θ is an involution. From (6.23) and (6.24) it follows that

$$\mathfrak{k}_0 \text{ and } \mathfrak{p}_0 \text{ are orthogonal under } B_{\mathfrak{g}_0} \text{ and under } B_\theta \tag{6.25}$$

In fact, if X is in $\mathfrak{k}_0$ and Y is in $\mathfrak{p}_0$, then $\operatorname{ad} X \operatorname{ad} Y$ carries $\mathfrak{k}_0$ to $\mathfrak{p}_0$ and $\mathfrak{p}_0$ to $\mathfrak{k}_0$. Thus it has trace 0, and $B_{\mathfrak{g}_0}(X, Y) = 0$; since $\theta Y = -Y$, $B_\theta(X, Y) = 0$ also.

Since B_θ is positive definite, the eigenspaces $\mathfrak{k}_0$ and $\mathfrak{p}_0$ in (6.23) have the property that

$$B_{\mathfrak{g}_0} \text{ is } \begin{cases} \text{negative definite on } \mathfrak{k}_0 \\ \text{positive definite on } \mathfrak{p}_0. \end{cases} \tag{6.26}$$

A decomposition (6.23) of $\mathfrak{g}_0$ that satisfies (6.24) and (6.26) is called a **Cartan decomposition** of $\mathfrak{g}_0$.

Conversely a Cartan decomposition determines a Cartan involution θ by the formula

$$\theta = \begin{cases} +1 & \text{on } \mathfrak{k}_0 \\ -1 & \text{on } \mathfrak{p}_0. \end{cases}$$

Here (6.24) shows that θ respects brackets, and (6.25) and (6.26) show that B_θ is positive definite. (B_θ is symmetric by Proposition 1.119 since θ has order 2.)

If $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ is a Cartan decomposition of $\mathfrak{g}_0$, then $\mathfrak{k}_0 \oplus i\mathfrak{p}_0$ is a compact real form of $\mathfrak{g} = (\mathfrak{g}_0)^{\mathbb{C}}$. Conversely if $\mathfrak{h}_0$ and $\mathfrak{q}_0$ are the $+1$ and -1 eigenspaces of an involution σ, then σ is a Cartan involution only if the real form $\mathfrak{h}_0 \oplus i\mathfrak{q}_0$ of $\mathfrak{g} = (\mathfrak{g}_0)^{\mathbb{C}}$ is compact.

If $\mathfrak{g}$ is a complex semisimple Lie algebra, then it follows from Corollary 6.22 that the most general Cartan decomposition of $\mathfrak{g}^{\mathbb{R}}$ is $\mathfrak{g}^{\mathbb{R}} = \mathfrak{u}_0 \oplus i\mathfrak{u}_0$, where $\mathfrak{u}_0$ is a compact real form of $\mathfrak{g}$.

Corollaries 6.18 and 6.19 have shown for an arbitrary real semisimple Lie algebra $\mathfrak{g}_0$ that Cartan decompositions exist and are unique up to conjugacy by $\operatorname{Int}\mathfrak{g}_0$. Let us see as a consequence that every real semisimple Lie algebra can be realized as a Lie algebra of real matrices closed under transpose.

Lemma 6.27. If $\mathfrak{g}_0$ is a real semisimple Lie algebra and θ is a Cartan involution, then

$$(\operatorname{ad} X)^* = -\operatorname{ad}\theta X \qquad \text{for all } X \in \mathfrak{g}_0,$$

where adjoint $(\,\cdot\,)^*$ is defined relative to the inner product B_θ.

PROOF. We have

$$\begin{aligned} B_\theta((\operatorname{ad}\theta X)Y, Z) &= -B([\theta X, Y], \theta Z) \\ &= B(Y, [\theta X, \theta Z]) = B(Y, \theta[X, Z]) \\ &= -B_\theta(Y, (\operatorname{ad} X)Z) = -B_\theta((\operatorname{ad} X)^* Y, Z). \end{aligned}$$

Proposition 6.28. If $\mathfrak{g}_0$ is a real semisimple Lie algebra, then $\mathfrak{g}_0$ is isomorphic to a Lie algebra of real matrices that is closed under transpose. If a Cartan involution θ of $\mathfrak{g}_0$ has been specified, then the isomorphism may be chosen so that θ is carried to negative transpose.

PROOF. Let θ be a Cartan involution of $\mathfrak{g}_0$ (existence by Corollary 6.18), and define the inner product B_θ on $\mathfrak{g}_0$ as in (6.13). Since $\mathfrak{g}_0$ is semisimple, $\mathfrak{g}_0 \cong \operatorname{ad}\mathfrak{g}_0$. The matrices of $\operatorname{ad}\mathfrak{g}_0$ in an orthonormal basis relative to B_θ will be the required Lie algebra of matrices. We have only to show that $\operatorname{ad}\mathfrak{g}_0$ is closed under adjoint. But this follows from Lemma 6.27 and the fact that $\mathfrak{g}_0$ is closed under θ.

Corollary 6.29. If $\mathfrak{g}_0$ is a real semisimple Lie algebra and θ is a Cartan involution, then any θ stable subalgebra $\mathfrak{s}_0$ of $\mathfrak{g}_0$ is reductive.

PROOF. Proposition 6.28 allows us to regard $\mathfrak{g}_0$ as a real Lie algebra of real matrices closed under transpose, and θ becomes negative transpose. Then $\mathfrak{s}_0$ is a Lie subalgebra of matrices closed under transpose, and the result follows from Proposition 1.59.

3. Cartan Decomposition on the Lie Group Level

In this section we turn to a consideration of groups. Let G be a semisimple Lie group, and let $\mathfrak{g}_0$ be its Lie algebra. The results of §2 established that $\mathfrak{g}_0$ has a Cartan involution and that any two Cartan involutions are conjugate by an inner automorphism. The theorem in this section lifts the corresponding Cartan decomposition $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ given in (6.23) to a decomposition of G.

In the course of the proof, we shall consider $\operatorname{Ad}(G)$ first, proving the theorem in this special case. Then we shall use the result for $\operatorname{Ad}(G)$ to obtain the theorem for G. The following proposition clarifies one detail about this process.

Proposition 6.30. If G is a semisimple Lie group and Z is its center, then G/Z has trivial center.

REMARK. The center Z is discrete, being a closed subgroup of G whose Lie algebra is 0.

PROOF. Let $\mathfrak{g}_0$ be the Lie algebra of G. For $x \in G$, $\operatorname{Ad}(x)$ is the differential of conjugation by x and is 1 if and only if x is in Z. Thus $G/Z \cong \operatorname{Ad}(G)$. If $g \in \operatorname{Ad}(G)$ is central, we have $g\operatorname{Ad}(x) = \operatorname{Ad}(x)g$ for all $x \in G$. Differentiation gives $g(\operatorname{ad} X) = (\operatorname{ad} X)g$ for $X \in \mathfrak{g}_0$, and application of both sides of this equation to $Y \in \mathfrak{g}_0$ gives $g([X, Y]) = [X, gY]$. Replacing Y by $g^{-1}Y$, we obtain $[gX, Y] = [X, Y]$. Interchanging X and Y gives $[X, gY] = [X, Y]$ and hence $g([X, Y]) = [X, Y]$. Since $[\mathfrak{g}_0, \mathfrak{g}_0] = \mathfrak{g}_0$ by

Corollary 1.55, the linear transformation g is 1 on all of $\mathfrak{g}_0$, i.e., $g = 1$. Thus $\mathrm{Ad}(G)$ has trivial center.

Theorem 6.31. Let G be a semisimple Lie group, let θ be a Cartan involution of its Lie algebra $\mathfrak{g}_0$, let $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ be the corresponding Cartan decomposition, and let K be the analytic subgroup of G with Lie algebra $\mathfrak{k}_0$. Then

(a) there exists a Lie group automorphism Θ of G with differential θ, and Θ has $\Theta^2 = 1$,
(b) the subgroup of G fixed by Θ is K,
(c) the mapping $K \times \mathfrak{p}_0 \to G$ given by $(k, X) \mapsto k \exp X$ is a diffeomorphism onto,
(d) K is closed,
(e) K contains the center Z of G,
(f) K is compact if and only if Z is finite,
(g) when Z is finite, K is a maximal compact subgroup of G.

REMARKS.

1) This theorem generalizes and extends Proposition 1.143, where (c) reduces to the polar decomposition of matrices. Proposition 1.143 therefore points to a host of examples of the theorem.

2) The automorphism Θ of the theorem will be called the **global Cartan involution**, and (c) is the **global Cartan decomposition**. Many authors follow the convention of writing θ for Θ, using the same symbol for the involution of G as for the involution of $\mathfrak{g}_0$, but we shall use distinct symbols for the two kinds of involution.

PROOF. Let $\overline{G} = \mathrm{Ad}(G)$. We shall prove the theorem for $\overline{G}$ and then deduce as a consequence the theorem for G. For the case of $\overline{G}$, we begin by constructing Θ as in (a), calling it $\overline{\Theta}$. Then we define $\overline{K}^{\#}$ to be the subgroup fixed by $\overline{\Theta}$, and we prove (c) with K replaced by $\overline{K}^{\#}$. The rest of the proof of the theorem for $\overline{G}$ is then fairly easy.

For $\overline{G}$, the Lie algebra is $\mathrm{ad}\,\mathfrak{g}_0$, and the Cartan involution $\bar{\theta}$ is $+1$ on $\mathrm{ad}_{\mathfrak{g}_0}(\mathfrak{k}_0)$ and -1 on $\mathrm{ad}_{\mathfrak{g}_0}(\mathfrak{p}_0)$. Let us write members of $\mathrm{ad}\,\mathfrak{g}_0$ with bars over them. Define the inner product B_θ on $\mathfrak{g}_0$ by (6.13), and let adjoint $(\,\cdot\,)^*$ be defined for linear maps of $\mathfrak{g}_0$ into itself by means of B_θ. Lemma 6.27 says that

$$(\mathrm{ad}\,W)^* = -\mathrm{ad}\,\theta W \qquad \text{for all } W \in \mathfrak{g}_0, \tag{6.32}$$

and therefore

$$\bar\theta\overline{W} = -\overline{W}^* \qquad \text{for all } \overline{W} \in \operatorname{ad}\mathfrak{g}_0. \tag{6.33}$$

If g is in $\operatorname{Aut}\mathfrak{g}_0$, we shall prove that g^* is in $\operatorname{Aut}\mathfrak{g}_0$. Since B_θ is definite, we are to prove that

$$B_\theta([g^*X, g^*Y], Z) \stackrel{?}{=} B_\theta(g^*[X,Y], Z) \tag{6.34}$$

for all $X, Y, Z \in \mathfrak{g}_0$. Using (6.32) three times, we have

$$\begin{aligned} B_\theta([g^*X, g^*Y], Z) &= -B_\theta(g^*Y, [\theta g^*X, Z]) = -B_\theta(Y, [g\theta g^*X, gZ]) \\ &= B_\theta((\operatorname{ad} gZ)g\theta g^*X, Y) = -B_\theta(g\theta g^*X, [\theta gZ, Y]) \\ &= B(g\theta g^*X, [gZ, \theta Y]) = -B_\theta(g^*X, g^{-1}[gZ, \theta Y]) \\ &= -B_\theta(X, [gZ, \theta Y]) = B_\theta(X, (\operatorname{ad}\theta Y)gZ) \\ &= B_\theta([X,Y], gZ) = B_\theta(g^*[X,Y], Z), \end{aligned}$$

and (6.34) is established.

We apply this fact when $g = \bar x$ is in $\operatorname{Ad}(G) = \overline{G}$. Then $\bar x^*\bar x$ is a positive definite element in $\operatorname{Aut}\mathfrak{g}_0$. By Lemma 6.15 the positive definite r^{th} power, which we write as $(\bar x^*\bar x)^r$, is in $\operatorname{Int}\mathfrak{g}_0 = \operatorname{Ad}(G) = \overline{G}$ for every real r. Hence

$$(\bar x^*\bar x)^r = \exp r\overline{X} \tag{6.35}$$

for some $\overline{X} \in \operatorname{ad}\mathfrak{g}_0$. Differentiating with respect to r and putting $r = 0$, we see that $\overline{X}^* = \overline{X}$. By (6.32), $\overline{X}$ is in $\operatorname{ad}_{\mathfrak{g}_0}(\mathfrak{p}_0)$.

Specializing to the case $r = 1$, we see that $\overline{G}$ is closed under adjoint. Hence we may define $\overline{\Theta}(\bar x) = (\bar x^*)^{-1}$, and $\overline{\Theta}$ is an automorphism of $\overline{G}$ with $\overline{\Theta}^2 = 1$. The differential of $\overline{\Theta}$ is $\overline{Y} \mapsto -\overline{Y}^*$, and (6.33) shows that this is $\bar\theta$. This proves (a) for $\overline{G}$.

The fixed group for $\overline{\Theta}$ is a closed subgroup of $\overline{G}$ that we define to be $\overline{K}^\#$. The members $\bar k$ of $\overline{K}^\#$ have $(\bar k^*)^{-1} = \bar k$ and hence are in the orthogonal group on $\mathfrak{g}_0$. Since $\overline{G} = \operatorname{Int}\mathfrak{g}_0$ and since Propositions 1.120 and 1.121 show that $\operatorname{Int}\mathfrak{g}_0 = (\operatorname{Aut}\mathfrak{g}_0)_0$, $\overline{K}^\#$ is closed in $GL(\mathfrak{g}_0)$. Since $\overline{K}^\#$ is contained in the orthogonal group, $\overline{K}^\#$ is compact. The Lie algebra of $\overline{K}^\#$ is the subalgebra of all $\overline{T} \in \operatorname{ad}\mathfrak{g}_0$ where $\bar\theta(\overline{T}) = \overline{T}$, and this is just $\operatorname{ad}_{\mathfrak{g}_0}(\mathfrak{k}_0)$.

Consider the smooth mapping $\varphi_{\bar G} : \overline{K}^\# \times \operatorname{ad}_{\mathfrak{g}_0}(\mathfrak{p}_0) \to \overline{G}$ given by $\varphi_{\bar G}(\bar k, \overline{S}) = \bar k \exp \overline{S}$. Let us prove that $\varphi_{\bar G}$ maps onto $\overline{G}$. Given $\bar x \in \overline{G}$, define $\overline{X} \in \mathrm{ad}_{\mathfrak{g}_0}(\mathfrak{p}_0)$ by (6.35), and put $\bar{p} = \exp \frac{1}{2}\overline{X}$. The element $\bar{p}$ is in $\mathrm{Ad}(G)$, and $\bar{p}^* = \bar{p}$. Put $\bar{k} = \bar{x}\bar{p}^{-1}$, so that $\bar{x} = \bar{k}\bar{p}$. Then $\bar{k}^*\bar{k} = (\bar{p}^{-1})^*\bar{x}^*\bar{x}\bar{p}^{-1} = (\exp -\frac{1}{2}\overline{X})(\exp \overline{X})(\exp -\frac{1}{2}\overline{X}) = 1$, and hence $\bar{k}^* = \bar{k}^{-1}$. Consequently $\overline{\Theta}(\bar{k}) = (\bar{k}^*)^{-1} = \bar{k}$, and we conclude that $\varphi_{\bar{G}}$ is onto.

Let us see that $\varphi_{\bar{G}}$ is one-one. If $\bar{x} = \bar{k}\exp \bar{X}$, then $\bar{x}^* = (\exp \overline{X}^*)\bar{k}^* = (\exp \overline{X})\bar{k}^* = (\exp \overline{X})\bar{k}^{-1}$. Hence $\bar{x}^*\bar{x} = \exp 2\overline{X}$. The two sides of this equation are equal positive-definite linear transformations. Their positive-definite r^{th} powers must be equal for all real r, necessarily to $\exp 2r\overline{X}$. Differentiating $(\bar{x}^*\bar{x})^r = \exp 2r\overline{X}$ with respect to r and putting $r = 0$, we see that $\bar{x}$ determines $\overline{X}$. Hence $\bar{x}$ determines also $\bar{k}$, and $\varphi_{\bar{G}}$ is one-one.

To complete the proof of (c) (but with K replaced by $\overline{K}^{\#}$), we are to show that the inverse map is smooth. It is enough to prove that the corresponding inverse map in the case of all n-by-n real nonsingular matrices is smooth, where $n = \dim \mathfrak{g}_0$. In fact, the given inverse map is a restriction of the inverse map for all matrices, and we recall from §I.10 that if M is an analytic subgroup of a Lie group M', then a smooth map into M' with image in M is smooth into M.

Thus we are to prove smoothness of the inverse for the case of matrices. The forward map is $O(n) \times \mathfrak{p}(n, \mathbb{R}) \to GL(n, \mathbb{R})$ with $(k, X) \mapsto ke^X$, where $\mathfrak{p}(n, \mathbb{R})$ denotes the vector space of n-by-n real symmetric matrices. It is enough to prove local invertibility of this mapping near $(1, X_0)$. Thus we examine the differential at $k = 1$ and $X = X_0$ of $(k, X) \mapsto ke^Xe^{-X_0}$, identifying tangent spaces as follows: At $k = 1$, we use the linear Lie algebra of $O(n)$, which is the space $\mathfrak{so}(n)$ of skew-symmetric real matrices. Near $X = X_0$, write $X = X_0 + S$, and use $\{S\} = \mathfrak{p}(n, \mathbb{R})$ as tangent space. In $GL(n, \mathbb{R})$, we use the linear Lie algebra, which consists of all real matrices.

To compute the differential, we consider restrictions of the forward map with each coordinate fixed in turn. The differential of $(k, X_0) \mapsto k$ is $(T, 0) \mapsto T$ for $T \in \mathfrak{so}(n)$. The map $(1, X) \mapsto e^Xe^{-X_0}$ has derivative at $t = 0$ along the curve $X = X_0 + tS$ equal to

$$\frac{d}{dt} e^{X_0+tS}e^{-X_0}\big|_{t=0}.$$

Thus we ask whether it is possible to have

(6.36a)
$$0 \overset{?}{=} T + \frac{d}{dt} e^{X_0+tS}e^{-X_0}\big|_{t=0}$$

$$= T + \frac{d}{dt}\big(1 + (X_0 + tS) + \tfrac{1}{2!}(X_0 + tS)^2 + \cdots\big)e^{-X_0}|_{t=0}$$
$$= T + \big(S + \tfrac{1}{2!}(SX_0 + X_0S) + \cdots + \tfrac{1}{(n+1)!}\sum_{k=0}^{n} X_0^k S X_0^{n-k} + \cdots\big)e^{-X_0}.$$

We left-bracket by X_0, noting that

$$\big[X_0,\ \sum_{k=0}^{n} X_0^k S X_0^{n-k}\big] = X_0^{n+1}S - SX_0^{n+1}.$$

Then we have

(6.36b)
$$\begin{aligned} 0 \overset{?}{=} [X_0, T] + \big((X_0S - SX_0) + \tfrac{1}{2!}(X_0^2S - SX_0^2) \\ + \cdots + \tfrac{1}{(n+1)!}(X_0^{n+1}S - SX_0^{n+1}) + \cdots\big)e^{-X_0} \\ = [X_0, T] + (e^{X_0}S - Se^{X_0})e^{-X_0} \\ = [X_0, T] + (e^{X_0}Se^{-X_0} - S). \end{aligned}$$

Since $[\mathfrak{p}(n,\mathbb{R}), \mathfrak{so}(n)] \subseteq \mathfrak{p}(n,\mathbb{R})$, we conclude that $e^{X_0}Se^{-X_0} - S$ is symmetric. Let v be an eigenvector, and let λ be the eigenvalue for v. Let $\langle\,\cdot\,,\,\cdot\,\rangle$ denote ordinary dot product on $\mathbb{R}^n$. Since e^{X_0} and S are symmetric, $e^{X_0}S - Se^{X_0}$ is skew symmetric, and we have

$$\begin{aligned} 0 &= \langle (e^{X_0}S - Se^{X_0})e^{-X_0}v,\ e^{-X_0}v\rangle \\ &= \langle (e^{X_0}Se^{-X_0} - S)v,\ e^{-X_0}v\rangle \\ &= \lambda\langle v, e^{-X_0}v\rangle. \end{aligned}$$

But e^{-X_0} is positive definite, and hence $\lambda = 0$. Thus

(6.37)
$$e^{X_0}Se^{-X_0} = S.$$

This equation forces

(6.38)
$$X_0S = SX_0.$$

In fact, there is no loss of generality is assuming that X_0 is diagonal with diagonal entries d_i. Then (6.37) implies $e^{d_i}S_{ij} = S_{ij}e^{d_j}$. Considering the two cases $S_{ij} = 0$ and $S_{ij} \neq 0$ separately, we deduce that $d_i S_{ij} = S_{ij} d_j$, and (6.38) is the result. Because of (6.37), (6.36a) collapses to

$$0 \stackrel{?}{=} T + S,$$

and we conclude that $T = S = 0$. Thus the differential is everywhere an isomorphism, and the proof of local invertibility of the forward map is complete. This completes the proof of (c) for $\overline{G}$, but with K replaced by $\overline{K}^{\#}$.

The homeomorphism $\overline{K}^{\#} \times \mathrm{ad}_{\mathfrak{g}_0}(\mathfrak{p}_0) \xrightarrow{\sim} \overline{G}$ of (c) forces $\overline{K}^{\#}$ to be connected. Thus $\overline{K}^{\#}$ is the analytic subgroup of $\overline{G}$ with Lie algebra $\mathrm{ad}_{\mathfrak{g}_0}(\mathfrak{k}_0)$, which we denote $\overline{K}$. This proves (c) for $\overline{K}$ and also (b).

To complete the proof for the adjoint group $\overline{G}$, we need to verify (d) through (g) with $\overline{K}$ in place of K. Since $\overline{K}$ is compact, (d) is immediate. Proposition 6.30 shows that $\overline{G}$ has trivial center, and then (e) and (f) follow.

For (g) suppose on the contrary that $\overline{K} \subsetneq \overline{K}_1$ with $\overline{K}_1$ compact. Let $\bar{x}$ be in $\overline{K}_1$ but not $\overline{K}$, and write $\bar{x} = \bar{k} \exp \overline{X}$ as in (c). Then $\exp \overline{X}$ is in $\overline{K}_1$ and is not 1. The powers of $\exp \overline{X}$ have unbounded eigenvalues, and this fact contradicts the compactness of $\overline{K}_1$. Thus (g) follows, and the proof of the theorem is complete for $\overline{G}$.

Now we shall prove the theorem for G. Write $e : G \to \overline{G}$ for the covering homomorphism $\mathrm{Ad}_{\mathfrak{g}_0}(\,\cdot\,)$. Let $\overline{K}$ be the analytic subgroup of $\overline{G}$ with Lie algebra $\mathrm{ad}\,\mathfrak{k}_0$, and let $K = e^{-1}(\overline{K})$. The subgroup K is closed in G since $\overline{K}$ is closed in $\overline{G}$.

From the covering homomorphism e, we obtain a smooth mapping ψ of G/K into $\overline{G}/\overline{K}$ by defining $\psi(gK) = e(g)\overline{K}$. The definition of K makes ψ one-one, and e onto makes ψ onto. Let us see that ψ^{-1} is continuous. Let $\lim \bar{g}_n = \bar{g}$ in $\overline{G}$, and choose g_n and g in G with $e(g_n) = \bar{g}_n$ and $e(g) = \bar{g}$. Then $e(g^{-1}g_n) = \bar{g}^{-1}\bar{g}_n$ tends to 1. Fix an open neighborhood N of 1 in $\overline{G}$ that is evenly covered by e. Then we can write $g^{-1}g_n = v_n z_n$ with $v_n \in N$ and $z_n \in Z$, and we have $\lim v_n = 1$. Since $Z \subseteq K$ by definition of K, $g_n K = g v_n K$ tends to gK. Therefore ψ^{-1} is continuous.

Hence G/K is homeomorphic with $\overline{G}/\overline{K}$. Conclusion (c) for $\overline{G}$ shows that $\overline{G}/\overline{K}$ is simply connected. Hence G/K is simply connected, and it follows that K is connected. Thus K is the analytic subgroup of G with Lie algebra $\mathfrak{k}_0$. This proves (d) and (e) for G. Since $Z \subseteq K$, the map $e|_K : K \to \overline{K}$ has kernel Z, and hence K is compact if and only if Z is finite. This proves (f) for G.

Now let us prove (c) for G. Define $\varphi_G : K \times \mathfrak{p}_0 \to G$ by $\varphi_G(k, X) = k \exp_G X$. From (1.82) we have

$$e\varphi_G(k, X) = e(k)e(\exp_G X) = e(k) \exp_{\bar{G}}(\mathrm{ad}_{\mathfrak{g}_0}(X)) = \varphi_{\bar{G}}(e(k), \mathrm{ad}_{\mathfrak{g}_0}(X)),$$

and therefore the diagram

$$\begin{tikzcd} K \times \mathfrak{p}_0 \arrow[r, "\varphi_G"] \arrow[d, "e|_K \times \mathrm{ad}_{\mathfrak{g}_0}"'] & G \arrow[d, "e"] \\ \overline{K} \times \mathrm{ad}_{\mathfrak{g}_0}(\mathfrak{p}_0) \arrow[r, "\varphi_{\bar{G}}"] & \overline{G} \end{tikzcd}$$

commutes. The maps on the sides are covering maps since K is connected, and $\varphi_{\bar{G}}$ is a diffeomorphism by (c) for $\overline{G}$. If we show that φ_G is one-one onto, then it follows that φ_G is a diffeomorphism, and (c) is proved for G.

First let us check that φ_G is one-one. Suppose $k \exp_G X = k' \exp_G X'$. Applying e, we have $e(k) \exp_{\bar{G}}(\mathrm{ad}_{\mathfrak{g}_0}(X)) = e(k') \exp_{\bar{G}}(\mathrm{ad}_{\mathfrak{g}_0}(X'))$. Then $X = X'$ from (c) for $\overline{G}$, and consequently $k = k'$.

Second let us check that φ_G is onto. Let $x \in G$ be given. Write $e(x) = \bar{k} \exp_{\bar{G}}(\mathrm{ad}_{\mathfrak{g}_0}(X))$ by (c) for $\overline{G}$, and let k be any member of $e^{-1}(\bar{k})$. Then $e(x) = e(k \exp_G X)$, and we see that $x = zk \exp_G X$ for some $z \in Z$. Since $Z \subseteq K$, $x = (zk) \exp_G X$ is the required decomposition. This completes the proof of (c) for G.

The next step is to construct Θ. Let $\widetilde{G}$ be a simply connected covering group of G, let $\widetilde{K}$ be the analytic subgroup of $\widetilde{G}$ with Lie algebra $\mathfrak{k}_0$, let $\widetilde{Z}$ be the center of $\widetilde{G}$, and let $\widetilde{e} : \widetilde{G} \to G$ be the covering homomorphism. Since $\widetilde{G}$ is simply connected, there exists a unique involution $\widetilde{\Theta}$ of $\widetilde{G}$ with differential θ. Since θ is 1 on $\mathfrak{k}_0$, $\widetilde{\Theta}$ is 1 on $\widetilde{K}$. By (e) for $\widetilde{G}$, $\widetilde{Z} \subseteq \widetilde{K}$. Therefore $\ker \widetilde{e} \subseteq \widetilde{K}$, and $\widetilde{\Theta}$ descends to an involution Θ of G with differential θ. This proves (a) for G.

Suppose that x is a member of G with $\Theta(x) = x$. Using (c), we can write $x = k \exp_G X$ and see that

$$k(\exp_G X)^{-1} = k \exp_G \theta X = k\Theta(\exp_G X) = \Theta(x) = x = k \exp_G X.$$

Then $\exp_G 2X = 1$, and it follows from (c) that $X = 0$. Thus x is in K, and (b) is proved for G.

Finally we are to prove (g) for G. Suppose that K is compact and that $K \subseteq K_1$ with K_1 compact. Applying e, we obtain a compact subgroup $e(K_1)$ of $\overline{G}$ that contains $\overline{K}$. By (g) for $\overline{G}$, $e(K_1) = e(K)$. Therefore $K_1 \subseteq ZK = K$, and we must have $K_1 = K$. This completes the proof of the theorem.

The Cartan decomposition on the Lie algebra level led in Proposition 6.28 to the conclusion that any real semisimple Lie algebra can be realized as a Lie algebra of real matrices closed under transpose. There is no corresponding proposition about realizing a semisimple Lie group as a group of real matrices. It is true that a semisimple Lie group of matrices is necessarily closed, and we shall prove this fact in Chapter VII. But the following example shows that a semisimple Lie group need not be realizable as a group of matrices.

EXAMPLE. By Proposition 1.143 the group $SL(2,\mathbb{R})$ has the same fundamental group as $SO(2)$, namely $\mathbb{Z}$, while $SL(2,\mathbb{C})$ has the same fundamental group as $SU(2)$, namely $\{1\}$. Then $SL(2,\mathbb{R})$ has a two-fold covering group G that is unique up to isomorphism. Let us see that G is not isomorphic to a group of n-by-n real matrices. If it were, then its linear Lie algebra $\mathfrak{g}_0$ would have the matrix Lie algebra $\mathfrak{g} = \mathfrak{g}_0 + i\mathfrak{g}_0$ as complexification. Let $G^{\mathbb{C}}$ be the analytic subgroup of $GL(n,\mathbb{C})$ with Lie algebra $\mathfrak{g}$. The diagram

$$\begin{array}{ccc} G & \longrightarrow & G^{\mathbb{C}} \\ \downarrow & & \uparrow \\ SL(2,\mathbb{R}) & \longrightarrow & SL(2,\mathbb{C}) \end{array} \tag{6.39}$$

has inclusions at the top and bottom, a two-fold covering map on the left, and a homomorphism on the right that exists since $SL(2,\mathbb{C})$ is simply connected and has Lie algebra isomorphic to $\mathfrak{g}$. The corresponding diagram of Lie algebras commutes, and hence so does the diagram (6.39) of Lie groups. However, the top map of (6.39) is one-one, while the composition of left, bottom, and right maps is not one-one. We have a contradiction, and we conclude that G is not isomorphic to a group of real matrices.

4. Iwasawa Decomposition

The Iwasawa decomposition is a second global decomposition of a semisimple Lie group. Unlike with the Cartan decomposition, the factors in the Iwasawa decomposition are closed subgroups. The prototype is the Gram–Schmidt orthogonalization process in linear algebra.

EXAMPLE. Let $G = SL(m, \mathbb{C})$. The group K from Proposition 1.143 or the global Cartan decomposition (Theorem 6.31) is $SU(m)$. Let A be the subgroup of G of diagonal matrices with positive diagonal entries, and let N be the upper-triangular group with 1 in each diagonal entry. The Iwasawa decomposition is $G = KAN$ in the sense that multiplication $K \times A \times N \to G$ is a diffeomorphism onto. To see that this decomposition of $SL(m, \mathbb{C})$ amounts to the Gram–Schmidt orthogonalization process, let $\{e_1, \dots, e_m\}$ be the standard basis of $\mathbb{C}^m$, let $g \in G$ be given, and form the basis $\{ge_1, \dots, ge_m\}$. The Gram–Schmidt process yields an orthonormal basis $v_1, \dots, v_m$ such that

$$\begin{aligned} \operatorname{span}\{ge_1, \dots, ge_j\} &= \operatorname{span}\{v_1, \dots, v_j\} \\ v_j &\in \mathbb{R}^+(ge_j) + \operatorname{span}\{v_1, \dots, v_{j-1}\} \end{aligned}$$

for $1 \le j \le m$. Define a matrix $k \in U(m)$ by $k^{-1}v_j = e_j$. Then $k^{-1}g$ is upper triangular with positive diagonal entries. Since g has determinant 1 and k has determinant of modulus 1, k must have determinant 1. Then k is in $K = SU(m)$, $k^{-1}g$ is in AN, and $g = k(k^{-1}g)$ exhibits g as in $K(AN)$. This proves that $K \times A \times N \to G$ is onto. It is one-one since $K \cap AN = \{1\}$, and the inverse is smooth because of the explicit formulas for the Gram–Schmidt process.

The decomposition in the example extends to all semisimple Lie groups. To prove such a theorem, we first obtain a Lie algebra decomposition, and then we lift the result to the Lie group.

Throughout this section, G will denote a semisimple Lie group. Changing notation from earlier sections of this chapter, we write $\mathfrak{g}$ for the Lie algebra of G. (We shall have relatively little use for the complexification of the Lie algebra in this section and write $\mathfrak{g}$ in place of $\mathfrak{g}_0$ to make the notation less cumbersome.) Let θ be a Cartan involution of $\mathfrak{g}$ (Corollary 6.18), let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the corresponding Cartan decomposition (6.23), and let K be the analytic subgroup of G with Lie algebra $\mathfrak{k}$.

Insistence on using the Killing form as our nondegenerate symmetric invariant bilinear form on $\mathfrak{g}$ will turn out to be inconvenient later when we want to compare the form on $\mathfrak{g}$ with a corresponding form on a semisimple subalgebra of $\mathfrak{g}$. Thus we shall allow some flexibility in choosing a form B. For now it will be enough to let B be any nondegenerate symmetric invariant bilinear form on $\mathfrak{g}$ such that $B(\theta X, \theta Y) = B(X, Y)$ for all X and Y in $\mathfrak{g}$ and such that the form B_θ defined in terms of B by (6.13) is positive definite. Then it follows that B is negative definite on the compact real form $\mathfrak{k} \oplus i\mathfrak{p}$. Therefore B is negative definite on a maximal abelian subspace of $\mathfrak{k} \oplus i\mathfrak{p}$, and we conclude as in the remarks with Corollary 2.38 that, for any Cartan subalgebra of $\mathfrak{g}^{\mathbb{C}}$, B is positive definite on the real subspace where all the roots are real valued.

The Killing form is one possible choice for B, but there are others. In any event, B_θ is an inner product on $\mathfrak{g}$, and we use it to define orthogonality and adjoints.

Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$. This exists because $\mathfrak{p}$ is finite dimensional. Since $(\operatorname{ad} X)^* = -\operatorname{ad} \theta X$ by Lemma 6.27, the set $\{\operatorname{ad} H \mid H \in \mathfrak{a}\}$ is a commuting family of self-adjoint transformations of $\mathfrak{g}$. Then $\mathfrak{g}$ is the orthogonal direct sum of simultaneous eigenspaces, all the eigenvalues being real. If we fix such an eigenspace and if λ_H is the eigenvalue of $\operatorname{ad} H$, then the equation $(\operatorname{ad} H)X = \lambda_H X$ shows that λ_H is linear in H. Hence the simultaneous eigenvalues are members of the dual space $\mathfrak{a}^*$. For $\lambda \in \mathfrak{a}^*$, we write

$$\mathfrak{g}_\lambda = \{X \in \mathfrak{g} \mid (\operatorname{ad} H)X = \lambda(H)X \text{ for all } H \in \mathfrak{a}\}.$$

If $\mathfrak{g}_\lambda \neq 0$ and $\lambda \neq 0$, we call λ a **restricted root** of $\mathfrak{g}$ or a **root** of $(\mathfrak{g}, \mathfrak{a})$. The set of restricted roots is denoted Σ. Any nonzero $\mathfrak{g}_\lambda$ is called a **restricted-root space**, and each member of $\mathfrak{g}_\lambda$ is called a **restricted-root vector** for the restricted root λ.

Proposition 6.40. The restricted roots and the restricted-root spaces have the following properties:

(a) $\mathfrak{g}$ is the orthogonal direct sum $\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\lambda \in \Sigma} \mathfrak{g}_\lambda$,
(b) $[\mathfrak{g}_\lambda, \mathfrak{g}_\mu] \subseteq \mathfrak{g}_{\lambda+\mu}$,
(c) $\theta \mathfrak{g}_\lambda = \mathfrak{g}_{-\lambda}$, and hence $\lambda \in \Sigma$ implies $-\lambda \in \Sigma$,
(d) $\mathfrak{g}_0 = \mathfrak{a} \oplus \mathfrak{m}$ orthogonally, where $\mathfrak{m} = Z_{\mathfrak{k}}(\mathfrak{a})$.

REMARK. The decomposition in (a) is called the **restricted-root space decomposition** of $\mathfrak{g}$.

PROOF. We saw (a) in the course of the construction of restricted-root spaces, and (b) follows from the Jacobi identity. For (c) let X be in $\mathfrak{g}_\lambda$; then $[H, \theta X] = \theta[\theta H, X] = -\theta[H, X] = -\lambda(H)\theta X$.

In (d) we have $\theta \mathfrak{g}_0 = \mathfrak{g}_0$ by (c). Hence $\mathfrak{g}_0 = (\mathfrak{k} \cap \mathfrak{g}_0) \oplus (\mathfrak{p} \cap \mathfrak{g}_0)$. Since $\mathfrak{a} \subseteq \mathfrak{p} \cap \mathfrak{g}_0$ and $\mathfrak{a}$ is maximal abelian in $\mathfrak{p}$, $\mathfrak{a} = \mathfrak{p} \cap \mathfrak{g}_0$. Also $\mathfrak{k} \cap \mathfrak{g}_0 = Z_{\mathfrak{k}}(\mathfrak{a})$. This proves (d).

EXAMPLES.

1) Let $G = SL(n, \mathbb{K})$, where $\mathbb{K}$ is $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$. The Lie algebra is $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{K})$ in the sense of §I.8. For a Cartan decomposition we can take $\mathfrak{k}$ to consist of the skew-Hermitian members of $\mathfrak{g}$ and $\mathfrak{p}$ to consist of the Hermitian members. The space of real diagonal matrices of trace 0 is a maximal abelian subspace of $\mathfrak{p}$, and we use it as $\mathfrak{a}$. Note that $\dim \mathfrak{a} = n-1$. The restricted-root space decomposition of $\mathfrak{g}$ is rather similar to Example 1 in §II.1. Let f_i be evaluation of the i^{th} diagonal entry of members of $\mathfrak{a}$. Then the restricted roots are all linear functionals $f_i - f_j$ with $i \neq j$, and $\mathfrak{g}_{f_i - f_j}$ consists of all matrices with all entries other than the $(i, j)^{\text{th}}$ equal to 0. The dimension of each restricted-root space is 1, 2, or 4 when $\mathbb{K}$ is $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$. The subalgebra $\mathfrak{m}$ of Proposition 6.40d consists of all skew-Hermitian diagonal matrices in $\mathfrak{g}$. For $\mathbb{K} = \mathbb{R}$ this is 0, and for $\mathbb{K} = \mathbb{C}$ it is all purely imaginary matrices of trace 0 and has dimension $n - 1$. For $\mathbb{K} = \mathbb{H}$, $\mathfrak{m}$ consists of all diagonal matrices whose diagonal entries x_j have $\bar{x}_j = -x_j$ and is isomorphic to the direct sum of n copies of $\mathfrak{su}(2)$; its dimension is $3n$.

2) Let $G = SU(p, q)$ with $p \geq q$. We can write the Lie algebra in block form as

$$\mathfrak{g} = \begin{array}{cc} \begin{array}{cc} p & q \end{array} & \\ \begin{pmatrix} a & b \\ b^* & d \end{pmatrix} & \begin{array}{c} p \\ q \end{array} \end{array} \tag{6.41}$$

with all entries complex, with a and d skew Hermitian, and with $\operatorname{Tr} a + \operatorname{Tr} d = 0$. We take $\mathfrak{k}$ to be all matrices in $\mathfrak{g}$ with $b = 0$, and we take $\mathfrak{p}$ to be all matrices in $\mathfrak{g}$ with $a = 0$ and $d = 0$. One way of forming a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$ is to allow b to have nonzero real entries only in the lower-left entry and the entries extending diagonally up from that one:

$$b = \begin{pmatrix} \vdots & \cdots & \vdots \\ 0 & \cdots & 0 \\ 0 & \cdots & a_q \\ & \iddots & \\ a_1 & \cdots & 0 \end{pmatrix}, \tag{6.42}$$

with $p - q$ rows of 0's at the top. Let f_i be the member of $\mathfrak{a}^*$ whose value on the $\mathfrak{a}$ matrix indicated in (6.42) is a_i. Then the restricted roots include all linear functionals $\pm f_i \pm f_j$ with $i \neq j$ and $\pm 2 f_i$ for all i. Also the $\pm f_i$ are restricted roots if $p \neq q$. The restricted-root spaces are described as follows: Let $i < j$, and let $J(z)$, $I_+(z)$, and $I_-(z)$ be the 2-by-2 matrices

$$J(z) = \begin{pmatrix} 0 & z \\ -\bar{z} & 0 \end{pmatrix}, \qquad I_+(z) = \begin{pmatrix} z & 0 \\ 0 & \bar{z} \end{pmatrix}, \qquad I_-(z) = \begin{pmatrix} z & 0 \\ 0 & -\bar{z} \end{pmatrix}.$$

Here z is any complex number. The restricted-root spaces for $\pm f_i \pm f_j$ are 2-dimensional and are nonzero only in the 16 entries corresponding to row and column indices $p - j + 1$, $p - i + 1$, $p + i$, $p + j$, where they are

$$\mathfrak{g}_{f_i - f_j} = \left\{ \begin{pmatrix} J(z) & -I_+(z) \\ -I_+(\bar{z}) & -J(\bar{z}) \end{pmatrix} \right\}, \qquad \mathfrak{g}_{-f_i + f_j} = \left\{ \begin{pmatrix} J(z) & I_+(z) \\ I_+(\bar{z}) & -J(\bar{z}) \end{pmatrix} \right\},$$

$$\mathfrak{g}_{f_i + f_j} = \left\{ \begin{pmatrix} J(z) & -I_-(z) \\ -I_-(\bar{z}) & J(\bar{z}) \end{pmatrix} \right\}, \qquad \mathfrak{g}_{-f_i - f_j} = \left\{ \begin{pmatrix} J(z) & I_-(z) \\ I_-(\bar{z}) & J(\bar{z}) \end{pmatrix} \right\}.$$

The restricted-root spaces for $\pm 2 f_i$ have dimension 1 and are nonzero only in the 4 entries corresponding to row and column indices $p - i + 1$ and $p + i$, where they are

$$\mathfrak{g}_{2 f_i} = i\mathbb{R} \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \quad \text{and} \quad \mathfrak{g}_{-2 f_i} = i\mathbb{R} \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}.$$

The restricted-root spaces for $\pm f_i$ have dimension $2(p-q)$ and are nonzero only in the entries corresponding to row and column indices 1 to $p - q$, $p - i + 1$, and $p + i$, where they are

$$\mathfrak{g}_{f_i} = \left\{ \begin{pmatrix} 0 & v & -v \\ -v^* & 0 & 0 \\ -v^* & 0 & 0 \end{pmatrix} \right\} \quad \text{and} \quad \mathfrak{g}_{-f_i} = \left\{ \begin{pmatrix} 0 & v & v \\ -v^* & 0 & 0 \\ v^* & 0 & 0 \end{pmatrix} \right\}.$$

Here v is any member of $\mathbb{C}^{p-q}$. The subalgebra $\mathfrak{m}$ of Proposition 6.40d consists of all skew-Hermitian matrices of trace 0 that are arbitrary in the upper left block of size $p - q$, are otherwise diagonal, and have the $(p-i+1)^{\text{st}}$ diagonal entry equal to the $(p+i)^{\text{th}}$ diagonal entry for $1 \leq i \leq q$; thus $\mathfrak{m} \cong \mathfrak{su}(p - q) \oplus \mathbb{R}^q$. In the next section we shall see that Σ is an abstract root system; this example shows that this root system need not be reduced.

3) Let $G = SO(p, q)_0$ with $p \geq q$. We can write the Lie algebra in block form as in (6.41) but with all entries real and with a and d skew symmetric. As in Example 2, we take $\mathfrak{k}$ to be all matrices in $\mathfrak{g}$ with $b = 0$,

and we take $\mathfrak{p}$ to be all matrices in $\mathfrak{g}$ with $a = 0$ and $d = 0$. We again choose $\mathfrak{a}$ as in (6.42). Let f_i be the member whose value on the matrix in (6.42) is a_i. Then the restricted roots include all linear functionals $\pm f_i \pm f_j$ with $i \neq j$. Also the $\pm f_i$ are restricted roots if $p \neq q$. The restricted-root spaces are the intersections with $\mathfrak{so}(p, q)$ of the restricted-root spaces in Example 2. Then the restricted-root spaces for $\pm f_i \pm f_j$ are 1-dimensional, and the restricted-root spaces for $\pm f_i$ have dimension $p-q$. The linear functionals $\pm 2 f_i$ are no longer restricted roots. The subalgebra $\mathfrak{m}$ of Proposition 6.40d consists of all skew-symmetric matrices that are nonzero only in the upper left block of size $p - q$; thus $\mathfrak{m} \cong \mathfrak{so}(p - q)$.

Choose a notion of positivity for $\mathfrak{a}^*$ in the manner of §II.5, as for example by using a lexicographic ordering. Let Σ^+ be the set of positive roots, and define $\mathfrak{n} = \bigoplus_{\lambda \in \Sigma^+} \mathfrak{g}_\lambda$. By Proposition 6.40b, $\mathfrak{n}$ is a Lie subalgebra of $\mathfrak{g}$ and is nilpotent.

Proposition 6.43 (Iwasawa decomposition of Lie algebra). With notation as above, $\mathfrak{g}$ is a vector-space direct sum $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$. Here $\mathfrak{a}$ is abelian, $\mathfrak{n}$ is nilpotent, $\mathfrak{a} \oplus \mathfrak{n}$ is a solvable Lie subalgebra of $\mathfrak{g}$, and $[\mathfrak{a} \oplus \mathfrak{n}, \mathfrak{a} \oplus \mathfrak{n}] = \mathfrak{n}$.

PROOF. We know that $\mathfrak{a}$ is abelian and that $\mathfrak{n}$ is nilpotent. Since $[\mathfrak{a}, \mathfrak{g}_\lambda] = \mathfrak{g}_\lambda$ for each $\lambda \neq 0$, we see that $[\mathfrak{a}, \mathfrak{n}] = \mathfrak{n}$ and that $\mathfrak{a} \oplus \mathfrak{n}$ is a solvable subalgebra with $[\mathfrak{a} \oplus \mathfrak{n}, \mathfrak{a} \oplus \mathfrak{n}] = \mathfrak{n}$.

To prove that $\mathfrak{k} + \mathfrak{a} + \mathfrak{n}$ is a direct sum, let X be in $\mathfrak{k} \cap (\mathfrak{a} \oplus \mathfrak{n})$. Then $\theta X = X$ with $\theta X \in \mathfrak{a} \oplus \theta\mathfrak{n}$. Since $\mathfrak{a} \oplus \mathfrak{n} \oplus \theta\mathfrak{n}$ is a direct sum (by (a) and (c) in Proposition 6.40), X is in $\mathfrak{a}$. But then X is in $\mathfrak{k} \cap \mathfrak{p} = 0$.

The sum $\mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ is all of $\mathfrak{g}$ because we can write any $X \in \mathfrak{g}$, using some $H \in \mathfrak{a}$, some $X_0 \in \mathfrak{m}$, and elements $X_\lambda \in \mathfrak{g}_\lambda$, as

$$\begin{aligned} X &= H + X_0 + \sum_{\lambda \in \Sigma} X_\lambda \\ &= \big(X_0 + \sum_{\lambda \in \Sigma^+} (X_{-\lambda} + \theta X_{-\lambda})\big) + H + \big(\sum_{\lambda \in \Sigma^+} (X_\lambda - \theta X_{-\lambda})\big), \end{aligned}$$

and the right side is in $\mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$.

To prepare to prove a group decomposition, we prove two lemmas.

Lemma 6.44. Let H be an analytic group with Lie algebra $\mathfrak{h}$, and suppose that $\mathfrak{h}$ is a vector-space direct sum of Lie subalgebras $\mathfrak{h} = \mathfrak{s} \oplus \mathfrak{t}$. If S and T denote the analytic subgroups of H corresponding to $\mathfrak{s}$ and $\mathfrak{t}$, then the multiplication map $\Phi(s, t) = st$ of $S \times T$ into H is everywhere regular.

PROOF. The tangent space at (s_0, t_0) in $S \times T$ can be identified by left translation within S and within T with $\mathfrak{s} \oplus \mathfrak{t} = \mathfrak{h}$, and the tangent space at $s_0 t_0$ in H can be identified by left translation within H with $\mathfrak{h}$. With these identifications we compute the differential $d\Phi$ at (s_0, t_0). Let X be in $\mathfrak{s}$ and Y be in $\mathfrak{t}$. Then

$$\Phi(s_0 \exp rX, t_0) = s_0 \exp(rX) t_0 = s_0 t_0 \exp(\mathrm{Ad}(t_0^{-1}) rX)$$

and
$$\Phi(s_0, t_0 \exp rY) = s_0 t_0 \exp rY,$$

from which it follows that

$$d\Phi(X) = \mathrm{Ad}(t_0^{-1})X$$

and
$$d\Phi(Y) = Y.$$

In matrix form, $d\Phi$ is therefore block triangular, and hence

$$\det d\Phi = \frac{\det \mathrm{Ad}_{\mathfrak{h}}(t_0^{-1})}{\det \mathrm{Ad}_{\mathfrak{t}}(t_0^{-1})} = \frac{\det \mathrm{Ad}_{\mathfrak{t}}(t_0)}{\det \mathrm{Ad}_{\mathfrak{h}}(t_0)}.$$

This is nonzero, and hence Φ is regular.

Lemma 6.45. There exists a basis $\{X_i\}$ of $\mathfrak{g}$ such that the matrices representing $\mathrm{ad}\,\mathfrak{g}$ have the following properties:

(a) the matrices of $\mathrm{ad}\,\mathfrak{k}$ are skew symmetric,
(b) the matrices of $\mathrm{ad}\,\mathfrak{a}$ are diagonal with real entries,
(c) the matrices of $\mathrm{ad}\,\mathfrak{n}$ are upper triangular with 0's on the diagonal.

PROOF. Let $\{X_i\}$ be an orthonormal basis of $\mathfrak{g}$ compatible with the orthogonal decomposition of $\mathfrak{g}$ in Proposition 6.40a and having the property that $X_i \in \mathfrak{g}_{\lambda_i}$ and $X_j \in \mathfrak{g}_{\lambda_j}$ with $i < j$ implies $\lambda_i \geq \lambda_j$. For $X \in \mathfrak{k}$, we have $(\mathrm{ad}\,X)^* = -\mathrm{ad}\,\theta X = -\mathrm{ad}\,X$ from Lemma 6.27, and this proves (a). Since each X_i is a restricted-root vector or is in $\mathfrak{g}_0$, the matrices of $\mathrm{ad}\,\mathfrak{a}$ are diagonal, necessarily with real entries. This proves (b). Conclusion (c) follows from Proposition 6.40b.

Theorem 6.46 (Iwasawa decomposition). Let G be a semisimple Lie group, let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ be an Iwasawa decomposition of the Lie algebra $\mathfrak{g}$ of G, and let A and N be the analytic subgroups of G with Lie algebras $\mathfrak{a}$ and $\mathfrak{n}$. Then the multiplication map $K \times A \times N \to G$ given by $(k, a, n) \mapsto kan$ is a diffeomorphism onto. The groups A and N are simply connected.

PROOF. Let $\overline{G} = \mathrm{Ad}(G)$, regarded as the closed subgroup $(\mathrm{Aut}\,\mathfrak{g})_0$ of $GL(\mathfrak{g})$ (Propositions 1.120 and 1.121). We shall prove the theorem for $\overline{G}$ and then lift the result to G.

We impose the inner product B_θ on $\mathfrak{g}$ and write matrices for elements of $\overline{G}$ and $\mathrm{ad}\,\mathfrak{g}$ relative to the basis in Lemma 6.45. Let $\overline{K} = \mathrm{Ad}_{\mathfrak{g}}(K)$, $\overline{A} = \mathrm{Ad}_{\mathfrak{g}}(A)$, and $\overline{N} = \mathrm{Ad}_{\mathfrak{g}}(N)$. Lemma 6.45 shows that the matrices of $\overline{K}$ are rotation matrices, those for $\overline{A}$ are diagonal with positive entries on the diagonal, and those for $\overline{N}$ are upper triangular with 1's on the diagonal. We know that $\overline{K}$ is compact (Proposition 6.30 and Theorem 6.31f). The diagonal subgroup of $GL(\mathfrak{g})$ with positive diagonal entries is simply connected abelian, and $\overline{A}$ is an analytic subgroup of it. By Corollary 1.134, $\overline{A}$ is closed in $GL(\mathfrak{g})$ and hence closed in $\overline{G}$. Similarly the upper-triangular subgroup of $GL(\mathfrak{g})$ with 1's on the diagonal is simply connected nilpotent, and $\overline{N}$ is an analytic subgroup of it. By Corollary 1.134, $\overline{N}$ is closed in $GL(\mathfrak{g})$ and hence closed in $\overline{G}$.

The map $\overline{A} \times \overline{N}$ into $GL(\mathfrak{g})$ given by $(\bar{a}, \bar{n}) \mapsto \bar{a}\bar{n}$ is one-one since we can recover $\bar{a}$ from the diagonal entries, and it is onto a subgroup $\overline{A}\,\overline{N}$ since $\bar{a}_1\bar{n}_1\bar{a}_2\bar{n}_2 = \bar{a}_1\bar{a}_2(\bar{a}_2^{-1}\bar{n}_1\bar{a}_2)\bar{n}_2$ and $(\bar{a}\bar{n})^{-1} = \bar{n}^{-1}\bar{a}^{-1} = \bar{a}^{-1}(\bar{a}\bar{n}\bar{a}^{-1})$. This subgroup is closed. In fact, if $\lim \bar{a}_m\bar{n}_m = x$, let $\bar{a}$ be the diagonal matrix with the same diagonal entries as x. Then $\lim \bar{a}_m = \bar{a}$, and $\bar{a}$ must be in $\overline{A}$ since $\overline{A}$ is closed in $GL(\mathfrak{g})$. Also $\bar{n}_m = \bar{a}_m^{-1}(\bar{a}_m\bar{n}_m)$ has limit $\bar{a}^{-1}x$, which has to be in $\overline{N}$ since $\overline{N}$ is closed in $\overline{G}$. Thus $\lim \bar{a}_m\bar{n}_m$ is in $\overline{A}\,\overline{N}$, and $\overline{A}\,\overline{N}$ is closed.

Clearly the closed subgroup $\overline{A}\,\overline{N}$ has Lie algebra $\mathfrak{a} \oplus \mathfrak{n}$. By Lemma 6.44, $\overline{A} \times \overline{N} \to \overline{A}\,\overline{N}$ is a diffeomorphism.

The subgroup $\overline{K}$ is compact, and thus the image of $\overline{K} \times \overline{A} \times \overline{N} \to \overline{K} \times \overline{A}\,\overline{N} \to \overline{G}$ is the product of a compact set and a closed set and is closed. Also the image is open since the map is everywhere regular (Lemma 6.44) and since the equality $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ shows that the dimensions add properly. Since the image of $\overline{K} \times \overline{A} \times \overline{N}$ is open and closed and since $\overline{G}$ is connected, the image is all of $\overline{G}$.

Thus the multiplication map is smooth, regular, and onto. Finally $\overline{K} \cap \overline{A}\,\overline{N} = \{1\}$ since a rotation matrix with positive eigenvalues is 1. Since $\overline{A} \times \overline{N} \to \overline{A}\,\overline{N}$ is one-one, it follows that $\overline{K} \times \overline{A} \times \overline{N} \to \overline{G}$ is one-one. This completes the proof for the adjoint group $\overline{G}$.

We now lift the above result to G. Let $e : G \to \overline{G} = \mathrm{Ad}(G)$ be the covering homomorphism. Using a locally defined inverse of e, we can write the map $(k, a, n) \mapsto kan$ locally as

$$(k, a, n) \mapsto (e(k), e(a), e(n)) \mapsto e(k)e(a)e(n) = e(kan) \mapsto kan,$$

and therefore the multiplication map is smooth and everywhere regular. Since A and N are connected, $e|_A$ and $e|_N$ are covering maps to $\overline{A}$ and $\overline{N}$, respectively. Since $\overline{A}$ and $\overline{N}$ are simply connected, it follows that e is one-one on A and on N and that A and N are simply connected.

Let us prove that the multiplication map is onto G. If $g \in G$ is given, write $e(g) = \bar{k}\bar{a}\bar{n}$. Put $a = (e|_A)^{-1}(\bar{a}) \in A$ and $n = (e|_N)^{-1}(\overline{N}) \in N$. Let k be in $e^{-1}(\bar{k})$. Then $e(kan) = \bar{k}\bar{a}\bar{n}$, so that $e(g(kan)^{-1}) = 1$. Thus $g(kan)^{-1} = z$ is in the center of G. By Theorem 6.31e, z is in K. Therefore $g = (zk)an$ exhibits g as in the image of the multiplication map.

Finally we show that the multiplication map is one-one. Since $\overline{A} \times \overline{N} \to \overline{A}\,\overline{N}$ is one-one, so is $A \times N \to AN$. The set of products AN is a group, just as in the adjoint case, and therefore it is enough to prove that $K \cap AN = \{1\}$. If x is in $K \cap AN$, then $e(x)$ is in $\overline{K} \cap \overline{A}\,\overline{N} = \{1\}$. Hence $e(x) = 1$. Write $x = an \in AN$. Then $1 = e(x) = e(an) = e(a)e(n)$, and the result for the adjoint case implies that $e(a) = e(n) = 1$. Since e is one-one on A and on N, $a = n = 1$. Thus $x = 1$. This completes the proof.

Recall from §IV.5 that a subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is called a **Cartan subalgebra** if $\mathfrak{h}^{\mathbb{C}}$ is a Cartan subalgebra of $\mathfrak{g}^{\mathbb{C}}$. The **rank** of $\mathfrak{g}$ is the dimension of any Cartan subalgebra; this is well defined since Proposition 2.15 shows that any two Cartan subalgebras of $\mathfrak{g}^{\mathbb{C}}$ are conjugate via $\operatorname{Int} \mathfrak{g}^{\mathbb{C}}$.

Proposition 6.47. If $\mathfrak{t}$ is a maximal abelian subspace of $\mathfrak{m} = Z_{\mathfrak{k}}(\mathfrak{a})$, then $\mathfrak{h} = \mathfrak{a} \oplus \mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}$.

PROOF. By Proposition 2.13 it is enough to show that $\mathfrak{h}^{\mathbb{C}}$ is maximal abelian in $\mathfrak{g}^{\mathbb{C}}$ and that $\operatorname{ad}_{\mathfrak{g}^{\mathbb{C}}} \mathfrak{h}^{\mathbb{C}}$ is simultaneously diagonable.

Certainly $\mathfrak{h}^{\mathbb{C}}$ is abelian. Let us see that it is maximal abelian. If $Z = X + iY$ commutes with $\mathfrak{h}^{\mathbb{C}}$, then so do X and Y. Thus there is no loss in generality in considering only X. The element X commutes with $\mathfrak{h}^{\mathbb{C}}$, hence commutes with $\mathfrak{a}$, and hence is in $\mathfrak{a} \oplus \mathfrak{m}$. The same thing is true of θX. Then $X + \theta X$, being in $\mathfrak{k}$, is in $\mathfrak{m}$ and commutes with $\mathfrak{t}$, hence is in $\mathfrak{t}$, while $X - \theta X$ is in $\mathfrak{a}$. Thus X is in $\mathfrak{a} \oplus \mathfrak{t}$, and we conclude that $\mathfrak{h}^{\mathbb{C}}$ is maximal abelian.

In the basis of Lemma 6.45, the matrices representing $\operatorname{ad} \mathfrak{t}$ are skew symmetric and hence are diagonable over $\mathbb{C}$, while the matrices representing $\operatorname{ad} \mathfrak{a}$ are already diagonal. Since all the matrices in question form a commuting family, the members of $\operatorname{ad} \mathfrak{h}^{\mathbb{C}}$ are diagonable.

With notation as in Proposition 6.47, $\mathfrak{h} = \mathfrak{a} \oplus \mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}$, and it is meaningful to speak of the set $\Delta = \Delta(\mathfrak{g}^{\mathbb{C}}, \mathfrak{h}^{\mathbb{C}})$ of roots of $\mathfrak{g}^{\mathbb{C}}$ with respect to $\mathfrak{h}^{\mathbb{C}}$. We can write the corresponding root-space decomposition as

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}} \oplus \bigoplus_{\alpha \in \Delta} (\mathfrak{g}^{\mathbb{C}})_\alpha. \tag{6.48a}$$

Then it is clear that

$$\mathfrak{g}_\lambda = \mathfrak{g} \cap \bigoplus_{\substack{\alpha \in \Delta, \\ \alpha|_{\mathfrak{a}} = \lambda}} (\mathfrak{g}^{\mathbb{C}})_\alpha \tag{6.48b}$$

and

$$\mathfrak{m}^{\mathbb{C}} = \mathfrak{t}^{\mathbb{C}} \oplus \bigoplus_{\substack{\alpha \in \Delta, \\ \alpha|_{\mathfrak{a}} = 0}} (\mathfrak{g}^{\mathbb{C}})_\alpha. \tag{6.48c}$$

That is, the restricted roots are the nonzero restrictions to $\mathfrak{a}$ of the roots, and $\mathfrak{m}$ arises from $\mathfrak{t}$ and the roots that restrict to 0 on $\mathfrak{a}$.

Corollary 6.49. If $\mathfrak{t}$ is a maximal abelian subspace of $\mathfrak{m} = Z_{\mathfrak{k}}(\mathfrak{a})$, then the Cartan subalgebra $\mathfrak{h} = \mathfrak{a} \oplus \mathfrak{t}$ of $\mathfrak{g}$ has the property that all of the roots are real on $\mathfrak{a} \oplus i\mathfrak{t}$. If $\mathfrak{m} = 0$, then $\mathfrak{g}$ is a split real form of $\mathfrak{g}^{\mathbb{C}}$.

PROOF. In view of (6.48) the values of the roots on a member H of $\mathfrak{h}$ are the eigenvalues of $\operatorname{ad} H$. For $H \in \mathfrak{a}$, these are real since $\operatorname{ad} H$ is self adjoint. For $H \in \mathfrak{t}$, they are purely imaginary since $\operatorname{ad} H$ is skew adjoint. The first assertion follows.

If $\mathfrak{m} = 0$, then $\mathfrak{t} = 0$. So the roots are real on $\mathfrak{h} = \mathfrak{a}$. Thus $\mathfrak{g}$ contains the real subspace of a Cartan subalgebra $\mathfrak{h}^{\mathbb{C}}$ of $\mathfrak{g}^{\mathbb{C}}$ where all the roots are real, and $\mathfrak{g}$ is a split real form of $\mathfrak{g}^{\mathbb{C}}$.

EXAMPLE. Corollary 6.49 shows that the Lie algebras $\mathfrak{so}(n+1, n)$ and $\mathfrak{so}(n, n)$ are split real forms of their complexifications, since Example 3 earlier in this section showed that $\mathfrak{m} = 0$ in each case. For any p and q, the complexification of $\mathfrak{so}(p, q)$ is conjugate to $\mathfrak{so}(p+q, \mathbb{C})$ by a diagonal matrix whose diagonal consists of p entries i and then q entries 1. Consequently $\mathfrak{so}(n+1, n)$ is isomorphic to a split real form of $\mathfrak{so}(2n+1, \mathbb{C})$, and $\mathfrak{so}(n, n)$ is isomorphic to a split real form of $\mathfrak{so}(2n, \mathbb{C})$.

With Δ as above, we can impose a positive system on Δ so that Δ^+ extends Σ^+. Namely we just take $\mathfrak{a}$ before $i\mathfrak{t}$ in forming a lexicographic ordering of $(\mathfrak{a} + i\mathfrak{t})^*$. If $\alpha \in \Delta$ is nonzero on $\mathfrak{a}$, then the positivity of α depends only on the $\mathfrak{a}$ part, and thus positivity for Σ has been extended to Δ.

5. Uniqueness Properties of the Iwasawa Decomposition

We continue with G as a semisimple Lie group, with $\mathfrak{g}$ as the Lie algebra of G, and with other notation as in §4. In this section we shall show that an Iwasawa decomposition of $\mathfrak{g}$ is unique up to conjugacy by $\operatorname{Int}\mathfrak{g}$; therefore an Iwasawa decomposition of G is unique up to inner automorphism.

We already know from Corollary 6.19 that any two Cartan decompositions are conjugate via $\operatorname{Int}\mathfrak{g}$. Hence $\mathfrak{k}$ is unique up to conjugacy. Next we show that with $\mathfrak{k}$ fixed, $\mathfrak{a}$ is unique up to conjugacy. Finally with $\mathfrak{k}$ and $\mathfrak{a}$ fixed, we show that the various possibilities for $\mathfrak{n}$ are conjugate.

Lemma 6.50. If $H \in \mathfrak{a}$ has $\lambda(H) \neq 0$ for all $\lambda \in \Sigma$, then $Z_{\mathfrak{g}}(H) = \mathfrak{m} \oplus \mathfrak{a}$. Hence $Z_{\mathfrak{p}}(H) = \mathfrak{a}$.

PROOF. Let X be in $Z_{\mathfrak{g}}(H)$, and use Proposition 6.40 to write $X = H_0 + X_0 + \sum_{\lambda\in\Sigma} X_\lambda$ with $H_0 \in \mathfrak{a}$, $X_0 \in \mathfrak{m}$, and $X_\lambda \in \mathfrak{g}_\lambda$. Then $0 = [H, X] = \sum \lambda(H)X_\lambda$, and hence $\lambda(H)X_\lambda = 0$ for all λ. Since $\lambda(H) \neq 0$ by assumption, $X_\lambda = 0$.

Theorem 6.51. If $\mathfrak{a}$ and $\mathfrak{a}'$ are two maximal abelian subspaces of $\mathfrak{p}$, then there is a member k of K with $\operatorname{Ad}(k)\mathfrak{a}' = \mathfrak{a}$. Consequently the space $\mathfrak{p}$ satisfies $\mathfrak{p} = \bigcup_{k\in K} \operatorname{Ad}(k)\mathfrak{a}$.

REMARKS.

1) In the case of $SL(m, \mathbb{C})$, this result amounts to the Spectral Theorem for Hermitian matrices.

2) The proof should be compared with the proof of Theorem 4.34.

PROOF. There are only finitely many restricted roots relative to $\mathfrak{a}$, and the union of their kernels therefore cannot exhaust $\mathfrak{a}$. By Lemma 6.50 we can find $H \in \mathfrak{a}$ such that $Z_{\mathfrak{p}}(H) = \mathfrak{a}$. Similarly we can find $H' \in \mathfrak{a}'$ such that $Z_{\mathfrak{p}}(H') = \mathfrak{a}'$. Choose by compactness of $\operatorname{Ad}(K)$ a member $k = k_0$ of K that minimizes $B(\operatorname{Ad}(k)H', H)$. For any $Z \in \mathfrak{k}$, $r \mapsto B(\operatorname{Ad}(\exp rZ)\operatorname{Ad}(k_0)H', H)$ is then a smooth function of r that is minimized for $r = 0$. Differentiating and setting $r = 0$, we obtain

$$0 = B((\operatorname{ad} Z)\operatorname{Ad}(k_0)H', H) = B(Z, [\operatorname{Ad}(k_0)H', H]).$$

Here $[\operatorname{Ad}(k_0)H', H]$ is in $\mathfrak{k}$, and Z is arbitrary in $\mathfrak{k}$. Since $B(\mathfrak{k}, \mathfrak{p}) = 0$ by (6.25) and since B is nondegenerate, we obtain $[\operatorname{Ad}(k_0)H', H] = 0$. Thus $\operatorname{Ad}(k_0)H'$ is in $Z_{\mathfrak{p}}(H) = \mathfrak{a}$. Since $\mathfrak{a}$ is abelian, this means

$$\mathfrak{a} \subseteq Z_{\mathfrak{p}}(\operatorname{Ad}(k_0)H') = \operatorname{Ad}(k_0)Z_{\mathfrak{p}}(H') = \operatorname{Ad}(k_0)\mathfrak{a}'.$$

Equality must hold since $\mathfrak{a}$ is maximal abelian in $\mathfrak{p}$. Thus $\mathfrak{a} = \mathrm{Ad}(k_0)\mathfrak{a}'$.

If X is any member of $\mathfrak{p}$, then we can extend $\mathbb{R}X$ to a maximal abelian subspace $\mathfrak{a}'$ of $\mathfrak{p}$. As above, we can write $\mathfrak{a}' = \mathrm{Ad}(k)\mathfrak{a}$, and hence X is in $\bigcup_{k\in K} \mathrm{Ad}(k)\mathfrak{a}$. Therefore $\mathfrak{p} = \bigcup_{k\in K} \mathrm{Ad}(k)\mathfrak{a}$.

Now we think of $\mathfrak{k}$ and $\mathfrak{a}$ as fixed and consider the various possibilities for $\mathfrak{n}$. The inner product B_θ on $\mathfrak{g}$ can be restricted to $\mathfrak{a}$ and transferred to $\mathfrak{a}^*$ to give an inner product and norm denoted by $\langle \cdot\,,\,\cdot \rangle$ and $|\cdot|$, respectively. We write H_λ for the element of $\mathfrak{a}$ that corresponds to $\lambda \in \mathfrak{a}^*$.

Proposition 6.52. Let λ be a restricted root, and let E_λ be a nonzero restricted-root vector for λ.

(a) $[E_\lambda, \theta E_\lambda] = B(E_\lambda, \theta E_\lambda)H_\lambda$, and $B(E_\lambda, \theta E_\lambda) < 0$.

(b) $\mathbb{R}H_\lambda \oplus \mathbb{R}E_\lambda \oplus \mathbb{R}\theta E_\lambda$ is a Lie subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}(2,\mathbb{R})$, and the isomorphism can be defined so that the vector $H'_\lambda = 2|\lambda|^{-2}H_\lambda$ corresponds to $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

(c) If E_λ is normalized so that $B(E_\lambda, \theta E_\lambda) = -2/|\lambda|^2$, then $k = \exp \frac{\pi}{2}(E_\lambda + \theta E_\lambda)$ is a member of the normalizer $N_K(\mathfrak{a})$, and $\mathrm{Ad}(k)$ acts as the reflection s_λ on $\mathfrak{a}^*$.

PROOF.

(a) By Proposition 6.40 the vector $[E_\lambda, \theta E_\lambda]$ is in $[\mathfrak{g}_\lambda, \mathfrak{g}_{-\lambda}] \subseteq \mathfrak{g}_0 = \mathfrak{a} \oplus \mathfrak{m}$, and $\theta[E_\lambda, \theta E_\lambda] = [\theta E_\lambda, E_\lambda] = -[E_\lambda, \theta E_\lambda]$. Thus $[E_\lambda, \theta E_\lambda]$ is in $\mathfrak{a}$. Then $H \in \mathfrak{a}$ gives

$$\begin{aligned} B([E_\lambda, \theta E_\lambda], H) &= B(E_\lambda, [\theta E_\lambda, H]) = \lambda(H) B(E_\lambda, \theta E_\lambda) \\ &= B(H_\lambda, H) B(E_\lambda, \theta E_\lambda) = B(B(E_\lambda, \theta E_\lambda)H_\lambda, H). \end{aligned}$$

By nondegeneracy of B on $\mathfrak{a}$, $[E_\lambda, \theta E_\lambda] = B(E_\lambda, \theta E_\lambda)H_\lambda$. Finally $B(E_\lambda, \theta E_\lambda) = -B_\theta(E_\lambda, E_\lambda) < 0$ since B_θ is positive definite.

(b) Put

$$H'_\lambda = \frac{2}{|\lambda|^2} H_\lambda, \qquad E'_\lambda = \frac{2}{|\lambda|^2 B(E_\lambda, \theta E_\lambda)} E_\lambda, \qquad E'_{-\lambda} = \theta E_\lambda.$$

Then (a) shows that

$$[H'_\lambda, E'_\lambda] = 2E'_\lambda, \quad [H'_\lambda, E'_{-\lambda}] = -2E'_{-\lambda}, \quad [E'_\lambda, E'_{-\lambda}] = H'_\lambda,$$

and (b) follows.

(c) Note from (a) that the normalization $B(E_\lambda, \theta E_\lambda) = -2/|\lambda|^2$ is allowable. If $\lambda(H) = 0$, then

$$\begin{aligned}\mathrm{Ad}(k)H &= \mathrm{Ad}(\exp \tfrac{\pi}{2}(E_\lambda + \theta E_\lambda))H \\ &= (\exp \mathrm{ad}\, \tfrac{\pi}{2}(E_\lambda + \theta E_\lambda))H \\ &= \sum_{n=0}^{\infty} \tfrac{1}{n!}(\mathrm{ad}\, \tfrac{\pi}{2}(E_\lambda + \theta E_\lambda))^n H \\ &= H.\end{aligned}$$

On the other hand, for the element H'_λ, we first calculate that

$$(\mathrm{ad}\, \tfrac{\pi}{2}(E_\lambda + \theta E_\lambda))H'_\lambda = \pi(\theta E_\lambda - E_\lambda)$$

and
$$(\mathrm{ad}\, \tfrac{\pi}{2}(E_\lambda + \theta E_\lambda))^2 H'_\lambda = -\pi^2 H'_\lambda.$$

Therefore

$$\begin{aligned}\mathrm{Ad}(k)H'_\lambda &= \sum_{n=0}^{\infty} \tfrac{1}{n!}(\mathrm{ad}\, \tfrac{\pi}{2}(E_\lambda + \theta E_\lambda))^n H'_\lambda \\ &= \sum_{m=0}^{\infty} \tfrac{1}{(2m)!}((\mathrm{ad}\, \tfrac{\pi}{2}(E_\lambda + \theta E_\lambda))^2)^m H'_\lambda \\ &\quad + \sum_{m=0}^{\infty} \tfrac{1}{(2m+1)!}(\mathrm{ad}\, \tfrac{\pi}{2}(E_\lambda + \theta E_\lambda))((\mathrm{ad}\, \tfrac{\pi}{2}(E_\lambda + \theta E_\lambda))^2)^m H'_\lambda \\ &= \sum_{m=0}^{\infty} \tfrac{1}{(2m)!}(-\pi^2)^m H'_\lambda + \sum_{m=0}^{\infty} \tfrac{1}{(2m+1)!}(-\pi^2)^m \pi(\theta E_\lambda - E_\lambda) \\ &= (\cos\pi)H'_\lambda + (\sin\pi)(\theta E_\lambda - E_\lambda) \\ &= -H'_\lambda,\end{aligned}$$

and (c) follows.

Corollary 6.53. Σ is an abstract root system in $\mathfrak{a}^*$.

REMARKS. Examples of Σ appear in §4 after Proposition 6.40. The example of $SU(p, q)$ for $p > q$ shows that the abstract root system Σ need not be reduced.

PROOF. We verify that Σ satisfies the axioms for an abstract root system. To see that Σ spans $\mathfrak{a}^*$, let $\lambda(H) = 0$ for all $\lambda \in \Sigma$. Then $[H, \mathfrak{g}_\lambda] = 0$ for all λ and hence $[H, \mathfrak{g}] = 0$. But $\mathfrak{g}$ has 0 center, and therefore $H = 0$. Thus Σ spans $\mathfrak{a}^*$.

Let us show that $2\langle\mu, \lambda\rangle/|\lambda|^2$ is an integer whenever μ and λ are in Σ. Consider the subalgebra of Proposition 6.52b, calling it $\mathfrak{sl}_\lambda$. This acts by ad on $\mathfrak{g}$ and hence on $\mathfrak{g}^{\mathbb{C}}$. Complexifying, we obtain a representation of $(\mathfrak{sl}_\lambda)^{\mathbb{C}} \cong \mathfrak{sl}(2, \mathbb{C})$ on $\mathfrak{g}^{\mathbb{C}}$. We know from Corollary 1.72 that the element $H'_\lambda = 2|\lambda|^{-2}H_\lambda$, which corresponds to h, has to act diagonably with integer eigenvalues. The action of H'_λ on $\mathfrak{g}_\mu$ is by the scalar $\mu(2|\lambda|^{-2}H_\lambda) = 2\langle\mu, \lambda\rangle/|\lambda|^2$. Hence $2\langle\mu, \lambda\rangle/|\lambda|^2$ is an integer.

Finally we are to show that $s_\lambda(\mu)$ is in Σ whenever μ and λ are in Σ. Define k as in Proposition 6.52c, let H be in $\mathfrak{a}$, and let X be in $\mathfrak{g}_\mu$. Then we have

$$\begin{aligned}[H, \mathrm{Ad}(k)X] &= \mathrm{Ad}(k)[\mathrm{Ad}(k)^{-1}H, X] = \mathrm{Ad}(k)[s_\lambda^{-1}(H), X] \\ &= \mu(s_\lambda^{-1}(H))\mathrm{Ad}(k)X = (s_\lambda\mu)(H)\mathrm{Ad}(k)X,\end{aligned} \tag{6.54}$$

and hence $\mathfrak{g}_{s_\lambda\mu}$ is not 0. This completes the proof.

The possibilities for the subalgebra $\mathfrak{n}$ are given by all possible Σ^+'s resulting from different orderings of $\mathfrak{a}^*$, and it follows from Corollary 6.53 that the Σ^+'s correspond to all possible simple systems for Σ. Any two such simple systems are conjugate by the Weyl group $W(\Sigma)$ of Σ, and it follows from Proposition 6.52c that the conjugation can be achieved by a member of $N_K(\mathfrak{a})$. The same computation as in (6.54) shows that if $k \in N_K(\mathfrak{a})$ represents the member s of $W(\Sigma)$, then $\mathrm{Ad}(k)\mathfrak{g}_\lambda = \mathfrak{g}_{s\lambda}$. We summarize this discussion in the following corollary.

Corollary 6.55. Any two choices of $\mathfrak{n}$ are conjugate by Ad of a member of $N_K(\mathfrak{a})$.

This completes our discussion of the conjugacy of different Iwasawa decompositions.

We now examine $N_K(\mathfrak{a})$ further. Define

$$W(G, A) = N_K(\mathfrak{a})/Z_K(\mathfrak{a}).$$

This is a group of linear transformations of $\mathfrak{a}$, telling all possible ways that members of K can act on $\mathfrak{a}$ by Ad. We have already seen that $W(\Sigma) \subseteq W(G, A)$, and we are going to prove that $W(\Sigma) = W(G, A)$.

We write M for the group $Z_K(\mathfrak{a})$. Modulo the center of G, M is a compact group (being a closed subgroup of K) with Lie algebra $Z_{\mathfrak{k}}(\mathfrak{a}) = \mathfrak{m}$. After Proposition 6.40 we saw examples of restricted-root space decompositions and the associated Lie algebras $\mathfrak{m}$. The following examples continue that discussion.

EXAMPLES.

1) Let $G = SL(n, \mathbb{K})$, where $\mathbb{K}$ is $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$. The subgroup M consists of all diagonal members of K. When $\mathbb{K} = \mathbb{R}$, the diagonal entries are ± 1, but there are only $n-1$ independent signs since the determinant is 1. Thus M is finite abelian and is the product of $n-1$ groups of order 2. When $\mathbb{K} = \mathbb{C}$, the diagonal entries are complex numbers of modulus 1, and again the determinant is 1. Thus M is a torus of dimension $n-1$. When $\mathbb{K} = \mathbb{H}$, the diagonal entries are quaternions of absolute value 1, and there is no restriction on the determinant. Thus M is the product of n copies of $SU(2)$.

2) Let $G = SU(p,q)$ with $p \geq q$. The group M consists of all unitary matrices of determinant 1 that are arbitrary in the upper left block of size $p-q$, are otherwise diagonal, and have the $(p-i+1)^{\text{st}}$ diagonal entry equal to the $(p+i)^{\text{th}}$ diagonal entry for $1 \leq i \leq q$. Let us abbreviate such a matrix as
$$m = \operatorname{diag}(\omega, e^{i\theta_q}, \dots, e^{i\theta_1}, e^{i\theta_1}, \dots, e^{i\theta_q}),$$
where ω is the upper left block of size $p-q$. When $p = q$, the condition that the determinant be 1 says that $\sum_{j=1}^{q} \theta_j \in \pi\mathbb{Z}$. Thus we can take $\theta_1, \dots, \theta_{q-1}$ to be arbitrary and use $e^{i\theta_q} = \pm e^{-i(\theta_1 + \cdots + \theta_{q-1})}$. Consequently M is the product of a torus of dimension $q-1$ and a 2-element group. When $p > q$, M is connected. In fact, the homomorphism that maps the above matrix m to the $2q$-by-$2q$ diagonal matrix
$$\operatorname{diag}(e^{i\theta_q}, \dots, e^{i\theta_1}, e^{i\theta_1}, \dots, e^{i\theta_q})$$
has a (connected) q-dimensional torus as image, and the kernel is isomorphic to the connected group $SU(p-q)$; thus M itself is connected.

3) Let $G = SO(p,q)_0$ with $p \geq q$. The subgroup M for this example is the intersection of $SO(p) \times SO(q)$ with the M of the previous example. Thus M here consists of matrices that are orthogonal matrices of total determinant 1, are arbitrary in the upper left block of size $p-q$, are otherwise diagonal, have q diagonal entries ± 1 after the upper left block, and then have those q diagonal entries ± 1 repeated in reverse order. For the lower right q entries to yield a matrix in $SO(q)$, the product of the q entries ± 1 must be 1. For the upper left p entries to yield a matrix in $SO(p)$, the orthogonal matrix in the upper left block of size $p-q$ must have determinant 1. Therefore M is isomorphic to the product of $SO(p-q)$ and the product of $q-1$ groups of order 2.

Lemma 6.56. The Lie algebra of $N_K(\mathfrak{a})$ is $\mathfrak{m}$. Therefore $W(G, A)$ is a finite group.

PROOF. The second conclusion follows from the first, since the first conclusion implies that $W(G, A)$ is 0-dimensional and compact, hence finite. For the first conclusion, the Lie algebra in question is $N_{\mathfrak{k}}(\mathfrak{a})$. Let $X = H_0 + X_0 + \sum_{\lambda\in\Sigma} X_\lambda$ be a member of $N_{\mathfrak{k}}(\mathfrak{a})$, with $H_0 \in \mathfrak{a}$, $X_0 \in \mathfrak{m}$, and $X_\lambda \in \mathfrak{g}_\lambda$. Since X is to be in $\mathfrak{k}$, θ must fix X, and we see that X may be rewritten as $X = X_0 + \sum_{\lambda\in\Sigma^+}(X_\lambda + \theta X_\lambda)$. When we apply $\operatorname{ad} H$ for $H \in \mathfrak{a}$, we obtain $[H, X] = \sum_{\lambda\in\Sigma^+} \lambda(H)(X_\lambda - \theta X_\lambda)$. This element is supposed to be in $\mathfrak{a}$, since we started with X in the normalizer of $\mathfrak{a}$, and that means $[H, X]$ is 0. But then $X_\lambda = 0$ for all λ, and X reduces to the member X_0 of $\mathfrak{m}$.

Theorem 6.57. The group $W(G, A)$ coincides with $W(\Sigma)$.

REMARK. This theorem should be compared with Theorem 4.54.

PROOF. Let us observe that $W(G, A)$ permutes the restricted roots. In fact, let k be in $N_K(\mathfrak{a})$, let λ be in Σ, and let E_λ be in $\mathfrak{g}_\lambda$. Then

$$\begin{aligned}[H, \operatorname{Ad}(k)E_\lambda] &= \operatorname{Ad}(k)[\operatorname{Ad}(k)^{-1}H, E_\lambda] = \operatorname{Ad}(k)(\lambda(\operatorname{Ad}(k)^{-1}H)E_\lambda)\\ &= \lambda(\operatorname{Ad}(k)^{-1}H)\operatorname{Ad}(k)E_\lambda = (k\lambda)(H)\operatorname{Ad}(k)E_\lambda\end{aligned}$$

shows that $k\lambda$ is in Σ and that $\operatorname{Ad}(k)E_\lambda$ is a restricted-root vector for $k\lambda$. Thus $W(G, A)$ permutes the restricted roots.

We have seen that $W(\Sigma) \subseteq W(G, A)$. Fix a simple system Σ^+ for Σ. In view of Theorem 2.63, it suffices to show that if $k \in N_K(\mathfrak{a})$ has $\operatorname{Ad}(k)\Sigma^+ = \Sigma^+$, then k is in $Z_K(\mathfrak{a})$.

The element $\operatorname{Ad}(k) = w$ acts as a permutation of Σ^+. Let 2δ denote the sum of the reduced members of Σ^+, so that w fixes δ. If λ_i is a simple restricted root, then Lemma 2.91 and Proposition 2.69 show that $2\langle\delta, \lambda_i\rangle/|\lambda_i|^2 = 1$. Therefore $\langle\delta, \lambda\rangle > 0$ for all $\lambda \in \Sigma^+$.

Let $\mathfrak{u} = \mathfrak{k} \oplus i\mathfrak{p}$ be the compact real form of $\mathfrak{g}^{\mathbb{C}}$ associated to θ, and let U be the adjoint group of $\mathfrak{u}$. Then $\operatorname{Ad}_{\mathfrak{g}^{\mathbb{C}}}(K) \subseteq U$, and in particular $\operatorname{Ad}(k)$ is a member of U. Form $S = \overline{\{\exp ir \operatorname{ad} H_\delta\}} \subseteq U$. Here S is a torus in U, and we let $\mathfrak{s}$ be the Lie algebra of S. The element $\operatorname{Ad}(k)$ is in $Z_U(S)$, and the claim is that every member of $Z_U(S)$ centralizes $\mathfrak{a}$. If so, then $\operatorname{Ad}(k)$ is 1 on $\mathfrak{a}$, and k is in $Z_K(\mathfrak{a})$, as required.

By Corollary 4.51 we can verify that $Z_U(S)$ centralizes $\mathfrak{a}$ by showing that $Z_{\mathfrak{u}}(\mathfrak{s})$ centralizes $\mathfrak{a}$. Here

$$Z_{\mathfrak{u}}(\mathfrak{s}) = \mathfrak{u} \cap Z_{\mathfrak{g}^{\mathbb{C}}}(\mathfrak{s}) = \mathfrak{u} \cap Z_{\mathfrak{g}^{\mathbb{C}}}(H_\delta).$$

To evaluate the right side, we complexify the statement of Lemma 6.50. Since $\langle \lambda, \delta \rangle \neq 0$, the centralizer $Z_{\mathfrak{g}^{\mathbb{C}}}(H_\delta)$ is just $\mathfrak{a}^{\mathbb{C}} \oplus \mathfrak{m}^{\mathbb{C}}$. Therefore

$$Z_{\mathfrak{u}}(\mathfrak{s}) = \mathfrak{u} \cap (\mathfrak{a}^{\mathbb{C}} \oplus \mathfrak{m}^{\mathbb{C}}) = i\mathfrak{a} \oplus \mathfrak{m}.$$

Every member of the right side centralizes $\mathfrak{a}$, and the proof is complete.

6. Cartan Subalgebras

Proposition 6.47 showed that every real semisimple Lie algebra has a Cartan subalgebra. But as we shall see shortly, not all Cartan subalgebras are conjugate. In this section and the next we investigate the conjugacy classes of Cartan subalgebras and some of their relationships to each other.

We revert to the use of subscripted Gothic letters for real Lie algebras and to unsubscripted letters for complexifications. Let $\mathfrak{g}_0$ be a real semisimple Lie algebra, let θ be a Cartan involution, and let $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ be the corresponding Cartan decomposition. Let $\mathfrak{g}$ be the complexification of $\mathfrak{g}_0$, and write $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ for the complexification of the Cartan decomposition. Let B be any nondegenerate symmetric invariant bilinear form on $\mathfrak{g}_0$ such that $B(\theta X, \theta Y) = B(X, Y)$ and such that B_θ, defined by (6.13), is positive definite.

All Cartan subalgebras of $\mathfrak{g}_0$ have the same dimension, since their complexifications are Cartan subalgebras of $\mathfrak{g}$ and are conjugate via $\operatorname{Int}\mathfrak{g}$, according to Theorem 2.15.

Let $K = \operatorname{Int}_{\mathfrak{g}_0}(\mathfrak{k}_0)$. This subgroup of $\operatorname{Int}\mathfrak{g}_0$ is compact.

EXAMPLE. Let $G = SL(2, \mathbb{R})$ and $\mathfrak{g}_0 = \mathfrak{sl}(2, \mathbb{R})$. A Cartan subalgebra $\mathfrak{h}_0$ complexifies to a Cartan subalgebra of $\mathfrak{sl}(2, \mathbb{C})$ and therefore has dimension 1. Therefore let us consider which 1-dimensional subspaces $\mathbb{R}X$ of $\mathfrak{sl}(2, \mathbb{R})$ are Cartan subalgebras. The matrix X has trace 0, and we divide matters into cases according to the sign of $\det X$. If $\det X < 0$, then X has real eigenvalues μ and $-\mu$, and X is conjugate via $SL(2, \mathbb{R})$ to a diagonal matrix. Thus, for some $g \in SL(2, \mathbb{R})$,

$$\mathbb{R}X = \{\operatorname{Ad}(g)\mathbb{R}h\}.$$

where $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ as usual. The subspace $\mathbb{R}h$ is maximal abelian in $\mathfrak{g}_0$ and $\operatorname{ad} h$ acts diagonably on $\mathfrak{g}$ with eigenvectors h, e, f. Since (1.82) gives

$$\operatorname{ad}(\operatorname{Ad}(g)h) = \operatorname{Ad}(g)(\operatorname{ad} h)\operatorname{Ad}(g)^{-1},$$

$\operatorname{ad}(\operatorname{Ad}(g)h)$ acts diagonably with eigenvectors $\operatorname{Ad}(g)h$, $\operatorname{Ad}(g)e$, $\operatorname{Ad}(g)f$. Therefore $\mathbb{R}X$ is a Cartan subalgebra when $\det X < 0$, and it is conjugate via $\operatorname{Int}\mathfrak{g}_0$ to $\mathbb{R}h$.

If $\det X > 0$, then X has purely imaginary eigenvalues μ and $-\mu$, and X is conjugate via $SL(2,\mathbb{R})$ to a real multiple of ih_B, where

(6.58a) $$h_B = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}.$$

Thus, for some $g \in SL(2,\mathbb{R})$,

$$\mathbb{R}X = \{\operatorname{Ad}(g)\mathbb{R}ih_B\}.$$

The subspace $\mathbb{R}ih_B$ is maximal abelian in $\mathfrak{g}_0$ and $\operatorname{ad} ih_B$ acts diagonably on $\mathfrak{g}$ with eigenvectors h_B, e_B, f_B, where

(6.58b) $$e_B = \frac{1}{2}\begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix} \quad \text{and} \quad f_B = \frac{1}{2}\begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}.$$

Then $\operatorname{ad}(\operatorname{Ad}(g)ih_B)$ acts diagonably with eigenvectors $\operatorname{Ad}(g)h_B$, $\operatorname{Ad}(g)e_B$, $\operatorname{Ad}(g)f_B$. Therefore $\mathbb{R}X$ is a Cartan subalgebra when $\det X > 0$, and it is conjugate via $\operatorname{Int}\mathfrak{g}_0$ to $\mathbb{R}ih_B$.

If $\det X = 0$, then X has both eigenvalues equal to 0, and X is conjugate via $SL(2,\mathbb{R})$ to a real multiple of $e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Thus, for some $g \in SL(2,\mathbb{R})$,

$$\mathbb{R}X = \{\operatorname{Ad}(g)\mathbb{R}e\}.$$

The subspace $\mathbb{R}e$ is maximal abelian in $\mathfrak{g}_0$, but the element $\operatorname{ad} e$ does not act diagonably on $\mathfrak{g}$. It follows that $\operatorname{ad}(\operatorname{Ad}(g)e)$ does not act diagonably. Therefore $\mathbb{R}X$ is not a Cartan subalgebra when $\det X = 0$.

In the above example every Cartan subalgebra is conjugate either to $\mathbb{R}h$ or to $\mathbb{R}ih_B$, and these two are θ stable. We shall see in Proposition 6.59 that this kind of conjugacy remains valid for all real semisimple Lie algebras $\mathfrak{g}_0$.

Another feature of the above example is that the two Cartan subalgebras $\mathbb{R}h$ and $\mathbb{R}ih_B$ are not conjugate. In fact, h has nonzero real eigenvalues, and ih_B has nonzero purely imaginary eigenvalues, and thus the two cannot be conjugate.

Proposition 6.59. Any Cartan subalgebra $\mathfrak{h}_0$ of $\mathfrak{g}_0$ is conjugate via $\operatorname{Int}\mathfrak{g}_0$ to a θ stable Cartan subalgebra.

PROOF. Let $\mathfrak{h}$ be the complexification of $\mathfrak{h}_0$, and let σ be the conjugation of $\mathfrak{g}$ with respect to $\mathfrak{g}_0$. Let $\mathfrak{u}_0$ be the compact real form constructed from $\mathfrak{h}$ and other data in Theorem 6.11, and let τ be the conjugation of $\mathfrak{g}$ with respect to $\mathfrak{u}_0$. The construction of $\mathfrak{u}_0$ has the property that $\tau(\mathfrak{h}) = \mathfrak{h}$.

The conjugations σ and τ are involutions of $\mathfrak{g}^{\mathbb{R}}$, and τ is a Cartan involution by Proposition 6.14. Theorem 6.16 shows that the element φ of $\operatorname{Int}\mathfrak{g}^{\mathbb{R}} = \operatorname{Int}\mathfrak{g}$ given by $\varphi = ((\sigma\tau)^2)^{1/4}$ has the property that the Cartan involution $\widetilde{\eta} = \varphi\tau\varphi^{-1}$ of $\mathfrak{g}^{\mathbb{R}}$ commutes with σ. Since $\sigma(\mathfrak{h}) = \mathfrak{h}$ and $\tau(\mathfrak{h}) = \mathfrak{h}$, it follows that $\varphi(\mathfrak{h}) = \mathfrak{h}$. Therefore $\widetilde{\eta}(\mathfrak{h}) = \mathfrak{h}$.

Since $\widetilde{\eta}$ and σ commute, it follows that $\widetilde{\eta}(\mathfrak{g}_0) = \mathfrak{g}_0$. Since $\mathfrak{h}_0 = \mathfrak{h}\cap\mathfrak{g}_0$, we obtain $\widetilde{\eta}(\mathfrak{h}_0) = \mathfrak{h}_0$.

Put $\eta = \widetilde{\eta}|_{\mathfrak{g}_0}$, so that $\eta(\mathfrak{h}_0) = \mathfrak{h}_0$. Since $\widetilde{\eta}$ is the conjugation of $\mathfrak{g}$ with respect to the compact real form $\varphi(\mathfrak{u}_0)$, the proof of Corollary 6.18 shows that η is a Cartan involution of $\mathfrak{g}_0$. Corollary 6.19 shows that η and θ are conjugate via $\operatorname{Int}\mathfrak{g}_0$, say $\theta = \psi\eta\psi^{-1}$ with $\psi \in \operatorname{Int}\mathfrak{g}_0$. Then $\psi(\mathfrak{h}_0)$ is a Cartan subalgebra of $\mathfrak{g}_0$, and

$$\theta(\psi(\mathfrak{h}_0)) = \psi\eta\psi^{-1}\psi(\mathfrak{h}_0) = \psi(\eta\mathfrak{h}_0) = \psi(\mathfrak{h}_0),$$

shows that it is θ stable.

Thus it suffices to study θ stable Cartan subalgebras. When $\mathfrak{h}_0$ is θ stable, we can write $\mathfrak{h}_0 = \mathfrak{t}_0\oplus\mathfrak{a}_0$ with $\mathfrak{t}_0 \subseteq \mathfrak{k}_0$ and $\mathfrak{a}_0 \subseteq \mathfrak{p}_0$. By the same argument as for Corollary 6.49, roots of $(\mathfrak{g}, \mathfrak{h})$ are real valued on $\mathfrak{a}_0 \oplus i\,\mathfrak{t}_0$. Consequently the **compact dimension** $\dim\mathfrak{t}_0$ and the **noncompact dimension** $\dim\mathfrak{a}_0$ of $\mathfrak{h}_0$ are unchanged when $\mathfrak{h}_0$ is conjugated via $\operatorname{Int}\mathfrak{g}_0$ to another θ stable Cartan subalgebra.

We say that a θ stable Cartan subalgebra $\mathfrak{h}_0 = \mathfrak{t}_0 \oplus \mathfrak{a}_0$ is **maximally compact** if its compact dimension is as large as possible, **maximally noncompact** if its noncompact dimension is as large as possible. In $\mathfrak{sl}(2,\mathbb{R})$, $\mathbb{R}h$ is maximally noncompact, and $\mathbb{R}ih_B$ is maximally compact. In any case $\mathfrak{a}_0$ is an abelian subspace of $\mathfrak{p}_0$, and thus Proposition 6.47 implies that $\mathfrak{h}_0$ is maximally noncompact if and only if $\mathfrak{a}_0$ is a maximal abelian subspace of $\mathfrak{p}_0$.

Proposition 6.60. Let $\mathfrak{t}_0$ be a maximal abelian subspace of $\mathfrak{k}_0$. Then $\mathfrak{h}_0 = Z_{\mathfrak{g}_0}(\mathfrak{t}_0)$ is a θ stable Cartan subalgebra of $\mathfrak{g}_0$ of the form $\mathfrak{h}_0 = \mathfrak{t}_0 \oplus \mathfrak{a}_0$ with $\mathfrak{a}_0 \subseteq \mathfrak{p}_0$.

PROOF. The subalgebra $\mathfrak{h}_0$ is θ stable and hence is a vector-space direct sum $\mathfrak{h}_0 = \mathfrak{t}_0 \oplus \mathfrak{a}_0$, where $\mathfrak{a}_0 = \mathfrak{h}_0 \cap \mathfrak{p}_0$. Since $\mathfrak{h}_0$ is θ stable, Proposition 6.29 shows that it is reductive. By Corollary 1.56, $[\mathfrak{h}_0, \mathfrak{h}_0]$ is semisimple.

We have $[\mathfrak{h}_0, \mathfrak{h}_0] = [\mathfrak{a}_0, \mathfrak{a}_0]$, and $[\mathfrak{a}_0, \mathfrak{a}_0] \subseteq \mathfrak{t}_0$ since $\mathfrak{a}_0 \subseteq \mathfrak{p}_0$ and $\mathfrak{h}_0 \cap \mathfrak{k}_0 = \mathfrak{t}_0$. Thus the semisimple Lie algebra $[\mathfrak{h}_0, \mathfrak{h}_0]$ is abelian and must be 0. Consequently $\mathfrak{h}_0$ is abelian.

It is clear that $\mathfrak{h} = (\mathfrak{h}_0)^{\mathbb{C}}$ is maximal abelian in $\mathfrak{g}$, and $\operatorname{ad}\mathfrak{h}_0$ is certainly diagonable on $\mathfrak{g}$ since the members of $\operatorname{ad}_{\mathfrak{g}_0}(\mathfrak{t}_0)$ are skew adjoint, the members of $\operatorname{ad}_{\mathfrak{g}_0}(\mathfrak{a}_0)$ are self adjoint, and $\mathfrak{t}_0$ commutes with $\mathfrak{a}_0$. By Proposition 2.13, $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$, and hence $\mathfrak{h}_0$ is a Cartan subalgebra of $\mathfrak{g}_0$.

With any θ stable Cartan subalgebra $\mathfrak{h}_0 = \mathfrak{t}_0 \oplus \mathfrak{a}_0$, $\mathfrak{t}_0$ is an abelian subspace of $\mathfrak{k}_0$, and thus Proposition 6.60 implies that $\mathfrak{h}_0$ is maximally compact if and only if $\mathfrak{t}_0$ is a maximal abelian subspace of $\mathfrak{k}_0$.

Proposition 6.61. Among θ stable Cartan subalgebras $\mathfrak{h}_0$ of $\mathfrak{g}_0$, the maximally noncompact ones are all conjugate via K, and the maximally compact ones are all conjugate via K.

PROOF. Let $\mathfrak{h}_0$ and $\mathfrak{h}'_0$ be given Cartan subalgebras. In the first case, as we observed above, $\mathfrak{h}_0 \cap \mathfrak{p}_0$ and $\mathfrak{h}'_0 \cap \mathfrak{p}_0$ are maximal abelian in $\mathfrak{p}_0$, and Theorem 6.51 shows that there is no loss of generality in assuming that $\mathfrak{h}_0 \cap \mathfrak{p}_0 = \mathfrak{h}'_0 \cap \mathfrak{p}_0$. Thus $\mathfrak{h}_0 = \mathfrak{t}_0 \oplus \mathfrak{a}_0$ and $\mathfrak{h}'_0 = \mathfrak{t}'_0 \oplus \mathfrak{a}_0$, where $\mathfrak{a}_0$ is maximal abelian in $\mathfrak{p}_0$. Define $\mathfrak{m}_0 = Z_{\mathfrak{k}_0}(\mathfrak{a}_0)$. Then $\mathfrak{t}_0$ and $\mathfrak{t}'_0$ are in $\mathfrak{m}_0$ and are maximal abelian there. Let $M = Z_K(\mathfrak{a}_0)$. This is a compact subgroup of K with Lie algebra $\mathfrak{m}_0$, and we let M_0 be its identity component. Theorem 4.34 says that $\mathfrak{t}_0$ and $\mathfrak{t}'_0$ are conjugate via M_0, and this conjugacy clearly fixes $\mathfrak{a}_0$. Hence $\mathfrak{h}_0$ and $\mathfrak{h}'_0$ are conjugate via K.

In the second case, as we observed above, $\mathfrak{h}_0 \cap \mathfrak{k}_0$ and $\mathfrak{h}'_0 \cap \mathfrak{k}_0$ are maximal abelian in $\mathfrak{k}_0$, and Theorem 4.34 shows that there is no loss of generality in assuming that $\mathfrak{h}_0 \cap \mathfrak{k}_0 = \mathfrak{h}'_0 \cap \mathfrak{k}_0$. Then Proposition 6.60 shows that $\mathfrak{h}_0 = \mathfrak{h}'_0$, and the proof is complete.

If we examine the proof of the first part of Proposition 6.61 carefully, we find that we can adjust it to obtain root data that determine a Cartan subalgebra up to conjugacy. As a consequence there are only finitely many conjugacy classes of Cartan subalgebras.

Lemma 6.62. Let $\mathfrak{h}_0$ and $\mathfrak{h}'_0$ be θ stable Cartan subalgebras of $\mathfrak{g}_0$ such that $\mathfrak{h}_0 \cap \mathfrak{p}_0 = \mathfrak{h}'_0 \cap \mathfrak{p}_0$. Then $\mathfrak{h}_0$ and $\mathfrak{h}'_0$ are conjugate via K.

PROOF. Since the $\mathfrak{p}_0$ parts of the two Cartan subalgebras are the same and since Cartan subalgebras are abelian, the $\mathfrak{k}_0$ parts $\mathfrak{h}_0 \cap \mathfrak{k}_0$ and $\mathfrak{h}_0' \cap \mathfrak{k}_0$ are both contained in $\widetilde{\mathfrak{m}}_0 = Z_{\mathfrak{k}_0}(\mathfrak{h}_0 \cap \mathfrak{p}_0)$. The Cartan subalgebras are maximal abelian in $\mathfrak{g}_0$, and therefore $\mathfrak{h}_0 \cap \mathfrak{k}_0$ and $\mathfrak{h}_0' \cap \mathfrak{k}_0$ are both maximal abelian in $\widetilde{\mathfrak{m}}_0$. Let $\widetilde{M} = Z_K(\mathfrak{h}_0 \cap \mathfrak{p}_0)$. This is a compact Lie group with Lie algebra $\widetilde{\mathfrak{m}}_0$, and we let $\widetilde{M}_0$ be its identity component. Theorem 4.34 says that $\mathfrak{h}_0 \cap \mathfrak{k}_0$ and $\mathfrak{h}_0' \cap \mathfrak{k}_0$ are conjugate via $\widetilde{M}_0$, and this conjugacy clearly fixes $\mathfrak{h}_0 \cap \mathfrak{p}_0$. Hence $\mathfrak{h}_0$ and $\mathfrak{h}_0'$ are conjugate via K.

Lemma 6.63. Let $\mathfrak{a}_0$ be a maximal abelian subspace of $\mathfrak{p}_0$, and let Σ be the set of restricted roots of $(\mathfrak{g}_0, \mathfrak{a}_0)$. Suppose that $\mathfrak{h}_0$ is a θ stable Cartan subalgebra such that $\mathfrak{h}_0 \cap \mathfrak{p}_0 \subseteq \mathfrak{a}_0$. Let $\Sigma' = \{\lambda \in \Sigma \mid \lambda(\mathfrak{h}_0 \cap \mathfrak{p}_0) = 0\}$. Then $\mathfrak{h}_0 \cap \mathfrak{p}_0$ is the common kernel of all $\lambda \in \Sigma'$.

PROOF. Let $\mathfrak{a}_0'$ be the common kernel of all $\lambda \in \Sigma'$. Then $\mathfrak{h}_0 \cap \mathfrak{p}_0 \subseteq \mathfrak{a}_0'$, and we are to prove that equality holds. Since $\mathfrak{h}_0$ is maximal abelian in $\mathfrak{g}_0$, it is enough to prove that $\mathfrak{h}_0 + \mathfrak{a}_0'$ is abelian.

Let $\mathfrak{g}_0 = \mathfrak{a}_0 \oplus \mathfrak{m}_0 \oplus \bigoplus_{\lambda\in\Sigma} (\mathfrak{g}_0)_\lambda$ be the restricted-root space decomposition of $\mathfrak{g}_0$, and let $X = H_0 + X_0 + \sum_{\lambda\in\Sigma} X_\lambda$ be an element of $\mathfrak{g}_0$ that centralizes $\mathfrak{h}_0 \cap \mathfrak{p}_0$. Bracketing the formula for X with $H \in \mathfrak{h}_0 \cap \mathfrak{p}_0$, we obtain $0 = \sum_{\lambda\in\Sigma-\Sigma'} \lambda(H) X_\lambda$, from which we conclude that $\lambda(H)X_\lambda = 0$ for all $H \in \mathfrak{h}_0 \cap \mathfrak{p}_0$ and all $\lambda \in \Sigma - \Sigma'$. Since the λ's in $\Sigma - \Sigma'$ have $\lambda(\mathfrak{h}_0 \cap \mathfrak{p}_0)$ not identically 0, we see that $X_\lambda = 0$ for all $\lambda \in \Sigma - \Sigma'$. Thus any X that centralizes $\mathfrak{h}_0 \cap \mathfrak{p}_0$ is of the form

$$X = H_0 + X_0 + \sum_{\lambda\in\Sigma'} X_\lambda.$$

Since $\mathfrak{h}_0$ is abelian, the elements $X \in \mathfrak{h}_0$ are of this form, and $\mathfrak{a}_0'$ commutes with any X of this form. Hence $\mathfrak{h}_0 + \mathfrak{a}_0'$ is abelian, and the proof is complete.

Proposition 6.64. Up to conjugacy by $\operatorname{Int}\mathfrak{g}_0$, there are only finitely many Cartan subalgebras of $\mathfrak{g}_0$.

PROOF. Fix a maximal abelian subspace $\mathfrak{a}_0$ of $\mathfrak{p}_0$. Let $\mathfrak{h}_0$ be a Cartan subalgebra. Proposition 6.59 shows that we may assume that $\mathfrak{h}_0$ is θ stable, and Theorem 6.51 shows that we may assume that $\mathfrak{h}_0 \cap \mathfrak{p}_0$ is contained in $\mathfrak{a}_0$. Lemma 6.63 associates to $\mathfrak{h}_0$ a subset of the set Σ of restricted roots that determines $\mathfrak{h}_0 \cap \mathfrak{p}_0$, and Lemma 6.62 shows that $\mathfrak{h}_0 \cap \mathfrak{p}_0$ determines $\mathfrak{h}_0$ up to conjugacy. Hence the number of conjugacy classes of Cartan subalgebras is bounded by the number of subsets of Σ.

7. Cayley Transforms

The classification of real semisimple Lie algebras later in this chapter will use maximally compact Cartan subalgebras, but much useful information about a semisimple Lie algebra $\mathfrak{g}_0$ comes about from a maximally noncompact Cartan subalgebra. To correlate this information, we need to be able to track down the conjugacy via $\mathfrak{g} = (\mathfrak{g}_0)^{\mathbb{C}}$ of a maximally compact Cartan subalgebra and a maximally noncompact one.

Cayley transforms are one-step conjugacies of θ stable Cartan subalgebras whose iterates explicitly relate any θ stable Cartan subalgebra with any other. We develop Cayley transforms in this section and show that in favorable circumstances we can see past the step-by-step process to understand the composite conjugation all at once.

There are two kinds of Cayley transforms, essentially inverse to each other. They are modeled on what happens in $\mathfrak{sl}(2,\mathbb{R})$. In the case of $\mathfrak{sl}(2,\mathbb{R})$, we start with the standard basis h, e, f for $\mathfrak{sl}(2,\mathbb{C})$ as in (1.5), as well as the members h_B, e_B, f_B of $\mathfrak{sl}(2,\mathbb{C})$ defined in (6.58). The latter elements satisfy the familiar bracket relations

$$[h_B, e_B] = 2e_B, \quad [h_B, f_B] = -2f_B, \quad [e_B, f_B] = h_B.$$

The definitions of e_B and f_B make $e_B + f_B$ and $i(e_B - f_B)$ be in $\mathfrak{sl}(2,\mathbb{R})$, while $i(e_B + f_B)$ and $e_B - f_B$ are in $\mathfrak{su}(2)$. The first kind of Cayley transform within $\mathfrak{sl}(2,\mathbb{C})$ is the mapping

$$\operatorname{Ad}\left(\frac{\sqrt{2}}{2}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}\right) = \operatorname{Ad}(\exp \tfrac{\pi}{4}(f_B - e_B)),$$

which carries h_B, e_B, f_B to complex multiples of h, e, f and carries the Cartan subalgebra $\mathbb{R}\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ to $i\mathbb{R}\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. When generalized below, this Cayley transform will be called $\mathbf{c}_\beta$.

The second kind of Cayley transform within $\mathfrak{sl}(2,\mathbb{C})$ is the mapping

$$\operatorname{Ad}\left(\frac{\sqrt{2}}{2}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}\right) = \operatorname{Ad}(\exp i\, \tfrac{\pi}{4}(-f - e)),$$

which carries h, e, f to complex multiples of h_B, e_B, f_B and carries the Cartan subalgebra $\mathbb{R}\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ to $i\mathbb{R}\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. In view of the explicit formula for the matrices of the Cayley transforms, the two transforms are inverse to one another. When generalized below, this second Cayley transform will be called $\mathbf{d}_\alpha$.

The idea is to embed each of these constructions into constructions in the complexification of our underlying semisimple algebra that depend upon a single root of a special kind, leaving fixed the part of the Cartan subalgebra that is orthogonal to the embedded copy of $\mathfrak{sl}(2,\mathbb{C})$.

Turning to the case of a general real semisimple Lie algebra, we continue with the notation of the previous section. We extend the inner product B_θ on $\mathfrak{g}_0$ to a Hermitian inner product on $\mathfrak{g}$ by the definition

$$B_\theta(Z_1, Z_2) = -B(Z_1, \theta\overline{Z}_2),$$

where bar denotes the conjugation of $\mathfrak{g}$ with respect to $\mathfrak{g}_0$. In this expression θ and bar commute.

If $\mathfrak{h}_0 = \mathfrak{t}_0 \oplus \mathfrak{a}_0$ is a θ stable Cartan subalgebra of $\mathfrak{g}_0$, we have noted that roots of $(\mathfrak{g}, \mathfrak{h})$ are imaginary on $\mathfrak{t}_0$ and real on $\mathfrak{a}_0$. A root is **real** if it takes on real values on $\mathfrak{h}_0$ (i.e., vanishes on $\mathfrak{t}_0$), **imaginary** if it takes on purely imaginary values on $\mathfrak{h}_0$ (i.e., vanishes on $\mathfrak{a}_0$), and **complex** otherwise.

For any root α, $\theta\alpha$ is the root $\theta\alpha(H) = \alpha(\theta^{-1}H)$. To see that $\theta\alpha$ is a root, we let E_α be a nonzero root vector for α, and we calculate

$$[H, \theta E_\alpha] = \theta[\theta^{-1}H, E_\alpha] = \alpha(\theta^{-1}H)\theta E_\alpha = (\theta\alpha)(H)\theta E_\alpha.$$

If α is imaginary, then $\theta\alpha = \alpha$. Thus $\mathfrak{g}_\alpha$ is θ stable, and we have $\mathfrak{g}_\alpha = (\mathfrak{g}_\alpha \cap \mathfrak{k}) \oplus (\mathfrak{g}_\alpha \cap \mathfrak{p})$. Since $\mathfrak{g}_\alpha$ is 1-dimensional, $\mathfrak{g}_\alpha \subseteq \mathfrak{k}$ or $\mathfrak{g}_\alpha \subseteq \mathfrak{p}$. We call an imaginary root α **compact** if $\mathfrak{g}_\alpha \subseteq \mathfrak{k}$, **noncompact** if $\mathfrak{g}_\alpha \subseteq \mathfrak{p}$.

We introduce two kinds of Cayley transforms, starting from a given θ stable Cartan subalgebra:

(i) Using an imaginary noncompact root β, we construct a new Cartan subalgebra whose intersection with $\mathfrak{p}_0$ goes up by 1 in dimension.

(ii) Using a real root α, we construct a new Cartan subalgebra whose intersection with $\mathfrak{p}_0$ goes down by 1 in dimension.

First we give the construction that starts from a Cartan subalgebra $\mathfrak{h}_0$ and uses an imaginary noncompact root β. Let E_β be a nonzero root vector. Since β is imaginary, $\overline{E_\beta}$ is in $\mathfrak{g}_{-\beta}$. Since β is noncompact, we have

$$0 < B_\theta(E_\beta, E_\beta) = -B(E_\beta, \overline{\theta E_\beta}) = B(E_\beta, \overline{E_\beta}).$$

Thus we are allowed to normalize E_β to make $B(E_\beta, \overline{E_\beta})$ be any positive constant. We choose to make $B(E_\beta, \overline{E_\beta}) = 2/|\beta|^2$. From Lemma 2.18a we have

$$[E_\beta, \overline{E_\beta}] = B(E_\beta, \overline{E_\beta})H_\beta = 2|\beta|^{-2}H_\beta.$$

Put $H'_\beta = 2|\beta|^{-2}H_\beta$. Then we have the bracket relations

$$[H'_\beta, E_\beta] = 2E_\beta, \quad [H'_\beta, \overline{E_\beta}] = -2\overline{E_\beta}, \quad [E_\beta, \overline{E_\beta}] = H'_\beta.$$

Also the elements $E_\beta + \overline{E_\beta}$ and $i(E_\beta - \overline{E_\beta})$ are fixed by bar and hence are in $\mathfrak{g}_0$. In terms of our discussion above of $\mathfrak{sl}(2,\mathbb{C})$, the correspondence is

$$\begin{aligned} H'_\beta &\leftrightarrow \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \\ E_\beta &\leftrightarrow \tfrac{1}{2}\begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix} \\ \overline{E_\beta} &\leftrightarrow \tfrac{1}{2}\begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix} \\ \overline{E_\beta} - E_\beta &\leftrightarrow \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}. \end{aligned}$$

Define

$$\mathbf{c}_\beta = \operatorname{Ad}(\exp \tfrac{\pi}{4}(\overline{E_\beta} - E_\beta)) \tag{6.65a}$$

and

$$\mathfrak{h}'_0 = \mathfrak{g}_0 \cap \mathbf{c}_\beta(\mathfrak{h}) = \ker(\beta|_{\mathfrak{h}_0}) \oplus \mathbb{R}(E_\beta + \overline{E_\beta}). \tag{6.65b}$$

The vector E_β is not uniquely determined by the conditions on it, and both formulas (6.65) depend on the particular choice we make for E_β. To see that (6.65b) is valid, we can use infinite series to calculate that

$$\mathbf{c}_\beta(H'_\beta) = E_\beta + \overline{E_\beta} \tag{6.66a}$$

$$\mathbf{c}_\beta(E_\beta - \overline{E_\beta}) = E_\beta - \overline{E_\beta} \tag{6.66b}$$

$$\mathbf{c}_\beta(E_\beta + \overline{E_\beta}) = -H'_\beta. \tag{6.66c}$$

Then (6.66a) implies (6.65b).

Next we give the construction that starts from a Cartan subalgebra $\mathfrak{h}'_0$ and uses a real root α. Let E_α be a nonzero root vector. Since α is real, $\overline{E_\alpha}$ is in $\mathfrak{g}_\alpha$. Adjusting E_α, we may therefore assume that E_α is in $\mathfrak{g}_0$. Since α is real, θE_α is in $\mathfrak{g}_{-\alpha}$, and we know from Proposition 6.52a that $[E_\alpha, \theta E_\alpha] = B(E_\alpha, \theta E_\alpha)H_\alpha$ with $B(E_\alpha, \theta E_\alpha) < 0$. We normalize E_α by a real constant to make $B(E_\alpha, \theta E_\alpha) = -2/|\alpha|^2$, and put $H'_\alpha = 2|\alpha|^{-2}H_\alpha$. Then we have the bracket relations

$$[H'_\alpha, E_\alpha] = 2E_\alpha, \quad [H'_\alpha, \theta E_\alpha] = -2\theta E_\alpha, \quad [E_\alpha, \theta E_\alpha] = -H'_\alpha.$$

In terms of our discussion above of $\mathfrak{sl}(2,\mathbb{C})$, the correspondence is

$$H'_\alpha \leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
$$E_\alpha \leftrightarrow \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$
$$\theta E_\alpha \leftrightarrow \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$$
$$i(\theta E_\alpha - E_\alpha) \leftrightarrow \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}.$$

Define

$$\mathbf{d}_\alpha = \operatorname{Ad}(\exp i\,\tfrac{\pi}{4}(\theta E_\alpha - E_\alpha)) \tag{6.67a}$$

and

$$\mathfrak{h}_0 = \mathfrak{g}_0 \cap \mathbf{d}_\alpha(\mathfrak{h}') = \ker(\alpha|_{\mathfrak{h}'_0}) \oplus \mathbb{R}(E_\alpha + \theta E_\alpha). \tag{6.67b}$$

To see that (6.67b) is valid, we can use infinite series to calculate that

$$\mathbf{d}_\alpha(H'_\alpha) = i(E_\alpha + \theta E_\alpha) \tag{6.68a}$$
$$\mathbf{d}_\alpha(E_\alpha - \theta E_\alpha) = E_\alpha - \theta E_\alpha \tag{6.68b}$$
$$\mathbf{d}_\alpha(E_\alpha + \theta E_\alpha) = iH'_\alpha. \tag{6.68c}$$

Then (6.68a) implies (6.67b).

Proposition 6.69. The two kinds of Cayley transforms are essentially inverse to each other in the following senses:

(a) If β is a noncompact imaginary root, then in the computation of $\mathbf{d}_{\mathbf{c}_\beta(\beta)} \circ \mathbf{c}_\beta$ the root vector $E_{\mathbf{c}_\beta(\beta)}$ can be taken to be $i\mathbf{c}_\beta(E_\beta)$ and this choice makes the composition the identity.

(b) If α is a real root, then in the the computation of $\mathbf{c}_{\mathbf{d}_\alpha(\alpha)} \circ \mathbf{d}_\alpha$ the root vector $E_{\mathbf{d}_\alpha(\alpha)}$ can be taken to be $-i\mathbf{d}_\alpha(E_\alpha)$ and this choice makes the composition the identity.

PROOF.

(a) By (6.66),

$$\mathbf{c}_\beta(E_\beta) = \tfrac{1}{2}\mathbf{c}_\beta(E_\beta + \overline{E_\beta}) + \tfrac{1}{2}\mathbf{c}_\beta(E_\beta - \overline{E_\beta}) = -\tfrac{1}{2}H'_\beta + \tfrac{1}{2}(E_\beta - \overline{E_\beta}).$$

Both terms on the right side are in $i\mathfrak{g}_0$, and hence $i\mathbf{c}_\beta(E_\beta)$ is in $\mathfrak{g}_0$. Since H'_β is in $\mathfrak{k}$ while E_β and $\overline{E_\beta}$ are in $\mathfrak{p}$,

$$\theta\mathbf{c}_\beta(E_\beta) = -\tfrac{1}{2}H'_\beta - \tfrac{1}{2}(E_\beta - \overline{E_\beta}).$$

Put $E_{\mathbf{c}_\beta(\beta)} = i\mathbf{c}_\beta(E_\beta)$. From $B(E_\beta, \overline{E_\beta}) = 2/|\beta|^2$, we obtain

$$B(E_{\mathbf{c}_\beta(\beta)}, \theta E_{\mathbf{c}_\beta(\beta)}) = -2/|\beta|^2 = -2/|\mathbf{c}_\beta(\beta)|^2.$$

Thus $E_{\mathbf{c}_\beta(\beta)}$ is properly normalized. Then $\mathbf{d}_{\mathbf{c}_\beta(\beta)}$ becomes

$$\begin{aligned}\mathbf{d}_{\mathbf{c}_\beta(\beta)} &= \mathrm{Ad}(\exp i\tfrac{\pi}{4}(\theta E_{\mathbf{c}_\beta(\beta)} - E_{\mathbf{c}_\beta(\beta)}))\\ &= \mathrm{Ad}(\exp \tfrac{\pi}{4}(\mathbf{c}_\beta(E_\beta) - \theta\mathbf{c}_\beta(E_\beta)))\\ &= \mathrm{Ad}(\exp \tfrac{\pi}{4}(E_\beta - \overline{E_\beta})),\end{aligned}$$

and this is the inverse of

$$\mathbf{c}_\beta = \mathrm{Ad}(\exp \tfrac{\pi}{4}(\overline{E_\beta} - E_\beta)).$$

(b) By (6.68),

$$\mathbf{d}_\alpha(E_\alpha) = \tfrac{1}{2}\mathbf{d}_\alpha(E_\alpha + \theta E_\alpha) + \tfrac{1}{2}\mathbf{d}_\alpha(E_\alpha - \theta E_\alpha) = \tfrac{1}{2}iH'_\alpha + \tfrac{1}{2}(E_\alpha - \theta E_\alpha).$$

Since H'_α, E_α, and θE_α are in $\mathfrak{g}_0$,

$$\overline{\mathbf{d}_\alpha(E_\alpha)} = -\tfrac{1}{2}iH'_\alpha + \tfrac{1}{2}(E_\alpha - \theta E_\alpha).$$

Put $E_{\mathbf{d}_\alpha(\alpha)} = -i\mathbf{d}_\alpha(E_\alpha)$. From $B(E_\alpha, \theta E_\alpha) = -2/|\alpha|^2$, we obtain

$$B(E_{\mathbf{d}_\alpha(\alpha)}, \overline{E_{\mathbf{d}_\alpha(\alpha)}}) = 2/|\alpha|^2 = 2/|\mathbf{d}_\alpha(\alpha)|^2.$$

Thus $E_{\mathbf{d}_\alpha(\alpha)}$ is properly normalized. Then $\mathbf{c}_{\mathbf{d}_\alpha(\alpha)}$ becomes

$$\begin{aligned}\mathbf{c}_{\mathbf{d}_\alpha(\alpha)} &= \mathrm{Ad}(\exp \tfrac{\pi}{4}(\overline{E_{\mathbf{d}_\alpha(\alpha)}} - E_{\mathbf{d}_\alpha(\alpha)}))\\ &= \mathrm{Ad}(\exp i\tfrac{\pi}{4}(\mathbf{d}_\alpha(E_\alpha) + \overline{\mathbf{d}_\alpha(E_\alpha)}))\\ &= \mathrm{Ad}(\exp i\tfrac{\pi}{4}(E_\alpha - \theta E_\alpha)),\end{aligned}$$

and this is the inverse of

$$\mathbf{d}_\alpha = \mathrm{Ad}(\exp i\tfrac{\pi}{4}(\theta E_\alpha - E_\alpha)).$$

Proposition 6.70. Let $\mathfrak{h}_0$ be a θ stable Cartan subalgebra of $\mathfrak{g}_0$. Then there are no noncompact imaginary roots if and only if $\mathfrak{h}_0$ is maximally noncompact, and there are no real roots if and only if $\mathfrak{h}_0$ is maximally compact.

PROOF. The Cayley transform construction $\mathbf{c}_\beta$ tells us that if $\mathfrak{h}_0$ has a noncompact imaginary root β, then $\mathfrak{h}_0$ is not maximally noncompact. Similarly the Cayley transform construction $\mathbf{d}_\alpha$ tells us that if $\mathfrak{h}_0$ has a real root α, then $\mathfrak{h}_0$ is not maximally compact.

For the converses write $\mathfrak{h}_0 = \mathfrak{t}_0 \oplus \mathfrak{a}_0$, and let $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$ be the set of roots. Form the expansion

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta} \mathfrak{g}_\alpha. \tag{6.71}$$

Suppose there are no noncompact imaginary roots. Then

$$Z_{\mathfrak{g}}(\mathfrak{a}_0) = \mathfrak{h} \oplus \bigoplus_{\substack{\alpha\in\Delta,\\ \alpha \text{ imaginary}}} \mathfrak{g}_\alpha = \mathfrak{h} \oplus \bigoplus_{\substack{\alpha\in\Delta,\\ \alpha \text{ compact}\\ \text{imaginary}}} \mathfrak{g}_\alpha$$

and $$\mathfrak{p}_0 \cap Z_{\mathfrak{g}}(\mathfrak{a}_0) = \mathfrak{g}_0 \cap (\mathfrak{p} \cap Z_{\mathfrak{g}}(\mathfrak{a}_0)) = \mathfrak{g}_0 \cap (\mathfrak{p} \cap \mathfrak{h}) = \mathfrak{a}_0.$$

Hence $\mathfrak{a}_0$ is maximal abelian in $\mathfrak{p}_0$, and $\mathfrak{h}_0$ is maximally noncompact.

Suppose there are no real roots. From the expansion (6.71) we obtain

$$Z_{\mathfrak{g}}(\mathfrak{t}_0) = \mathfrak{h} \oplus \bigoplus_{\substack{\alpha\in\Delta,\\ \alpha \text{ real}}} \mathfrak{g}_\alpha = \mathfrak{h}$$

and $\mathfrak{k}_0 \cap Z_{\mathfrak{g}}(\mathfrak{t}_0) = \mathfrak{k}_0 \cap \mathfrak{h} = \mathfrak{t}_0$. Therefore $\mathfrak{t}_0$ is maximal abelian in $\mathfrak{k}_0$, and $\mathfrak{h}_0$ is maximally compact.

The Cayley transforms and the above propositions give us a method of finding all Cartan subalgebras up to conjugacy. In fact, if we start with a θ stable Cartan subalgebra, we can apply various Cayley transforms $\mathbf{c}_\beta$ as long as there are noncompact imaginary roots, and we know that the resulting Cartan subalgebra will be maximally noncompact when we are done. Consequently if we apply various Cayley transforms $\mathbf{d}_\alpha$ in the reverse order, starting from a maximally noncompact Cartan subalgebra, we obtain all Cartan subalgebras up to conjugacy.

Alternatively if we start with a θ stable Cartan subalgebra, we can apply various Cayley transforms $\mathbf{d}_\alpha$ as long as there are real roots, and we know that the resulting Cartan subalgebra will be maximally compact when we are done. Consequently if we apply various Cayley transforms $\mathbf{c}_\beta$ in the reverse order, starting from a maximally compact Cartan subalgebra, we obtain all Cartan subalgebras up to conjugacy.

EXAMPLE. Let $\mathfrak{g}_0 = \mathfrak{sp}(2,\mathbb{R})$ with θ given by negative transpose. We can take the Iwasawa $\mathfrak{a}_0$ to be the diagonal subalgebra

$$\mathfrak{a}_0 = \{\operatorname{diag}(s, t, -s, -t)\}.$$

Let f_1 and f_2 be the linear functionals on $\mathfrak{a}_0$ that give s and t on the indicated matrix. For this example, $\mathfrak{m}_0 = 0$. Thus Proposition 6.47 shows that $\mathfrak{a}_0$ is a maximally noncompact Cartan subalgebra. The roots are $\pm 2f_1$, $\pm 2f_2$, $\pm(f_1 + f_2)$, $\pm(f_1 - f_2)$. All of them are real. We begin with a $\mathbf{d}_\alpha$ type Cayley transform, noting that $\pm\alpha$ give the same thing. The data for $2f_1$ and $2f_2$ are conjugate within $\mathfrak{g}_0$, and so are the data for $f_1 + f_2$ and $f_1 - f_2$. So there are only two essentially different first steps, say $\mathbf{d}_{2f_2}$ and $\mathbf{d}_{f_1-f_2}$. After $\mathbf{d}_{2f_2}$, the only real roots are $\pm 2f_1$ (or more precisely $\mathbf{d}_{2f_2}(\pm 2f_1)$). A second Cayley transform $\mathbf{d}_{2f_1}$ leads to all roots imaginary, hence to a maximally compact Cartan subalgebra, and we can go no further. Similarly after $\mathbf{d}_{f_1-f_2}$, the only real roots are $\pm(f_1 + f_2)$, and the second Cayley transform $\mathbf{d}_{f_1+f_2}$ leads to all roots imaginary. A little computation shows that we have produced

$$\begin{pmatrix} s & 0 & 0 & 0 \\ 0 & t & 0 & 0 \\ 0 & 0 & -s & 0 \\ 0 & 0 & 0 & -t \end{pmatrix}, \quad \begin{pmatrix} s & 0 & 0 & 0 \\ 0 & 0 & 0 & \theta \\ 0 & 0 & -s & 0 \\ 0 & -\theta & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} t & \theta & 0 & 0 \\ -\theta & t & 0 & 0 \\ 0 & 0 & -t & \theta \\ 0 & 0 & -\theta & -t \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & \theta_1 & 0 \\ 0 & 0 & 0 & \theta_2 \\ -\theta_1 & 0 & 0 & 0 \\ 0 & -\theta_2 & 0 & 0 \end{pmatrix}.$$

The second Cartan subalgebra results from the first by applying $\mathbf{d}_{2f_2}$, the third results from the first by applying $\mathbf{d}_{f_1-f_2}$, and the fourth results from the first by applying $\mathbf{d}_{2f_1}\mathbf{d}_{2f_2}$.

As in the example, when we pass from $\mathfrak{h}_0'$ to $\mathfrak{h}_0$ by $\mathbf{d}_\alpha$, we can anticipate what roots will be real for $\mathfrak{h}_0$. What we need in order to do a succession of such Cayley transforms is a sequence of real roots that become imaginary one at a time. In other words, we can do a succession of such Cayley transforms with ease if we have an orthogonal sequence of real roots.

Similarly when we apply $\mathbf{c}_\alpha$ to pass from $\mathfrak{h}_0$ to $\mathfrak{h}_0'$, we can anticipate what roots will be imaginary for $\mathfrak{h}_0'$. But a further condition on a root beyond "imaginary" is needed to do a Cayley transform $\mathbf{c}_\alpha$; we need the imaginary root to be noncompact. The following proposition tells how to anticipate which imaginary roots are noncompact after a Cayley transform.

Proposition 6.72. Let α be a noncompact imaginary root. Let β be a root orthogonal to α, so that the α string containing β is symmetric about β. Let E_α and E_β be nonzero roots vectors for α and β, and normalize E_α as in the definition of the Cayley transform $\mathbf{c}_\alpha$.

(a) If $\beta \pm \alpha$ are not roots, then $\mathbf{c}_\alpha(E_\beta) = E_\beta$. Thus if β is imaginary, then β is compact if and only if $\mathbf{c}_\alpha(\beta)$ is compact.

(b) If $\beta \pm \alpha$ are roots, then $\mathbf{c}_\alpha(E_\beta) = \frac{1}{2}([\overline{E_\alpha}, E_\beta] - [E_\alpha, E_\beta])$. Thus if β is imaginary, then β is compact if and only if $\mathbf{c}_\alpha(\beta)$ is noncompact.

PROOF. Recall that $\mathbf{c}_\alpha = \mathrm{Ad}(\exp \frac{\pi}{4}(\overline{E_\alpha} - E_\alpha))$ with $[E_\alpha, \overline{E_\alpha}] = H'_\alpha$.

(a) In this case $\mathbf{c}_\alpha(E_\beta) = E_\beta$ clearly. If β is imaginary, then the equal vectors $\mathbf{c}_\alpha(E_\beta)$ and E_β are both in $\mathfrak{k}$ or both in $\mathfrak{p}$.

(b) Here we use Corollary 2.37 and Proposition 2.48g to calculate that

$$\begin{aligned}
\mathrm{ad}\,\tfrac{\pi}{4}(\overline{E_\alpha} - E_\alpha)E_\beta &= \tfrac{\pi}{4}([\overline{E_\alpha}, E_\beta] - [E_\alpha, E_\beta]) \\
\mathrm{ad}^2(\tfrac{\pi}{4}(\overline{E_\alpha} - E_\alpha))E_\beta &= -(\tfrac{\pi}{4})^2([E_\alpha, [\overline{E_\alpha}, E_\beta]] + [\overline{E_\alpha}, [E_\alpha, E_\beta]]) \\
&= -(\tfrac{\pi}{4})^2(2E_\beta + 2E_\beta) \\
&= -(\tfrac{\pi}{2})^2 E_\beta.
\end{aligned}$$

Then we have

$$\begin{aligned}
\mathbf{c}_\alpha(E_\beta) &= \sum_{n=0}^{\infty} \tfrac{1}{(2n)!}\mathrm{ad}^{2n}(\tfrac{\pi}{4}(\overline{E_\alpha} - E_\alpha))E_\beta \\
&\quad + \sum_{n=0}^{\infty} \tfrac{1}{(2n+1)!}\mathrm{ad}(\tfrac{\pi}{4}(\overline{E_\alpha} - E_\alpha))\mathrm{ad}^{2n}(\tfrac{\pi}{4}(\overline{E_\alpha} - E_\alpha))E_\beta \\
&= \sum_{n=0}^{\infty} \tfrac{1}{(2n)!}(-1)^n(\tfrac{\pi}{2})^{2n}E_\beta \\
&\quad + \sum_{n=0}^{\infty} \tfrac{1}{(2n+1)!}(-1)^n(\tfrac{\pi}{2})^{2n}(\tfrac{\pi}{4})([\overline{E_\alpha}, E_\beta] - [E_\alpha, E_\beta]) \\
&= (\cos\tfrac{\pi}{2})E_\beta + \tfrac{1}{2}(\sin\tfrac{\pi}{2})([\overline{E_\alpha}, E_\beta] - [E_\alpha, E_\beta]) \\
&= \tfrac{1}{2}([\overline{E_\alpha}, E_\beta] - [E_\alpha, E_\beta]).
\end{aligned}$$

If β is imaginary, then $\mathbf{c}_\alpha(E_\beta)$ is in $\mathfrak{k}$ if and only if E_β is in $\mathfrak{p}$ since E_α and $\overline{E_\alpha}$ are in $\mathfrak{p}$.

We say that two orthogonal roots α and β are **strongly orthogonal** if $\beta \pm \alpha$ are not roots. Proposition 6.72 indicates that we can do a succession of

Cayley transforms $\mathbf{c}_\beta$ with ease if we have a strongly orthogonal sequence of noncompact imaginary roots.

If α and β are orthogonal but not strongly orthogonal, then

$$|\beta \pm \alpha|^2 = |\beta|^2 + |\alpha|^2 \tag{6.73}$$

shows that there are at least two root lengths. Actually we must have $|\beta|^2 = |\alpha|^2$, since otherwise (6.73) would produce three root lengths, which is forbidden within a simple component of a reduced root system. Thus (6.73) becomes $|\beta \pm \alpha|^2 = 2|\alpha|^2$, and the simple component of the root system containing α and β has a double line in its Dynkin diagram. In other words, whenever the Dynkin diagram of the root system has no double line, then orthogonal roots are automatically strongly orthogonal.

8. Vogan Diagrams

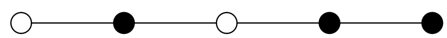
To a real semisimple Lie algebra $\mathfrak{g}_0$, in the presence of some other data, we shall associate a diagram consisting of the Dynkin diagram of $\mathfrak{g} = (\mathfrak{g}_0)^{\mathbb{C}}$ with some additional information superimposed. This diagram will be called a "Vogan diagram." We shall see that the same Vogan diagram cannot come from two nonisomorphic $\mathfrak{g}_0$'s and that every diagram that looks formally like a Vogan diagram comes from some $\mathfrak{g}_0$. Thus Vogan diagrams give us a handle on the problem of classification, and all we need to do is to sort out which Vogan diagrams come from the same $\mathfrak{g}_0$.

Let $\mathfrak{g}_0$ be a real semisimple Lie algebra, let $\mathfrak{g}$ be its complexification, let θ be a Cartan involution, let $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ be the corresponding Cartan decomposition, and let B be as in §§6–7. We introduce a maximally compact θ stable Cartan subalgebra $\mathfrak{h}_0 = \mathfrak{t}_0 \oplus \mathfrak{a}_0$ of $\mathfrak{g}_0$, with complexification $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}$, and we let $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$ be the set of roots. By Proposition 6.70 there are no real roots, i.e., no roots that vanish on $\mathfrak{t}$.

Choose a positive system Δ^+ for Δ that takes $i\mathfrak{t}_0$ before $\mathfrak{a}$. For example, Δ^+ can be defined in terms of a lexicographic ordering built from a basis of $i\mathfrak{t}_0$ followed by a basis of $\mathfrak{a}_0$. Since θ is $+1$ on $\mathfrak{t}_0$ and -1 on $\mathfrak{a}_0$ and since there are no real roots, $\theta(\Delta^+) = \Delta^+$. Therefore θ permutes the simple roots. It must fix the simple roots that are imaginary and permute in 2-cycles the simple roots that are complex.

By the **Vogan diagram** of the triple $(\mathfrak{g}_0, \mathfrak{h}_0, \Delta^+)$, we mean the Dynkin diagram of Δ^+ with the 2-element orbits under θ so labeled and with the 1-element orbits painted or not, according as the corresponding imaginary simple root is noncompact or compact.

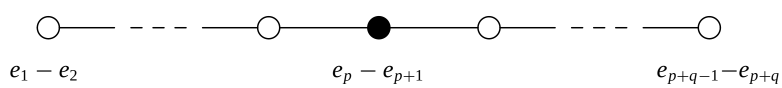
For example if $\mathfrak{g}_0 = \mathfrak{su}(3,3)$, let us take θ to be negative conjugate transpose, $\mathfrak{h}_0 = \mathfrak{t}_0$ to be the diagonal subalgebra, and Δ^+ to be determined by the conditions $e_1 \geq e_2 \geq e_4 \geq e_5 \geq e_3 \geq e_6$. The Dynkin diagram is of type A_5, and all simple roots are imaginary since $\mathfrak{a}_0 = 0$. In particular, θ acts as the identity in the Dynkin diagram. The compact roots $e_i - e_j$ are those with i and j in the same set $\{1, 2, 3\}$ or $\{4, 5, 6\}$, while the noncompact roots are those with i and j in opposite sets. Then among the simple roots, $e_1 - e_2$ is compact, $e_2 - e_4$ is noncompact, $e_4 - e_5$ is compact, $e_5 - e_3$ is noncompact, and $e_3 - e_6$ is noncompact. Hence the Vogan diagram is



Here are two infinite classes of examples.

EXAMPLES.

1) Let $\mathfrak{g}_0 = \mathfrak{su}(p, q)$ with negative conjugate transpose as Cartan involution. We take $\mathfrak{h}_0 = \mathfrak{t}_0$ to be the diagonal subalgebra. Then θ is 1 on all the roots. We use the standard ordering, so that the positive roots are $e_i - e_j$ with $i < j$. A positive root is compact if i and j are both in $\{1, \ldots, p\}$ or both in $\{p+1, \ldots, p+q\}$. It is noncompact if i is in $\{1, \ldots, p\}$ and j is in $\{p+1, \ldots, p+q\}$. Thus among the simple roots $e_i - e_{i+1}$, the root $e_p - e_{p+1}$ is noncompact, and the others are compact. The Vogan diagram is



2) Let $\mathfrak{g}_0 = \mathfrak{sl}(2n, \mathbb{R})$ with negative transpose as Cartan involution, and define

$$\mathfrak{h}_0 = \left\{ \begin{pmatrix} x_1 & \theta_1 & & & \\ -\theta_1 & x_1 & & & \\ & & \ddots & & \\ & & & x_n & \theta_n \\ & & & -\theta_n & x_n \end{pmatrix} \right\}.$$

The matrices here are understood to be built from 2-by-2 blocks and to have $\sum_{j=1}^n x_j = 0$. The subspace $\mathfrak{t}_0$ corresponds to the θ_j part, $1 \leq j \leq n$, i.e., it is the subspace where all x_j are 0. The subspace $\mathfrak{a}_0$ similarly corresponds to the x_j part, $1 \leq j \leq n$. We define linear functionals e_j and f_j to depend only on the j^{th} block, the dependence being

$$e_j\begin{pmatrix} x_j & -iy_j \\ iy_j & x_j \end{pmatrix} = y_j \quad \text{and} \quad f_j\begin{pmatrix} x_j & -iy_j \\ iy_j & x_j \end{pmatrix} = x_j.$$

Computation shows that

$$\Delta = \{\pm e_j \pm e_k \pm (f_j - f_k) \mid j \neq k\} \cup \{\pm 2e_l \mid 1 \leq l \leq n\}.$$

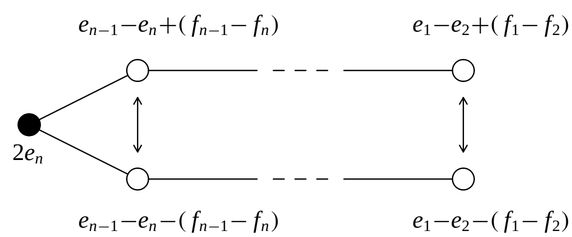
Roots that involve only e_j's are imaginary, those that involve only f_j's are real, and the remainder are complex. It is apparent that there are no real roots, and therefore $\mathfrak{h}_0$ is maximally compact. The involution θ acts as $+1$ on the e_j's and -1 on the f_j's. We define a lexicographic ordering by using the spanning set

$$e_1, \ldots, e_n,\ f_1, \ldots, f_n,$$

and we obtain

$$\Delta^+ = \begin{cases} e_j + e_k \pm (f_j - f_k), & \text{all } j \neq k \\ e_j - e_k \pm (f_j - f_k), & j < k \\ 2e_l, & 1 \leq l \leq n. \end{cases}$$

The Vogan diagram is



Theorem 6.74. Let $\mathfrak{g}_0$ and $\mathfrak{g}_0'$ be real semisimple Lie algebras. With notation as above, if two triples $(\mathfrak{g}_0, \mathfrak{h}_0, \Delta^+)$ and $(\mathfrak{g}_0', \mathfrak{h}_0', (\Delta')^+)$ have the same Vogan diagram, then $\mathfrak{g}_0$ and $\mathfrak{g}_0'$ are isomorphic.

REMARK. This theorem is an analog for real semisimple Lie algebras of the Isomorphism Theorem (Theorem 2.108) for complex semisimple Lie algebras.

PROOF. Since the Dynkin diagrams are the same, the Isomorphism Theorem (Theorem 2.108) shows that there is no loss of generality in assuming that $\mathfrak{g}_0$ and $\mathfrak{g}_0'$ have the same complexification $\mathfrak{g}$. Let $\mathfrak{u}_0 = \mathfrak{k}_0 \oplus i\mathfrak{p}_0$

and $\mathfrak{u}_0' = \mathfrak{k}_0' \oplus i\mathfrak{p}_0'$ be the associated compact real forms of $\mathfrak{g}$. By Corollary 6.20, there exists $x \in \operatorname{Int}\mathfrak{g}$ such that $x\mathfrak{u}_0' = \mathfrak{u}_0$. The real form $x\mathfrak{g}_0'$ of $\mathfrak{g}$ is isomorphic to $\mathfrak{g}_0'$ and has Cartan decomposition $x\mathfrak{g}_0' = x\mathfrak{k}_0' \oplus x\mathfrak{p}_0'$. Since $x\mathfrak{k}_0' \oplus ix\mathfrak{p}_0' = x\mathfrak{u}_0' = \mathfrak{u}_0$, there is no loss of generality in assuming that $\mathfrak{u}_0' = \mathfrak{u}_0$ from the outset. Then

$$\theta(\mathfrak{u}_0) = \mathfrak{u}_0 \quad \text{and} \quad \theta'(\mathfrak{u}_0) = \mathfrak{u}_0. \tag{6.75}$$

Let us write the effect of the Cartan decompositions on the Cartan subalgebras as $\mathfrak{h}_0 = \mathfrak{t}_0 \oplus \mathfrak{a}_0$ and $\mathfrak{h}_0' = \mathfrak{t}_0' \oplus \mathfrak{a}_0'$. Then $\mathfrak{t}_0 \oplus i\mathfrak{a}_0$ and $\mathfrak{t}_0' \oplus i\mathfrak{a}_0'$ are maximal abelian subspaces of $\mathfrak{u}_0$. By Theorem 4.34 there exists $k \in \operatorname{Int}\mathfrak{u}_0$ with $k(\mathfrak{t}_0' \oplus i\mathfrak{a}_0') = \mathfrak{t}_0 \oplus i\mathfrak{a}_0$. Replacing $\mathfrak{g}_0'$ by $k\mathfrak{g}_0'$ and arguing as above, we may assume that $\mathfrak{t}_0' \oplus i\mathfrak{a}_0' = \mathfrak{t}_0 \oplus i\mathfrak{a}_0$ from the outset. Therefore $\mathfrak{h}_0$ and $\mathfrak{h}_0'$ have the same complexification, which we denote $\mathfrak{h}$. The space

$$\mathfrak{u}_0 \cap \mathfrak{h} = \mathfrak{t}_0 \oplus i\mathfrak{a}_0 = \mathfrak{t}_0' \oplus i\mathfrak{a}_0'$$

is a maximal abelian subspace of $\mathfrak{u}_0$.

Now that the complexifications $\mathfrak{g}$ and $\mathfrak{h}$ have been aligned, the root systems are the same. Let the positive systems given in the respective triples be Δ^+ and $\Delta^{+\prime}$. By Theorems 4.54 and 2.63 there exists $k' \in \operatorname{Int}\mathfrak{u}_0$ normalizing $\mathfrak{u}_0 \cap \mathfrak{h}$ with $k'\Delta^{+\prime} = \Delta^+$. Replacing $\mathfrak{g}_0'$ by $k'\mathfrak{g}_0'$ and arguing as above, we may assume that $\Delta^{+\prime} = \Delta^+$ from the outset.

The next step is to choose normalizations of root vectors relative to $\mathfrak{h}$. For this proof let B be the Killing form of $\mathfrak{g}$. We start with root vectors X_α produced from $\mathfrak{h}$ as in Theorem 6.6. Using (6.12), we construct a compact real form $\widetilde{\mathfrak{u}_0}$ of $\mathfrak{g}$. The subalgebra $\widetilde{\mathfrak{u}_0}$ contains the real subspace of $\mathfrak{h}$ where the roots are imaginary, which is just $\mathfrak{u}_0 \cap \mathfrak{h}$. By Corollary 6.20, there exists $g \in \operatorname{Int}\mathfrak{g}$ such that $g\widetilde{\mathfrak{u}_0} = \mathfrak{u}_0$. Then $g\widetilde{\mathfrak{u}_0} = \mathfrak{u}_0$ is built by (6.12) from $g(\mathfrak{u}_0 \cap \mathfrak{h})$ and the root vectors gX_α. Since $\mathfrak{u}_0 \cap \mathfrak{h}$ and $g(\mathfrak{u}_0 \cap \mathfrak{h})$ are maximal abelian in $\mathfrak{u}_0$, Theorem 4.34 produces $u \in \operatorname{Int}\mathfrak{u}_0$ with $ug(\mathfrak{u}_0 \cap \mathfrak{h}) = \mathfrak{u}_0 \cap \mathfrak{h}$. Then $\mathfrak{u}_0$ is built by (6.12) from $ug(\mathfrak{u}_0 \cap \mathfrak{h})$ and the root vectors ugX_α. For $\alpha \in \Delta$, put $Y_\alpha = ugX_\alpha$. Then we have established that

$$\mathfrak{u}_0 = \sum_{\alpha\in\Delta} \mathbb{R}(iH_\alpha) + \sum_{\alpha\in\Delta} \mathbb{R}(Y_\alpha - Y_{-\alpha}) + \sum_{\alpha\in\Delta} \mathbb{R}i(Y_\alpha + Y_{-\alpha}). \tag{6.76}$$

We have not yet used the information that is superimposed on the Dynkin diagram of Δ^+. Since the automorphisms of Δ^+ defined by θ and θ' are the same, θ and θ' have the same effect on $\mathfrak{h}^*$. Thus

$$\theta(H) = \theta'(H) \qquad \text{for all } H \in \mathfrak{h}. \tag{6.77}$$

If α is an imaginary simple root, then

(6.78a) $$\theta(Y_\alpha) = Y_\alpha = \theta'(Y_\alpha) \qquad \text{if } \alpha \text{ is unpainted,}$$

(6.78b) $$\theta(Y_\alpha) = -Y_\alpha = \theta'(Y_\alpha) \qquad \text{if } \alpha \text{ is painted.}$$

We still have to deal with the complex simple roots. For $\alpha \in \Delta$, write $\theta Y_\alpha = a_\alpha Y_{\theta\alpha}$. From (6.75) we know that

$$\theta(\mathfrak{u}_0 \cap \operatorname{span}\{Y_\alpha, Y_{-\alpha}\}) \subseteq \mathfrak{u}_0 \cap \operatorname{span}\{Y_{\theta\alpha}, Y_{-\theta\alpha}\}.$$

In view of (6.76) this inclusion means that

$$\theta(\mathbb{R}(Y_\alpha - Y_{-\alpha}) + \mathbb{R}i(Y_\alpha + Y_{-\alpha})) \subseteq \mathbb{R}(Y_{\theta\alpha} - Y_{-\theta\alpha}) + \mathbb{R}i(Y_{\theta\alpha} + Y_{-\theta\alpha}).$$

If x and y are real and if $z = x + yi$, then we have

$$x(Y_\alpha - Y_{-\alpha}) + yi(Y_\alpha + Y_{-\alpha}) = zY_\alpha - \bar{z}Y_{-\alpha}.$$

Thus the expression $\theta(zY_\alpha - \bar{z}Y_{-\alpha}) = za_\alpha Y_{\theta\alpha} - \bar{z}a_{-\alpha}Y_{-\theta\alpha}$ must be of the form $wY_{\theta\alpha} - \bar{w}Y_{-\theta\alpha}$, and we conclude that

(6.79) $$a_{-\alpha} = \overline{a_\alpha}.$$

Meanwhile

(6.80) $$a_\alpha a_{-\alpha} = B(a_\alpha Y_{\theta\alpha}, a_{-\alpha}Y_{-\theta\alpha}) = B(\theta Y_\alpha, \theta Y_{-\alpha}) = B(Y_\alpha, Y_{-\alpha}) = 1.$$

Combining (6.79) and (6.80), we see that

(6.81) $$|a_\alpha| = 1.$$

Next we observe that

(6.82) $$a_\alpha a_{\theta\alpha} = 1$$

since $Y_\alpha = \theta^2 Y_\alpha = \theta(a_\alpha Y_{\theta\alpha}) = a_\alpha a_{\theta\alpha} Y_\alpha$.

For each pair of complex simple roots α and $\theta\alpha$, choose square roots $a_\alpha^{1/2}$ and $a_{\theta\alpha}^{1/2}$ so that

(6.83) $$a_\alpha^{1/2} a_{\theta\alpha}^{1/2} = 1.$$

This is possible by (6.82).

Similarly write $\theta' Y_\alpha = b_\alpha Y_{\theta\alpha}$ with

$$|b_\alpha| = 1, \tag{6.84}$$

and define $b_\alpha^{1/2}$ and $b_{\theta\alpha}^{1/2}$ for α and $\theta\alpha$ simple so that

$$b_\alpha^{1/2} b_{\theta\alpha}^{1/2} = 1. \tag{6.85}$$

By (6.81) and (6.84), we can define H and H' in $\mathfrak{u}_0 \cap \mathfrak{h}$ by the conditions that $\alpha(H) = \alpha(H') = 0$ for α imaginary simple and

$$\begin{aligned} \exp\big(\tfrac{1}{2}\alpha(H)\big) &= a_\alpha^{1/2}, \quad \exp\big(\tfrac{1}{2}\theta\alpha(H)\big) = a_{\theta\alpha}^{1/2}, \\ \exp\big(\tfrac{1}{2}\alpha(H')\big) &= b_\alpha^{1/2}, \quad \exp\big(\tfrac{1}{2}\theta\alpha(H')\big) = b_{\theta\alpha}^{1/2} \end{aligned}$$

for α and $\theta\alpha$ complex simple.

We shall show that

$$\theta' \circ \mathrm{Ad}(\exp \tfrac{1}{2}(H - H')) = \mathrm{Ad}(\exp \tfrac{1}{2}(H - H')) \circ \theta. \tag{6.86}$$

In fact, the two sides of (6.86) are equal on $\mathfrak{h}$ and also on each X_α for α imaginary simple, by (6.77) and (6.78), since the Ad factor drops out from each side. If α is complex simple, then

$$\begin{aligned} \theta' \circ \mathrm{Ad}(\exp \tfrac{1}{2}(H - H'))Y_\alpha &= \theta'(e^{\frac{1}{2}\alpha(H-H')}Y_\alpha) \\ &= b_\alpha a_\alpha^{1/2} b_\alpha^{-1/2} Y_{\theta\alpha} \\ &= b_\alpha^{1/2} a_\alpha^{-1/2} \theta Y_\alpha \\ &= b_{\theta\alpha}^{-1/2} a_{\theta\alpha}^{1/2} \theta Y_\alpha \quad \text{by (6.83) and (6.85)} \\ &= \mathrm{Ad}(\exp \tfrac{1}{2}(H - H')) \circ \theta Y_\alpha. \end{aligned}$$

This proves (6.86).

Applying (6.86) to $\mathfrak{k}$ and then to $\mathfrak{p}$, we see that

$$\begin{aligned} \mathrm{Ad}(\exp \tfrac{1}{2}(H - H'))(\mathfrak{k}) &\subseteq \mathfrak{k}' \\ \mathrm{Ad}(\exp \tfrac{1}{2}(H - H'))(\mathfrak{p}) &\subseteq \mathfrak{p}', \end{aligned} \tag{6.87}$$

and then equality must hold in each line of (6.87). Since the element $\mathrm{Ad}(\exp \frac{1}{2}(H-H'))$ carries $\mathfrak{u}_0$ to itself, it must carry $\mathfrak{k}_0 = \mathfrak{u}_0 \cap \mathfrak{k}$ to $\mathfrak{k}_0' = \mathfrak{u}_0 \cap \mathfrak{k}'$ and $\mathfrak{p}_0 = \mathfrak{u}_0 \cap \mathfrak{p}$ to $\mathfrak{p}_0' = \mathfrak{u}_0 \cap \mathfrak{p}'$. Hence it must carry $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ to $\mathfrak{g}_0' = \mathfrak{k}_0' \oplus \mathfrak{p}_0'$. This completes the proof.

Now let us address the question of existence. We define an **abstract Vogan diagram** to be an abstract Dynkin diagram with two pieces of additional structure indicated: One is an automorphism of order 1 or 2 of the diagram, which is to be indicated by labeling the 2-element orbits. The other is a subset of the 1-element orbits, which is to be indicated by painting the vertices corresponding to the members of the subset. Every Vogan diagram is of course an abstract Vogan diagram.

Theorem 6.88. If an abstract Vogan diagram is given, then there exist a real semisimple Lie algebra $\mathfrak{g}_0$, a Cartan involution θ, a maximally compact θ stable Cartan subalgebra $\mathfrak{h}_0 = \mathfrak{t}_0 \oplus \mathfrak{a}_0$, and a positive system Δ^+ for $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$ that takes $i\mathfrak{t}_0$ before $\mathfrak{a}_0$ such that the given diagram is the Vogan diagram of $(\mathfrak{g}_0, \mathfrak{h}_0, \Delta^+)$.

REMARK. Briefly the theorem says that any abstract Vogan diagram comes from some $\mathfrak{g}_0$. Thus the theorem is an analog for real semisimple Lie algebras of the Existence Theorem (Theorem 2.111) for complex semisimple Lie algebras.

PROOF. By the Existence Theorem (Theorem 2.111) let $\mathfrak{g}$ be a complex semisimple Lie algebra with the given abstract Dynkin diagram as its Dynkin diagram, and let $\mathfrak{h}$ be a Cartan subalgebra (Theorem 2.9). Put $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$, and let Δ^+ be the positive system determined by the given data. Introduce root vectors X_α normalized as in Theorem 6.6, and define a compact real form $\mathfrak{u}_0$ of $\mathfrak{g}$ in terms of $\mathfrak{h}$ and the X_α by (6.12). The formula for $\mathfrak{u}_0$ is

$$\mathfrak{u}_0 = \sum_{\alpha\in\Delta} \mathbb{R}(iH_\alpha) + \sum_{\alpha\in\Delta} \mathbb{R}(X_\alpha - X_{-\alpha}) + \sum_{\alpha\in\Delta} \mathbb{R}i(X_\alpha + X_{-\alpha}). \tag{6.89}$$

The given data determine an automorphism θ of the Dynkin diagram, which extends linearly to $\mathfrak{h}^*$ and is isometric. Let us see that $\theta(\Delta) = \Delta$. It is enough to see that $\theta(\Delta^+) \subseteq \Delta$. We prove that $\theta(\Delta^+) \subseteq \Delta$ by induction on the level $\sum n_i$ of a positive root $\alpha = \sum n_i\alpha_i$. If the level is 1, then the root α is simple and we are given that $\theta\alpha$ is a simple root. Let $n > 1$, and assume inductively that $\theta\alpha$ is in Δ if $\alpha \in \Delta^+$ has level $< n$. Let α have level n. If we choose α_i simple with $\langle\alpha, \alpha_i\rangle > 0$, then $s_{\alpha_i}(\alpha)$ is a positive root β with smaller level than α. By inductive hypothesis, $\theta\beta$ and $\theta\alpha_i$ are in Δ. Since θ is isometric, $\theta\alpha = s_{\theta\alpha_i}(\theta\beta)$, and therefore $\theta\alpha$ is in Δ. This completes the induction. Thus $\theta(\Delta) = \Delta$.

We can then transfer θ to $\mathfrak{h}$, retaining the same name θ. Define θ on the root vectors X_α for simple roots by

$$\theta X_\alpha = \begin{cases} X_\alpha & \text{if } \alpha \text{ is unpainted and forms a 1-element orbit} \\ -X_\alpha & \text{if } \alpha \text{ is painted and forms a 1-element orbit} \\ X_{\theta\alpha} & \text{if } \alpha \text{ is in a 2-element orbit.} \end{cases}$$

By the Isomorphism Theorem (Theorem 2.108), θ extends to an automorphism of $\mathfrak{g}$ consistently with these definitions on $\mathfrak{h}$ and on the X_α's for α simple. The uniqueness in Theorem 2.108 implies that $\theta^2 = 1$.

The main step is to prove that $\theta\mathfrak{u}_0 = \mathfrak{u}_0$. Let B be the Killing form of $\mathfrak{g}$. For $\alpha \in \Delta$, define a constant a_α by $\theta X_\alpha = a_\alpha X_{\theta\alpha}$. Then $a_\alpha a_{-\alpha} = B(a_\alpha X_{\theta\alpha}, a_{-\alpha} X_{-\theta\alpha}) = B(\theta X_\alpha, \theta X_{-\alpha}) = B(X_\alpha, X_{-\alpha}) = 1$ shows that

$$a_\alpha a_{-\alpha} = 1. \tag{6.90}$$

We shall prove that

$$a_\alpha = \pm 1 \qquad \text{for all } \alpha \in \Delta. \tag{6.91}$$

To prove (6.91), it is enough because of (6.90) to prove the result for $\alpha \in \Delta^+$. We do so by induction on the level of α. If the level is 1, then $a_\alpha = \pm 1$ by definition. Thus it is enough to prove that if (6.91) holds for positive roots α and β and if $\alpha + \beta$ is a root, then it holds for $\alpha + \beta$. In the notation of Theorem 6.6, we have

$$\begin{aligned} \theta X_{\alpha+\beta} &= N_{\alpha,\beta}^{-1} \theta [X_\alpha, X_\beta] = N_{\alpha,\beta}^{-1} [\theta X_\alpha, \theta X_\beta] \\ &= N_{\alpha,\beta}^{-1} a_\alpha a_\beta [X_{\theta\alpha}, X_{\theta\beta}] = N_{\alpha,\beta}^{-1} N_{\theta\alpha,\theta\beta} a_\alpha a_\beta X_{\theta\alpha+\theta\beta}. \end{aligned}$$

Therefore

$$a_{\alpha+\beta} = N_{\alpha,\beta}^{-1} N_{\theta\alpha,\theta\beta} a_\alpha a_\beta.$$

Here $a_\alpha a_\beta = \pm 1$ by assumption, while Theorem 6.6 and the fact that θ is an automorphism of Δ say that $N_{\alpha,\beta}$ and $N_{\theta\alpha,\theta\beta}$ are real with

$$N_{\alpha,\beta}^2 = \tfrac{1}{2} q(1+p)|\alpha|^2 = \tfrac{1}{2} q(1+p)|\theta\alpha|^2 = N_{\theta\alpha,\theta\beta}^2.$$

Hence $a_{\alpha+\beta} = \pm 1$, and (6.91) is proved.

Let us see that

(6.92)

$$\theta(\mathbb{R}(X_\alpha - X_{-\alpha}) + \mathbb{R}i(X_\alpha + X_{-\alpha})) \subseteq \mathbb{R}(X_{\theta\alpha} - X_{-\theta\alpha}) + \mathbb{R}i(X_{\theta\alpha} + X_{-\theta\alpha}).$$

If x and y are real and if $z = x + yi$, then we have

$$x(X_\alpha - X_{-\alpha}) + yi(X_\alpha + X_{-\alpha}) = zX_\alpha - \bar{z}X_{-\alpha}.$$

Thus (6.92) amounts to the assertion that the expression

$$\theta(zX_\alpha - \bar{z}X_{-\alpha}) = za_\alpha X_{\theta\alpha} - \bar{z}a_{-\alpha}X_{-\theta\alpha}$$

is of the form $wX_{\theta\alpha} - \bar{w}X_{-\theta\alpha}$, and this follows from (6.91) and (6.90). Since θ carries roots to roots,

$$\theta\Big(\sum_{\alpha\in\Delta}\mathbb{R}(iH_\alpha)\Big) = \sum_{\alpha\in\Delta}\mathbb{R}(iH_\alpha). \tag{6.93}$$

Combining (6.92) and (6.93) with (6.89), we see that $\theta\mathfrak{u}_0 = \mathfrak{u}_0$.

Let $\mathfrak{k}$ and $\mathfrak{p}$ be the $+1$ and -1 eigenspaces for θ in $\mathfrak{g}$, so that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Since $\theta\mathfrak{u}_0 = \mathfrak{u}_0$, we have

$$\mathfrak{u}_0 = (\mathfrak{u}_0 \cap \mathfrak{k}) \oplus (\mathfrak{u}_0 \cap \mathfrak{p}).$$

Define $\mathfrak{k}_0 = \mathfrak{u}_0 \cap \mathfrak{k}$ and $\mathfrak{p}_0 = i(\mathfrak{u}_0 \cap \mathfrak{p})$, so that

$$\mathfrak{u}_0 = \mathfrak{k}_0 \oplus i\mathfrak{p}_0.$$

Since $\mathfrak{u}_0$ is a real form of $\mathfrak{g}$ as a vector space, so is

$$\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0.$$

Since $\theta\mathfrak{u}_0 = \mathfrak{u}_0$ and since θ is an involution, we have the bracket relations

$$[\mathfrak{k}_0, \mathfrak{k}_0] \subseteq \mathfrak{k}_0, \quad [\mathfrak{k}_0, \mathfrak{p}_0] \subseteq \mathfrak{p}_0, \quad [\mathfrak{p}_0, \mathfrak{p}_0] \subseteq \mathfrak{k}_0.$$

Therefore $\mathfrak{g}_0$ is closed under brackets and is a real form of $\mathfrak{g}$ as a Lie algebra. The involution θ is $+1$ on $\mathfrak{k}_0$ and is -1 on $\mathfrak{p}_0$; it is a Cartan involution of $\mathfrak{g}_0$ by the remarks following (6.26), since $\mathfrak{k}_0 \oplus i\mathfrak{p}_0 = \mathfrak{u}_0$ is compact.

Formula (6.93) shows that θ maps $\mathfrak{u}_0 \cap \mathfrak{h}$ to itself, and therefore

$$\begin{aligned}
\mathfrak{u}_0 \cap \mathfrak{h} &= (\mathfrak{u}_0 \cap \mathfrak{k} \cap \mathfrak{h}) \oplus (\mathfrak{u}_0 \cap \mathfrak{p} \cap \mathfrak{h}) \\
&= (\mathfrak{k}_0 \cap \mathfrak{h}) \oplus (i\mathfrak{p}_0 \cap \mathfrak{h}) \\
&= (\mathfrak{k}_0 \cap \mathfrak{h}) \oplus i(\mathfrak{p}_0 \cap \mathfrak{h}).
\end{aligned}$$

The abelian subspace $\mathfrak{u}_0 \cap \mathfrak{h}$ is a real form of $\mathfrak{h}$, and hence so is

$$\mathfrak{h}_0 = (\mathfrak{k}_0 \cap \mathfrak{h}) \oplus (\mathfrak{p}_0 \cap \mathfrak{h}).$$

The subspace $\mathfrak{h}_0$ is contained in $\mathfrak{g}_0$, and it is therefore a θ stable Cartan subalgebra of $\mathfrak{g}_0$.

A real root α relative to $\mathfrak{h}_0$ has the property that $\theta\alpha = -\alpha$. Since θ preserves positivity relative to Δ^+, there are no real roots. By Proposition 6.70, $\mathfrak{h}_0$ is maximally compact.

Let us verify that Δ^+ results from a lexicographic ordering that takes $i(\mathfrak{k}_0 \cap \mathfrak{h})$ before $\mathfrak{p}_0 \cap \mathfrak{h}$. Let $\{\beta_i\}_{i=1}^l$ be the set of simple roots of Δ^+ in 1-element orbits under θ, and let $\{\gamma_i, \theta\gamma_i\}_{i=1}^m$ be the set of simple roots of Δ^+ in 2-element orbits. Relative to basis $\{\alpha_i\}_{i=1}^{l+2m}$ consisting of all simple roots, let $\{\omega_i\}$ be the dual basis defined by $\langle \omega_i, \alpha_j\rangle = \delta_{ij}$. We shall write ω_{β_j} or ω_{γ_j} or $\omega_{\theta\gamma_j}$ in place of ω_i in what follows. We define a lexicographic ordering by using inner products with the ordered basis

$$\omega_{\beta_1}, \dots, \omega_{\beta_l}, \omega_{\gamma_1} + \omega_{\theta\gamma_1}, \dots, \omega_{\gamma_m} + \omega_{\theta\gamma_m}, \omega_{\gamma_1} - \omega_{\theta\gamma_1}, \dots, \omega_{\gamma_m} - \omega_{\theta\gamma_m},$$

which takes $i(\mathfrak{k}_0 \cap \mathfrak{h})$ before $\mathfrak{p}_0 \cap \mathfrak{h}$. Let α be in Δ^+, and write

$$\alpha = \sum_{i=1}^{l} n_i\beta_i + \sum_{j=1}^{m} r_j\gamma_j + \sum_{j=1}^{m} s_j\theta\gamma_j.$$

Then $$\langle \alpha, \omega_{\beta_j}\rangle = n_j \geq 0$$

and $$\langle \alpha, \omega_{\gamma_j} + \omega_{\theta\gamma_j}\rangle = r_j + s_j \geq 0.$$

If all these inner products are 0, then all coefficients of α are 0, contradiction. Thus α has positive inner product with the first member of our ordered basis for which the inner product is nonzero, and the lexicographic ordering yields Δ^+ as positive system. Consequently $(\mathfrak{g}_0, \mathfrak{h}_0, \Delta^+)$ is a triple.

Our definitions of θ on $\mathfrak{h}^*$ and on the X_α for α simple make it clear that the Vogan diagram of $(\mathfrak{g}_0, \mathfrak{h}_0, \Delta^+)$ coincides with the given data. This completes the proof.

9. Complexification of a Simple Real Lie Algebra

This section deals with some preliminaries for the classification of simple real Lie algebras. Our procedure in the next section is to start from a complex semisimple Lie algebra and pass to all possible real forms that are simple. In order to use this method effectively, we need to know what complex semisimple Lie algebras can arise in this way.

Theorem 6.94. Let $\mathfrak{g}_0$ be a simple Lie algebra over $\mathbb{R}$, and let $\mathfrak{g}$ be its complexification. Then there are just two possibilities:

(a) $\mathfrak{g}_0$ is complex, i.e., is of the form $\mathfrak{s}^{\mathbb{R}}$ for some complex $\mathfrak{s}$, and then $\mathfrak{g}$ is $\mathbb{C}$ isomorphic to $\mathfrak{s} \oplus \mathfrak{s}$,

(b) $\mathfrak{g}_0$ is not complex, and then $\mathfrak{g}$ is simple over $\mathbb{C}$.

PROOF.
(a) Let J be multiplication by $\sqrt{-1}$ in $\mathfrak{g}_0$, and define an $\mathbb{R}$ linear map $L : \mathfrak{g} \to \mathfrak{s} \oplus \mathfrak{s}$ by $L(X + iY) = (X + JY, X - JY)$ for X and Y in $\mathfrak{g}_0$. We readily check that L is one-one and respects brackets. Since the domain and range have the same real dimension, L is an $\mathbb{R}$ isomorphism.

Moreover L satisfies

$$\begin{aligned} L(i(X + iY)) &= L(-Y + iX) \\ &= (-Y + JX, -Y - JX) \\ &= (J(X + JY), -J(X - JY)). \end{aligned}$$

This equation exhibits L as a $\mathbb{C}$ isomorphism of $\mathfrak{g}$ with $\mathfrak{s} \oplus \bar{\mathfrak{s}}$, where $\bar{\mathfrak{s}}$ is the same real Lie algebra as $\mathfrak{g}_0$ but where the multiplication by $\sqrt{-1}$ is defined as multiplication by $-i$.

To complete the proof of (a), we show that $\bar{\mathfrak{s}}$ is $\mathbb{C}$ isomorphic to $\mathfrak{s}$. By Theorem 6.11, $\mathfrak{s}$ has a compact real form $\mathfrak{u}_0$. The conjugation τ of $\mathfrak{s}$ with respect to $\mathfrak{u}_0$ is $\mathbb{R}$ linear and respects brackets, and the claim is that τ is a $\mathbb{C}$ isomorphism of $\mathfrak{s}$ with $\bar{\mathfrak{s}}$. In fact, if U and V are in $\mathfrak{u}_0$, then

$$\begin{aligned} \tau(J(U + JV)) &= \tau(-V + JU) = -V - JU \\ &= -J(U - JV) = -J\tau(U + JV), \end{aligned}$$

and (a) follows.

(b) Let bar denote conjugation of $\mathfrak{g}$ with respect to $\mathfrak{g}_0$. If $\mathfrak{a}$ is a simple ideal in $\mathfrak{g}$, then $\mathfrak{a} \cap \bar{\mathfrak{a}}$ and $\mathfrak{a} + \bar{\mathfrak{a}}$ are ideals in $\mathfrak{g}$ invariant under conjugation and hence are complexifications of ideals in $\mathfrak{g}_0$. Thus they are 0 or $\mathfrak{g}$. Since $\mathfrak{a} \neq 0$, $\mathfrak{a} + \bar{\mathfrak{a}} = \mathfrak{g}$.

If $\mathfrak{a} \cap \bar{\mathfrak{a}} = 0$, then $\mathfrak{g} = \mathfrak{a} \oplus \bar{\mathfrak{a}}$. The inclusion of $\mathfrak{g}_0$ into $\mathfrak{g}$, followed by projection to $\mathfrak{a}$, is an $\mathbb{R}$ homomorphism φ of Lie algebras. If $\ker\varphi$ is nonzero, then $\ker\varphi$ must be $\mathfrak{g}_0$. In this case $\mathfrak{g}_0$ is contained in $\bar{\mathfrak{a}}$. But conjugation fixes $\mathfrak{g}_0$, and thus $\mathfrak{g}_0 \subseteq \mathfrak{a} \cap \bar{\mathfrak{a}} = 0$, contradiction. We conclude that φ is one-one. A count of dimensions shows that φ is an $\mathbb{R}$ isomorphism of $\mathfrak{g}_0$ onto $\mathfrak{a}$. But then $\mathfrak{g}_0$ is complex, contradiction.

We conclude that $\mathfrak{a} \cap \bar{\mathfrak{a}} = \mathfrak{g}$ and hence $\mathfrak{a} = \mathfrak{g}$. Therefore $\mathfrak{g}$ is simple, as asserted.

Proposition 6.95. If $\mathfrak{g}$ is a complex Lie algebra simple over $\mathbb{C}$, then $\mathfrak{g}^{\mathbb{R}}$ is simple over $\mathbb{R}$.

PROOF. Suppose that $\mathfrak{a}$ is an ideal in $\mathfrak{g}^{\mathbb{R}}$. Since $\mathfrak{g}^{\mathbb{R}}$ is semisimple, $[\mathfrak{a}, \mathfrak{g}^{\mathbb{R}}] \subseteq \mathfrak{a} = [\mathfrak{a}, \mathfrak{a}] \subseteq [\mathfrak{a}, \mathfrak{g}^{\mathbb{R}}]$. Therefore $\mathfrak{a} = [\mathfrak{a}, \mathfrak{g}^{\mathbb{R}}]$. Let X be in $\mathfrak{a}$, and write $X = \sum_j [X_j, Y_j]$ with $X_j \in \mathfrak{a}$ and $Y_j \in \mathfrak{g}$. Then

$$iX = \sum_j i[X_j, Y_j] = \sum [X_j, iY_j] \in [\mathfrak{a}, \mathfrak{g}^{\mathbb{R}}] = \mathfrak{a}.$$

So $\mathfrak{a}$ is a complex ideal in $\mathfrak{g}$. Since $\mathfrak{g}$ is complex simple, $\mathfrak{a} = 0$ or $\mathfrak{a} = \mathfrak{g}$. Thus $\mathfrak{g}^{\mathbb{R}}$ is simple over $\mathbb{R}$.

10. Classification of Simple Real Lie Algebras

Before taking up the problem of classification, a word of caution is in order. The virtue of classification is that it provides a clear indication of the scope of examples in the subject. It is rarely a sound idea to prove a theorem by proving it case-by-case for all simple real Lie algebras. Instead the important thing about classification is the techniques that are involved. Techniques that are subtle enough to identify all the examples are probably subtle enough to help in investigating all semisimple Lie algebras simultaneously.

Theorem 6.94 divided the simple real Lie algebras into two kinds, and we continue with that distinction in this section.

The first kind is a complex simple Lie algebra that is regarded as a real Lie algebra and remains simple when regarded that way. Proposition 6.95 shows that every complex simple Lie algebra may be used for this purpose. In view of the results of Chapter II, the classification of this kind is complete. We obtain complex Lie algebras of the usual types A_n through G_2. Matrix realizations of the complex Lie algebras of the classical types A_n through D_n are listed in (2.43).

The other kind is a noncomplex simple Lie algebra $\mathfrak{g}_0$, and its complexification is then simple over $\mathbb{C}$. Since the complexification is simple, any Vogan diagram for $\mathfrak{g}_0$ will have its underlying Dynkin diagram connected. Conversely any real semisimple Lie algebra $\mathfrak{g}_0$ with a Vogan diagram having connected Dynkin diagram has $(\mathfrak{g}_0)^{\mathbb{C}}$ simple, and therefore $\mathfrak{g}_0$ has to be simple. We know from Theorem 6.74 that the same Vogan diagram cannot come from nonisomorphic $\mathfrak{g}_0$'s, and we know from Theorem 6.88 that every abstract Vogan diagram is a Vogan diagram. Therefore the classification of this type of simple real Lie algebra comes down to classifying abstract Vogan diagrams whose underlying Dynkin diagram is connected.

Thus we want to eliminate the redundancy in connected Vogan diagrams. There is no redundancy from the automorphism. The only connected Dynkin diagrams admitting nontrivial automorphisms of order 2 are A_n, D_n, and E_6. In these cases a nontrivial automorphism of order 2 of the Dynkin diagram is unique up to an automorphism of the diagram (and is absolutely unique except in D_4). A Vogan diagram for $\mathfrak{g}_0$ incorporates a nontrivial automorphism of order 2 if and only if there exist complex roots, and this condition depends only on $\mathfrak{g}_0$.

The redundancy comes about through having many allowable choices for the positive system Δ^+. The idea, partly but not completely, is that we can always change Δ^+ so that at most one imaginary simple root is painted.

Theorem 6.96 (Borel and de Siebenthal Theorem). Let $\mathfrak{g}_0$ be a noncomplex simple real Lie algebra, and let the Vogan diagram of $\mathfrak{g}_0$ be given that corresponds to the triple $(\mathfrak{g}_0, \mathfrak{h}_0, \Delta^+)$. Then there exists a simple system Π' for $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$, with corresponding positive system $\Delta^{+\prime}$, such that $(\mathfrak{g}_0, \mathfrak{h}_0, \Delta^{+\prime})$ is a triple and there is at most one painted simple root in its Vogan diagram. Furthermore suppose that the automorphism associated with the Vogan diagram is the identity, that $\Pi' = \{\alpha_1, \dots, \alpha_l\}$, and that $\{\omega_1, \dots, \omega_l\}$ is the dual basis given by $\langle \omega_j, \alpha_k \rangle = \delta_{jk}$. Then the single painted simple root α_i may be chosen so that there is no i' with $\langle \omega_i - \omega_{i'}, \omega_{i'} \rangle > 0$.

REMARKS.

1) The proof will be preceded by two lemmas. The main conclusion of the theorem is that we can arrange that at most one simple root is painted. The second conclusion (concerning ω_i and therefore limiting which simple root can be painted) is helpful only when the Dynkin diagram is exceptional (E_6, E_7, E_8, F_4, or G_2).

2) The proof simplifies somewhat when the automorphism marked as part of the Vogan diagram is the identity. This is the case that $\mathfrak{h}_0$ is contained in $\mathfrak{k}_0$, and most examples will turn out to have this property.

Lemma 6.97. Let Δ be an irreducible abstract reduced root system in a real vector space V, let Π be a simple system, and let ω and ω' be nonzero members of V that are dominant relative to Π. Then $\langle \omega, \omega' \rangle > 0$.

PROOF. The first step is to show that in the expansion $\omega = \sum_{\alpha \in \Pi} a_\alpha \alpha$, all the a_α are ≥ 0. Let us enumerate Π as $\alpha_1, \dots, \alpha_l$ so that

$$\omega = \sum_{i=1}^{r} a_i \alpha_i - \sum_{i=r+1}^{s} b_i \alpha_i = \omega^+ - \omega^-$$

with all $a_i \geq 0$ and all $b_i > 0$. We shall show that $\omega^- = 0$. Since $\omega^- = \omega^+ - \omega$, we have

$$0 \leq |\omega^-|^2 = \langle \omega^+, \omega^- \rangle - \langle \omega^-, \omega \rangle = \sum_{i=1}^{r} \sum_{j=r+1}^{s} a_i b_j \langle \alpha_i, \alpha_j \rangle - \sum_{j=r+1}^{l} b_j \langle \omega, \alpha_j \rangle.$$

The first term on the right side is ≤ 0 by Lemma 2.51, and the second term on the right side (with the minus sign included) is term-by-term ≤ 0 by hypothesis. Therefore the right side is ≤ 0, and we conclude that $\omega^- = 0$.

Thus we can write $\omega = \sum_{j=1}^{l} a_j \alpha_j$ with all $a_j \geq 0$. The next step is to show from the irreducibility of Δ that $a_j > 0$ for all j. Assuming the contrary, suppose that $a_i = 0$. Then

$$0 \leq \langle \omega, \alpha_i \rangle = \sum_{j \neq i} a_j \langle \alpha_j, \alpha_i \rangle,$$

and every term on the right side is ≤ 0 by Lemma 2.51. Thus $a_j = 0$ for every α_j such that $\langle \alpha_j, \alpha_i \rangle < 0$, i.e., for all neighbors of α_i in the Dynkin diagram. Since the Dynkin diagram is connected (Proposition 2.54), iteration of this argument shows that all coefficients are 0 once one of them is 0.

Now we can complete the proof. For at least one index i, $\langle \alpha_i, \omega' \rangle > 0$, since $\omega' \neq 0$. Then

$$\langle \omega, \omega' \rangle = \sum_j a_j \langle \alpha_j, \omega' \rangle \geq a_i \langle \alpha_i, \omega' \rangle,$$

and the right side is > 0 since $a_i > 0$. This proves the lemma.

Lemma 6.98. Let $\mathfrak{g}_0$ be a noncomplex simple real Lie algebra, and let the Vogan diagram of $\mathfrak{g}_0$ be given that corresponds to the triple $(\mathfrak{g}_0, \mathfrak{h}_0, \Delta^+)$. Write $\mathfrak{h}_0 = \mathfrak{t}_0 \oplus \mathfrak{a}_0$ as usual. Let V be the span of the simple roots that are imaginary, let Δ_0 be the root system $\Delta \cap V$, let $\mathcal{H}$ be the subset of $i\mathfrak{t}_0$ paired with V, and let Λ be the subset of $\mathcal{H}$ where all roots of Δ_0 take integer values and all noncompact roots of Δ_0 take odd-integer values. Then Λ is nonempty. In fact, if $\alpha_1, \dots, \alpha_m$ is any simple system for Δ_0 and if $\omega_1, \dots, \omega_m$ in V are defined by $\langle \omega_j, \alpha_k \rangle = \delta_{jk}$, then the element

$$\omega = \sum_{\substack{i \text{ with } \alpha_i \\ \text{noncompact}}} \omega_i.$$

is in Λ.

PROOF. Fix a simple system $\alpha_1, \dots, \alpha_m$ for Δ_0, and let Δ_0^+ be the set of positive roots of Δ_0. Define $\omega_1, \dots, \omega_m$ by $\langle \omega_j, \alpha_k \rangle = \delta_{jk}$. If $\alpha = \sum_{i=1}^m n_i \alpha_i$ is a positive root of Δ_0, then $\langle \omega, \alpha \rangle$ is the sum of the n_i for which α_i is noncompact. This is certainly an integer.

We shall prove by induction on the level $\sum_{i=1}^m n_i$ that $\langle \omega, \alpha \rangle$ is even if α is compact, odd if α is noncompact. When the level is 1, this assertion is true by definition. In the general case, let α and β be in Δ_0^+ with $\alpha + \beta$ in Δ, and suppose that the assertion is true for α and β. Since the sum of the n_i for which α_i is noncompact is additive, we are to prove that imaginary roots satisfy

$$\begin{aligned} \text{compact} + \text{compact} &= \text{compact} \\ \text{compact} + \text{noncompact} &= \text{noncompact} \qquad (6.99) \\ \text{noncompact} + \text{noncompact} &= \text{compact}. \end{aligned}$$

But this is immediate from Corollary 2.35 and the bracket relations (6.24).

PROOF OF THEOREM 6.96. Define V, Δ_0, and Λ as in Lemma 6.98. Before we use Lemma 6.97, it is necessary to observe that the Dynkin diagram of Δ_0 is connected, i.e., that the roots in the Dynkin diagram of Δ fixed by the given automorphism form a connected set. There is no problem when the automorphism is the identity, and we observe the connectedness in the other cases one at a time by inspection.

Let $\Delta_0^+ = \Delta^+ \cap V$. The set Λ is discrete, being a subset of a lattice, and Lemma 6.98 has just shown that it is nonempty. Let H_0 be a member of Λ with norm as small as possible. By Proposition 2.67 we can choose a new positive system $\Delta_0^{+\prime}$ for Δ_0 that makes H_0 dominant. The main step is to show that

$$\text{at most one simple root of } \Delta_0^{+\prime} \text{ is painted.} \qquad (6.100)$$

Suppose $H_0 = 0$. If α is in Δ_0, then $\langle H_0, \alpha \rangle$ is 0 and is not an odd integer. By definition of Λ, α is compact. Thus all roots of Δ_0 are compact, and (6.100) is true.

Now suppose $H_0 \neq 0$. Let $\alpha_1, \dots, \alpha_m$ be the simple roots of Δ_0 relative to $\Delta_0^{+\prime}$, and define $\omega_1, \dots, \omega_m$ by $\langle \omega_j, \alpha_k \rangle = \delta_{jk}$. We can write $H_0 = \sum_{j=1}^m n_j \omega_j$ with $n_j = \langle H_0, \alpha_j \rangle$. The number n_j is an integer since H_0 is in Λ, and it is ≥ 0 since H_0 is dominant relative to $\Delta_0^{+\prime}$.

Since $H_0 \neq 0$, we have $n_i > 0$ for some i. Then $H_0 - \omega_i$ is dominant relative to $\Delta_0^{+\prime}$, and Lemma 6.97 shows that $\langle H_0 - \omega_i, \omega_i \rangle \geq 0$ with equality only if $H_0 = \omega_i$. If strict inequality holds, then the element $H_0 - 2\omega_i$ is in Λ and satisfies

$$|H_0 - 2\omega_i|^2 = |H_0|^2 - 4\langle H_0 - \omega_i, \omega_i\rangle < |H_0|^2,$$

in contradiction with the minimal-norm condition on H_0. Hence equality holds, and $H_0 = \omega_i$.

Since H_0 is in Λ, a simple root α_j in $\Delta_0^{+\prime}$ is noncompact only if $\langle H_0, \alpha_j\rangle$ is an odd integer. Since $\langle H_0, \alpha_j\rangle = 0$ for $j \neq i$, the only possible noncompact simple root in $\Delta_0^{+\prime}$ is α_i. This proves (6.100).

If the automorphism associated with the Vogan diagram is the identity, then (6.100) proves the first conclusion of the theorem. For the second conclusion we are assuming that $H_0 = \omega_i$; then an inequality $\langle \omega_i - \omega_{i'}, \omega_{i'}\rangle > 0$ would imply that

$$|H_0 - 2\omega_{i'}|^2 = |H_0|^2 - 4\langle \omega_i - \omega_{i'}, \omega_{i'}\rangle < |H_0|^2,$$

in contradiction with the minimal-norm condition on H_0.

To complete the proof of the theorem, we have to prove the first conclusion when the automorphism associated with the Vogan diagram is not the identity. Choose by Theorem 2.63 an element $s \in W(\Delta_0)$ with $\Delta_0^{+\prime} = s\Delta_0^+$, and define $\Delta^{+\prime} = s\Delta^+$. With $\mathfrak{h}_0 = \mathfrak{t}_0 \oplus \mathfrak{a}_0$ as usual, the element s maps $i\mathfrak{t}_0$ to itself. Since Δ^+ is defined by an ordering that takes $i\mathfrak{t}_0$ before $\mathfrak{a}_0$, so is $\Delta^{+\prime}$. Let the simple roots of Δ^+ be $\beta_1, \dots, \beta_l$ with $\beta_1, \dots, \beta_m$ in Δ_0. Then the simple roots of $\Delta^{+\prime}$ are $s\beta_1, \dots, s\beta_l$. Among these, $s\beta_1, \dots, s\beta_m$ are the simple roots $\alpha_1, \dots, \alpha_m$ of $\Delta_0^{+\prime}$ considered above, and (6.100) says that at most one of them is noncompact. The roots $s\beta_{m+1}, \dots, s\beta_l$ are complex since $\beta_{m+1}, \dots, \beta_l$ are complex and s carries complex roots to complex roots. Thus $\Delta^{+\prime}$ has at most one simple root that is noncompact imaginary. This completes the proof.

Now we can mine the consequences of the theorem. To each connected abstract Vogan diagram that survives the redundancy tests of Theorem 6.96, we associate a noncomplex simple real Lie algebra. If the underlying Dynkin diagram is classical, we find a known Lie algebra of matrices with that Vogan diagram, and we identify any isomorphisms among the Lie algebras obtained. If the underlying Dynkin digram is exceptional, we give the Lie algebra a name, and we eliminate any remaining redundancy.

As we shall see, the data at hand from a Vogan diagram for $\mathfrak{g}_0$ readily determine the Lie subalgebra $\mathfrak{k}_0$ in the Cartan decomposition $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$.

This fact makes it possible to decide which of the Lie algebras obtained are isomorphic to one another.

First suppose that the automorphism of the underlying Dynkin diagram is trivial. When no simple root is painted, then $\mathfrak{g}_0$ is a compact real form. For the classical Dynkin diagrams, the compact real forms are as follows:

(6.101)

Diagram	Compact Real Form
A_n	$\mathfrak{su}(n+1)$
B_n	$\mathfrak{so}(2n+1)$
C_n	$\mathfrak{sp}(n)$
D_n	$\mathfrak{so}(2n)$

For the situation in which one simple root is painted, we treat the classical Dynkin diagrams separately from the exceptional ones. Let us begin with the classical cases. For each classical Vogan diagram with just one simple root painted, we attach a known Lie algebra of matrices to that diagram. The result is that we are associating a Lie algebra of matrices to each simple root of each classical Dynkin diagram. We can assemble all the information for one Dynkin diagram in one picture by labeling each root of the Dynkin diagram with the associated Lie algebra of matrices. Those results are in Figure 6.1.

Verification of the information in Figure 6.1 is easy for the most part. For A_n, Example 1 in §8 gives the outcome, which is that $\mathfrak{su}(p,q)$ results when $p+q=n+1$ and the p^{th} simple root from the left is painted.

For B_n, suppose that $p+q = 2n+1$ and that p is even. Represent $\mathfrak{so}(p,q)$ by real matrices $\begin{pmatrix} a & b \\ b^* & d \end{pmatrix}$ with a and d skew symmetric. For $\mathfrak{h}_0$, we use block-diagonal matrices whose first n blocks are $\mathbb{R}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ of size 2-by-2 and whose last block is of size 1-by-1. With linear functionals on $(\mathfrak{h}_0)^{\mathbb{C}}$ as in Example 2 of §II.1 and with the positive system as in that example, the Vogan diagram is as indicated by Figure 6.1.

For C_n, the analysis for the first $n-1$ simple roots uses $\mathfrak{sp}(p,q)$ with $p+q=n$ in the same way that the analysis for A_n uses $\mathfrak{su}(p,q)$ with $p+q=n+1$. The analysis for the last simple root is different. For this case we take the Lie algebra to be $\mathfrak{sp}(n,\mathbb{R})$. Actually it is more convenient to use the isomorphic Lie algebra $\mathfrak{g}_0 = \mathfrak{su}(n,n)\cap\mathfrak{sp}(n,\mathbb{C})$, which is conjugate to $\mathfrak{sp}(n,\mathbb{R})$ by the matrix given in block form as $\dfrac{\sqrt{2}}{2}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$.

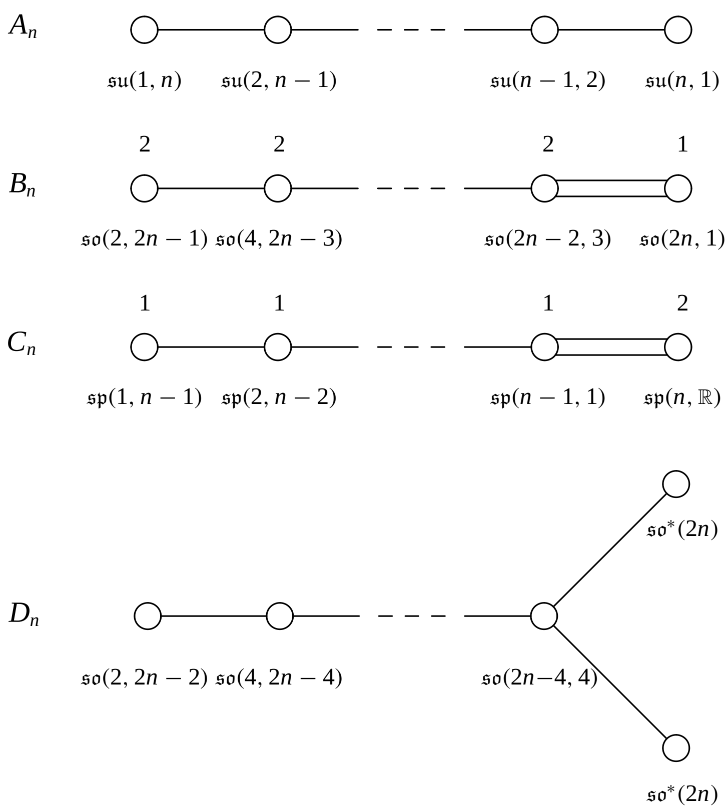


FIGURE 6.1. Association of classical matrix algebras to Vogan diagrams with the trivial automorphism

Within $\mathfrak{g}_0$, we take

$$\mathfrak{h}_0 = \{\operatorname{diag}(iy_1, \ldots, iy_n, -iy_1, \ldots, -iy_n)\}. \tag{6.102}$$

If we define e_j of the indicated matrix to be iy_j, then the roots are those of type C_n on (2.43), and we choose as positive system the customary one given in (2.50). The roots $e_i - e_j$ are compact, and the roots $\pm(e_i + e_j)$ and $\pm 2e_j$ are noncompact. Thus $2e_n$ is the unique noncompact simple root.

For D_n, the analysis for the first $n-2$ simple roots uses $\mathfrak{so}(p,q)$ with p and q even and $p+q=2n$. It proceeds in the same way as with B_n. The analysis for either of the last two simple roots is different. For one of the two simple roots we take $\mathfrak{g}_0 = \mathfrak{so}^*(2n)$. We use the same $\mathfrak{h}_0$ and e_j as in

(6.102). Then the roots are those of type D_n in (2.43), and we introduce the customary positive system (2.50). The roots $e_i - e_j$ are compact, and the roots $\pm(e_i + e_j)$ are noncompact. Thus $e_{n-1} + e_n$ is the unique noncompact simple root. The remaining Vogan diagram is isomorphic to the one we have just considered, and hence it too must correspond to $\mathfrak{so}^*(2n)$.

For the exceptional Dynkin diagrams we make use of the additional conclusion in Theorem 6.96; this says that we can disregard the case in which α_i is the unique simple noncompact root if $\langle \omega_i - \omega_{i'}, \omega_{i'} \rangle > 0$ for some i'. First let us see how to apply this test in practice. Write $\alpha_i = \sum_k d_{ik}\omega_k$. Taking the inner product with α_j shows that $d_{ij} = \langle \alpha_i, \alpha_j \rangle$. If we put $\omega_j = \sum_l c_{lj}\alpha_l$, then

$$\delta_{ij} = \langle \alpha_i, \omega_j \rangle = \sum_{k,l} d_{ik} c_{lj} \langle \omega_k, \alpha_l \rangle = \sum_k d_{ik} c_{kj}.$$

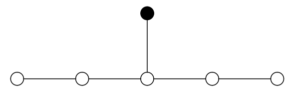
Thus the matrix (c_{ij}) is the inverse of the matrix (d_{ij}). Finally the quantity of interest is just $\langle \omega_j, \omega_{j'} \rangle = c_{j'j}$.

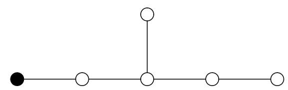
The Cartan matrix will serve as (d_{ij}) if all roots have the same length because we can assume that $|\alpha_i|^2 = 2$ for all i; then the coefficients c_{ij} are obtained by inverting the Cartan matrix. When there are two root lengths, (d_{ij}) is a simple modification of the Cartan matrix.

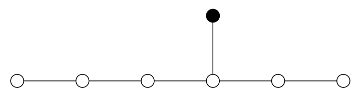
Appendix C gives all the information necessary to make the computations quickly. Let us indicate details for E_6. Let the simple roots be $\alpha_1, \dots, \alpha_6$ as in (2.86c). Then Appendix C gives

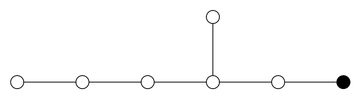
$$\begin{aligned}
\omega_1 &= \tfrac{1}{3}(4\alpha_1 + 3\alpha_2 + 5\alpha_3 + 6\alpha_4 + 4\alpha_5 + 2\alpha_6) \\
\omega_2 &= \phantom{\tfrac{1}{3}(}1\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + 1\alpha_6 \\
\omega_3 &= \tfrac{1}{3}(5\alpha_1 + 6\alpha_2 + 10\alpha_3 + 12\alpha_4 + 8\alpha_5 + 4\alpha_6) \\
\omega_4 &= \phantom{\tfrac{1}{3}(}2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 4\alpha_5 + 2\alpha_6 \\
\omega_5 &= \tfrac{1}{3}(4\alpha_1 + 6\alpha_2 + 8\alpha_3 + 12\alpha_4 + 10\alpha_5 + 5\alpha_6) \\
\omega_6 &= \tfrac{1}{3}(2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6).
\end{aligned}$$

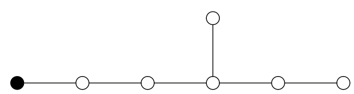
Let us use Theorem 6.96 to rule out $i = 3$, 4, and 5. For $i = 3$, we take $i' = 1$; we have $\langle \omega_3, \omega_1 \rangle = \frac{5}{3}$ and $\langle \omega_1, \omega_1 \rangle = \frac{4}{3}$, so that $\langle \omega_3 - \omega_1, \omega_1 \rangle > 0$. For $i = 4$, we take $i' = 1$; we have $\langle \omega_4, \omega_1 \rangle = 2$ and $\langle \omega_1, \omega_1 \rangle = \frac{4}{3}$, so that $\langle \omega_4 - \omega_1, \omega_1 \rangle > 0$. For $i = 5$, we take $i' = 6$; we have $\langle \omega_5, \omega_6 \rangle = \frac{5}{3}$ and $\langle \omega_6, \omega_6 \rangle = \frac{4}{3}$, so that $\langle \omega_5 - \omega_6, \omega_6 \rangle > 0$. Although there are six abstract

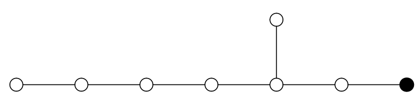
E II  $\mathfrak{k}_0 = \mathfrak{su}(6) \oplus \mathfrak{su}(2)$

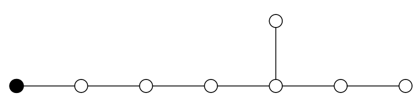
E III  $\mathfrak{k}_0 = \mathfrak{so}(10) \oplus \mathbb{R}$

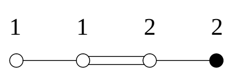
E V  $\mathfrak{k}_0 = \mathfrak{su}(8)$

E VI  $\mathfrak{k}_0 = \mathfrak{so}(12) \oplus \mathfrak{su}(2)$

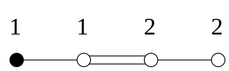
E VII  $\mathfrak{k}_0 = \mathfrak{e}_6 \oplus \mathbb{R}$

E VIII  $\mathfrak{k}_0 = \mathfrak{so}(16)$

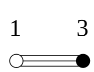
E IX  $\mathfrak{k}_0 = \mathfrak{e}_7 \oplus \mathfrak{su}(2)$

F I 

$\mathfrak{k}_0 = \mathfrak{sp}(3) \oplus \mathfrak{su}(2)$

F II 

$\mathfrak{k}_0 = \mathfrak{so}(9)$

G 

$\mathfrak{k}_0 = \mathfrak{su}(2) \oplus \mathfrak{su}(2)$

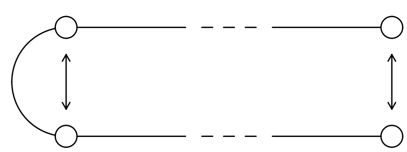
FIGURE 6.2. Noncompact noncomplex exceptional simple real Lie algebras with the trivial automorphism in the Vogan diagram

Vogan diagrams of E_6 with trivial automorphism and with one noncompact simple root, Theorem 6.96 says that we need to consider only the three where the simple root is α_1, α_2, or α_6. Evidently α_6 yields a result isomorphic to that for α_1 and may be disregarded.

By similar computations for the other exceptional Dynkin diagrams, we find that we may take α_i to be an endpoint vertex of the Dynkin diagram. Moreover, in G_2, α_i may be taken to be the long simple root, while in E_8, we do not have to consider α_2 (the endpoint vertex on the short branch). Thus we obtain the 10 Vogan diagrams in Figure 6.2. We have given each of them its name from the Cartan listing [1927a]. Computing $\mathfrak{k}_0$ is fairly easy. As a Lie algebra, $\mathfrak{k}_0$ is reductive by Corollary 4.25. The root system of its semisimple part is the system of compact roots, which we can compute from the Vogan diagram if we remember (6.99) and use the tables in Appendix C that tell which combinations of simple roots are roots. Then we convert the result into a compact Lie algebra using (6.101), and we add $\mathbb{R}$ as center if necessary to make the dimension of the Cartan subalgebra work out correctly. A glance at Figure 6.2 shows that when the Vogan diagrams for two $\mathfrak{g}_0$'s have the same underlying Dynkin diagram, then the $\mathfrak{k}_0$'s are different; by Corollary 6.19 the $\mathfrak{g}_0$'s are nonisomorphic.

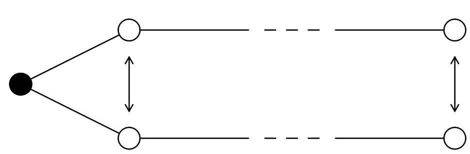
Now we suppose that the automorphism of the underlying Dynkin diagram is nontrivial. We already observed that the Dynkin diagram has to be of type A_n, D_n, or E_6.

For type A_n, we distinguish n even from n odd. For n even there is just one abstract Vogan diagram, namely

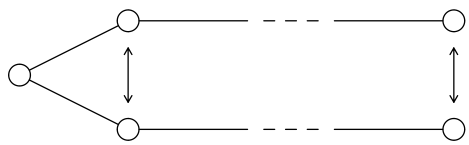


It must correspond to $\mathfrak{sl}(n+1, \mathbb{R})$ since we have not yet found a Vogan diagram for $\mathfrak{sl}(n+1, \mathbb{R})$ and since the equality $\mathfrak{sl}(n+1, \mathbb{R})^{\mathbb{C}} = \mathfrak{sl}(n+1, \mathbb{C})$ determines the underlying Dynkin diagram as being A_n.

For A_n with n odd, there are two abstract Vogan diagrams, namely



and



The first of these, according to Example 2 in §8, comes from $\mathfrak{sl}(n+1,\mathbb{R})$. The second one comes from $\mathfrak{sl}(\frac{1}{2}(n+1),\mathbb{H})$. In the latter case we take

$$\mathfrak{h}_0 = \left\{\operatorname{diag}(x_1+iy_1,\dots,x_{\frac{1}{2}(n+1)}+iy_{\frac{1}{2}(n+1)}) \,\middle|\, \sum x_m = 0\right\}.$$

If e_m and f_m on the indicated member of $\mathfrak{h}_0$ are iy_m and x_m, respectively, then Δ is the same as in Example 2 of §8. The imaginary roots are the $\pm 2e_m$, and they are compact. (The root vectors for $\pm 2e_m$ generate the complexification of the $\mathfrak{su}(2)$ in the j^{th} diagonal entry formed by the skew-Hermitian quaternions there.)

For type D_n, the analysis uses $\mathfrak{so}(p,q)$ with p and q odd and with $p+q=2n$. Represent $\mathfrak{so}(p,q)$ by real matrices $\begin{pmatrix} a & b \\ b^* & d\end{pmatrix}$ with a and d skew symmetric. For $\mathfrak{h}_0$, we use block-diagonal matrices with all blocks of size 2-by-2. The first $\frac{1}{2}(p-1)$ and the last $\frac{1}{2}(q-1)$ blocks are $\mathbb{R}\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}$, and the remaining one is $\mathbb{R}\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}$. The blocks $\mathbb{R}\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}$ contribute to $\mathfrak{t}_0$, while $\mathbb{R}\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}$ contributes to $\mathfrak{a}_0$. The linear functionals e_j for $j\neq\frac{1}{2}(p+1)$ are as in Example 4 of §II.1, and $e_{\frac{1}{2}(p+1)}$ on the embedded $\begin{pmatrix} 0 & t \\ t & 0\end{pmatrix}\in\mathbb{R}\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}$ is just t. The roots are $\pm e_i\pm e_j$ with $i\neq j$, and those involving index $\frac{1}{2}(p+1)$ are complex.

Suppose $q=1$. Then the standard ordering takes $i\mathfrak{t}_0$ before $\mathfrak{a}_0$. The simple roots as usual are

$$e_1-e_2,\ \dots,\ e_{n-2}-e_{n-1},\ e_{n-1}-e_n,\ e_{n-1}+e_n.$$

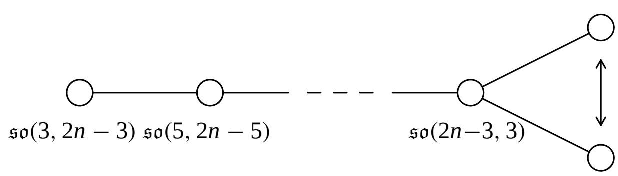
The last two are complex, and the others are compact imaginary. Similarly if $p=1$, we can use the reverse of the standard ordering and conclude that all imaginary roots are compact.

Now suppose $p > 1$ and $q > 1$. In this case we cannot use the standard ordering. To have $i\mathfrak{t}_0$ before $\mathfrak{a}_0$ in defining positivity, we take $\frac{1}{2}(p+1)$ last, and the simple roots are

$$e_1 - e_2,\ \ldots,\ e_{\frac{1}{2}(p-1)-1} - e_{\frac{1}{2}(p-1)},\ e_{\frac{1}{2}(p-1)} - e_{\frac{1}{2}(p+1)+1},$$
$$e_{\frac{1}{2}(p+1)+1} - e_{\frac{1}{2}(p+1)+2},\ \ldots,\ e_{n-1} - e_n,\ e_n - e_{\frac{1}{2}(p+1)},\ e_n + e_{\frac{1}{2}(p+1)}.$$

The last two are complex, and the others are imaginary. Among the imaginary simple roots, $e_{\frac{1}{2}(p-1)} - e_{\frac{1}{2}(p+1)+1}$ is the unique noncompact simple root.

We can assemble our results for D_n in a diagram like that in Figure 6.1. As we observed above, the situation with all imaginary roots unpainted corresponds to $\mathfrak{so}(1, 2n-1) \cong \mathfrak{so}(2n-1, 1)$. If one imaginary root is painted, the associated matrix algebra may be seen from the diagram



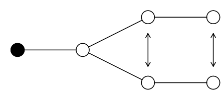
For type E_6, Theorem 6.96 gives us three diagrams to consider. As in (2.86c) let α_2 be the simple root corresponding to the endpoint vertex of the short branch in the Dynkin diagram, and let α_4 correspond to the triple point. The Vogan diagram in which α_4 is painted gives the same $\mathfrak{g}_0$ (up to isomorphism) as the Vogan diagram with α_2 painted. In fact, the Weyl group element $s_{\alpha_4}s_{\alpha_2}$ carries the one with α_2 painted to the one with α_4 painted. Thus there are only two Vogan diagrams that need to be considered, and they are in Figure 6.3. The figure also gives the names of the Lie algebras $\mathfrak{g}_0$ in the Cartan listing [1927a] and identifies $\mathfrak{k}_0$.

To compute $\mathfrak{k}_0$ for each case of Figure 6.3, we regroup the root-space decomposition of $\mathfrak{g}$ as

$$\begin{aligned} \mathfrak{g} &= \Big(\mathfrak{t} \oplus \bigoplus_{\substack{\alpha \text{ imaginary} \\ \text{compact}}} \mathfrak{g}_\alpha \oplus \bigoplus_{\substack{\text{complex pairs} \\ \{\alpha, \theta\alpha\}}} (X_\alpha + \theta X_\alpha)\Big) \\ &\quad \oplus \Big(\mathfrak{a} \oplus \bigoplus_{\substack{\alpha \text{ imaginary} \\ \text{noncompact}}} \mathfrak{g}_\alpha \oplus \bigoplus_{\substack{\text{complex pairs} \\ \{\alpha, \theta\alpha\}}} (X_\alpha - \theta X_\alpha)\Big), \end{aligned} \tag{6.103}$$

and it is clear that the result is $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Therefore the roots in $\Delta(\mathfrak{k}, \mathfrak{t})$ are the restrictions to $\mathfrak{t}$ of the imaginary compact roots in $\Delta(\mathfrak{g}, \mathfrak{h})$, together with the restrictions to $\mathfrak{t}$ of each pair $\{\alpha, \theta\alpha\}$ of complex roots in $\Delta(\mathfrak{g}, \mathfrak{h})$. Also the dimension of $\mathfrak{a}_0$ is the number of 2-element orbits in the Vogan diagram and is therefore 2 in each case.

We can tell which roots are complex, and we need to know how to decide which imaginary roots are compact. This determination can be carried out by induction on the level in the expansion in terms of simple roots. Thus suppose that α and β are positive roots with β simple, and

E I  $\mathfrak{k}_0 = \mathfrak{sp}(4)$

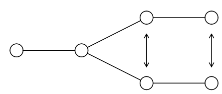
E IV  $\mathfrak{k}_0 = \mathfrak{f}_4$

FIGURE 6.3. Noncompact noncomplex exceptional simple real Lie algebras with a nontrivial automorphism in the Vogan diagram

suppose $\alpha + \beta$ is an imaginary root. If β is imaginary, then (6.99) settles matters. Otherwise β is complex simple, and Figure 6.3 shows that $\langle \beta, \theta\beta \rangle = 0$. Therefore the following proposition settles matters for $\mathfrak{g}_0$ as in Figure 6.3 and allows us to complete the induction.

Proposition 6.104. For a connected Vogan diagram involving a nontrivial automorphism, suppose that α and β are positive roots, that β is complex simple, that β is orthogonal to $\theta\beta$, and that $\alpha + \beta$ is an imaginary root. Then $\alpha - \theta\beta$ is an imaginary root, and $\alpha - \theta\beta$ and $\alpha + \beta$ have the same type, compact or noncompact.

PROOF. Taking the common length of all roots to be 2, we have

$$\begin{aligned} 1 = 2 - 1 &= \langle \beta, \beta \rangle + \langle \beta, \alpha \rangle = \langle \beta, \alpha + \beta \rangle \\ &= \langle \theta\beta, \theta(\beta + \alpha) \rangle = \langle \theta\beta, \alpha + \beta \rangle = \langle \theta\beta, \alpha \rangle + \langle \theta\beta, \beta \rangle = \langle \theta\beta, \alpha \rangle. \end{aligned}$$

Thus $\alpha - \theta\beta$ is a root, and we have

$$\alpha + \beta = \theta\beta + (\alpha - \theta\beta) + \beta.$$

Since $\alpha + \beta$ is imaginary, $\alpha - \theta\beta$ is imaginary. Therefore we can write $\theta X_{\alpha - \theta\beta} = s X_{\alpha - \theta\beta}$ with $s = \pm 1$. Write $\theta X_\beta = t X_{\theta\beta}$ and $\theta X_{\theta\beta} = t X_\beta$ with

$t = \pm 1$. Then we have

$$\begin{aligned}\theta[[X_{\theta\beta}, X_{\alpha-\theta\beta}], X_\beta] &= [[\theta X_{\theta\beta}, \theta X_{\alpha-\theta\beta}], \theta X_\beta] \\ &= st^2[[X_\beta, X_{\alpha-\theta\beta}], X_{\theta\beta}] \\ &= -s[[X_{\alpha-\theta\beta}, X_{\theta\beta}], X_\beta] - s[[X_{\theta\beta}, X_\beta], X_{\alpha-\theta\beta}] \\ &= -s[[X_{\alpha-\theta\beta}, X_{\theta\beta}], X_\beta] \\ &= s[[X_{\theta\beta}, X_{\alpha-\theta\beta}], X_\beta],\end{aligned}$$

and the proof is complete.

Let us summarize our results.

Theorem 6.105 (classification). Up to isomorphism every simple real Lie algebra is in the following list, and everything in the list is a simple real Lie algebra:

(a) the Lie algebra $\mathfrak{g}^{\mathbb{R}}$, where $\mathfrak{g}$ is complex simple of type A_n for $n \geq 1$, B_n for $n \geq 2$, C_n for $n \geq 3$, D_n for $n \geq 4$, E_6, E_7, E_8, F_4, or G_2,
(b) the compact real form of any $\mathfrak{g}$ as in (a),
(c) the classical matrix algebras

$$\begin{array}{lrl} \mathfrak{su}(p,q) & \text{with} & p \geq q > 0,\ p+q \geq 2 \\ \mathfrak{so}(p,q) & \text{with} & p > q > 0,\ p+q \text{ odd},\ p+q \geq 5 \\ & \text{or with} & p \geq q > 0,\ p+q \text{ even},\ p+q \geq 8 \\ \mathfrak{sp}(p,q) & \text{with} & p \geq q > 0,\ p+q \geq 3 \\ \mathfrak{sp}(n,\mathbb{R}) & \text{with} & n \geq 3 \\ \mathfrak{so}^*(2n) & \text{with} & n \geq 4 \\ \mathfrak{sl}(n,\mathbb{R}) & \text{with} & n \geq 3 \\ \mathfrak{sl}(n,\mathbb{H}) & \text{with} & n \geq 2, \end{array}$$

(d) the 12 exceptional noncomplex noncompact simple Lie algebras given in Figures 6.2 and 6.3.

The only isomorphism among Lie algebras in the above list is $\mathfrak{so}^*(8) \cong \mathfrak{so}(6,2)$.

REMARKS. The restrictions on rank in (a) prevent coincidences in Dynkin diagrams. These restrictions are maintained in (b) and (c) for the same reason. In the case of $\mathfrak{sl}(n,\mathbb{R})$ and $\mathfrak{sl}(n,\mathbb{H})$, the restrictions on n force the automorphism to be nontrivial. In (c) there are no isomorphisms within a series because the $\mathfrak{k}_0$'s are different. To have an isomorphism between members of two series, we need at least two series with the same

Dynkin diagram and automorphism. Then we examine the possibilities and are led to compare $\mathfrak{so}^*(8)$ with $\mathfrak{so}(6,2)$. The standard Vogan diagrams for these two Lie algebras are isomorphic, and hence the Lie algebras are isomorphic by Theorem 6.74.

11. Restricted Roots in the Classification

Additional information about the simple real Lie algebras of §10 comes by switching from a maximally compact Cartan subalgebra to a maximally noncompact Cartan subalgebra. The switch exposes the system of restricted roots, which governs the Iwasawa decomposition and some further structure theory that will be developed in Chapter VII.

According to §7 the switch in Cartan subalgebra is best carried out when we can find a maximal strongly orthogonal sequence of noncompact imaginary roots such that, after application of the Cayley transforms, no noncompact imaginary roots remain. If $\mathfrak{g}_0$ is a noncomplex simple real Lie algebra and if we have a Vogan diagram for $\mathfrak{g}_0$ as in Theorem 6.96, such a sequence is readily at hand by an inductive construction. We start with a noncompact imaginary simple root, form the set of roots orthogonal to it, label their compactness or noncompactness by means of Proposition 6.72, and iterate the process.

EXAMPLE. Let $\mathfrak{g}_0 = \mathfrak{su}(p, n-p)$ with $p \le n-p$. The distinguished Vogan diagram is of type A_{n-1} with $e_p - e_{p+1}$ as the unique noncompact imaginary simple root. Since the Dynkin diagram does not have a double line, orthogonality implies strong orthogonality. The above process yields the sequence of noncompact imaginary roots

$$\begin{aligned} 2f_p &= e_p - e_{p+1} \\ 2f_{p-1} &= e_{p-1} - e_{p+2} \\ &\ \vdots \\ 2f_1 &= e_1 - e_{2p}. \end{aligned} \tag{6.106}$$

We do a Cayley transform with respect to each of these. The order is irrelevant; since the roots are strongly orthogonal, the individual Cayley transforms commute. It is helpful to use the same names for roots before and after Cayley transform but always to remember what Cartan subalgebra is being used. After Cayley transform the remaining imaginary roots are those roots involving only indices $2p+1, \ldots, n$, and such roots are compact. Thus a maximally noncompact Cartan subalgebra has noncompact dimension p. The restricted roots are obtained by projecting all $e_k - e_l$ on the linear span of (6.106). If $1 \le k < l \le p$, we have

$$\begin{aligned} e_k - e_l &= \tfrac{1}{2}(e_k - e_{2p+1-k}) - \tfrac{1}{2}(e_l - e_{2p+1-l}) + (\text{orthogonal to (6.106)}) \\ &= (f_k - f_l) + (\text{orthogonal to (6.106)}). \end{aligned}$$

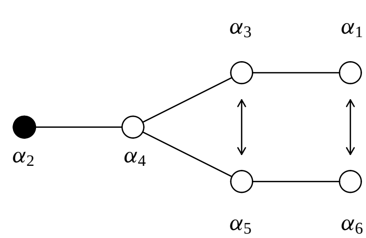
Thus $f_k - f_l$ is a restricted root. For the same k and l, $e_k - e_{2p+1-l}$ restricts to $f_k + f_l$. In addition, if $k + l = 2p+1$, then $e_k - e_l$ restricts to $2f_k$, while if $k \le p$ and $l > 2p$, then $e_k - e_l$ restricts to f_k. Consequently the set of restricted roots is

$$\Sigma = \begin{cases} \{\pm f_k \pm f_l\} \cup \{\pm 2f_k\} \cup \{\pm f_k\} & \text{if } 2p < n \\ \{\pm f_k \pm f_l\} \cup \{\pm 2f_k\} & \text{if } 2p = n. \end{cases}$$

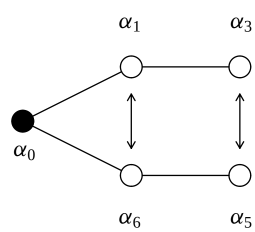
Thus Σ is of type $(BC)_p$ if $2p < n$ and of type C_p if $2p = n$.

We attempt to repeat the construction in the above example for all of the classical matrix algebras and exceptional algebras in Theorem 6.105, parts (c) and (d). There is no difficulty when the automorphism in the Vogan diagram is trivial. However, the cases where the automorphism is nontrivial require special comment. Except for $\mathfrak{sl}(2n+1, \mathbb{R})$, which we can handle manually, each of these Lie algebras has β orthogonal to $\theta\beta$ whenever β is a complex simple root. Then it follows from Proposition 6.104 that any positive imaginary root is the sum of imaginary simple roots and a number of pairs $\beta, \theta\beta$ of complex simple roots and that the complex simple roots can be disregarded in deciding compactness or noncompactness. In particular, $\mathfrak{sl}(n, \mathbb{H})$ and E IV have no noncompact imaginary roots.

EXAMPLE. Let $\mathfrak{g}_0 = $ E I. The Vogan diagram is



Let α_2 be the first member in the orthogonal sequence of imaginary noncompact roots. From the theory for D_4, a nonobvious root orthogonal to α_2 is $\alpha_0 = \alpha_2 + 2\alpha_4 + \alpha_3 + \alpha_5$. This root is imaginary, and no smaller imaginary root is orthogonal to α_2. We can disregard the complex pair α_3, α_5 in deciding compactness or noncompactness (Proposition 6.104), and we see that α_0 is noncompact. Following our algorithm, we can expand our list to α_2, α_0. The Vogan diagram of the system orthogonal to α_2 is



This is the Vogan diagram of $\mathfrak{sl}(6, \mathbb{R})$, and we therefore know that the list extends to

$$\alpha_2,\ \alpha_0,\ \alpha_1 + \alpha_0 + \alpha_6,\ \alpha_3 + (\alpha_1 + \alpha_0 + \alpha_6) + \alpha_5.$$

Thus the Cayley transforms increase the noncompact dimension of the Cartan subalgebra by 4 from 2 to 6, and it follows that E I is a split real form.

It is customary to refer to the noncompact dimension of a maximal noncompact Cartan subalgebra of $\mathfrak{g}_0$ as the **real rank** of $\mathfrak{g}_0$. We are led to the following information about restricted roots. In the case of the classical matrix algebras, the results are

(6.107)

$\mathfrak{g}_0$	Condition	Real Rank	Restricted Roots
$\mathfrak{su}(p,q)$	$p \geq q$	q	$(BC)_q$ if $p > q$, C_q if $p = q$
$\mathfrak{so}(p,q)$	$p \geq q$	q	B_q if $p > q$, D_q if $p = q$
$\mathfrak{sp}(p,q)$	$p \geq q$	q	$(BC)_q$ if $p > q$, C_q if $p = q$
$\mathfrak{sp}(n,\mathbb{R})$		n	C_n
$\mathfrak{so}^*(2n)$		$[\frac{n}{2}]$	$C_{\frac{1}{2}n}$ if n even, $(BC)_{\frac{1}{2}(n-1)}$ if n odd
$\mathfrak{sl}(n,\mathbb{R})$		$n-1$	A_{n-1}
$\mathfrak{sl}(n,\mathbb{H})$		$n-1$	A_{n-1}

For the exceptional Lie algebras the results are

(6.108)

$\mathfrak{g}_0$	Real Rank	Restricted Roots
E I	6	E_6
E II	4	F_4
E III	2	$(BC)_2$
E IV	2	A_2
E V	7	E_7
E VI	4	F_4
E VII	3	C_3
E VIII	8	E_8
E IX	4	F_4
F I	4	F_4
F II	1	$(BC)_1$
G	2	G_2

For the Lie algebras in Theorem 6.105a, the above analysis simplifies. Here $\mathfrak{g}$ is complex simple, and we take $\mathfrak{g}_0 = \mathfrak{g}^{\mathbb{R}}$. Let J be multiplication by $\sqrt{-1}$ within $\mathfrak{g}^{\mathbb{R}}$. If θ is a Cartan involution of $\mathfrak{g}^{\mathbb{R}}$, then Corollary 6.22 shows that θ comes from conjugation of $\mathfrak{g}$ with respect to a compact real form $\mathfrak{u}_0$. In other words, $\mathfrak{g}^{\mathbb{R}} = \mathfrak{u}_0 \oplus J\mathfrak{u}_0$ with $\theta(X + JY) = X - JY$. Let $\mathfrak{h}_0 = \mathfrak{t}_0 \oplus \mathfrak{a}_0$ be a θ stable Cartan subalgebra of $\mathfrak{g}^{\mathbb{R}}$. Since $\mathfrak{t}_0$ commutes with $\mathfrak{a}_0$, $\mathfrak{t}_0$ commutes with $J\mathfrak{a}_0$. Also $\mathfrak{a}_0$ commutes with $J\mathfrak{a}_0$. Since $\mathfrak{h}_0$ is maximal abelian, $J\mathfrak{a}_0 \subseteq \mathfrak{t}_0$. Similarly $J\mathfrak{t}_0 \subseteq \mathfrak{a}_0$. Therefore $J\mathfrak{t}_0 = \mathfrak{a}_0$, and $\mathfrak{h}_0$ is actually a complex subalgebra of $\mathfrak{g}$. By Proposition 2.7, $\mathfrak{h}_0$ is a (complex) Cartan subalgebra of $\mathfrak{g}$. Let

$$\mathfrak{g} = \mathfrak{h}_0 \oplus \bigoplus_{\alpha\in\Delta} \mathfrak{g}_\alpha$$

be the root-space decomposition. Here each α is complex linear on the complex vector space $\mathfrak{h}_0$. Thus distinct α's have distinct restrictions to $\mathfrak{a}_0$. Hence

$$\mathfrak{g}^{\mathbb{R}} = \mathfrak{a}_0 \oplus \mathfrak{t}_0 \oplus \bigoplus_{\alpha\in\Delta} \mathfrak{g}_\alpha$$

is the restricted-root space decomposition, each restricted-root space being 2-dimensional over $\mathbb{R}$. Consequently the real rank of $\mathfrak{g}^{\mathbb{R}}$ equals the rank of $\mathfrak{g}$, and the system of restricted roots of $\mathfrak{g}^{\mathbb{R}}$ is canonically identified (by restriction or complexification) with the system of roots of $\mathfrak{g}$. In particular the system Σ of restricted roots is of the same type (A_n through G_2) as the system Δ of roots.

The simple real Lie algebras of real-rank one will play a special role in Chapter VII. From Theorem 6.105 and our determination above of the real rank of each example, the full list of such Lie algebras is

$$\begin{array}{ll} \mathfrak{su}(p,1) & \text{with } p \geq 1 \\ \mathfrak{so}(p,1) & \text{with } p \geq 3 \\ \mathfrak{sp}(p,1) & \text{with } p \geq 2 \\ \text{F II} & \end{array} \tag{6.109}$$

Low-dimensional isomorphisms show that other candidates are redundant:

$$\begin{aligned} \mathfrak{sl}(2,\mathbb{C}) &\cong \mathfrak{so}(3,1) \\ \mathfrak{so}(2,1) &\cong \mathfrak{su}(1,1) \\ \mathfrak{sp}(1,1) &\cong \mathfrak{so}(4,1) \\ \mathfrak{sp}(1,\mathbb{R}) &\cong \mathfrak{su}(1,1) \\ \mathfrak{so}^*(4) &\cong \mathfrak{su}(2) \oplus \mathfrak{su}(1,1) \\ \mathfrak{so}^*(6) &\cong \mathfrak{su}(3,1) \\ \mathfrak{sl}(2,\mathbb{R}) &\cong \mathfrak{su}(1,1) \\ \mathfrak{sl}(2,\mathbb{H}) &\cong \mathfrak{so}(5,1). \end{aligned} \tag{6.110}$$

12. Problems

1. Prove that if $\mathfrak{g}$ is a complex semisimple Lie algebra, then any two split real forms of $\mathfrak{g}$ are conjugate via $\operatorname{Aut}\mathfrak{g}$.
2. Let $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ be a Cartan decomposition of a real semisimple Lie algebra. Prove that $\mathfrak{k}_0$ is compactly embedded in $\mathfrak{g}_0$ and that it is maximal with respect to this property.
3. Let G be semisimple, let $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ be a Cartan decomposition of the Lie algebra, and let X and Y be in $\mathfrak{p}_0$. Prove that $\exp X \ \exp Y \ \exp X$ is in $\exp \mathfrak{p}_0$.
4. Let $g \in SL(m,\mathbb{C})$ be positive definite. Prove that g can be decomposed as $g = lu$, where l is lower triangular and u is upper triangular.
5. In the development of the Iwasawa decomposition for $SO(p,1)_0$ and $SU(p,1)$, make particular choices of a positive system for the restricted roots, and compute N in each case.
6. (a) Prove that $\mathfrak{g}_0 = \mathfrak{so}^*(2n)$ consists in block form of all complex matrices $\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$ with a skew Hermitian and b skew symmetric.

(b) In $\mathfrak{g}_0$, let $\mathfrak{h}_0$ be the Cartan subalgebra in (6.102). Assuming that the roots are $\pm e_i \pm e_j$, find the root vectors. Show that $e_i - e_j$ is compact and $e_i + e_j$ is noncompact.

(c) Show that a choice of maximal abelian subspace of $\mathfrak{p}_0$ is to take a to be 0 and take b to be block diagonal and real with blocks of sizes $2, \dots, 2$ if n is even and $1, 2, \dots, 2$ if n is odd.

(d) Find the restricted-root space decomposition of $\mathfrak{g}_0$ relative to the maximal abelian subspace of $\mathfrak{p}_0$ given in (c).

7. Let $\mathfrak{h}_0 = \mathfrak{t}_0 \oplus \mathfrak{a}_0$ be a maximally noncompact θ stable Cartan subalgebra, and let $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}$ be the complexification. Fix a positive system Σ^+ for the restricted roots, and introduce a positive system Δ^+ for the roots so that a nonzero restriction to $\mathfrak{a}_0$ of a member of Δ^+ is always in Σ^+.

   (a) Prove that every simple restricted root for Σ^+ is the restriction of a simple root for Δ^+.

   (b) Let V be the span of the imaginary simple roots. Prove for each simple α_i not in V that $-\theta\alpha_i$ is in $\alpha_{i'} + V$ for a unique simple $\alpha_{i'}$, so that $\alpha_i \mapsto \alpha_{i'}$ defines a permutation of order 2 on the simple roots not in V.

   (c) For each orbit $\{i, i'\}$ of one or two simple roots not in V, define an element $H = H_{\{i,i'\}} \in \mathfrak{h}$ by $\alpha_i(H) = \alpha_{i'}(H) = 1$ and $\alpha_j(H) = 0$ for all other j. Prove that H is in $\mathfrak{a}$.

   (d) Using the elements constructed in (c), prove that the linear span of the restrictions to $\mathfrak{a}_0$ of the simple roots has dimension equal to the number of orbits.

   (e) Conclude from (d) that the nonzero restriction to $\mathfrak{a}_0$ of a simple root for Δ^+ is simple for Σ^+.

8. The group K for $G = SL(3, \mathbb{R})$ is $K = SO(3)$, which has a double cover $\widetilde{K}$. Therefore G itself has a double cover $\widetilde{G}$. The group $M = Z_K(A)$ is known from Example 1 of §5 to be the direct sum of two 2-element groups. Prove that $\widetilde{M} = Z_{\widetilde{K}}(A)$ is isomorphic to the subgroup $\{\pm 1, \pm i, \pm j, \pm k\}$ of the unit quaternions.

9. Suppose that D and D' are Vogan diagrams corresponding to $\mathfrak{g}_0$ and $\mathfrak{g}_0'$, respectively. Prove that an inclusion $D \subseteq D'$ induces a one-one Lie algebra homomorphism $\mathfrak{g}_0 \to \mathfrak{g}_0'$.

10. Let G be a semisimple Lie group with Lie algebra $\mathfrak{g}_0$. Fix a Cartan involution θ and Cartan decomposition $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$, and let K be the analytic subgroup of G with Lie algebra $\mathfrak{k}_0$. Suppose that $\mathfrak{g}_0$ has a Cartan subalgebra contained in $\mathfrak{k}_0$.

    (a) Prove that there exists $k \in K$ such that $\theta = \mathrm{Ad}(k)$.

    (b) Prove that if Σ is the system of restricted roots of $\mathfrak{g}_0$, then -1 is in the Weyl group of Σ.

11. Let G be a semisimple Lie group with Lie algebra $\mathfrak{g}_0$. Fix a Cartan involution θ and Cartan decomposition $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$, and let K be the analytic subgroup of G with Lie algebra $\mathfrak{k}_0$. Prove that if $\mathfrak{g}_0$ does not have a Cartan subalgebra contained in $\mathfrak{k}_0$, then there does not exist $k \in K$ such that $\theta = \mathrm{Ad}(k)$.

12. Let $\mathfrak{t}_0 \oplus \mathfrak{a}_0$ be a maximally noncompact θ stable Cartan subalgebra. Prove that if α is a root, then $\alpha + \theta\alpha$ is not a root.

13. For $\mathfrak{g}_0 = \mathfrak{sl}(2n, \mathbb{R})$, let $\mathfrak{h}_0^{(i)}$ consist of all block-diagonal matrices whose first i blocks are of size 2 of the form $\left\{\begin{pmatrix} t_j & \theta_j \\ -\theta_j & t_j \end{pmatrix}\right\}$, for $1 \le j \le i$, and whose remaining blocks are $2(n-i)$ blocks of size 1.
    (a) Prove that the $\mathfrak{h}_0^{(i)}$, $0 \le i \le n$, form a complete set of nonconjugate Cartan subalgebras of $\mathfrak{g}_0$.
    (b) Relate $\mathfrak{h}_0^{(i)}$ to the maximally compact θ stable Cartan subalgebra of Example 2 in §8, using Cayley transforms.
    (c) Relate $\mathfrak{h}_0^{(i)}$ to the maximally noncompact θ stable Cartan subalgebra of diagonal matrices, using Cayley transforms.

14. The example in §7 constructs four Cartan subalgebras for $\mathfrak{sp}(2, \mathbb{R})$. The first one $\mathfrak{h}_0$ is maximally noncompact, and the last one $\mathfrak{h}_0'$ is maximally compact. The second one has noncompact part contained in $\mathfrak{h}_0$ and compact part contained in $\mathfrak{h}_0'$, but the third one does not. Show that the third one is not even conjugate to a Cartan subalgebra whose noncompact part is contained in $\mathfrak{h}_0$ and whose compact part is contained in $\mathfrak{h}_0'$.

15. Let a $(2n)$-by-$(2n)$ matrix be given in block form by $\dfrac{\sqrt{2}}{2}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$. Define a mapping $X \mapsto Y$ of the set of $(2n)$-by-$(2n)$ complex matrices to itself by $Y = \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} X \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}^{-1}$.
    (a) Prove that the map carries $\mathfrak{su}(n, n)$ to an image whose members Y are characterized by $\mathrm{Tr}\, Y = 0$ and $JY + Y^* J = 0$, where J is as in Example 2 of §I.8.
    (b) Prove that the mapping exhibits $\mathfrak{su}(n, n) \cap \mathfrak{sp}(n, \mathbb{C})$ as isomorphic with $\mathfrak{sp}(n, \mathbb{R})$.
    (c) Within $\mathfrak{g}_0 = \mathfrak{su}(n, n) \cap \mathfrak{sp}(n, \mathbb{C})$, let θ be negative conjugate transpose. Define $\mathfrak{h}_0$ to be the Cartan subalgebra in (6.102). Referring to Example 3 in §II.1, find all root vectors and identify which are compact and which are noncompact. Interpret the above mapping on $(\mathfrak{g}_0)^{\mathbb{C}}$ as a product of Cayley transforms $\mathbf{c}_\beta$. Which roots β are involved?

16. (a) Prove that every element of $SL(2, \mathbb{R})$ is conjugate to at least one matrix of the form

$$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, \quad \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} -1 & t \\ 0 & -1 \end{pmatrix}, \quad \text{or} \quad \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

Here a is nonzero, and t and θ are arbitrary in $\mathbb{R}$.

(b) Prove that the exponential map from $\mathfrak{sl}(2, \mathbb{R})$ into $SL(2, \mathbb{R})$ has image

$$\{X \mid \operatorname{Tr} X > -2\} \cup \{-1\}.$$

17. Let $\mathfrak{g}$ be a simple complex Lie algebra. Describe the Vogan diagram of $\mathfrak{g}^{\mathbb{R}}$.

18. This problem examines the effect on the painting in a Vogan diagram when the positive system is changed from Δ^+ to $s_\alpha\Delta^+$, where α is an imaginary simple root.
   (a) Show that the new diagram is a Vogan diagram with the same Dynkin diagram and automorphism and with the painting unchanged at the position of α and at all positions not adjacent to α.
   (b) If α is compact, show that there is no change in the painting of imaginary roots in positions adjacent to α.
   (c) If α is noncompact, show that the painting of an imaginary root at a position adjacent to α is reversed unless the root is connected by a double line to α and is long, in which case it is unchanged.
   (d) Devise an algorithm for a Vogan diagram of type A_n for a step-by-step change of positive system so that ultimately at most one simple root is painted (as is asserted to be possible by Theorem 6.96).

19. In the Vogan diagram from Theorem 6.96 for the Lie algebra F II of §10, the simple root $\frac{1}{2}(e_1 - e_2 - e_3 - e_4)$ is noncompact, and the simple roots $e_2 - e_3$, $e_3 - e_4$, and e_4 are compact.
   (a) Verify that $\frac{1}{2}(e_1 - e_2 + e_3 + e_4)$ is noncompact.
   (b) The roots $\frac{1}{2}(e_1 - e_2 - e_3 - e_4)$ and $\frac{1}{2}(e_1 - e_2 + e_3 + e_4)$ are orthogonal and noncompact, yet (6.108) says that F II has real rank one. Explain.

20. The Vogan diagram of F I, as given by Theorem 6.96, has $e_2 - e_3$ as its one and only noncompact simple root. What strongly orthogonal set of noncompact roots is produced by the algorithm of §11?

21. Verify the assertion in (6.108) that E VII has real rank 3 and restricted roots of type C_3.

Problems 22–24 give further information about the Cartan decomposition $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ of a real semisimple Lie algebra. Let B be the Killing form of $\mathfrak{g}_0$.

22. Let $\mathfrak{p}_0'$ be an $\operatorname{ad} \mathfrak{k}_0$ invariant subspace of $\mathfrak{p}_0$, and define ${\mathfrak{p}_0'}^{\perp}$ to be the set of all $X \in \mathfrak{p}_0$ such that $B(X, \mathfrak{p}_0') = 0$. Prove that $B([\mathfrak{p}_0', {\mathfrak{p}_0'}^{\perp}], \mathfrak{k}_0) = 0$, and conclude that $[\mathfrak{p}_0', {\mathfrak{p}_0'}^{\perp}] = 0$.

23. If $\mathfrak{p}_0'$ is an $\operatorname{ad}\mathfrak{k}_0$ invariant subspace of $\mathfrak{p}_0$, prove that $[\mathfrak{p}_0', \mathfrak{p}_0] \oplus \mathfrak{p}_0'$ is an ideal in $\mathfrak{g}_0$.

24. Under the additional assumption that $\mathfrak{g}_0$ is simple but not compact, prove that
    (a) $[\mathfrak{p}_0, \mathfrak{p}_0] = \mathfrak{k}_0$
    (b) $\mathfrak{k}_0$ is a maximal proper Lie subalgebra of $\mathfrak{g}_0$.

Problems 25–27 deal with low-dimensional isomorphisms.

25. Establish the following isomorphisms by using Vogan diagrams:
    (a) the isomorphisms in (6.110)
    (b) $\mathfrak{sl}(4, \mathbb{R}) \cong \mathfrak{so}(3, 3)$, $\mathfrak{su}(2, 2) \cong \mathfrak{so}(4, 2)$, $\mathfrak{sp}(2, \mathbb{R}) \cong \mathfrak{so}(3, 2)$
    (c) $\mathfrak{sp}(2) \cong \mathfrak{so}(5)$, $\mathfrak{su}(4) \cong \mathfrak{so}(6)$, $\mathfrak{su}(2) \oplus \mathfrak{su}(2) \cong \mathfrak{so}(4)$.

26. (a) Prove that the mapping of Problem 36 of Chapter II gives an isomorphism of $\mathfrak{sl}(4, \mathbb{R})$ onto $\mathfrak{so}(3, 3)$.
    (b) Prove that the mapping of Problem 38 of Chapter II gives an isomorphism of $\mathfrak{sp}(2, \mathbb{R})$ onto $\mathfrak{so}(3, 2)$.

27. Prove that the Lie algebra isomorphisms of Problem 25b induce Lie group homomorphisms $SL(4, \mathbb{R}) \to SO(3, 3)_0$, $SU(2, 2) \to SO(4, 2)_0$, and $Sp(2, \mathbb{R}) \to SO(3, 2)_0$. What is the kernel in each case?

Problems 28–35 concern quasisplit Lie algebras and inner forms. They use facts about Borel subalgebras, which are defined in Chapter V. Let $\mathfrak{g}_0$ be a real semisimple Lie algebra with complexification $\mathfrak{g}$, and let σ be the conjugation of $\mathfrak{g}$ with respect to $\mathfrak{g}_0$: $\sigma(X+iY) = X-iY$ for X and Y in $\mathfrak{g}_0$. The Lie algebra $\mathfrak{g}_0$ is said to be **quasisplit** if $\mathfrak{g}$ has a Borel subalgebra $\mathfrak{b}$ such that $\sigma(\mathfrak{b}) = \mathfrak{b}$. Any split real form of $\mathfrak{g}$ is quasisplit. Two real forms $\mathfrak{g}_0$ and $\mathfrak{g}_0'$ of $\mathfrak{g}$, with respective conjugations σ and σ', are said to be **inner forms** of one another if there exists $g \in \operatorname{Int}\mathfrak{g}$ such that $\sigma' = \operatorname{Ad}(g) \circ \sigma$; this is an equivalence relation. This sequence of problems shows that any real form of $\mathfrak{g}$ is an inner form of a quasisplit form, the quasisplit form being unique up to the action of $\operatorname{Int}\mathfrak{g}$. The problems also give a useful criterion for deciding which real forms are quasisplit.

28. Show that the conjugation $\sigma_{m,n}$ of $\mathfrak{sl}(m+n, \mathbb{C})$ with respect to $\mathfrak{su}(m, n)$ is $\sigma_{m,n}(X) = -I_{m,n}X^*I_{m,n}$. Deduce that $\mathfrak{su}(m, n)$ and $\mathfrak{su}(m', n')$ are inner forms of one another if $m + n = m' + n'$.

29. Let $\mathfrak{g}_0$ be a real form of $\mathfrak{g}$, and let σ be the corresponding conjugation of $\mathfrak{g}$. Prove that there exists an automorphism Σ of $\operatorname{Int}(\mathfrak{g}^{\mathbb{R}})$ whose differential is σ.

30. Problem 35 of Chapter V dealt with a triple $(\mathfrak{b}, \mathfrak{h}, \{X_\alpha\})$ consisting of a Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$, a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ that lies in $\mathfrak{b}$, and a system of nonzero root vectors for the simple roots in the positive system of roots defining $\mathfrak{b}$. Let $(\mathfrak{b}', \mathfrak{h}', \{X_{\alpha'}\})$ be another such triple. Under the assumption

that there is a compact Lie algebra $\mathfrak{u}_0$ that is a real form of $\mathfrak{g}$ and has the property that $\mathfrak{h}_0 = \mathfrak{h} \cap \mathfrak{u}_0$ is a maximal abelian subalgebra of $\mathfrak{u}_0$, that problem showed that there exists an element $g \in \operatorname{Int}\mathfrak{g}$ such that $\operatorname{Ad}(g)\mathfrak{b} = \mathfrak{b}'$, $\operatorname{Ad}(g)\mathfrak{h} = \mathfrak{h}'$, and $\operatorname{Ad}(g)\{X_\alpha\} = \{X_{\alpha'}\}$. Prove that the assumption about the existence of $\mathfrak{u}_0$ is automatically satisfied and that the element g is unique.

31. Let $\mathfrak{g}_0$ be a real form of $\mathfrak{g}$, let σ be the corresponding conjugation of $\mathfrak{g}$, and let $(\mathfrak{b}, \mathfrak{h}, \{X_\alpha\})$ be a triple as in Problem 30. Choose $g \in \operatorname{Int}\mathfrak{g}$ as in that problem carrying the triple $(\mathfrak{b}, \mathfrak{h}, \{X_\alpha\})$ to the triple $\sigma(\mathfrak{b}, \mathfrak{h}, \{X_\alpha\}) = (\sigma(\mathfrak{b}), \sigma(\mathfrak{h}), \sigma\{X_\alpha\})$, and let $\sigma' = \operatorname{Ad}(g)^{-1} \circ \sigma$. Prove that $(\sigma')^2$ is in $\operatorname{Int}\mathfrak{g}$, deduce that $(\sigma')^2 = 1$, and conclude that σ' is the conjugation of $\mathfrak{g}$ with respect to a quasisplit real form $\mathfrak{g}_0'$ of $\mathfrak{g}$ such that $\mathfrak{g}_0$ and $\mathfrak{g}_0'$ are inner forms of one another.

32. Let $\mathfrak{g}_0$ be a quasisplit real form of $\mathfrak{g}$, let σ be the corresponding conjugation of $\mathfrak{g}$, and let $\mathfrak{b}$ be a Borel subalgebra of $\mathfrak{g}$ such that $\sigma(\mathfrak{b}) = \mathfrak{b}$. Write $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$ for a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, where $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$. Let H_δ be the member of $\mathfrak{h}$ corresponding to half the sum of the positive roots, and let $\mathfrak{h}'$ be the centralizer $Z_{\mathfrak{b}}(H_\delta + \sigma(H_\delta))$. Using Problem 34 of Chapter V, prove that $\mathfrak{h}'$ is a Cartan subalgebra of $\mathfrak{g}$ such that $\mathfrak{b} = \mathfrak{h}' \oplus \mathfrak{n}$ and $\sigma(\mathfrak{h}') = \mathfrak{h}'$.

33. Let $\mathfrak{g}_0$ be a real form of $\mathfrak{g}$, and let θ be a Cartan involution of $\mathfrak{g}_0$. Prove that the following are equivalent:
    (a) The real form $\mathfrak{g}_0$ is quasisplit.
    (b) If $\mathfrak{h}_0 = \mathfrak{a}_0 \oplus \mathfrak{t}_0$ is a maximally noncompact θ stable Cartan subalgebra of $\mathfrak{g}_0$ and if $\mathfrak{h} = \mathfrak{h}_0^{\mathbb{C}}$, then $\Delta(\mathfrak{g}, \mathfrak{h})$ has no imaginary roots.
    (c) If $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ is the Cartan decomposition of $\mathfrak{g}_0$ with respect to θ and if $\mathfrak{a}_0$ is maximal abelian in $\mathfrak{p}_0$, then $\mathfrak{m}_0 = Z_{\mathfrak{k}_0}(\mathfrak{a}_0)$ is abelian.

34. Let $\mathfrak{g}_0$ be a quasisplit real form of $\mathfrak{g}$, let σ be the corresponding conjugation of $\mathfrak{g}$, and let $\mathfrak{b}$ be a Borel subalgebra of $\mathfrak{g}$ such that $\sigma(\mathfrak{b}) = \mathfrak{b}$. Using Problem 32, write $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$ for a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ with $\sigma(\mathfrak{h}) = \mathfrak{h}$, where $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$. Prove that the set $\{X_\alpha\}$ of root vectors for simple roots can be chosen so that $\sigma\{X_\alpha\} = \{X_\alpha\}$.

35. Let $\mathfrak{g}_0$ and $\mathfrak{g}_0'$ be quasisplit real forms of $\mathfrak{g}$, let σ and σ' be their corresponding conjugations of $\mathfrak{g}$, and suppose that $(\mathfrak{b}, \mathfrak{h}, \{X_\alpha\})$ and $(\mathfrak{b}', \mathfrak{h}', \{X_{\alpha'}\})$ are triples as in Problem 30 such that $\sigma(\mathfrak{b}, \mathfrak{h}, \{X_\alpha\}) = (\mathfrak{b}, \mathfrak{h}, \{X_\alpha\})$ and $\sigma'(\mathfrak{b}', \mathfrak{h}', \{X_{\alpha'}\}) = (\mathfrak{b}', \mathfrak{h}', \{X_{\alpha'}\})$. Choose $g \in \operatorname{Int}\mathfrak{g}$ by that problem such that $\operatorname{Ad}(g)(\mathfrak{b}, \mathfrak{h}, \{X_\alpha\}) = (\mathfrak{b}', \mathfrak{h}', \{X_{\alpha'}\})$. Prove that if $\mathfrak{g}_0$ and $\mathfrak{g}_0'$ are inner forms of one another, then the automorphism $\operatorname{Ad}(g) \circ \sigma \circ \operatorname{Ad}(g)^{-1} \circ \sigma'$ of $\mathfrak{g}$ sends $(\mathfrak{b}', \mathfrak{h}', \{X_{\alpha'}\})$ to itself and is inner, and conclude that $\mathfrak{g}_0$ and $\mathfrak{g}_0'$ are conjugate via $\operatorname{Int}\mathfrak{g}$.