Hints for Solutions of Problems, 715-792

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Basic Real Analysis

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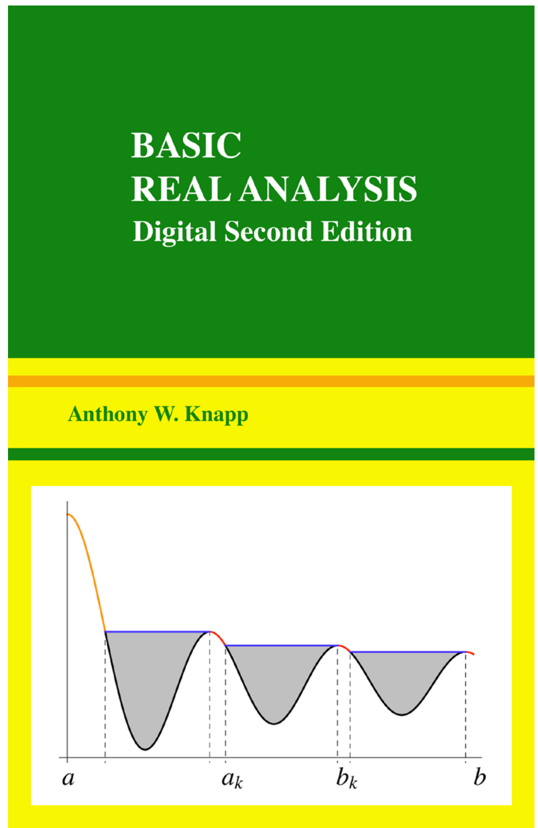

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HINTS FOR SOLUTIONS OF PROBLEMS

Chapter I

1. The derivation for (a) is similar to the proof of Corollary 1.3. For (b), let E be a nonempty set that is bounded above. Start with a member s_1 of E. Choose if possible an s_2 in E with $s_2 - s_1 \geq 1$. Continue with $s_3 - s_2 \geq 1$, $s_4 - s_3 \geq 1$, etc., until this is no longer possible; the existence of an upper bound forces the process to stop at some stage. Suppose that s_k has been constructed at this stage. Define s_{k+n} inductively for $n \geq 1$ to be a member of E with $s_{k+n} - s_{k+n-1} \geq 2^{-n}$ if possible; otherwise define $s_{k+n} = s_{k+n-1}$. Then $\{s_n\}$ is bounded and monotone increasing. To complete the problem, one has only to show that $\lim_n s_n$ is the least upper bound of E. Doing so makes use of (a).

2. Show that $x_1 \geq \sqrt{a}$ and that $\sqrt{a} \leq x_{n+1} \leq x_n$ for $n \geq 1$. Then $\lim_n x_n = c$ exists by Corollary 1.6, and c must satisfy $c = \frac{1}{2}(c^2 + a)/c$.

3. Write out a few cases and guess that the pattern is $a_{2n} = \frac{1}{2}(1 - 2^{-(n-1)})$ for $n \geq 1$ and $a_{2n+1} = 1 - 2^{-n}$ for $n \geq 0$. Prove each of these statements by induction. Since $a_{2n} \to \frac{1}{2}$ and $a_{2n+1} \to 1$ and since these two subsequences use all the terms of the sequence, the only subsequential limits of $\{a_k\}$ are $\frac{1}{2}$ and 1. Therefore $\limsup a_k = 1$ and $\liminf a_k = \frac{1}{2}$.

4. The argument without paying attention to finiteness is that $a_n + b_n \leq \sup_{r \geq k} a_r + \sup_{r \geq k} b_r$ for $n \geq k$, then that $\sup_{r \geq k}(a_r + b_r) \leq \sup_{r \geq k} a_r + \sup_{r \geq k} b_r$ for all r, and then that the limit of the sum is the sum of the limits.

5. Only (ii) converges uniformly, the reason being that $0 \leq x^n/n \leq 1/n$ and that $\lim 1/n = 0$. There is uniform convergence in (i) on $[0, 1-\epsilon]$ because $0 \leq x^n \leq (1-\epsilon)^n$, and there is uniform convergence in (iii) on $[0, 1-\epsilon]$ because the Weierstrass M test applies with $|x^k|/k \leq (1-\epsilon)^k$ and $\sum_k (1-\epsilon)^k < +\infty$.

6. The uniform convergence of $\sum_{n=0}^{\infty} a_n(x)$ follows from Corollary 1.18, and the pointwise convergence of $\sum_{n=0}^{\infty} |a_n(x)|$ follows because $(1-x)\sum_{n=0}^{\infty} x^n = 1$ for $0 \leq x < 1$ and because every $a_n(x)$ is 0 for $x = 1$. The convergence of $\sum_{n=0}^{\infty} |a_n(x)|$ cannot be uniform because the sum is discontinuous and Theorem 1.21 says that it would have to be continuous.

7. Put $g_n = f - f_n$, so that g_n is continuous and decreases pointwise to the 0 function. Let $x = x_n$ be a point where $g_n(x)$ is a maximum, and let $M_n = g_n(x_n)$. We are to prove that M_n tends to 0. Suppose it does not. If $k \geq n$, then $M_k =$

$g_k(x_k) \ge g_k(x_n) \ge g_n(x_n) = M_n$. So M_n decreases to some $M > 0$. Passing to a subsequence if necessary, we may assume by the Bolzano–Weierstrass Theorem that $\lim_n x_n = x'$. For $k \ge n$, we have $g_k(x_n) \ge g_n(x_n) = M_n \ge M$. Letting n tend to infinity gives $g_k(x') \ge M$ since g_k is continuous. This inequality for all k contradicts the assumption that $\lim_k g_k(x') = 0$.

8. The idea is to prove the four inequalities

$$\sum_{k=0}^{2m} (-1)^k x^{2k+1}/(2k+1)! > \sin x, \qquad \sum_{k=0}^{2m+1} (-1)^k x^{2k}/(2k)! < \cos x,$$
$$\sum_{k=0}^{2m+1} (-1)^k x^{2k+1}/(2k+1)! < \sin x, \qquad \sum_{k=0}^{2m+2} (-1)^k x^{2k}/(2k)! > \cos x$$

together by an induction. They are to be proved in order for $m = 0$, then in order for $m = 1$, and so on. In each case of the inductive step, the left side minus the right side is 0 at $x = 0$ and has derivative equal to the previous left side minus right side. The Mean Value Theorem says that each left side minus right side at $x > 0$ equals the product of x and the left side minus right side at ξ with $0 < \xi < x$. Substituting the previously proved inequality at ξ then gives the result. In other words, everything comes down to proving the first inequality, namely $x > \sin x$ for $x > 0$. Arguing in the same style, we have $x - \sin x = 1 - \cos\xi$ with $0 < \xi < x$. So at least $x - \sin x \ge 0$. For $0 < x \le \pi$, we actually obtain $x - \sin x > 0$. Since $\frac{d}{dx}(x - \sin x) \ge 0$, we have $x - \sin x \ge \pi - \sin\pi$ for $\pi \le x$. Thus $x - \sin x > 0$ for all $x > 0$.

9. The thing to prove is that the remainder term $\frac{1}{n!}\int_0^x (x-t)^n f^{(n+1)}(t)\,dt$ tends to 0 for each x as n tends to ∞. If $x \ge 0$, the absolute value is $\le (n!)^{-1}\int_0^x (x-t)^n\,dt = x^{n+1}/(n+1)!$, which tends to 0 for any fixed x. If $x \le 0$, one argues in a similar fashion.

10. By a diagonal process we can find a subsequence $\{F_{n_k}\}$ convergent for each rational x. Let F be the resulting limit function, carrying the rationals in $[-1, 1]$ into $[0, 1]$. If r and s are rationals with $r \le s$, then $F(r) = \lim_k F_{n_k}(r) \le \lim_k F_{n_k}(s) = F(s)$. Thus F is nondecreasing on the rationals. For each real x with $-1 < x < 1$, define $F(x^-)$ to be the limit of $F(r)$ with r rational as r increases to 1, and define $F(x^+)$ to be the limit of $F(r)$ with r rational as r decreases to 1. Then $F(x^-) \le F(x^+)$ for each x, and $F(x^+) \le F(y^-)$ if $x < y$. For each $N > 0$, it follows that there can be only finitely many x's for which $F(x^+) - F(x^-) \ge 1/N$, and hence there can be at most countably many x's for which $F(x^-) \ne F(x^+)$. Let this exceptional set be denoted by C. For x not in C, define $F(x) = F(x^+) = F(x^-)$.

For x not in C, let us show that $\lim_k F_{n_k}(x)$ exists and equals $F(x)$. If $r < x$ is rational, we have $F(r) = \liminf_k F_{n_k}(r) \le \liminf_k F_{n_k}(x)$; taking the supremum over r gives $F(x) = F(x^-) \le \liminf_k F_{n_k}(x)$. Arguing similarly with s rational and $x < s$, we have $\limsup_k F_{n_k}(x) \le \limsup_k F_{n_k}(s) = F(s)$, and hence

$\limsup_k F_{n_k}(x) \le F(x^+) = F(x)$. Combining these two conclusions, we see that $\liminf_k F_{n_k}(x) = \limsup_k F_{n_k}(x)$ and that the common value of these limits is $F(x)$.

Thus $\{F_{n_k}(x)\}$ converges except possibly for x in C. At each point of C, the sequence is bounded. Since C is countable, another use of a diagonal process produces a subsequence of F_{n_k} that converges at every point of C, hence at every point of $[-1, 1]$.

11. If $|x| > 1/\limsup \sqrt[n]{|a_n|}$, then $\sqrt[n]{|a_n|} \ge 1/|x|$ for infinitely many n. Thus $|a_n x^n| \ge 1$ for infinitely many n, and the terms of the series do not tend to 0. Hence the series cannot converge. In the reverse direction we want to see that the inequality $|x| < 1/\limsup \sqrt[n]{|a_n|}$ implies convergence of the series. We rewrite this as $\limsup \sqrt[n]{|a_n|} < 1/|x|$. Choose a number r with $\limsup \sqrt[n]{|a_n|} < r < 1/|x|$. Then $\sqrt[n]{|a_n|} \le r$ for all sufficiently large n, $\sqrt[n]{|a_n|}\,|x| \le r|x| < 1$ for all n sufficiently large, and $|a_n x^n| \le (r|x|)^n$ for all n sufficiently large. Thus $\sum |a_n x^n|$ is dominated term-by-term (from some point on) by the geometric series $\sum s^n$, where $s = r|x|$. Since $s < 1$, the geometric series converges, and hence so does $\sum |a_n x^n|$.

12. $1/(1-x)^2 = \sum_{n=0}^\infty (n+1)x^n$, $\log(1-x) = -\sum_{n=1}^\infty x^n/n$, $1/(1+x^2) = \sum_{n=0}^\infty (-1)^n x^{2n}$, and $\arctan x = \sum_{n=0}^\infty (-1)^n x^{2n+1}/(2n+1)$. All these series have radius of convergence 1.

13. The proof of existence of $\arccos x$ uses the proposition in Section A3 of Appendix A. The result of the calculation of the derivative is that $\frac{d}{dx}\arccos x = -1/\sqrt{1-x^2}$ for $|x| < 1$. Then $\arcsin x + \arccos x$ has derivative 0 on $(-1, 1)$ and hence is constant. The constant is evaluated by putting $x = 0$, and the result is that $\arcsin x + \arccos x = \pi/2$ on $(-1, 1)$.

14. The uniform version of Abel's Theorem is this: Let $\{a_n(x)\}_{n\ge 0}$ be a sequence of complex-valued functions with $\sum_{n=0}^\infty a_n(x)$ converging uniformly to the limit $s(x)$. Then $\lim_{r\uparrow 1} \sum_{n\ge 0} a_n(x) r^n = s(x)$ uniformly in x. The proof is just a matter of seeing that the estimates in the proof of Theorem 1.48 can be made uniform in x under the stated assumptions. The result about Cesàro sums is handled similarly.

15. Write $\cos n\theta = \frac{1}{2}(e^{in\theta} + e^{-in\theta})$ and $\sin n\theta = \frac{1}{2i}(e^{in\theta} - e^{-in\theta})$. Then $\sum_{n=1}^N \cos n\theta = \frac{1}{2}\sum_{n=1}^N e^{in\theta} + \frac{1}{2}\sum_{n=1}^N e^{-in\theta} = \frac{1}{2}\,\dfrac{1-e^{i(N+1)\theta}}{1-e^{i\theta}} + \frac{1}{2}\,\dfrac{1-e^{-i(N+1)\theta}}{1-e^{-i\theta}}$. Each numerator is bounded by 2, and each denominator gets close to 0 only as θ tends to a multiple of 2π. This proves the estimate for the cosines, and the estimate for the sines works in the same way.

17. For (a), the relevant result is that when all a_n are 0, $\sum_{n=1}^\infty |b_n|^2$ equals $\dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} |f(x)|^2\,dx$. Here $\displaystyle\sum_{n=1}^\infty |b_n|^2$ is $(4/\pi)^2 \displaystyle\sum_{n=1}^\infty \frac{1}{(2n-1)^2}$, and $\dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} |f(x)|^2\,dx$ is just $\dfrac{2\pi}{\pi} = 2$. Hence $\displaystyle\sum_{n=1}^\infty \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$.

18. We have $F(x)f(y) = \int_0^x f(t)f(y)\,dt = \int_0^x f(t+y)\,dt = \int_y^{x+y} f(t)\,dt = F(x+y) - F(y)$. If $F(x) \ne 0$ for some x, we can divide and use the Fundamental The-

orem of Calculus to see that $f(y)$ has a continuous derivative everywhere. (If $F(x) = 0$ for all x, then differentiation gives $f(x) = 0$ for all x.) Differentiating the original identity in x gives $f'(x)f(y) = f'(x+y)$. When $x = 0$, we obtain $f'(0)f(y) = f'(y)$. Then $\frac{d}{dy}\big(f(y)e^{-f'(0)y}\big) = f'(y)e^{-f'(0)y} + f(y)(-f'(0)e^{-f'(0)y}) = 0$, and hence $f(y)e^{-f'(0)y}$ is constant. Thus $f(y) = ae^{f'(0)y}$. In the original identity $f(x)f(y) = f(x+y)$, if we put $x = 0$ and choose y such that $f(y) \neq 0$, then we see that $f(0) = 1$. Hence $f(y) = e^{f'(0)y}$ if f is not identically 0.

19. We may assume that f is not identically 0. As in Problem 18, we have $f(0) = 1$. By continuity of f, choose x_0 such that $|f(x) - 1| \leq \frac{1}{10}$ when $|x| \leq |x_0|$. Then $\operatorname{Re} f(x_0) > 0$, and we can choose a unique c with $|\operatorname{Im}(cx_0)| < \pi/2$ such that $e^{cx_0} = f(x_0)$. The equation for f shows that $f\big(\frac{1}{2}x_0\big)^2 = f(x_0)$, and hence $f\big(\frac{1}{2}x_0\big)$ equals $e^{cx_0/2}$ or $-e^{cx_0/2}$. From $|f\big(\frac{1}{2}x_0\big) - 1| \leq \frac{1}{10}$, we have $\operatorname{Re} f\big(\frac{1}{2}x_0\big) > 0$. Since $|\operatorname{Im}(cx_0/2)| < \pi/2$, $e^{cx_0/2}$ is the choice of square root of e^{cx_0} with positive real part, and we conclude that $f\big(\frac{1}{2}x_0\big) = e^{cx_0/2}$. Iterating this argument, we obtain $f(2^{-n}x_0) = e^{c2^{-n}x_0}$ for all $n \geq 0$. The equation for f shows that $f(kx) = f(x)^k$ for all integers $k \geq 0$, and thus $f(qx_0) = e^{cqx_0}$ for every rational q of the form $k/2^n$ with k an integer ≥ 0. From $f(x)f(-x) = f(0) = 1$, we have $f(x^{-1}) = f(x)^{-1}$, and thus $f(qx_0) = e^{cqx_0}$ for every rational number of the form $k/2^n$ with k any integer. Using continuity and passing to the limit, we obtain $f(r) = e^{cr}$ for all real r.

21. This uses the discussion at the end of Section A2 of Appendix A. For $x \neq 0$, we compute that $g'(x) = (R(x)/S(x))e^{-1/x^2}$ for polynomials R and S with S not the 0 polynomial. Then $\lim_{x\to 0} g'(x) = 0$ by Problem 20, and the appendix shows that $g'(0)$ exists and equals 0.

22. Use Problem 21 and induction.

23. Since $\{s_n\}$ is convergent, it is bounded. Say $|s_n| \leq K$ for all n. Let $\epsilon > 0$ be given, and choose N such that $n \geq N$ implies $|s_n - s| < \epsilon/2$. Write $t_n - s = \sum_j M_{nj}s_j - s = \sum_j M_{nj}(s_j - s)$ by (i). A second application of (i) gives

$$\begin{aligned} |t_n - s| &\leq \sum_{j=0}^{N} M_{nj}(|s_j| + |s|) + \sum_{j=N+1}^{\infty} M_{nj}|s_j - s| \\ &\leq 2K\sum_{j=0}^{N} M_{nj} + \sum_{j=N+1}^{\infty} M_{nj}\epsilon/2 \leq 2K\sum_{j=0}^{N} M_{nj} + \epsilon/2. \end{aligned}$$

Since N is fixed, (ii) shows that $2K\sum_{j=0}^{N} M_{nj} < \epsilon/2$ for n sufficiently large. For those n, $|t_n - s| < \epsilon$.

24. For Cesàro summability the i^{th} row, for $i \geq 1$, has its first i entries equal to $1/i$ and its remaining entries equal to 0. For Abel summability the row going with r_i has j^{th} entry $(1 - r_i)(r_i)^j$ for $j \geq 0$.

25. Certainly $M_{ij} \geq 0$ for all i and j. The power series in Problem 12a shows that $\sum_j M_{ij} = 1$ for all i, and (ii) holds because $\lim_{r\uparrow 1}(j+1)r^j(1-r)^2 = (j+1)\cdot 1\cdot 0 = 0$.

26. Check that M as in the previous problem transforms the Cesàro sums into the Abel sums, and apply Problem 23.

27. This is handled by the same kind of computation as with the Fejér kernel.

28. The formula for $P_r(\theta)$ comes from summing the two geometric series for $n \geq 0$ and $n < 0$ and then adding the results. Properties (i) and (iii) are then immediate by inspection. For property (ii) we use the series expansion of $P_r(\theta)$. Theorem 1.31 allows the integration to be done term by term, and the result follows.

29. This is proved in the same way as Fejér's Theorem (Theorem 1.59).

30. Corollary 1.38 shows that $f_k'(x) = \sum_{n=0}^{\infty} c_{n,k} n x^{n-1}$ and that $f_k''(x) = \sum_{n=0}^{\infty} c_{n,k} n(n-1)x^{n-2}$ for $|x| < R$. The point is to show that $\{f_k'(x)\}$ is uniformly bounded and uniformly equicontinuous for $|x| \leq r$, and then Ascoli's Theorem produces the required subsequence. For proving the equicontinuity, it is enough to prove that $\{f_k''(x)\}$ is uniformly bounded for $|x| \leq r$.

Fix $r < R$, and choose r_1 with $r < r_1 < R$. Since $\lim f_k(x) = f(x)$ uniformly for $|x| \leq r_1$, there is an M such that $|f_k(r_1)| \leq M$ for all k. Thus $|\sum_n c_{n,k} r_1^n| \leq M$ for all k. Since $c_{n,k} \geq 0$ for all n and k, $c_{n,k} \leq M r_1^{-n}$ for all n and k. Since $r < r_1$, choose N such that $n \geq N$ implies $n(r/r_1)^{n-1} \leq 1$ and $n(n-1)(r/r_1)^{n-2} \leq 1$ for $n \geq N$. Since $c_{n,k} \geq 0$ for all n and k, $c_{n,k} n|x|^{n-1} \leq c_{n,k} n r^{n-1} \leq (c_{n,k} r_1^{n-1})(n(r/r_1)^{n-1}) \leq c_{n,k} r_1^{n-1}$ for $n \geq N$ and $|x| \leq r$. Summing on $n \geq N$ and taking Corollary 1.38 into account, we see that

$$\Big|f_k'(x) - \sum_{n=0}^{N-1} n c_{n,k} x^{n-1}\Big| \leq r_1^{-1}\Big(f_k(r_1) - \sum_{n=0}^{N-1} c_{n,k} r_1^n\Big) \leq r_1^{-1} f_k(r_1) \leq r_1^{-1} M$$

for $|x| \leq r$. Thus $|x| \leq r$ implies that $|f_k'(x)|$ is

$$\leq r_1^{-1}M + \sum_{n=0}^{N-1} n c_{n,k}|x|^{n-1} \leq r_1^{-1}M + \sum_{n=0}^{N-1} n c_{n,k} r_1^{n-1} \leq r_1^{-1}M + N(N-1)Mr_1^{-1},$$

and $\{f_k'(x)\}$ is uniformly bounded for $|x| \leq r$.

A similar argument with f_k'' shows that

$$\Big|f_k''(x) - \sum_{n=0}^{N-1} n(n-1) c_{n,k} x^{n-2}\Big| \leq r_1^{-2} M,$$

and we find similarly that $\{f_k''(x)\}$ is uniformly bounded for $|x| \leq r$. This completes the proof.

31. Theorem 1.23 shows that the limit of the subsequence of first derivatives is the first derivative of the limit, the limit being differentiable. In other words, f is differentiable for $|x| < r$, and the subsequence converges to $f'(x)$ there. Since $r < R$ is arbitrary, f is differentiable for $|x| < R$. Now we can induct, replacing f and the sequence f_k in Problem 30 by f' and a subsequence of f_k' on a smaller disk, then passing to f'', and so on. The result is that f is infinitely differentiable for $|x| < R$.

32. This is proved in the same way as in Problem 9.

33. $|\frac{1}{N+k}z^k| \le r^{N+k}$ if $|z| \le r$, and $\sum_{k=0}^{\infty} r^{N+k} = r^N/(1-r)$. Thus $|\frac{1}{N}z^N + \frac{1}{N+1}z^{N+1} + \cdots|$ tends uniformly to 0 for $|z| \le r$. Since $t \mapsto \exp(t)$ is continuous at $t = 0$, the required convergence follows.

34. Corollary 1.38 shows from the behavior for z real that all c_n are 0.

35. Write

$$\exp\big(z + \tfrac{1}{2}z^2 + \tfrac{1}{3}z^3 + \cdots\big) = \big(\textstyle\prod_{k=1}^{N-1} \exp(\frac{1}{k}z^k)\big) \exp\big(\frac{1}{N}z^N + \frac{1}{N+1}z^{N+1} + \cdots\big).$$

Problem 33 shows that the left side is the uniform limit of $\prod_{k=1}^{N-1} \exp(\frac{1}{k}z^k)$ for $|z| \le r$ if $r < 1$. Each factor of the finite product is given by a convergent power series with nonnegative coefficients, and Theorem 1.40 shows that the finite product is given by a convergent power series with nonnegative coefficients. By Problem 32, $\exp\big(z + \frac{1}{2}z^2 + \frac{1}{3}z^3 + \cdots\big)$ is given by a convergent power series for $|z| < 1$. Hence $\exp\big(z + \frac{1}{2}z^2 + \frac{1}{3}z^3 + \cdots\big) - 1/(1-z)$ is given by a convergent power series for $|z| < 1$. For $z = x$ real with $|x| < 1$, the series expansion of Problem 12b shows that our expression is $\exp\big(-\log(1-x)\big) - 1/(1-x) = 0$. Thus our power series sums to 0 on the real axis. By Problem 34, it sums to 0 everywhere.

Chapter II

1. Let us compare $d(x, y)$ with $d(x, z) + d(z, y)$. If j contributes to $d(x, y)$, then $x_j \ne y_j$. Hence $x_j \ne z_j$ or $z_j \ne y_j$. Thus j contributes to at least one of $d(x, z)$ and $d(z, y)$. In other words, the contribution of j to $d(x, y)$ is $\le$ the contribution of j to $d(x, z) + d(z, y)$. Summing on j gives the desired result.

2. Let (X, d) be the given separable metric space, define E to be the subset of members x of X such that every neighborhood of x is uncountable, and let F be the complement of E. If x is in F, we can associate to x some open neighborhood N_x containing at most countably many elements, and N_x is entirely contained in F. As x varies in F, the sets N_x form an open cover of F. By Proposition 2.32b, some subcollection of the N_x that is at most countable covers F. The union of these sets is open and is at most countable, and it equals F.

3. Let $f(x) = 1/x$ for $0 < x \le 1$, and let $f(0) = 0$.

4. Suppose that x is in U. Since A is dense, the set $A \cap B(1/n; x)$ is nonempty for each $n \ge 1$. Let x_n be a member of it. Since U is open, $B(1/n; x)$ is contained in U if n is $\ge N$ for a suitable N. Thus x_n is in $A \cap U$ for $n \ge N$ and converges to x. By Proposition 2.22b, either $x_n = x$ infinitely often, in which case x is in $A \cap U$, or x is a limit point of $A \cap U$. In either case, $U \subset (A \cap U)^{\text{cl}}$.

5. For (a), the sets E_n are compact by the Heine–Borel Theorem. Then each $E_n - U$ is compact. Their intersection is $\bigcap_{n=1}^{\infty} (E_n \cap U^c) = \big(\bigcap_{n=1}^{\infty} E_n\big) \cap U^c \subseteq U \cap U^c = \varnothing$. By Proposition 2.35 the system $\{E_n - U\}$ does not have the finite-intersection property.

Thus $\bigcap_{n=1}^{N}(E_n - U) = \varnothing$ for some N. Since $E_1 \supseteq E_2 \supseteq \cdots$, we find that $E_N - U = \varnothing$. Therefore $E_N \subseteq U$.

For (b), let U be empty, and let $E_n = \mathbb{Q} \cap [\sqrt{2}, \sqrt{2} + 1/n]$.

6. In both parts of the problem, let the metrics be d_X, d_Y, d_Z. For (a), use continuity of F to choose for each (x_0, y) some $\delta_{1,y} > 0$ and $\delta_{2,y} > 0$ such that the two inequalities $d_X(x, x_0) < \delta_{1,y}$ and $d_Y(y', y) < \delta_{2,y}$ together imply $d_Z(F(x, y'), F(x_0, y)) < \epsilon/2$. As y varies, the open balls $B(\delta_{2,y}; y)$ cover Y. Since Y is compact, a finite number of them suffice to cover Y, say $B(\delta_{2,y_1}; y_1), \dots, B(\delta_{2,y_n}; y_n)$. Put $\delta_1 = \min\{\delta_{1,y_1}, \dots, \delta_{1,y_n}\}$. Suppose now that $d_X(x, x_0) < \delta_1$ and that y' is in Y. Then y' is in some $B(\delta_{2,y_j}; y_j)$. Hence we have $d_X(x, x_0) < \delta_1 \le \delta_{1,y_j}$ and $d(y', y_j) < \delta_{2,y_j}$, and we therefore obtain $d_Z(F(x, y'), F(x_0, y_j)) < \epsilon/2$. Since also $d_X(x_0, x_0) = 0$ and $d(y', y_j) < \delta_{2,y_j}$, we obtain also $d_Z(F(x_0, y'), F(x_0, y_j)) < \epsilon/2$. Combining these two results gives $d_Z(F(x, y'), F(x_0, y')) < \epsilon$.

For (b), consider $d_Z(F(x, y), F(x_0, y_0))$, and let $\epsilon > 0$ be given. By uniform convergence, choose $\delta_1 > 0$ such that $d_X(x, x_0) < \delta_1$ implies $d_Z(F(x, y), F(x_0, y)) < \epsilon/2$ for all y. Proposition 2.21 gives us continuity of $F(x_0, \cdot)$, and thus there exists $\delta_2 > 0$ such that $d_Y(y, y_0) < \delta_2$ implies $d_Z(F(x_0, y), F(x_0, y_0)) < \epsilon/2$. Then $d_X(x, x_0) < \delta_1$ and $d_Y(y, y_0) < \delta_2$ together imply $d_Z(F(x, y), F(x_0, y_0)) \le d_Z(F(x, y), F(x_0, y)) + d_Z(F(x_0, y), F(x_0, y_0)) < \epsilon/2 + \epsilon/2 = \epsilon$.

7. Let $f : (0, 1) \to \mathbb{R}$ be defined by $f(x) = 1/x$. Then the Cauchy sequence $\{1/n\}$ is carried to a sequence that is not Cauchy in $\mathbb{R}$.

8. Define inductively $f^{(0)}$ to be the identity and $f^{(k)} = f \circ f^{(k-1)}$ for $k > 0$. For existence we see inductively that $d(f^{(k)}(x), f^{(k)}(y)) \le r^k d(x, y)$ for all x and y. If $n \ge m$ and if x is arbitrary but fixed, we then have $d(f^{(n)}(x), f^{(m)}(x)) \le \sum_{k=m}^{n-1} d(f^{(k+1)}(x), f^{(k)}(x)) \le \sum_{k=m}^{n-1} r^k d(f(x), x) \le r^m d(f(x), x)/(1 - r)$. Hence the sequence $\{f^{(n)}(x)\}$ is Cauchy. Let x' be its limit. Since

$$d(f(f^{(n)}(x)), f^{(n)}(x)) = d(f^{(n+1)}(x), f^{(n)}(x)) \le r^n d(f(x), x)/(1 - r)$$

and since d and f are continuous, $d(f(x'), x') \le \limsup_n r^n d(f(x), x)/(1-r) = 0$. Thus $f(x') = x'$.

For uniqueness, let $f(x'') = x''$ also. Then $d(x'', x') = d(f(x''), f(x'))$ since f fixes x' and x'', and $d(f(x''), f(x')) \le r d(x'', x')$ by the contraction property. Then $(1 - r)d(x'', x') \le 0$ and we conclude that $d(x'', x') = 0$. Thus $x'' = x'$.

9. If no point is isolated, each one-point set is closed nowhere dense. The countable union of these sets is the whole space, in contradiction to the Baire Category Theorem. An alternative argument is to appeal to Problem 2.

10. The set is closed and bounded, hence compact, and it is pathwise connected, hence connected. It is not, however, locally connected. Take, for example, the point $p = [c, 1/2]$ in X, where c is in C. The open ball of radius 1/4 around p has the property that no open subneighborhood of p is connected.

11. Fix x_0 in X, and let U be the set of all points in X that can be connected to x_0 by paths. The set U is nonempty, and we prove that it is open and closed. Being connected, it must then be all of X. It is open because the local pathwise connectedness means that any x in U can be connected to every point in some neighborhood of x by a path; hence U contains a neighborhood of each of its points and is open. To see that U is closed, let y be a limit point of U. If V is a pathwise connected open neighborhood of y, the set $U \cap V$ is nonempty because y is a limit point of U. Let z be in $U \cap V$. Then x_0 can be connected to z by a path because of the defining property of U, and z can be connected to y by a path because V is pathwise connected. Hence x_0 can be connected to y by a path, and y is in U.

12. Any open subset of $\mathbb{R}^n$ is locally pathwise connected. So the desired conclusion follows from the previous problem.

13. Let the open set be U. For each x in U, let U_x be the union of all connected subsets of U containing x. It was shown in Section 8 that this is connected. For x and y in U, either $U_x = U_y$ or $U_x \cap U_y = \varnothing$ for the same reason. Then U is the disjoint union of its subsets U_x, which are connected. These are intervals, being connected, and they must be open in order not to be contained in larger connected subsets of U.

14. Same as for Proposition 2.21.

15. Suppose $\{f_t\}$ is totally bounded. Let $\epsilon > 0$ be given. Find, by total boundedness, real numbers $t_1, \ldots, t_n$ such that for any t, there is an index $j = j(t)$ with $\|f_t - f_{t_j}\| < \epsilon$. Put $L/2 = \max\{|t_1|, \ldots, |t_n|\}$. If we are given an interval of length $\geq L$, take t to be its center, so that the interval contains $[t - L/2, t + L/2]$. Choose j by total boundedness with $\|f_t - f_{t_j}\} < \epsilon$. Then $\|f_{t-t_j} - f_0\| < \epsilon$. So $t - t_j$ is an ϵ almost period, and this lies in $[t - L/2, t + L/2]$. Thus the Bohr condition holds.

Conversely suppose that the Bohr condition holds and f is uniformly continuous. Let $\epsilon > 0$ be given, and find L as in the Bohr condition for $\epsilon/2$ almost periods. Also, find some δ for uniform continuity of f and the number $\epsilon/2$. Choose $t_1, \ldots, t_n$ in $I = [-L/2, L/2]$ such that any point in I is within δ of one of $t_1, \ldots, t_n$. Let us see that the open balls of radius ϵ around $f_{t_1}, \ldots, f_{t_n}$ together cover the set $\{f_t\}$ of all translates. If t is given, find an $L/2$ almost period $t - s$ in $[t - L/2, t + L/2]$. Here $|s| < L/2$, so that $\|f_{t-s} - f_0\| < \epsilon/2$ and $\|f_t - f_s\| < \epsilon/2$. Since $|s - t_j| < \delta$ for some j, we have $\|f_s - f_{t_j}\| < \epsilon/2$ by uniform continuity. Thus $\|f_t - f_{t_j}\| < \epsilon$.

16. Let T_f be the closure of the set of translates of f. This is complete by Problem 14. Theorem 2.36 shows that T_f is compact if and only if every sequence in it has a convergent subsequence, and this is the definition of Bochner almost periodicity. Theorem 1.46 shows that T_f is compact if and only if it is totally bounded, and this is equivalent to Bohr almost periodicity by Problem 15.

17. This is easier with the Bochner definition. For an example of closure under the various operations, consider closure under multiplication. Suppose that f and g are given and that we want a convergent subsequence from the sequence of translates $(fg)_{t_n}$. First choose a subsequence of $\{t_n\}$ such that those translates of f converge

uniformly, and then choose a subsequence of that such that the translates of g converge uniformly. These sequences of translates of f and g will be uniformly bounded, and then it follows that the sequence of products converges uniformly.

For closure under uniform limits, we argue similarly with translates of each of the functions $\{f_n\}$ when $\lim f_n = f$ uniformly. A Cantor diagonal process is used to extract the sequence of translates to use for f.

18. If $\epsilon > 0$ is given, let U_n be the set where $|f(x) - f_n(x)| < \epsilon$. This is open by the assumed continuity, and $\bigcup_{n=1}^{\infty} U_n = X$ by the assumed convergence. Since X is compact, some finite collection of U_n's suffices. Since the f_n's are pointwise increasing with n, the U_n's are increasing, and thus $X = U_N$ for some N. For that N, $|f(x) - f_N(x)| < \epsilon$. Then $|f(x) - f_n(x)| < \epsilon$ for $n \geq N$ since the f_n's are pointwise increasing.

19. If $0 \leq P_n(x) \leq \sqrt{x} \leq 1$, then $x \geq P_n(x)^2$ and the recursion shows that $P_{n+1}(x) \geq P_n(x)$. Also, $P_{n+1}(x) = P_n(x) + \frac{1}{2}(\sqrt{x} + P_n(x))(\sqrt{x} - P_n(x)) \leq P_n(x) + \frac{1}{2}(1+1)(\sqrt{x} - P_n(x)) = \sqrt{x}$.

20. By Problem 19, $P_n(x)$ increases pointwise to some $f(x)$. Passing to the limit in the recursion gives $f(x) = f(x) + \frac{1}{2}(x - f(x)^2)$, and thus $f(x)^2 = x$ and $f(x) = \sqrt{x}$. Since $\sqrt{x}$ is continuous and $[0, 1]$ is compact, Dini's Theorem (Problem 18) shows that the convergence is uniform.

21. If x and y are given with $x \neq y$, then we are given three relevant functions in $\mathcal{A}$, possibly not all distinct. They are h_1 with $h_1(x) \neq h_1(y)$, h_2 with $h_2(x) \neq 0$, and h_3 with $h_3(y) \neq 0$. If $h_1(x)$ or $h_1(y)$ is 0, we can add a multiple of h_2 or h_3 to h_1 to obtain an h_4 with $h_4(x) \neq h_4(y)$, $h_4(x) \neq 0$, and $h_4(y) \neq 0$. The restrictions of h_4 and h_4^2 to the two-element set $\{x, y\}$ are linearly independent and therefore form a basis for the 2-dimensional space of restrictions. Hence some linear combination of h_4 and h_4^2 equals the given f at x and y.

22. Let f be in $C_{\mathbb{R}}(S)$ with $f(s_0) = 0$. Since $\mathcal{B}^{\text{cl}} = C_{\mathbb{R}}(S)$, there exists a sequence $\{g_n\}$ in $\mathcal{B}$ with $\lim g_n = f$ uniformly. Then $\lim g_n(s_0) = f(s_0) = 0$ in particular. Put $f_n(s) = g_n(s) - g_n(s_0)$. Then $f_n(s_0) = 0$. The inequality $|f_n(s) - f(s)| = |g_n(s) - f(s) - g_n(s_0)| \leq |g_n(s) - f(s)| + |g_n(s_0)|$ shows that $\{f_n\}$ converges uniformly to f. The members of $\mathcal{A}$ are the members of $\mathcal{B}$ that vanish at s_0. The functions f_n have this property, and thus $\{f_n\}$ is a sequence in $\mathcal{A}$ converging uniformly to f.

24. For (a), we identify $C_0([0, +\infty), \mathbb{R})$ with the subalgebra of $C([0, +\infty], \mathbb{R})$ of continuous functions equal to 0 at $+\infty$. The function e^{-x} separates points on $[0, +\infty]$. Apply Problem 22 to the algebra it generates, namely the algebra of all finite linear combinations of e^{-nx} for n a positive integer.

For (b), let $\epsilon > 0$ be given, and choose $g(x) = \sum c_n e^{-nx}$ by (a) such that $\sup_{0 \leq x < +\infty} |f(x) - g(x)| \leq \epsilon$. The hypothesis forces $\int_0^b f(x)g(x)\,dx = 0$, and this is $\int_0^b f(x)^2\,dx - \int_0^b f(x)\big(f(x)-g(x)\big)\,dx$. Thus

$$0 \ge \int_0^b f(x)^2\,dx - \Big|\int_0^b f(x)\big(f(x)-g(x)\big)\,dx\Big|.$$

So $\int_0^b f(x)^2\,dx \le \epsilon \int_0^b |f(x)|\,dx$. Since ϵ is arbitrary, $\int_0^b f(x)^2\,dx = 0$. Therefore $f = 0$.

25. Isometries are uniformly continuous. Applying Proposition 2.47 to the uniformly continuous function $\varphi_2 \circ \big(\varphi_1^{-1}\big|_{\varphi_1(X)}\big)$ of the dense subset $\varphi_1(X)$ of X_1^* into X_2^*, we obtain an isometry $\Psi : X_1^* \to X_2^*$ extending $\varphi_2 \circ \big(\varphi_1^{-1}\big|_{\varphi_1(X)}\big)$. Reversing the roles of X_1^* and X_2^*, we obtain an isometry $\Phi : X_2^* \to X_1^*$ extending $\varphi_1 \circ \big(\varphi_2^{-1}\big|_{\varphi_2(X)}\big)$. Then $\Phi \circ \Psi$ is a continuous extension of the composition $\varphi_1 \circ \big(\varphi_2^{-1}\big|_{\varphi_2(X)}\big) \circ \varphi_2 \circ \big(\varphi_1^{-1}\big|_{\varphi_1(X)}\big)$, which is the identity map on $\varphi_1(X)$. Hence $\Phi \circ \Psi$ is the identity on X_1^*. Similarly $\Psi \circ \Phi$ is the identity on X_2^*. Thus Ψ is onto. This proves existence.

For uniqueness let Ψ and Ψ^* be two such maps. Then $\Psi^{-1} \circ \Psi^*$ is a continuous extension of the identity map on the dense subset $\varphi_1(X)$ of X_1^*, and hence it is the identity. Therefore $\Psi = \Psi^*$.

26. Theorem 2.60 says that X is dense in X^*. Then $X = X^*$ if and only if X is closed, and this happens if and only if X is complete, by Proposition 2.43.

27. The only one of these that requires explanation is (iv). We may assume that none of r, s, and $r+s$ is 0. Write $r = mp^k/n$ and $s = up^l/v$ with p not dividing any of r, s, u, v. Without loss of generality, we may assume $k \le l$, so that $\max\{|r|_p, |s|_p\} = |r|_p = p^{-k}$. We have

$$r + s = mp^k/n + up^l/v = p^k\big(\tfrac{m}{n} + \tfrac{up^{l-k}}{v}\big) = p^k\big(\tfrac{mv+p^{l-k}nu}{nv}\big).$$

The denominator nv is not divisible by p. The part of the numerator within the parentheses is an integer, and we factor out any factors of p from it as p^a with $a \ge 0$. Then we have $|r+s|_p = p^{-(k+a)}$ and this is $\le p^{-k}$ as required.

28. For the triangle inequality, let r, s, t be given. Then Problem 27 gives $d(r,t) = |r-t|_p = |(r-s)+(s-t)|_p \le \max\{|r-s|_p, |s-t|_p\} \le |r-s|_p + |s-t|_p = d(r,s) + d(s,t)$.

29. Part (a) will be illustrated by the more difficult (b) and (c). Multiplication by a member r of $\mathbb{Q}$ is a uniformly continuous function from $\mathbb{Q}$ into $\mathbb{Q}_p$; in fact, the equality $|r(s-s_0)|_p = |r|_p|s-s_0|_p$ shows that if ϵ is given, then the δ of uniform continuity can be taken as $|r|_p^{-1}\epsilon$. Proposition 2.47 then tells us how to form products rs for r in $\mathbb{Q}$ and s in $\mathbb{Q}_p$. For fixed s, the result is a uniformly continuous map of $\mathbb{Q}$ into $\mathbb{Q}_p$ since $|\cdot|_p$ extends continuously to $\mathbb{Q}_p$ and we have $|(r-r_0)s|_p = |r-r_0|_p|s|_p$. A second application of Proposition 2.47 extends the operation to a mapping of $\mathbb{Q}_p \times \mathbb{Q}_p$ into $\mathbb{Q}_p$ that is uniformly continuous in each variable when the other variable is held fixed. In

fact, it is continuous in both variables since $|rs-r_0s_0|_p = |(r-r_0)s+r_0(s-s_0)|_p \le |r-r_0|_p|s|_p + |r_0|_p|s-s_0| \le |r-r_0|_p|s-s_0|_p + |r-r_0|_p|s_0|_p + |r_0|_p|s-s_0|$.

For (c), take a shell $A_{kn} = \{r \in \mathbb{Q}_p \mid p^{-k} \le |r|_p \le p^n\}$. This is a closed subset of $\mathbb{Q}_p$, hence complete. Reciprocal is a mapping from $A_{nk} \cap \mathbb{Q}$ into A_{kn} that is uniformly continuous because r and s in $A_{nk} \cap \mathbb{Q}$ implies $|r^{-1} - s^{-1}|_p = |(s-r)/rs|_p = |s-r|_p|r|_p^{-1}|s|_p^{-1} \le p^{2n}|s-r|_p$. Hence reciprocal extends to a uniformly continuous mapping from A_{nk} to A_{kn}. These mappings are consistent as n and k tend to infinity, and thus reciprocal is a well-defined function from $\mathbb{Q}_p^\times$ to itself. It is continuous because the same computation as just given shows that $|r^{-1} - r_0^{-1}|_p = |r-r_0|_p|r|_p^{-1}|r_0|_p^{-1}$. If we write $|r|_p \ge \big||r_0|_p - |r-r_0|_p\big|$ and require that $|r-r_0|_p \le \frac{1}{2}|r_0|_p$, then $|r^{-1} - r_0^{-1}|_p = |r-r_0|_p\big(\frac{1}{2}|r_0|_p\big)^{-1}|r_0|_p^{-1}$, and continuity of reciprocal at r_0 follows.

The abelian group axioms in (c) are associativity, commutativity, existence of the two-sided identity 1, and existence of two-sided reciprocals. To complete (c), we need associativity and commutativity. We can regard associativity as asserting the equality of two continuous functions from $\mathbb{Q}_p \times \mathbb{Q}_p \times \mathbb{Q}_p$ to $\mathbb{Q}_p$. These are equal on $\mathbb{Q} \times \mathbb{Q} \times \mathbb{Q}$, and this subset is dense. Hence the two functions are equal everywhere. Commutativity is proved similarly.

The distributive law in (d) is proved by the same technique used for associativity in (c). Thus $\mathbb{Q}_p$ is a field.

30. For (a), it is enough to prove that $S = \{t \in \mathbb{Q} \mid |t|_p \le 1\}$ is totally bounded. For x in $\mathbb{Q}$, let $C(\delta; x) = \{t \in \mathbb{Q} \mid |t-x|_p \le \delta\}$. It is enough to show for each integer $l \ge 0$ that $S \subseteq \bigcup_{r=0}^{p^l-1} C(p^{-l}; r)$. If t is given in S, t is of the form $t = m/n$ with m and n in $\mathbb{Z}$ and n nondivisible by p. Let n^{-1} denote the integer from 0 to $p^l - 1$ such that $nn^{-1} \equiv 1 \bmod p^l$, and let r denote the integer from 0 to $p^l - 1$ such that $n^{-1}m \equiv r \bmod p^l$. Then $m - nr \equiv 0 \bmod p^l$, and so $|m-nr|_p \le p^{-l}$. Since $|n|_p = 1$, $\big|\frac{m}{n} - r\big|_p \le p^{-l}$. Thus t is in $C(p^{-l}; r)$.

For (b), compact sets are closed and bounded by Proposition 2.34a. Conversely let E be closed and bounded. The set $T = \{t \in \mathbb{Q}_p \mid |t|_p \le 1\}$ is certainly closed. Since $\mathbb{Q}_p$ is complete, T is complete. Part (a) shows that T is totally bounded. By Theorem 2.46, T is compact. The given set E is contained in some set $T_n = \{t \in \mathbb{Q}_p \mid |t|_p \le p^n\}$. Multiplication by the member p^{-n} of $\mathbb{Q}_p$ carries T continuously onto T_n, and T_n is compact by Proposition 2.38. Since E is a closed subset of the compact set T_n, Proposition 2.34b shows that E is compact.

31. The first two assertions are routine consequences of (ii), (iii), and (iv). Let us consider the quotient $\mathbb{Z}_p/P$. We show that P is a maximal ideal. In fact, if I is an ideal in $\mathbb{Z}_p$ properly containing P, then I contains some element t with $|t|_p = 1$. Then (iii) shows that t^{-1} has $|t^{-1}|_p = 1$ and lies in $\mathbb{Z}_p$. Since t is in I and t^{-1} is in $\mathbb{Z}_p$, their product 1 is in I. Thus $I = \mathbb{Z}_p$. In other words, P is a maximal ideal. Hence $\mathbb{Z}_p/P$ is a field. To complete the argument, we show that $\mathbb{Z}_p/P$ has exactly p elements. Given x in $\mathbb{Z}_p$, choose m/n in $\mathbb{Q}$ with $\big|x - \frac{m}{n}\big|_p \le p^{-1}$, by denseness of

$\mathbb{Q}$ in $\mathbb{Q}_p$. Here $\left|\frac{m}{n}\right|_p \le 1$, and we may assume that n is nondivisible by p. Arguing as in Problem 30a, we can find r in $\{0, 1, \dots, p-1\}$ such that $\left|\frac{m}{n} - r\right|_p \le p^{-1}$. Then $|x - r|_p \le \max\left\{\left|x - \frac{m}{n}\right|_p, \left|\frac{m}{n} - r\right|_p\right\} \le p^{-1}$ by the ultrametric inequality. So $x = (x - r) + r$ with $x - r$ in P. Thus $\{0, 1, \dots, p-1\}$ represents all cosets of $\mathbb{Z}_p/P$. Finally no two distinct elements r and r' in $\{0, 1, \dots, p-1\}$ have $|r - r'|_p \le p^{-1}$ because this inequality would entail having $r - r'$ divisible by p.

Chapter III

1. For (a), $|TS|^2 = \sum_j |TS(e_j)|^2 = \sum_j |\sum_i (S(e_j), e_i)T(e_i)|^2$. Use of the triangle inequality and then the Schwarz inequality shows that this expression is $\le \sum_j \big(\sum_i |(S(e_j), e_i)|\,|T(e_i)|\big)^2 \le \sum_j \big(\big(\sum_i |(S(e_j), e_i)|^2\big)^{1/2}\big(\sum_i |T(e_i)|^2\big)^{1/2}\big)^2 = \sum_j |S(e_j)|^2 |T|^2 = |S|^2\,|T|^2$. Part (b) is routine.

2. The member of $L(\mathbb{R}^n, \mathbb{R}^m)$ with matrix A.

3. $\limsup_{h\to 0}\big(|h|^{-1}|f(h) - 0 - 0|\big) \le \limsup_{h\to 0}(|h|^{-1}|h|^2) = 0$.

4. The formula is $\frac{d}{dt} f(x + tu)\big|_{t=0} = \sum_j u_j \frac{\partial f}{\partial x_k}(x)$. The argument is written out within the proof of Theorem 3.11.

5. $\begin{pmatrix} e^t & 0 \\ 0 & e^{-t}\end{pmatrix}$, $e^t\begin{pmatrix} 1 & t \\ 0 & 1\end{pmatrix}$, $\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t\end{pmatrix}$, $\begin{pmatrix} \cos t & i\sin t \\ i\sin t & \cos t\end{pmatrix}$, $\begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t\end{pmatrix}$.

7. The equality is false because the left side is positive and the right side is negative. In fact, the left side is $\int_0^1 \big[\lim \int_1^N (e^{-xy} - 2e^{-xy})\,dx\big]\,dy$, which equals $\int_0^1 \lim\big[-e^{-xy}/y + e^{-2xy}/y\big]_1^N\,dy = \int_0^1 \frac{1}{y}\big[e^{-y} - e^{-2y}\big]\,dy$; since $e^{-y} > e^{-2y}$ on $(0, 1)$, the left side is > 0. Meanwhile, the right side is $\int_1^\infty \big[-e^{-xy}/x + e^{-2xy}/x\big]_0^1\,dx = \int_1^\infty \frac{1}{x}\big[e^{-2x} - e^{-x}\big]\,dx$; since $e^{-2x} < e^{-x}$ on $(1, \infty)$, the right side is < 0.

8. Define $\|\cdot\|_2$ as in Section I.10, and let $f_x(t) = f(x - t)$; the latter definition is not the one used earlier in the book. For (a), the Schwarz inequality gives

$$\begin{aligned}|f * g(x) - f * g(x_0)| &= \big|\tfrac{1}{2\pi}\textstyle\int_{-\pi}^{\pi}[f(x - t) - f(x_0 - t)]g(t)\,dt\big| \\ &= \|f_x - f_{x_0}\|_2\|g\|_2 \le \|g\|_2 \sup_t |f(x - t) - f(x_0 - t)|,\end{aligned}$$

and the right side tends to 0 as x tends to x_0 by uniform continuity of f. This proves that $f * g$ is continuous. The periodicity is evident. The proof that $f * g = g * f$ is the same as the proof in Section I.10 that $f * D_N = D_N * f$.

For (b), an application of Fubini's Theorem (Corollary 3.33) and a change of variables gives $\frac{1}{2\pi}\int_\pi^\pi f * g(x)e^{-inx}\,dx = \big(\frac{1}{2\pi}\big)^2 \int_\pi^\pi \int_\pi^\pi f(x - t)g(t)e^{-inx}\,dt\,dx = \big(\frac{1}{2\pi}\big)^2 \int_\pi^\pi \int_\pi^\pi f(x - t)g(t)e^{-inx}\,dx\,dt = \big(\frac{1}{2\pi}\big)^2 \int_\pi^\pi \int_\pi^\pi f(x)g(t)e^{-in(x+t)}\,dx\,dt = \big(\frac{1}{2\pi}\big)^2 \int_\pi^\pi \int_\pi^\pi f(x)g(t)e^{-inx}e^{-int}\,dx\,dt = c_n d_n$.

For (c), we apply the Weierstrass M test. It is enough to prove that $\sum_n |c_n d_n| < +\infty$, and the Schwarz and Bessel inequalities together do this:

$$\sum_n |c_n d_n| \le \Big(\sum_n |c_n|^2\Big)^{1/2}\Big(\sum_n |d_n|^2\Big)^{1/2} \le \|f\|_2\|g\|_2 < +\infty.$$

9. Write out each side as an iterated integral, and apply Fubini's Theorem (Corollary 3.33).

10. For the partial derivatives, $\frac{\partial x}{\partial x}(0,0) = \frac{d}{dx} f\big(\frac{x0}{x^2+0}\big)\big|_{x=0} = 0$ and $\frac{\partial f}{\partial y}(0,0) = 0$ similarly. The fact that f is not continuous at $(0,0)$ is a special case of Problem 11a.

11. For (a), the homogeneity says in particular that $f(rx) = f(x)$ for $r > 0$ and $|x| = 1$. Then $\sup_{y\neq 0} |f(y)| = \sup_{|x|=1} |f(x)|$, and the right side is finite, being the maximum value of a continuous function on a compact set. If $f(y)$ is continuous at $y = 0$, then $f(0) = \lim_{r\downarrow 0} f(rx) = f(x)$ for every x with $|x| = 1$ and so f must be constantly equal to $f(0)$.

For (b), $\limsup_{rx\to 0} |f(rx)| = \limsup_{rx\to 0} r^d |f(x)| = 0$ if $d > 0$ since $f(x)$ is bounded for $|x| = 1$. Thus f is continuous at 0 if $d > 0$ and $f(0) = 0$. If $d < 0$, then $\limsup_{rx\to 0} r^d |f(x)| = +\infty$ if $d < 0$ and $f(x) \neq 0$.

For (c), we have $f(rx) = r^d f(x)$ for any $x = (x_1, \dots, x_n) \neq 0$. Put $g = f \circ m_r$, where m_r refers to multiplication by r. The homogeneity gives $g = r^d f$, and thus $\frac{\partial g}{\partial x_j}(x) = r^d \frac{\partial f}{\partial x_j}(x)$. On the other hand, the chain rule gives $\frac{\partial g}{\partial x_j}(x) = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(rx)\frac{\partial (rx_i)}{\partial x_j}(x) = r\frac{\partial f}{\partial x_j}(rx)$. So $r^d \frac{\partial f}{\partial x_j}(x) = r\frac{\partial f}{\partial x_j}(rx)$, and (c) follows.

For (d), the given conditions say that $f(tx) = tf(x)$ for all real t. Then $\frac{\partial f}{\partial x_j}(0) = \lim_{t\to 0} t^{-1}\big(f(0+te_j) - 0\big) = \lim_{t\to 0} t^{-1} t f(e_j) = f(e_j)$. On the other hand, (c) says that $\partial f/\partial x_j$ is homogeneous of degree 0, and (a) says that $\partial f/\partial x_j$ cannot be continuous at 0 unless it is constant.

12. Part (a) follows from Problem 11b. In (b), $\frac{\partial f}{\partial x}(0) = \frac{d}{dt} f(0 + t(1,0))\big|_{t=0} = \frac{d}{dt} t\big|_{t=0} = 1$ and $\frac{\partial f}{\partial y}(0) = \frac{d}{dt} f(0+ty)\big|_{t=0} = \frac{d}{dt} 0\big|_{t=0} = 0$. The failure of continuity is by parts (a) and (c) of Problem 11.

For (c), we have $\frac{d}{dt} f(0+tu)\big|_{t=0} = \frac{d}{dt} t\cos^3\theta\big|_{t=0} = \cos^3\theta$. If f were differentiable at $x = 0$, the chain rule would give $\frac{d}{dt} f(0+tu)\big|_{t=0} = u_1\frac{\partial f}{\partial x}(0) + u_2\frac{\partial f}{\partial y}(0) = \cos\theta$. Since $\cos^3\theta$ is not identically equal to $\cos\theta$, f is not differentiable at 0.

13. Part (a) follows from (a), (b), and (c) of Problem 11. About 0, the function f is even in x and even in y, and hence the first partial derivatives are odd about 0. Then part (b) follows from Problem 11d. To calculate the results for (c), we need to compute $\frac{\partial f}{\partial y}(x,0)$ for $x \neq 0$ and $\frac{\partial f}{\partial x}(0,y)$ for $y \neq 0$. The first of these is x, and the second is $-y$. The formulas for the second partial derivatives follow.

14. For $n \ge 0$, $r^n e^{in\theta} = (x+iy)^n$ is of class C^∞, and so is $r^n e^{-in\theta} = (x-iy)^n$. For the first of these functions, $\frac{\partial^2}{\partial x^2}(x+iy)^n = n(n-1)(x+iy)^{n-2}$, while $\frac{\partial^2}{\partial y^2}(x+iy)^n = i^2 n(n-1)(x+iy)^{n-2}$. Hence $\Delta(x+iy)^n = 0$. The result for $(x-iy)^n$ follows

by taking complex conjugates. The final conclusion is a routine consequence of Theorem 1.37, the complex-valued version of Theorem 1.23, and the fact that each term is harmonic.

15. This follows by direct calculation.

16. In the notation of Theorem 3.17, $\varphi(x, y)$ is $\begin{pmatrix} x^4 y + x \\ x + y^3 \end{pmatrix}$, a is $(1, 1)$, and b is $(2, 2)$. One checks that $\varphi'(1, 1) = \begin{pmatrix} 5 & 1 \\ 1 & 3 \end{pmatrix}$. The locally defined inverse function f near $(2, 2)$ has $f'(2, 2) = \varphi'(1, 1)^{-1} = \begin{pmatrix} 3/14 & -1/14 \\ -1/14 & 5/14 \end{pmatrix}$, and $\frac{\partial F}{\partial u}(2, 2)$ is the upper left entry of this, namely $3/14$.

17. All 6 derivatives of possible interest are given by the matrix product $\begin{pmatrix} 2 & -1 & 0 \\ 2 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 0 & 0 \\ 0 & -\pi/2 \\ 0 & 0 \end{pmatrix} = \frac{1}{6} \begin{pmatrix} 0 & -\pi/2 \\ 0 & -\pi \\ 0 & 3\pi/2 \end{pmatrix}$. Then $\frac{\partial x}{\partial u}(\pi/2, 0) = 0$ and $\frac{\partial x}{\partial v}(\pi/2, 0) = -\pi/12$. The function $x(u, v)$ is of class C^∞ by Corollary 3.21.

18. The map in question is $X \mapsto X^2$ and is the composition of $X \mapsto (X, X)$ followed by $(U, V) \mapsto UV$. Here we can write $UV = L(U)V = R(V)U$, where $L(U)$ is the linear function "left multiplication by U" on matrix space and $R(V)$ is the linear function "right multiplication by V." The derivative of $(U, V) \mapsto UV$ is then $(\, R(V) \quad L(U)\,)$ by Problem 2. Hence the derivative of $X \mapsto X^2$, by the chain rule, is

$$(\, R(V) \quad L(U)\,) \begin{pmatrix} 1 \\ 1 \end{pmatrix} \bigg|_{U=V=X} = (R(V) + L(U))\big|_{U=V=X} = R(X) + L(X).$$

At $X = 1$, this is $R(1) + L(1)$, which is "multiplication by 2" and is invertible. The Inverse Function Theorem thus applies.

19. We may assume that $g'(x_0) \neq 0$, thus that $\frac{\partial g}{\partial x_i}(x_0) \neq 0$ for some i. We take this i to be $i = n$; the other cases involve only notational changes. Write $x = (x', x_n)$ with $x' \in \mathbb{R}^{n-1}$, and write $x_0 = (a, b)$ similarly. Then the Implicit Function Theorem produces a real-valued C^1 function $h(x')$ defined on an open set V about the point a in $\mathbb{R}^{n-1}$ such that $h(a) = b$, $g(x', h(x')) = 0$ for all x' in V, and $\frac{\partial h}{\partial x_j}(a) = -\big(\frac{\partial g}{\partial x_n}(a, b)\big)^{-1}\big(\frac{\partial g}{\partial x_j}(a, b)\big)$ for $1 \leq j < n$. Let $H(x) = (x', h(x'))$. Form $f \circ H$, which has a local maximum or minimum at $x' = a$ in V. All the first partial derivatives of this function must be 0 at $x' = a$. Thus, for $1 \leq j \leq n-1$, $0 = \frac{\partial (f \circ H)}{\partial x_j}(a) = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(x_0) \frac{\partial H_i}{\partial x_j}(a)$. Since $H_i(x) = x_i$ for $i < n$, all the terms of this sum are 0 except possibly for the j^{th} and the n^{th}. Thus $0 = \frac{\partial f}{\partial x_j}(x_0) + \frac{\partial f}{\partial x_n}(x_0) \frac{\partial h}{\partial x_j}(a)$ $= \frac{\partial f}{\partial x_j}(x_0) - \big(\frac{\partial f}{\partial x_n}\big)(x_0)\big(\frac{\partial g}{\partial x_n}(a, b)\big)^{-1}\big(\frac{\partial g}{\partial x_j}(a, b)\big)$ for $j < n$. The right side is 0 trivially for $j = n$, and thus the result follows with $\lambda = -\big(\frac{\partial g}{\partial x_n}(a, b)\big)^{-1}$.

20. The difficulty in handling this inequality as a maximum-minimum problem is the question of existence. Lagrange multipliers can constrain matters to a compact set, and then existence is no longer an obstacle. The domain D initially will be the set where $a_1 \geq 0, \ldots, a_n \geq 0$. Fix a number c, and let $g(a_1, \ldots, a_n) = \frac{1}{n}(a_1 + \cdots + a_n) - c$ and $f(a_1, \ldots, a_n) = \sqrt[n]{a_1 \cdots a_n}$. The subset of D where $g(a_1, \ldots, a_n) = 0$ is compact, and f must have an absolute maximum on it. This maximum cannot occur where any a_j equals 0 since f is 0 at such points. So it is at a point in the set U where all a_j are > 0. Apply Lagrange multipliers on U. The resulting equations are $\frac{1}{n}(a_1 \cdots a_n)^{1/n}/a_j = 1/n$ for $1 \leq j \leq n$, as well as the constraint equation $\frac{1}{n}(a_1 + \cdots + a_n) = c$. The first n equations show that all a_j's must be equal, and the constraint equation shows that they must equal c. The desired inequality is true in this case and hence is true in all cases.

21. Write $x(\theta) = r(\theta)\cos\theta$ and $y(\theta) = r(\theta)\sin\theta$, differentiate with respect to θ, and form $x'(\theta)^2 + y'(\theta)^2$. The result is that $x'(\theta)^2 + y'(\theta)^2 = r'(\theta)^2 + r^2$. Substitution into the result of Theorem 3.42 gives the desired formula.

22. For (a), $s(t) = \int_0^t \sqrt{\big(\frac{d}{du}\cos u\big)^2 + \big(\frac{d}{du}\sin u\big)^2 + \big(\frac{d}{du}u\big)^2}\,du = \sqrt{2}\int_0^t du = t\sqrt{2}$.

For (b), $s(x) = \int_0^x \sqrt{\big(\frac{d}{du}u\big)^2 + \big(\frac{d}{du}\frac{1}{2}(e^u + e^{-u})\big)^2}\,du$. Here $\frac{d}{du}\big(\frac{1}{2}(e^u + e^{-u})\big) = \frac{1}{2}(e^u - e^{-u})$, and the sum of 1 and the square of this is the square of $\frac{1}{2}(e^u + e^{-u})$. Thus $s(x) = \int_0^x \frac{1}{2}(e^u + e^{-u})\,du = \frac{1}{2}(e^x - e^{-x})$.

For (c), $s(x) = \int_0^x \sqrt{\big(\frac{d}{du}u\big)^2 + \big(\frac{d}{du}u^{3/2}\big)^2}\,du = \int_0^t \sqrt{1 + \frac{9}{4}u}\,du$, and this equals $\frac{8}{27}\big[\big(1 + \frac{9}{4}t\big)^{3/2} - 1\big]$.

For (d), the integral in question is $s(x) = \int_0^x \sqrt{1 + y'(t)^2}\,dt$. Since $y'(t) = 2t$, the right side is equal to $\int_0^x \sqrt{1 + 4t^2}\,dt$. The substitution $2t = \tan u$ leads to an integral of a multiple of $\sec^3 u = \cos u/\cos^4 u = (\cos u)(1 - \sin^2 u)^{-2}$. Then the substitution $v = \sin u$ leads to a definite integral of $(1 - v^2)^{-2}$, which can be handled by partial fractions.

For (e), we have $r(t) = t$ and $r'(t) = 1$. Problem 21 shows that the integral is $s(t) = \int_{\theta_0}^{\theta} \sqrt{t^2 + 1}\,dt$. This is treated the same way as in (d).

For (f), we have $x(\theta) = \theta\cos\theta$ and $y(\theta) = \theta\sin\theta$. These are both C^1 functions in an interval about 0, and thus $x'(\theta)$ and $y'(\theta)$ have finite limits at $\theta = 0$. Hence the curve is tamely behaved at 0.

23. $\ell(\gamma) = \int_0^1 \sqrt{4t^2 + 1}\,dt + \int_1^2 \sqrt{2}\,dt + \int_2^3 \sqrt{1 + 4(t-2)^2}\,dt$, and if one wants, these integrals can be evaluated exactly.

24. The first line of inequalities is proved in the same way as for Lemmas 1.24 and 1.25. Any two partitions have a common refinement, and thus the second line of inequalities follows. Taking the infimum over P_1 and then the infimum over P_2 yields the third inequality.

25. Let $\epsilon > 0$ be given. Choose $\delta > 0$ so that $|f(x) - f(x')| \le \epsilon$ whenever $|x - x'| \le \delta$. If $\mu(P) \le \delta$, then $\max_{x_{j-1} \le x \le x_j} f(x) - \min_{x_{j-1} \le x \le x_j} f(x) \le \epsilon$. Hence

$$\begin{aligned} U(P, f, \alpha) - L(P, f, \alpha) &= \sum_{j=1}^{m} \Big(\max_{x_{j-1} \le x \le x_j} f(x) - \min_{x_{j-1} \le x \le x_j} f(x) \Big) \big(\alpha(x_j) - \alpha(x_{j-1})\big) \\ &\le \sum_{j=1}^{m} \epsilon \big(\alpha(x_j) - \alpha(x_{j-1})\big) = \epsilon \big(\alpha(b) - \alpha(a)\big). \end{aligned}$$

26. Let $A = \sup_{P'} L(P', f, \alpha)$. From Problem 24 it follows that $U(P, f, \alpha) \ge A \ge L(P, f, \alpha)$ for every P. Combining this inequality with Problem 25 shows that $\lim_{\mu(P)\to 0} U(P, f, \alpha) = A = \lim_{\mu(P)\to 0} L(P, f, \alpha)$.

27. With $\gamma(t) = (1 - t)(x_1, y_1) + t(x_2, y_2)$, we have $x(t) = x_1 + t(x_2 - x_1)$, $dx = (x_2 - x_1)\,dt$, $y(t) = y_1 + (y_2 - y_1)t$, and $dy = (y_2 - y_1)\,dt$. Then $\int_\gamma x\,dy = \int_0^1 (x_1 + (x_2 - x_1)t)(y_2 - y_1)\,dt = x_1(y_2 - y_1) + \frac{1}{2}(x_2 - x_1)(y_2 - y_1)$, and similarly $\int_\gamma y\,dx = y_1(x_2 - x_1) + \frac{1}{2}(x_2 - x_1)(y_2 - y_1)$. Subtraction gives $\int_\gamma x\,dy - y\,dx = x_1(y_2 - y_1) - y_1(x_2 - x_1) = x_1 y_2 - x_2 y_1$.

28. In (b), take $f(x, y) = \frac{1}{2}\log(x^2 + y^2)$.

29. In (b), $\int_\gamma F \cdot d\mathbf{s} = \int_0^{2\pi} (P(\cos t, \sin t)(-\sin t) + Q(\cos t, \sin t)(\cos t))\,dt = \int_0^{2\pi} (-\sin^2 t - \cos^2 t)\,dt = \int_0^{2\pi} (-1)\,dt = -2\pi$.

In (c), if there were such a function, then Proposition 3.46 would say that $\int_\gamma F \cdot d\mathbf{s} = 0$, in contradiction to the result of (b).

30. $\int_0^1 t\,dt + \int_0^1 2t^5\,dt + \int_0^1 3t^{11}\,dt$, etc.

31. Since $\begin{pmatrix} e^x \cos y \\ -e^x \sin y \end{pmatrix} = \nabla(e^x \cos y)$, the line integral equals $\int_\gamma \begin{pmatrix} y \\ -x \end{pmatrix} \cdot d\mathbf{s} = \int_0^{2\pi} ((\sin t)(-\sin t) + (-\cos t)(\cos t))\,dt = -2\pi$.

32. In Green's Theorem with $P(x, y) = -\frac{1}{2}y$ and $Q(x, y) = \frac{1}{2}x$, we have $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial x} = 1$. Thus $\int_\gamma \frac{1}{2}x\,dy - \frac{1}{2}y\,dx = \iint_U \big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial x}\big)\,dx\,dy = \iint_U 1\,dx\,dy = \text{Area}(U)$.

33. The integral over the polygon of $\frac{1}{2}(x\,dy - y\,dx)$ is the sum of terms as in Problem 27, and this expression equals $\sum_{j=0}^{m} (x_j y_{j+1} - y_j x_{j+1})$. Green's Theorem applies in this situation, according to Corollary 3.50, and the line integral therefore equals the double integral over the inside of the polygon. The integrand is 1, according to Problem 32, and thus the double integral gives the area of the inside.

Chapter IV

1. For (a), $\frac{1}{2}y^2 = -\frac{1}{2}t^2 + c$. Adjusting c, we have $y^2 = -t^2 + c$. Then $y(t) = \pm\sqrt{c - t^2}$. For (b), the exceptional points are $(t_0, 0)$. For (c), a solution with $y(t_0) = y_0$ is $y(t) = \operatorname{sgn}(y_0)\sqrt{y_0^2 + t_0^2 - t^2}$.

2. In Theorem 4.1, take $a = 1$ and $b = 1$. Then $M = 2$ and $a' = \frac{1}{2}$. The theorem therefore gives a solution for $|t| < 1/2$.

3. To be an integral curve, $(x(t), y(t))$ must satisfy $x'(t) = \sqrt{x}$ and $y'(t) = 1/2$. Then $2\sqrt{x(t)} = t + c_1$ and $y(t) = \frac{1}{2}t + c_2$. At some unspecified time t_0, the curve is to pass through $(1, 1)$. Then $x(t_0) = 1$ and $y(t_0) = 1$; these force $2 = t_0 + c_1$ and $1 = \frac{1}{2}t_0 + c_2$. So $(x(t), y(t)) = \big(\frac{1}{4}(t - t_0 + 2)^2, \frac{1}{2}(t - t_0 + 2)\big)$. If $t_0 = 0$, for example, the curve is $(x(t), y(t)) = \big(\frac{1}{4}(t + 2)^2, \frac{1}{2}(t + 2)\big)$.

4. This uses the multivariable chain rule, Proposition 3.28b, and the Fundamental Theorem of Calculus. The derivative in question is

$$= (2t)(1/t^2)\sin(t^3) + \int_0^{t^2} (\partial/\partial t)(s^{-1}\sin(st))\,ds = (2/t)\sin(t^3) + \int_0^{t^2}\cos(st)\,ds$$

$$= (2/t)\sin(t^3) + \big[t^{-1}\sin(st)\big]_{s=0}^{t^2} = (2/t)\sin(t^3) + t^{-1}\sin(t^3).$$

5. $y(t) = 2 + c_1e^t + c_2e^{2t}$.

6. For (a), $J = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$ for the first, and $J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{pmatrix}$ and $B = \begin{pmatrix} 0 & i & -i \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$ for the second. For (b), the bases are $e^{3t}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $e^{3t}\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} + t\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right)$ for the first, and $e^t\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, e^{it}\begin{pmatrix} i \\ 0 \\ 1 \end{pmatrix}, e^{-it}\begin{pmatrix} -i \\ 0 \\ 1 \end{pmatrix}$ for the second.

Part (b) can be solved directly without solving part (a) first. Consider the 2-by-2 example. The only root of the characteristic polynomial is 3, and it has multiplicity 2. We solve $(A - 3\cdot 1)k_0 = 0$ and get $k_0 = \begin{pmatrix} c \\ 2c \end{pmatrix}$. Then we solve $(A - 3\cdot 1)l_0 = \begin{pmatrix} c \\ 2c \end{pmatrix}$ and get $l_0 = \begin{pmatrix} d \\ c + 2d \end{pmatrix}$. Choose any $c \neq 0$ and any d, say $c = 1$ and $d = 0$. Then $k_0 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, and $l_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, and we obtain the solutions in the form given above. For more complicated examples, the choice of these constants can get tricky, but this method works quickly for easy examples.

7. For $n = 1$, $\det(\lambda - (-a_0)) = \lambda + a_0$. Assume the result for $n - 1$, and expand the n^{th}-order determinant by cofactors about the first column. Then

$$\det(\lambda 1 - A) = \det \begin{pmatrix} \lambda & -1 & 0 & 0 & \cdots & 0 & 0 \\ & \lambda & -1 & 0 & \cdots & 0 & 0 \\ & & \lambda & -1 & \cdots & 0 & 0 \\ & & & \ddots & \ddots & \vdots & \vdots \\ & & & & \lambda & -1 & 0 \\ & & & & & \lambda & -1 \\ a_0 & a_1 & a_2 & & \cdots & & a_{n-1} \end{pmatrix}$$

$$= \lambda \det \begin{pmatrix} \lambda & -1 & 0 & \cdots & 0 & 0 \\ & \lambda & -1 & \cdots & 0 & 0 \\ & & \ddots & \ddots & \vdots & \vdots \\ & & & \lambda & -1 & 0 \\ & & & & \lambda & -1 \\ a_1 & a_2 & & \cdots & & a_{n-1} \end{pmatrix} + (-1)^{n-1} a_0 \det \begin{pmatrix} -1 & & & \\ & -1 & \cdots & 0 \\ & & \ddots & \\ & * & \cdots & \ddots & \\ & & & & -1 \end{pmatrix}$$

$$= \lambda(\lambda^{n-1} + a_{n-1}\lambda^{n-2} + \cdots + a_1) + (-1)^{n-1} a_0 (-1)^{n-1}$$

$$= \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_0,$$

the next-to-last equality following by induction.

8. In (a), let $|f_n(t) \le M$ for all t and n. Then $|F_n(t) - F_n(t')| = \left| \int_{t'}^t f_n(s)\,ds \right| \le M|t - t'|$. Thus equicontinuity holds with $\delta = \epsilon/M$.

In (b), we solve the equation explicitly, using variation of parameters. The solutions of the homogeneous equation are $c_1 \cos t + c_2 \sin t$, and computation shows that the unique solution of the inhomogeneous equation with the given initial condition is $y^*(t) = -(\cos t) \int_0^t (\sin s) f(s)\,ds + (\sin t) \int_0^t (\cos s) f(s)\,ds$. Each integral is equicontinuous by the same argument as in (a), and the operations of multiplication by a bounded continuous function and addition preserve the equicontinuity.

In (c), we do not know explicit formulas for the solutions of the homogeneous equation, but the same argument as in (b) with variation of parameters will work anyway.

10. For any C^2 periodic function f, the n^{th} Fourier coefficient c_n of f has $|c_n| \le n^{-2} \sup |f''|$. The function $v(r, \theta)$, being a composition of two C^2 functions, is C^2 for $0 \le r < 1$ and $|\theta| \le \pi$, and hence $\sup \left| \frac{\partial^2 v}{\partial \theta^2} \right|$ is bounded by some M for $0 \le r \le 1 - \delta$. Then we obtain $|c_n(r)| \le M/n^2$.

11. The function $(u \circ R_\varphi)(x, y) e^{-ik\varphi}$ is of class C^2 jointly in x, y, φ. By Proposition 3.28 we can pass the second derivatives with respect to x and y under the given integral sign with respect to φ. The integrand is harmonic in (x, y) for each φ, and therefore the integral itself is harmonic. The integral itself is given by

$$\tfrac{1}{2\pi} \int_{-\pi}^{\pi} v(r, \theta + \varphi) e^{-ik\varphi}\,d\varphi = \tfrac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{n=-\infty}^{\infty} c_n(r) e^{in\theta} e^{i(n-k)\varphi}\,d\varphi.$$

The series in the integrand is uniformly convergent as a function of φ, by the estimate in Problem 10 and by the Weierstrass M-test. Theorem 1.31 says that we can interchange sum and integral, and then the right side above collapses to $c_k(r) e^{in\theta}$.

12. Starting from $v(r,\theta)=u(r\cos\theta, r\sin\theta)$, we compute $\frac{\partial v}{\partial r}$ and $\frac{\partial v}{\partial \theta}$ by the chain rule and obtain

$$\frac{\partial v}{\partial r}=\cos\theta\,\frac{\partial u}{\partial x}+\sin\theta\,\frac{\partial u}{\partial y}\quad\text{and}\quad\frac{\partial v}{\partial \theta}=-r\sin\theta\,\frac{\partial u}{\partial x}+r\cos\theta\,\frac{\partial u}{\partial y}.$$

Using the same technique, we form $\frac{\partial^2 v}{\partial r^2}$ and $\frac{\partial^2 v}{\partial \theta^2}$ in terms of the partial derivatives of u, and we find that

$$\Delta u=\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}+\frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2}.$$

Substituting $v(r,\theta)=c_k(r)e^{ik\theta}$ and taking into account that $\Delta u=0$, we obtain

$$0=e^{ik\theta}\big(c_k''+r^{-1}c_k'-k^2r^{-2}c_k\big).$$

Thus $r^2c_k''+rc_k'-k^2c_k=0$. This is an Euler equation. The solutions are $c_k(r)=a_kr^{|k|}+b_kr^{-|k|}$ if $k\neq 0$ and are $a_0+b_0\log r$ if $k=0$. Taking into account that $c_k(r)$ is differentiable at $r=0$, we obtain $c_k(r)=a_kr^{|k|}$ for all k. Substitution gives $v(r,\theta)=\sum_{n=-\infty}^{\infty}c_nr^{|n|}e^{in\theta}$.

13. Since $f_R(\theta)=\sum_{n=-\infty}^{\infty}c_nR^{|n|}e^{in\theta}$ and $P_{r/R}(\theta)=\sum_{n=-\infty}^{\infty}(r/R)^{|n|}e^{in\theta}$, the result follows immediately from Problem 8b at the end of Chapter III.

15. For (a), substitute $y=uv$, $y'=u'v+uv'$, and $y''=u''v+2u'v'+uv''$ into the equation for y, take into account that $u''+Pu'+Qu=0$, and get $2u'v'+uv''+Puv'=0$. Put $w=v'$. We can rewrite our equation as $w'=(-P-2u'/u)w$ since u is assumed nonvanishing. Then Problem 14 gives $w(t)=ce^{-\int P\,dt-2\int(u'/u)\,dt}=ce^{-\int P\,dt}e^{\log(|u|^{-2})}=cu(t)^{-2}e^{-\int P(t)\,dt}$.

For (b), the formula in (a) gives $v'(t)=ce^{-t^2/2}$, and hence $y(t)=u(t)v(t)=e^{t^2/2}\int_0^t e^{-s^2/2}\,ds$.

16. The substitution leads to $uv''+(2u'+Pu)v'+(u''+Pu'+Qu)v=0$. Thus the condition is $2u'+Pu=0$. By Problem 14, $u(t)$ is a multiple of $e^{-\int(P/2)\,dt}$. The computation of $R(t)$ is then routine.

17. Substitution of $v=ur^{-1/2}$ shows that $L(v)=r^{1/2}L_0(u)$ with L_0 of the indicated form.

18. For (a), the formula is $d_n=-\sum_{k=1}^{n}c_kd_{n-k}$, with $d_0=1$. For (b), we have $d_1=-c_1d_0=-c_1$, so that $|d_1|=|c_1|\le Mr^1$. Thus $|d_n|\le M(M+1)^{n-1}r^n$ for $n=1$. Assume that $|d_k|\le Mr^k$ for $1\le k<n$. Then $|d_n|\le\sum_{k=0}^{n-1}|c_{n-k}||d_k|\le|c_n|+\sum_{k=1}^{n-1}(Mr^{n-k})(M(M+1)^{k-1}r^k)\le Mr^n+M^2r^n\sum_{k=1}^{n-1}(M+1)^{k-1}$. This is

$$\begin{aligned}&=Mr^n\big(1+M\textstyle\sum_{k=1}^{n-1}(M+1)^{k-1}\big)\\&=Mr^n(1+M((M+1)^{n-1}-1)/((M+1)-1)\\&=Mr^n(1+(M+1)^{n-1}-1)=M(M+1)^{n-1}r^n.\end{aligned}$$

For (c), we may assume that $f(0) = 1$. Write $f(x) = \sum_{n=0}^{\infty} c_n x^n$, and define d_n as in the answer to (a). The estimate in (b) shows that the power series $g(x) = \sum_{n=0}^{\infty} d_n x^n$ has positive radius of convergence, and Theorem 1.40 shows that $f(x)g(x) = 1$ on the common region of convergence. Then $g(x) = 1/f(x)$, and $1/f(x)$ is exhibited as the sum of a convergent power series.

19. The indicial equation is $s(s-1) + a_0 s + b_0 = 0$, where $a_0 = P(0)$ and $b_0 = Q(0)$. Thus $s_1 + s_2 = 1 - a_0$.

In (a), we apply Problem 15a with $u(t) = t^{s_1} \sum_{n=0}^{\infty} c_n t^n$. The expression $P(t)$ in that problem has become $t^{-1}P(t)$ here, and we obtain $v'(t) = u(t)^{-2} e^{-\int t^{-1}P(t)\,dt}$. In the integrand of the exponent, we separate the term $-a_0/t$ from the power series, and we see that $v'(t) = u(t)^{-2} e^{-a_0 \log t} \times$ power series $= t^{-a_0} u(t)^{-2} \times$ power series, the power series having nonzero constant term since exponentials are nowhere vanishing. This is of the form $t^{-2s_1 - a_0} \times$ power series as a consequence of Problem 18 and Theorem 1.40, the power series having nonzero constant term. When this expression is integrated to form $v(t)$, the t^{-1} produces a logarithm, and the rest produces powers of t. Thus $v(t)$ equals $c \log t + t^{-2s_1 - a_0 + 1} \times$ power series; here the power series has nonzero constant term. Then $u(t)v(t) = cu(t) \log t + t^{s_1} t^{-2s_1 - a_0 + 1} \times$ power series; once again the power series has nonzero constant term. The exponent of t in the second term is $-s_1 + 1 - a_0 = -s_1 + (s_1 + s_2) = s_2$, and (a) is done.

In (b), we know that there is only one solution beginning with t^{s_1}, and thus we must have $c \neq 0$ in (a). Another way to see this conclusion is to recognize that the exponent of $t^{-2s_1 - a_0}$ in $v'(t)$ is just -1 since $2s_1 = s_1 + s_2$. Thus the coefficient of t^{-1} in integrating to form $v(t)$ is not 0, and the logarithm occurs.

In (c), we know from a computation in the text that no series solution begins with t^{-p} except when $p = 0$, and thus the first argument for (b) applies.

20. When $t = t_{k-1}$ is substituted into the formula valid for $t_{k-1} < t \leq t_k$, we get $y(t) = y(t_{k-1})$; so the formula is valid also at t_{k-1}.

We induct on k. For $k = 0$, $y(t_0) = y_0$. Assume inductively for $k > 0$ that $|y(t_{k-1}) - y(t_0)| \leq M|t_{k-1} - t_0| \leq Ma' \leq b$. For $t_{k-1} \leq t \leq t_k$, the displayed formula in the problem implies $|y(t) - y(t_{k-1})| = |F(t_{k-1}, y(t_{k-1}))|\,|t - t_{k-1}|$. Since $(t_{k-1}, y(t_{k-1}))$ lies in R', $|F|$ is $\leq M$ on it. Thus $|y(t) - y(t_{k-1})| \leq M|t - t_{k-1}| \leq Ma' \leq b$. If $t_{l-1} \leq t \leq t_l$, then adding such inequalities gives $|y(t) - y(t_0)| \leq M|t_1 - t_0| + \cdots + M|t_{l-1} - t_{l-2}| + M|t - t_{l-1}| = M|t - t_0|$ as required. Since $|t - t_0| \leq a'$, we have $M|t - t_0| \leq Ma' \leq b$. Thus $(t, y(t))$ is in R'.

21. We may assume that $t' \leq t$. If t' and t lie in the same interval $[t_{k-1}, t_k]$ of the partition, then $y(t) - y(t') = F(t_{k-1}, y(t_{k-1}))(t - t')$. Taking absolute values gives $|y(t) - y(t')| \leq M|t - t'|$.

Otherwise let $t' \leq t_l \leq t_{k-1} \leq t$. Then each pair of points (t', t_l),(t_l, t_{l+1}), $\ldots$,(t_{k-2}, t_{k-1}),(t_{k-1}, t) lies in a single interval of the partition. Adding the estimates for each and taking into account that each difference of t values is ≥ 0, we obtain $|y(t) - y(t')| \leq M|t - t'|$.

22. Let $t_{k-1} \le t \le t_k$. Then $\int_{t_0}^{t} y'(s)\,ds = \sum_{j=1}^{k-1} \int_{t_{j-1}}^{t_j} y'(s)\,ds + \int_{t_{k-1}}^{t} y'(s)\,ds = (y(t_1) - y(t_0)) + \cdots + (y(t_{k-1}) - y(t_{k-2})) + (y(t) - y(t_{k-1})) = y(t) - y(t_0)$, by an application of the Fundamental Theorem of Calculus on each interval. If $t_{k-1} < s < t_k$, we have $|y'(s) - F(s, y(s))| = |F(t_{k-1}, y(t_{k-1})) - F(s, y(s))|$. Here $|s - t_{k-1}| \le |t_k - t_{k-1}| \le \delta$ by the choice of the partition. Again by the choice of the partition, $|y(s) - y(t_{k-1})| \le M|s - t_{k-1}| \le M(\delta/M) = \delta$. By the definition of δ in terms of ϵ and the uniform continuity of F, we conclude that $|y'(s) - F(s, y(s))| \le \epsilon$.

23. We have $\big|y(t) - \big(y_0 + \int_{t_0}^{t} F(s, y(s))\,ds\big)\big| = \big|\int_{t_0}^{t}[y'(s) - F(s, y(s))]\,ds\big| \le \int_{t_0}^{t} |y'(s) - F(s, y(s))|\,ds \le \int_{t_0}^{t} \epsilon\,ds \le \epsilon|t - t_0| \le \epsilon a'$.

24. The statement of Problem 21 proves uniform equicontinuity with $\delta = \epsilon/M$. If we specialize to $t' = t_0$, it implies uniform boundedness.

25. Let $y(t) = \lim y_{n_k}(t)$ uniformly. The functions $y_{n_k}(t)$ are continuous, and the uniform limit of continuous functions is continuous. Hence $y(t)$ is continuous. By Problem 23 we have $\big|y_{n_k}(t) - \big(y_0 + \int_{t_0}^{t} F(s, y_{n_k}(s))\,ds\big)\big| \le \epsilon_{n_k} a'$ for each k. We take the limsup of this expression as k tends to infinity. We know that $y_{n_k}(t)$ tends uniformly to $y(t)$. Then $y_{n_k}(s)$ tends uniformly to $y(s)$ uniformly for $t_0 \le s \le t$. By uniform continuity of F, $F(s, y_{n_k}(s))$ tends uniformly to $F(s, y(s))$. By Theorem 1.31, $\int_{t_0}^{t} F(s, y_{n_k}(s))\,ds$ tends to $\int_{t_0}^{t} F(s, y(s))\,ds$.

26. For some analytic $f(z)$, we can write $u(x, y) = \operatorname{Re} f(z)$ in the unit disk by Problem 70 in Appendix B. Also $f(z) = \sum_{n=0}^{\infty} C_n z^n$ in the unit disk by Taylor's Theorem (Theorem B.21). In polar coordinates, $C_n z^n$ takes the form $C_n r^n e^{in\theta}$, and $\operatorname{Re}(C_n r^n e^{in\theta}) = \operatorname{Re} C_n \cos n\theta - \operatorname{Im} C_n \sin n\theta = \big(\tfrac{1}{2} \operatorname{Re} C_n - \tfrac{1}{2i} \operatorname{Im} C_n\big) e^{in\theta} + \big(\tfrac{1}{2} \operatorname{Re} C_n + \tfrac{1}{2i} \operatorname{Im} C_n\big) e^{-in\theta}$, as required.

27. The function $f(z)$ is analytic for $|z| < R$ and is nonzero at $z = 0$. If $f(z)$ is nowhere 0 for $|z| < \varepsilon$ with $\varepsilon < R$, then $1/f(z)$ is analytic for $|z| < \varepsilon$ and equals the sum of its Taylor series for $|z| < \varepsilon$.

28. (a) This is an instance of Corollary B.15.

(b) For the expansion we have $e^{iz\sin\theta} = \sum_{p=0}^{\infty} \frac{1}{p!}(iz)^p (e^{i\theta} - e^{-i\theta})^p (2i)^{-p} = \sum_{p=0}^{\infty} \frac{1}{p!}(z/2)^p (e^{i\theta} - e^{-i\theta})^p$. For each fixed z, the series is uniformly convergent in θ. Thus when we integrate the product of the two sides with $e^{-in\theta}$, we can interchange the sum and integral to get the asserted expression for $c_n(z)$.

(c) Since the only integer power of $e^{in\theta}$ that has nonzero integral is the 0^{th} power, $\frac{1}{2\pi}\int_{-\pi}^{\pi} (e^{i\theta} - e^{-i\theta})^p e^{-in\theta}$ is nonzero only for $n = p, p-2, \ldots, p-2p$, i.e., only when n is of the form $p - 2k$ with $k = 0, 1, \ldots, p$. When $n = p - 2k$ with $k \ge 0$, we have $(e^{i\theta} - e^{-i\theta})^p e^{-in\theta} = e^{i(p-n)\theta}(1 - e^{-2i\theta})^p = e^{2ik\theta}(1 - e^{-2i\theta})^p = e^{2ik\theta} \sum_{l=0}^{p} (-1)^l e^{-2il\theta} \binom{p}{l}$. The only term that contributes to the integral is the one for $l = k$, and its contribution is $(-1)^k \binom{p}{k}$. Thus $I_{n,p}$ is nonzero except when $p - n = 2k$ with $0 \le k \le p$, and then it contributes $(-1)^k \binom{p}{k}$. This formula for $I_{n,p}$ remains correct when $p - n = 2k$ for all $k \ge 0$ because the binomial coefficient $\binom{p}{k}$ is 0 when

$k > p$. Thus $c_n(z) = \sum_{p=0}^{\infty} \frac{1}{p!}(z/2)^p I_{n,p} = \sum_{k=0}^{\infty} \frac{1}{(n+2k)!}(z/2)^{n+2k}(-1)^k\binom{n+2k}{k}$, and the desired formula for $c_n(z)$ follows.

(d) For $n \geq 0$, the series for $c_n(z)$ matches that for $J_n(z)$. For $n \leq 0$, we replace θ by $-\theta$ in the integral defining $c_{-n}(z)$ and find that $c_{-n}(z) = c_n(-z) = J_n(-z)$, and this equals $(-1)^n J_n(z)$ by inspection.

(e) The function $e^{i\sin\theta}$ has a uniformly convergent Fourier series by Proposition 1.56 since $e^{i\sin\theta}$ has a continuous derivative in θ, and it converges to the function by Dini's test (Theorem 1.57) or by Fejér's Theorem (Theorem 1.59).

Chapter V

1. For (a) and (c), the answer is 2^k for $1 \leq k \leq n$. However, the assertion in (d) is false; for a counterexample, take $X = \{1, 2, 3, 4\}$, and let $\mathcal{B}$ consist of all sets with an even number of elements. For (b), the associativity is proved by observing that $A \,\Delta\, B \,\Delta C$ is the set of all elements that lie in an odd number of the sets A, B, C.

2. Let $X = \{1, 2, 3\}$ with the σ-algebra consisting of all subsets. Take $\rho(\{1\}) = \rho(\{3\}) = +2$, $\rho(\{2\}) = -3$, $A = \{1, 2\}$, and $B = \{2, 3\}$.

4. This can be worked out carefully, but it is easier to use Problem 3 and apply dominated convergence to see that the measure of the left side is $\limsup \mu(E_n)$, and the measure of the right side is $\liminf \mu(E_n)$.

5. Part (a) is proved the same way as for Lebesgue measure. In (b), the interval I of rationals from 0 to 1 has $\mu(I) = 1$, and it is a countable union of one-point sets $\{p\}$, each of which has $\mu(\{p\}) = 0$.

6. Argue by contradiction. If E^c is not dense, then there is a nonempty open interval U in $[0, 1]$ with $U \cap E^c = \varnothing$ and hence $U \subseteq E$. Since $\mu(U) > 0$, we must have $\mu(E) > 0$.

7. As soon as $\sup \mu(A)$ is known to be finite, B can be constructed as the union of a sequence of sets whose measures increase to the supremum. Thus assume that the supremum of $\mu(A)$ over all sets of finite measure is infinite. Then we can choose a disjoint sequence of sets A_n with each $\mu(A_n)$ finite and with $\sum \mu(A_n) = +\infty$. A little argument allows us to partition the terms of the series into two subsets, with the series obtained from each subset divergent. Say the terms of one subset are $\mu(B_i)$ and the terms of the other are $\mu(C_j)$. Since $\sum \mu(B_i) = +\infty$, the hypothesis makes $\mu\big(\big(\bigcup_i B_i\big)^c\big)$ finite. A contradiction arises because $\big(\bigcup_i B_i\big)^c \supseteq \bigcup_j C_j$ and $\bigcup_j C_j$ has infinite measure.

8. Consider the set $\mathcal{A}$ of all Borel sets E such that $f^{-1}(E)$ is measurable. The set $\mathcal{A}$ is closed under complements and countable unions, and it contains all intervals. So it is a σ-algebra containing all intervals and must consist of all Borel sets.

10. This problem can be done via dominated convergence, but let us do it from scratch in order to be able to quote it in solving Problem 18 and other problems. We

have

$$\left|\int_X f_n\,d\mu - \int_X f\,d\mu\right| \le \int_X |f_n - f|\,d\mu \le \mu(X)\ \sup_x |f_n(x) - f(x)|,$$

and the right side tends to 0 by the uniform convergence. Thus $\lim_X f_n\,d\mu = \int_X f\,d\mu$, the limit existing.

11. In (a) the approximating sets are finite unions of intervals, and we can add their lengths to obtain $\prod_{n=1}^N (1 - r_n)$. Then apply Corollary 5.3. For (b), the set C^c is open, and every point of C^c has an open interval about it where I_C is identically 0; this proves the continuity at points of C^c. To have continuity of I_C at a point x_0 of C, we would need $I_C > 1/2$ on some interval about x_0, and this would mean that I_C equals 1 on that interval and hence that the interval is contained in C. But C contains no intervals of positive length. Part (c) is handled by the same argument as (b). For (d), part (c) says that I_C cannot be redefined on a Lebesgue measurable set of measure 0 so as to be continuous except on a set of measure 0. Theorem 3.29 says that no f obtained by redefining I_C on a set of Lebesgue measure 0 can be Riemann integrable. On the other hand, I_C is measurable, being the indicator function of a compact set, and hence it is Lebesgue integrable.

12. Argue for indicator functions and then simple functions. Then pass to the limit to handle nonnegative functions.

13. Let $\mathcal{B}$ be the set of all subsets E of $X \times X$ such that there exists a set S_E in $\mathcal{A}$ with $E_x = S_E$ for all but countably many x in X. Every rectangle in $\mathcal{A} \times \mathcal{A}$ is in $\mathcal{B}$. In fact, there are two kinds of sets to check, sets $E = A \times B$ with A countable, in which case E_x is empty except for x in the countable set A, and sets $A^c \times B$ with A countable, in which case $E_x = B$ except for x in A. Also $\mathcal{B}$ is a σ-algebra. In fact, let sets E_n in $\mathcal{B}$ be given with associated sets S_{E_n}. Then $\left(\bigcup E_n\right)_x = \bigcup((E_n)_x) = \bigcup S_{E_n}$ except when x is in the countable exceptional set for some n; also if E and S_E are given, then $(E^c)_x = (E_x)^c = (S_E)^c$ except when x lies in the exceptional set for E. Finally the diagonal D is not in $\mathcal{B}$ and therefore cannot be in $\mathcal{A} \times \mathcal{A}$. In fact, $D_x = \{x\}$ for each x, and there can be at most one x with $D_x = S_D$, whatever S_D is.

14. To prove that R is measurable, one first proves the assertion for simple functions ≥ 0 and then passes to the limit. For the rest Fubini's Theorem gives

$$\begin{aligned}\int_{X\times[0,+\infty]} I_R\,d(\mu \times m) &= \int_X \left[\int_{[0,+\infty]} I_R(x, y)\,dm(y)\right] d\mu(x)\\ &= \int_X \left[\int_{[0,f(x))} dm(y)\right] d\mu(x) = \int_X f(x)\,d\mu(x).\end{aligned}$$

15. This is proved in the same way as Proposition 5.52a.

16. The measure space is the unit interval with Lebesgue measure, and each f_n is an indicator function. The set of which f_n is the indicator function is the subset of $\mathbb{R}$ between $\sum_{k=1}^{n-1} a_k$ and $\sum_{k=1}^{n} a_k$ written modulo 1, i.e., the set of fractional parts of each of these rational numbers. The divergence of the series forces these sets to cycle through the unit interval infinitely often, and thus $f_n(x)$ is 1 infinitely often and 0 infinitely often.

17. From the definition of E_{MN}, we see that $\bigcup_N E_{MN} = X$ and $\bigcap_N E^c_{MN} = \varnothing$. The sets E_{MN} are increasing as a function of N, and their complements are decreasing with empty intersection. Corollary 5.3 produces an integer $C(M)$ such that $\mu(E^c_{M,C(M)}) < \epsilon/2^M$. Put $E = \bigcup_M E^c_{M,C(M)}$. Then $\mu(E) < \epsilon$ by Proposition 5.1g. If $\epsilon' > 0$ is given, we are to produce K such that $|f_k(x) - f(x)| < \epsilon'$ for all $k \geq K$ and all x in E^c. Choose M_0 with $1/M_0 < \epsilon'$. The integer K will be $C(M_0)$. Since x is in $E^c = \bigcap_M E_{M,C(M)}$, x is in $E_{M_0,C(M_0)}$ in particular. Then $|f_k(x) - f(x)| < 1/M_0 < \epsilon'$ for $k \geq C(M_0)$.

18. In (a), we may take the set of integration to be X. Let S be the set of measure 0 on which any of f_n and f is infinite, and redefine all the functions to be 0 on S. Given $\epsilon > 0$, choose $\delta > 0$ by Corollary 5.24 such that $\mu(F) < \delta$ implies $\int_F g\,d\mu < \epsilon$. Let E be as in Egoroff's Theorem for the number δ. Problem 10 shows that $\lim \int_{E^c} f_n\,d\mu = \int_{E^c} f\,d\mu$, the limit existing. Also, $\big|\int_E f_n\,d\mu\big| \leq \int_E |f_n|\,d\mu \leq \int_E g\,d\mu < \epsilon$ for all n, and similarly for f. Hence $\limsup_n \big|\int_X f_n\,d\mu - \int_X f\,d\mu\big| \leq 2\epsilon$. Since ϵ is arbitrary, the result follows.

In (b), consider the measure $g\,d\mu$ and the sequence of functions $\{h_n\}$ with $h_n(x) = f_n(x)/g(x)$ when $g(x) > 0$, $h_n(x) = 0$ when $g(x) = 0$. After checking that h_n is measurable, use Corollary 5.28 and apply (a). The constant that bounds the sequence is 1.

19. By Fatou's Lemma, $\int_{E^c} f\,d\mu \leq \liminf_n \int_{E^c} f_n\,d\mu$. Subtracting this from $\int_X f\,d\mu = \lim \int_X f_n\,d\mu$ gives $\int_E f\,d\mu \geq \limsup_n \int_E f_n\,d\mu$. Another application of Fatou's Lemma gives $\liminf_n \int_E f_n\,d\mu \geq \int_E f\,d\mu$, and we conclude that $\liminf_n \int_E f_n\,d\mu = \limsup_n \int_E f_n\,d\mu = \int_E f\,d\mu$, from which the result follows.

20. Let $\epsilon > 0$ be given. Choose $\delta > 0$ by Corollary 5.24 such that $\mu(F) \leq \delta$ implies $\int_F f\,d\mu \leq \epsilon$. Then choose E with $\mu(E) < \delta$ such that f_n converges to f uniformly off E. Problem 10 shows that there is an N such that $\int_{E^c} |f_n - f|\,d\mu < \epsilon$ for $n \geq N$, and Problem 19 shows that there is an N' such that $\int_E |f_n - f|\,d\mu \leq \int_E f_n\,d\mu + \int_E f\,d\mu \leq 2\int_E f\,d\mu + \epsilon$ for $n \geq N'$. Since $\mu(E) < \delta$, $2\int_E f\,d\mu + \epsilon \leq 3\epsilon$. Then $n \geq \max\{N, N'\}$ implies $\int_X |f_n - f|\,d\mu \leq 4\epsilon$.

21. Suppose that $\lim \int_X f_n\,d\mu = \int_X f\,d\mu$. Given $\epsilon > 0$, choose $\delta > 0$ by Corollary 5.24 such that $\mu(E) < \delta$ implies $\int_E f\,d\mu < \epsilon$. Then choose N such that $N^{-1}\big(\int_X f\,d\mu + \epsilon\big) < \delta$. For any n, the convergence of $\int_X f_n\,d\mu$ to $\int_X f\,d\mu$ implies that $N\mu(\{x \mid f_n(x) \geq N\}) \leq \int_{\{x \mid f_n(x) \geq N\}} f_n\,d\mu \leq \int_X f_n\,d\mu \leq \int_X f\,d\mu + \epsilon$ if n is sufficiently large. Hence $\mu(\{x \mid f_n(x) \geq N\}) \leq N^{-1}\big(\int_X f\,d\mu + \epsilon\big) < \delta$ for large n, and therefore $\int_{\{x \mid f_n(x) \geq N\}} f\,d\mu < \epsilon$. Problem 20 shows that $\int_X |f_n - f|\,d\mu \leq \epsilon$ if n is large enough, and then also $\int_{\{x \mid f_n(x) \geq N\}} |f_n - f|\,d\mu \leq \epsilon$. So we have $\int_{\{x \mid f_n(x) \geq N\}} f_n\,d\mu \leq \int_{\{x \mid f_n(x) \geq N\}} |f_n - f|\,d\mu + \int_{\{x \mid f_n(x) \geq N\}} f\,d\mu \leq \epsilon + \epsilon = 2\epsilon$ for n large, say $n \geq N'$. By increasing N and taking the integrability of $f_1, \ldots, f_{N'-1}$ into account, we can achieve the inequality $\int_{\{x \mid f_n(x) \geq N\}} f_n\,d\mu \leq 2\epsilon$ for all n.

Conversely suppose that $\{f_n\}$ is uniformly integrable. Given $\epsilon > 0$, find the N of uniform integrability, put $\delta = \epsilon/N$, and choose E_0 by Egoroff's Theorem such that $\mu(E_0) < \delta$ and f_n converges uniformly off E_0. Then $\lim \int_{E_0^c} f_n \, d\mu = \int_{E_0^c} f \, d\mu$ by Problem 10. Fatou's Lemma gives $\int_{E_0} f \, d\mu \le \liminf \int_{E_0} f_n \, d\mu$, and we have

$$\int_{E_0} f_n \, d\mu = \int_{E_0 \cap \{x \mid f_n(x) \ge N\}} f_n \, d\mu + \int_{E_0 - \{x \mid f_n(x) \ge N\}} f_n \, d\mu.$$

The first term on the right side is $\le \int_{\{x \mid f_n(x) \ge N\}} f_n \, d\mu$, which is $\le \epsilon$ by uniform integrability, and the second term on the right side is $\le N\delta = \epsilon$ because $\mu(E_0) < \delta$ and $f_n(x) \le N$ on the set of integration. Thus $\limsup \int_{E_0} f_n \, d\mu \le 2\epsilon$, and we obtain $\limsup_n \left| \int_{E_0} f_n \, d\mu - \int_{E_0} f \, d\mu \right| \le 4\epsilon$.

22. In the notation of Section 5, $\mathcal{K} = \mathcal{U} = \mathcal{A}$ since $\mathcal{A}$ is now assumed to be a σ-algebra. Thus $\mu_*(E) = \sup_{K \in \mathcal{A},\, K \subseteq E} \mu(K)$ and $\mu^*(E) = \inf_{U \in \mathcal{A},\, U \supseteq E} \mu(U)$. Take a sequence of sets K_n in $\mathcal{A}$ with $\lim \mu(K_n) = \mu_*(E)$; without loss of generality, the sets K_n may be assumed increasing. Then we may take K to be the union of the K_n. The construction of U is similar.

The set K is any member of $\mathcal{A}$ such that $\mu(K)$ is the supremum of $\mu(S)$ for all S in $\mathcal{A}$ with $S \subseteq E$. Then $\mu(K^c)$ is the infimum of all $\mu(S^c) = \mu(X) - \mu(S)$ for all S^c in $\mathcal{A}$ with $S^c \supseteq E^c$. A similar argument applies to U and U^c. The result is that $U^c \subseteq E^c \subseteq K^c$, $\mu_*(E^c) = \mu(U^c)$, and $\mu^*(E^c) = \mu(K^c)$.

23. Lemma 5.33 gives $\mu(A \cap K) \le \mu_*(A \cap E)$, $\mu(A^c \cap K) \le \mu_*(A^c \cap E)$, and $\mu_*(E) = \mu(K) = \mu(A \cap K) + \mu(A^c \cap K) \le \mu_*(A \cap E) + \mu_*(A^c \cap E) \le \mu_*(E)$, from which we obtain $\mu_*(A \cap E) = \mu(A \cap K)$. The argument that $\mu^*(A \cap E) = \mu(A \cap U)$ is similar.

24. The right side of the definition of σ depends only on $A \cap E$ and $B \cap E^c$, and hence σ is well defined. The formulas

$$\bigcup_n \left[(A_n \cap E) \cup (B_n \cap E^c) \right] = \Big(\Big(\bigcup_n A_n \Big) \cap E \Big) \cup \Big(\Big(\bigcup_n B_n \Big) \cap E^c \Big)$$

and $[(A \cap E) \cup (B \cap E^c)]^c = (A^c \cap E) \cup (B^c \cap E^c)$ show that the sets in question form a σ-algebra $\mathcal{C}$. Taking $A = B$ shows that $\mathcal{A} \subseteq \mathcal{C}$, and taking $A = X$ and $B = \varnothing$ shows that E is in $\mathcal{C}$. Therefore $\mathcal{B} \subseteq \mathcal{C}$, and σ is defined on all of $\mathcal{B}$.

The complete additivity of σ results from the complete additivity of each of the four terms in the definition of σ. Specifically let a disjoint sequence $(A_n \cap E) \cup (B_n \cap E)$ be given, and let $A = \bigcup_n A_n$ and $B = \bigcup_n B_n$. We have $\mu_*(A_n \cap E) = \mu(A_n \cap K)$, and the sets $A_n \cap K$ are disjoint; thus $\sum \mu_*(A_n \cap E) = \mu_*(A \cap E)$. The next term is $\mu^*(A_n \cap E) = \mu(A_n \cap U)$, and the sets $A_n \cap U$ may not be disjoint. However, $\mu^*(A_m \cap E) + \mu^*(A_n \cap E) = \mu(A_m \cap U) + \mu(A_n \cap U) = \mu(A_m \cap A_n \cap E) + \mu((A_1 \cup A_2) \cap E)$, and $\mu(A_m \cap A_n \cap U) = \mu^*(A_m \cap A_n \cap E) = \mu^*(\varnothing) = 0$. Thus the term with $\mu^*(A_n \cap E)$ behaves in additive fashion. Consequently $\mu^*(A \cap E) \ge \mu^*\big(\big(\bigcup_{k=1}^n A_k \big) \cap E \big) = \sum_{k=1}^n \mu^*(A_k \cap E)$. Letting n tend to infinity gives $\mu^*(A \cap E) \ge \sum_{k=1}^\infty \mu^*(A_k \cap E)$. The reverse inequality follows from Lemma 5.33a, and thus the term $\mu^*(A_n \cap E)$ is completely additive. The terms with the B_n's are handled similarly, and σ is completely additive.

Taking $A = X$ and $B = \varnothing$, we see immediately that the formula for $\sigma(E)$ is as asserted.

To prove that $\sigma(A) = \mu(A)$ for A in $\mathcal{A}$, we take $A = B$. Then we see that $\sigma(A) = t\mu(A \cap K) + (1-t)\mu(A \cap U) + t\mu(A \cap K^c) + (1-t)\mu(A \cap U^c) = t\mu(A) + (1-t)\mu(A) = \mu(A)$.

25. Each member of the countable set has only countably many ordinals less than it, and the countable union of countable sets is countable. Therefore some member of Ω is not accounted for and is an upper bound for the countable set. Application of (iii) completes the argument.

27. For (a), if $U_n \uparrow U$ and $V_n \uparrow V$, then $U_n \cup V_n \uparrow U \cup V$ and $U_n \cap V_n \uparrow U \cap V$. Similar remarks apply to $\mathcal{K}_\alpha$. Then the assertion follows by transfinite induction.

For (b), we know that $\mathcal{K}_\alpha$ is closed under finite unions and intersections, and we readily see that the complement of any set occurs at most one step later. Now let an increasing sequence of sets in various $\mathcal{K}_\alpha$'s be given. Say that U_n is in $\mathcal{K}_{\alpha_n}$. Problem 25 shows that there is a countable ordinal α_0 that is $\geq$ all the α_n, and then all the U_n are in $\mathcal{K}_{\alpha_0}$. The union is then in $\mathcal{U}_{\alpha_0+1}$ and necessarily in $\mathcal{K}_{\alpha_0+1}$. Hence the union is in the union of the $\mathcal{K}_\alpha$'s. So the union of the $\mathcal{K}_\alpha$'s is a σ-algebra and must contain $\mathcal{B}$. All the set-theoretic operations take place within $\mathcal{B}$, and thus the union must actually equal $\mathcal{B}$.

28. Proposition 5.2 and Corollary 5.3 show that the value of the measure is determined on all the new sets that are constructed in terms of the values on the previous sets. Problem 27 shows that all members of $\mathcal{B}$ are obtained by the construction, and hence μ is completely determined on $\mathcal{B}$.

29. Same argument as for Problem 27b.

30. At every stage of taking limits, we have closure under addition and scalar multiplication. Pointwise decreasing limits produce the indicator functions of finite unions of closed intervals, and pointwise increasing limits of them produce the indicator functions of arbitrary finite unions of intervals. Since the constants are present as continuous functions, we have the indicator function of every elementary set and its complement. These sets form an algebra. Going through the construction of Problem 27, we obtain the indicator function of every Borel set. Since we have closure under addition and scalar multiplication at each step, we obtain all simple functions. One increasing limit gives us all nonnegative Borel measurable functions, and a subtraction (allowable without another passage to the limit) gives us all Borel measurable functions.

32. To see that C has the same cardinality as $\mathbb{R}$, we can make an identification of the disjoint union of $\mathbb{R}$ and a countable set. To do so, we write C as the members of [0, 1] whose base-3 expansions involve no 1's. For each such infinite sequence of 0's and 2's, we change all the 2's to 1's and regard the result as the base-2 expansion of

some real number. This identification is onto [0, 1], and it is one-one if we discard from C all the sequences of 0's and 2's that end in all 2's.

The standard Cantor set has Lebesgue measure 0, and thus any subset of it is Lebesgue measurable of measure 0. The cardinality of this set of subsets is the same as the cardinality of the set of subsets of $\mathbb{R}$. In Section A.10 of Appendix A, it is shown for any set S that the cardinality of S is less than the cardinality of the set of all subsets of S. So the cardinality of the set of Lebesgue measurable sets is at least that of the set of all subsets of $\mathbb{R}$.

33. Since C^c is open, any member x of C^c has the property that $I_{C'}$ is 0 on some open interval about x. Thus $I_{C'}$ is continuous at x. Since C has Lebesgue measure 0, $I_{C'}$ is continuous except on a Lebesgue measurable set of measure 0. Theorem 3.29 shows that $I_{C'}$ is Riemann integrable. Hence the cardinality of the set of Riemann integrable functions is at least that of the set of all subsets of $\mathbb{R}$.

35. If $\mathcal{F}$ is the given filter, form the partially ordered set consisting of all filters on X containing $\mathcal{F}$, with inclusion as the partial ordering. The union of the members of a chain is readily verified to be an upper bound for the chain, and Zorn's Lemma produces a maximal element. This maximal element is readily seen to be an ultrafilter.

36. The filter in question consists of all supersets of finite intersections of members of $\mathcal{C}$.

37–38. Suppose that $\mathcal{F}$ is an ultrafilter, $A \cup B$ is in $\mathcal{F}$, A is not in $\mathcal{F}$, and B is not in $\mathcal{F}$. Let $\mathcal{F}'$ consist of all sets in $\mathcal{F}$ and all sets $B \cap F$ with F in $\mathcal{F}$. Since B is not in $\mathcal{F}$, $\mathcal{F}'$ properly contains $\mathcal{F}$. Since $\mathcal{F}$ is an ultrafilter, $\mathcal{F}'$ must fail to be a filter. On the other hand, by inspection, $\mathcal{F}'$ satisfies properties (i) and (ii) in the definition of filter. We conclude that $\varnothing$ is in $\mathcal{F}'$, hence that there is a set F in $\mathcal{F}$ with $B \cap F = \varnothing$. Since $\mathcal{F}$ satisfies (ii), the set $(A \cup B) \cap F = (A \cap F) \cup (B \cap F) = A \cap F$ is in $\mathcal{F}$. By (i), A is in $\mathcal{F}$, contradiction.

Conversely suppose that $\mathcal{F}$ is a filter such that either A or A^c is in $\mathcal{F}$ for each subset A of X. If $\mathcal{F}$ is not maximal, let B be a set that lies in some filter $\mathcal{F}'$ properly containing $\mathcal{F}$ while B is not itself in $\mathcal{F}$. By hypothesis, B^c is in $\mathcal{F}$ and hence is in $\mathcal{F}'$. But then $B \cap B^c = \varnothing$ lies in $\mathcal{F}'$, in contradiction to (iii).

39. If an ultrafilter $\mathcal{F}$ is given, define $\mu(E) = 1$ if E is in $\mathcal{F}$ and define $\mu(E) = 0$ otherwise. Then μ is defined on all subsets, and we have $\mu(\varnothing) = 0$ and $\mu(X) = 1$. If E and E' are disjoint, we are to show that

$$\mu(E) + \mu(E') = \mu(E \cup E').$$

If $E \cup E'$ is not in $\mathcal{F}$, then all terms in the displayed equation are 0 since $\mathcal{F}$ is closed under supersets. If $E \cup E'$ is in $\mathcal{F}$, then Problem 37 shows that E or E' is in $\mathcal{F}$; on the other hand, they cannot both be in $\mathcal{F}$ because $\mathcal{F}$ is closed under finite intersections and the empty set is not in $\mathcal{F}$. Thus exactly one term on the left side of the displayed equation is 1, and the right side is 1. This proves additivity.

Conversely if an additive set function μ is given on all subsets of X that takes only the values 0 and 1 and is not the 0 set function, let $\mathcal{F}$ consist of the sets E for which

$\mu(E) = 1$. It is immediate that (i) and (iii) hold in the definition of filter. For (ii), let E and E' be in $\mathcal{F}$. Then $E \cup E'$ is in $\mathcal{F}$. Hence $\mu(E \cap E') + 1 = \mu(E) + \mu(E') = 1 + 1$, and $\mu(E \cap E') = 1$. Hence $\mathcal{F}$ is closed under finite intersections and (ii) holds. Thus $\mathcal{F}$ is a filter. If A is given, we have $1 = \mu(X) = \mu(A) + \mu(A^c)$, and hence exactly one of the sets A and A^c is in $\mathcal{F}$. By Problem 38, $\mathcal{F}$ is an ultrafilter.

The statement that complete additivity is equivalent to closure of the ultrafilter under countable intersections is a routine consequence of Corollary 5.3.

40. This follows from Problems 34d and 35.

41. Let S_n be the set of all integers $\geq n$. Since $S_1 = X$, S_1 is in the ultrafilter. Since the ultrafilter is not trivial, $\{n\}$ is not in it, and thus Problem 37 shows that S_n is in it if S_{n-1} is in it. Hence S_n is in the ultrafilter for all n. The countable intersection $\bigcap_n S_n$ is empty, and the empty set is not in any filter. Hence the ultrafilter is not closed under countable intersections. Corollary 5.3 shows that the corresponding set function is not completely additive.

43. The proof of Proposition 5.26 shows that the result holds for simple functions ≥ 0. If $f \geq 0$ and $g \geq 0$, choose the standard sequences t_n and u_n of simple functions increasing to f and g. These converge uniformly. Hence so does the sum $s_n = t_n + u_n$. The same argument as for Problem 10 shows that $\lim \int_E s_n \, d\mu = \int_E (f + g) \, d\mu$, $\lim \int_E t_n \, d\mu = \int_E f \, d\mu$, and $\lim \int_E u_n \, d\mu = \int_E g \, d\mu$. Thus the result holds for bounded nonnegative f and g. The passage to general bounded f and g is achieved as in Proposition 5.26.

Chapter VI

1. In additive notation, the sets $E + t$ for t in T are disjoint, and their countable union is S^1. Since Lebesgue measure is translation invariant, these sets all have the same measure c. Then complete additivity gives $c\,\infty = 2\pi$, which is impossible.

2. Parts (b) and (c) are easy. For (a), expand the Jacobian determinant $J(N)$ in cofactors about the first row, obtaining two terms—one each from the first two entries of the first row. The first term is $\cos\theta_1$ times a determinant of size $N - 1$ whose first column has a common factor of $r\cos\theta_1$ and whose second column has a common factor of $\sin\theta_1$, the remaining part of the determinant being $J(N - 1)$; thus the first term gives $(r\cos^2\theta_1 \sin\theta_1)J(N - 1)$. The second term is $-(-r\sin\theta_1)$ times a determinant of size $N - 1$ whose first column has a common factor of $\sin\theta_1$ and whose second column has a common factor of $r\sin\theta_1$, the remaining part of the determinant being $J(N-1)$; thus the second term gives $(r\sin^3\theta_1)J(N-1)$. Adding the two terms gives $J(N) = (r\sin\theta_1)J(N - 1)$, and an induction readily proves the formula.

3. Replace f in Theorem 6.32 by $f \circ L$, and use $\varphi = L^{-1}$. Since $\varphi'(x) = L^{-1}$ for each x, the result follows.

4. In the result of Problem 3, use $L(x) = yx$ and replace $f(z)$ by $f(z)/|\det z|^N$. Then the left side in Problem 3 is $\int_{M_N} f(yx)/|\det(yx)|^N\,dx$, while the right side is $|\det L|^{-1}\int_{M_N} f(x)/|\det x|^N\,dx$. Thus $|\det y|^{-N|}\int_{M_N} f(yx)/|\det(x)|^N\,dx = |\det L|^{-1}\int_{M_N} f(x)/|\det x|^N\,dx$, and the problem reduces to showing that $\det L = (\det y)^N$. One way of doing this is to verify that this formula is true if y is the matrix of an elementary row operation and then to multiply the results. But a faster way is to let $x_1, \dots, x_n$ be the columns of x, so that $L(x_1, \dots, x_n) = (yx_1, \dots, yx_n)$. Then L as a matrix is given in block diagonal form by a copy of y in each block. Hence $\det L = (\det y)^n$. In a little more detail, the matrix of L is being formed relative to the following basis of M_N: if E_{ij} is the N-by-N matrix with 1 in the $(i, j)^{\text{th}}$ entry and 0 elsewhere, the basis is $E_{11}, E_{21}, \dots, E_{N1}, E_{12}, \dots, E_{NN}$.

5. For (a), we have, for $n \neq 0$,

$$2\pi c_n = \int_{-\pi}^{\pi} f(x)e^{-inx}\,dx = \int_{|x|\le\frac{1}{|n|}} f(x)e^{-inx}\,dx + \int_{\frac{1}{|n|}\le|x|\le\pi} f(x)e^{-inx}\,dx.$$

Let us call these terms I and II. Since $|f(x)| \le C|x|^\alpha$ for $|x| \le 1$,

$$|I| \le \int_{|x|\le\frac{1}{|n|}} |f(x)|\,dx \le C\int_{|x|\le\frac{1}{|n|}} |x|^\alpha\,dx = \tfrac{2C}{1+\alpha}\,\tfrac{1}{|n|^{1+\alpha}}.$$

For II, we use integration by parts and take into account that the terms at π and $-\pi$ cancel by periodicity:

$$\begin{aligned} II &= \Big(\int_{-\pi}^{-1/|n|} + \int_{1/|n|}^{\pi}\Big) f(x)\,dx \\ &= \Big[\tfrac{f(x)e^{-inx}}{-in}\Big]_{-\pi}^{-1/|n|} + \Big[\tfrac{f(x)e^{-inx}}{-in}\Big]_{1/|n|}^{\pi} + \tfrac{1}{in}\int_{\frac{1}{|n|}\le|x|\le\pi} f'(x)e^{-inx}\,dx \\ &= \tfrac{1}{in}\big\{f\big(\tfrac{1}{n}\big)e^{-in/|n|} - f\big(-\tfrac{1}{n}\big)e^{+in/|n|}\big\} + \tfrac{1}{in}\int_{\frac{1}{|n|}\le|x|\le\pi} f'(x)e^{-inx}\,dx. \end{aligned}$$

Let us call the terms on the right III and IV. Since $|f(x)| \le C|x|^\alpha$ for $|x| \le 1$,

$$|III| \le \tfrac{1}{|n|}\big(\big|f\big(\tfrac{1}{n}\big)\big| + \big|f\big(-\tfrac{1}{n}\big)\big|\big) \le 2C\,\tfrac{1}{|n|^{1+\alpha}}.$$

The derivation of the formula for II, when applied to f' instead of f, gives the following value for IV:

$$IV = -\tfrac{1}{n^2}\big\{f'\big(\tfrac{1}{n}\big)e^{-in/|n|} - f'\big(-\tfrac{1}{n}\big)e^{+in/|n|}\big\} - \tfrac{1}{n^2}\int_{\frac{1}{|n|}\le|x|\le\pi} f''(x)e^{-inx}\,dx.$$

Let us call the terms on the right V and VI. Since $|f'(x)| \le C|x|^{\alpha-1}$ for $|x| \le 1$,

$$|V| \le \tfrac{1}{n^2}\big(\big|f'\big(\tfrac{1}{n}\big)\big| + \big|f'\big(-\tfrac{1}{n}\big)\big|\big) \le 2C\,\tfrac{1}{|n|^{1+\alpha}}.$$

Since $f''(x)$ is bounded for $1 \le |x| \le \pi$, we can write $|f''(x)| \le C'|x|^{\alpha-2}$ for $0 < |x| \le \pi$, in view of the assumption on f''. Therefore

$$|VI| \le \tfrac{1}{n^2} \int_{\frac{1}{|n|} \le |x| \le \pi} C'|x|^{\alpha-2}\, dx = \tfrac{2C'}{n^2} \int_{1/|n|}^{\pi} x^{\alpha-2}\, dx$$
$$= \tfrac{2C'}{1-\alpha}\, \tfrac{1}{n^2} \Big(\tfrac{1}{|n|^{\alpha-1}} - \pi^{\alpha-1} \Big) \le \tfrac{2C'}{1-\alpha}\, \tfrac{1}{|n|^{1+\alpha}}.$$

Since $2\pi|c_n| \le |I| + |III| + |V| + |VI|$, we obtain $|c_n| \le K/|n|^{1+\alpha}$.

For (b), the uniform convergence follows by applying the Weierstrass M-test, and the limit is f as a consequence of the uniqueness theorem.

In (c), a proof is called for. The crux of the matter is to show, under the assumption that f is real valued, that the variation V_ε of f on $[\varepsilon, 1]$, which gets larger as ε decreases to 0, is bounded. If $x_0 < \cdots < x_n$ is a partition P of $[\varepsilon, 1]$, then

$$\textstyle\sum_{i=1}^n |f(x_i) - f(x_{i-1})| = \sum_{i=1}^n |f'(\xi_i)|(x_i - x_{i-1}) \le C \sum_{i=1}^n \xi_i^{\alpha-1}(x_i - x_{i-1})$$

with $x_{i-1} < \xi_i < x_i$. With ε fixed, the right side is a Riemann sum for the bounded function $x^{\alpha-1}$ on $[\varepsilon, 1]$ and is $\le$ the corresponding upper sum $U(P, x^{\alpha-1}\big|_{[\varepsilon,1]})$. As we insert points into the partition, the left sides increase and the right sides decrease to the limit $\int_\varepsilon^1 x^{\alpha-1}\, dx = \alpha^{-1}(1-\varepsilon^\alpha)$. Hence $V_\varepsilon \le C\alpha^{-1}(1-\varepsilon^\alpha)$, and $\sup_{\varepsilon>0} V_\varepsilon \le C/\alpha$.

6. The distribution function F of μ must have $F(b) - F(a)$ equal to 0 or 1 for all a and b. If c is the supremum of the x's for which there exists $y > x$ with $F(x) < F(y)$, then F has to be k on $(-\infty, c)$ and $k+1$ on $[c, +\infty)$ for the value of k that makes $F(0) = 0$. Hence μ is a point mass at c with $\mu(\{c\}) = 1$.

7. Let K be compact, and let f and g both be equal to the members of a sequence $\{f_n\}$ of continuous functions of compact support decreasing to the indicator function I_K of K. Applying the identity to f_n and passing to the limit, we obtain $\nu(K) = \nu(K)^2$. Thus $\nu(K)$ is 0 or 1 for each compact set. By regularity ν takes on only the values 0 and 1 on Borel sets. Then the argument (but not the statement) of Problem 6 applies, and there is some c with ν equal to a point mass at c with $\nu(\{c\}) = 1$.

8. In (a), if the complement of the set in question is not dense, it omits an open set. However, nonempty open sets have positive measure.

In (b), form $\int_{\mathbb{R}^1} \big[\int_{\mathbb{R}^1} I_E(x-t)\, dt \big]\, d\mu(x)$. The inner integral equals the Lebesgue measure of E for every x since Lebesgue measure is invariant under translations and the map $t \mapsto -t$. Hence the iterated integral is 0. The integral in the other order is $0 = \int_{\mathbb{R}^1} \big[\int_{\mathbb{R}_1} I_E(x-t)\, d\mu(x) \big]\, dt = \int_{\mathbb{R}^1} \big[\int_{\mathbb{R}_1} I_{E+t}(x)\, d\mu(x) \big]\, dt = \int_{\mathbb{R}^1} \mu(E+t)\, dt$, and Corollary 5.23 shows that $\mu(E+t)$ is 0 almost everywhere.

In (c), the same computation applies, and $\mu(E+t)$ is 0 almost everywhere. Under the assumption that $\lim_{t\to 0} \mu(E+t)$ exists, the limit must be 0, by (a).

9. Write $1/|x|$ as a sum $F_1 + F_\infty$, where F_1 is $1/|x|$ for $|x| < 1$ and is 0 for $|x| \ge 1$. Then $\int_{\mathbb{R}^3} F_\infty(x-y)\, d\mu(y)$ is bounded by $\mu(\mathbb{R}^3)$, and it is enough to handle the contribution from F_1. For that we have $\int_{\mathbb{R}^3} \big[\int_{\mathbb{R}^3} F_1(x-y)\, d\mu(y) \big]\, dx = \int_{\mathbb{R}^3} \big[\int_{\mathbb{R}^3} F_1(x-y)\, dx \big]\, d\mu(y) = \int_{\mathbb{R}^3} \big[\int_{\mathbb{R}^3} F_1(x)\, dx \big]\, d\mu(y) = \mu(\mathbb{R}^3) \int_{|x|\le 1} |x|^{-1}\, dx$, and this is finite in $\mathbb{R}^3$. Hence the inner integral $\int_{\mathbb{R}^3} F_1(x-y)\, d\mu(y)$ is finite almost everywhere.

10. We proceed by induction on n, the case $n = 1$ following since finite sets have Lebesgue measure 0. Assume the result in $n-1$ variables, and let $P(x_1, \dots, x_n) \not\equiv 0$ be given. Let E be the set where $P = 0$. This is closed, hence Borel measurable in $\mathbb{R}^n$. Fix $(x_1', \dots, x_n')$ with $P(x_1', \dots, x_n') \neq 0$. The polynomial in one variable $R(x) = P(x_1', \dots, x_{n-1}', x)$ is not identically 0, being nonzero at $x = x_n'$, and hence it vanishes only finitely often, say for x in the finite set F. Fix $x' \notin F$. Then the polynomial $Q(x_1, \dots, x_{n-1}) = P(x_1, \dots, x_{n-1}, x')$ in $n-1$ variables is not identically 0, being nonzero at $(x_1', \dots, x_{n-1}')$, and its set $E_{x'}$ of zeros has measure 0 by inductive hypothesis. If m_n denotes n-dimensional Lebesgue measure, then Fubini's Theorem applied to I_E gives

$$m_n(E) = \int_{\mathbb{R}} m_{n-1}(E_{x'})\,dx = \int_F m_{n-1}(E_{x'})\,dx' + \int_{F^c} m_{n-1}(E_{x'})\,dx'.$$

On the right side the first term is 0 since the 1-dimensional measure of F is 0, while the second term is 0 since the integrand is 0. Thus $m(E) = 0$.

11. $\Gamma(x+y)\int_0^1 t^{x-1}(1-t)^{y-1}\,dt = \int_0^\infty e^{-s}s^{x+y-1}\,ds \int_0^1 t^{x-1}(1-t)^{y-1}\,dt = \int_0^\infty \big[\int_0^s u^{x-1}(s-u)^{y-1}e^{-s}\,du\big]\,ds = \int_0^\infty \big[\int_u^\infty u^{x-1}(s-u)^{y-1}e^{-s}\,ds\big]\,du = \int_0^\infty \big[\int_0^\infty u^{x-1}s^{y-1}e^{-s}e^{-u}\,ds\big]\,du = \Gamma(x)\Gamma(y)$.

12. In Cartesian coordinates we obtain 1^N, hence 1. In spherical coordinates we obtain $\Omega_{N-1}\int_0^\infty r^{N-1}e^{-\pi r^2}\,dr$. Putting $\pi r^2 = s$ shows that $\int_0^\infty r^{N-1}e^{-\pi r^2}\,dr = \int_0^\infty (s/\pi)^{(N-2)/2}e^{-s}\frac{1}{2\pi}\,ds = \frac{1}{2}\pi^{-N/2}\Gamma(N/2)$. Hence $\Omega_{N-1} = 2\pi^{N/2}/\Gamma(N/2)$.

13. Part (a) is carried out by showing by induction on k that $\sum_{i=1}^k x_i = 1 - \prod_{i=1}^k (1-u_i)$. The case $k = n$ is the desired result.

In (b), let $0 < u_i < 1$ for all i. Then $x_i > 0$ for all i, and (a) makes it clear that $\sum_{i=1}^n x_i < 1$. Therefore φ carries I into S. Define $u = \widetilde{\varphi}(x)$ by the formula in (b). If all $x_i > 0$ and $\sum_{i=1}^n x_i < 1$, then certainly $u_i > 0$. Also, $\sum_{j=1}^i x_j < 1$ implies $x_i < 1 - \sum_{j=1}^{i-1} x_j$, so that $u_i = x_i\big/\big(1 - \sum_{j=1}^{i-1} x_j\big) < 1$. Therefore $\widetilde{\varphi}$ carries S into I. To complete the proof, we show that $\widetilde{\varphi}\circ\varphi$ is the identity on I and $\varphi\circ\widetilde{\varphi}$ is the identity on S. For $\widetilde{\varphi}\circ\varphi$, we pass from u to x to v. Thus we start with v_i, substitute the x's, use the inductive version of (a) to substitute the u's, and then sort matters out to see that $v_i = u_i$. For $\varphi\circ\widetilde{\varphi}$, we pass from x to u to y. Then we start with y_i and substitute the u's to obtain $y_i = \big(\prod_{l=1}^{i-1}(1-u_i)\big)u_i$. To substitute for the u's in terms of the x's, we use the inductive version of (a) in the form $\sum_{l=1}^{i-1} y_l = 1 - \prod_{l=1}^{i-1}(1-u_l)$. This gives $\big(\prod_{l=1}^{i-1}(1-u_i)\big)u_i = \big(1 - \sum_{l=1}^{i-1} y_l\big)x_i\big/\big(1 - \sum_{l=1}^{i-1} x_l\big)$. Then an induction on i shows that $y_i = x_i$, and hence $\varphi\circ\widetilde{\varphi}$ is the identity on S.

In (c), routine computation shows that $\varphi'(u)$ is lower triangular with diagonal entries 1, $(1-u_1)$, $(1-u_1)(1-u_2)$, $\dots$, $(1-u_1)\cdots(1-u_{n-1})$, and hence the determinant is the product of these diagonal entries. Similarly $\widetilde{\varphi}'(x)$ is lower triangular with diagonal entries 1, $(1-x_1)^{-1}$, $(1-x_1-x_2)^{-1}, \dots, (1-x_1-x_2-\cdots-x_{n-1})^{-1}$, and its determinant is the product of these diagonal entries.

14. The change of variables in Problem 13 gives

$$\begin{aligned}\int_S x_1^{a_1-1}\cdots x_n^{a_n-1}dx &= \int_I u_1^{a_1-1}[(1-u_1)u_2]^{a_2-1}\cdots[(1-u_1)\cdots(1-u_{n-1})u_n]^{a_n-1}\\ &\qquad \times (1-u_1)^{n-1}\cdots(1-u_{n-1})\,du\\ &= \int_I u_1^{a_1-1}(1-u_1)^{a_2+\cdots+a_n-(n-1)+(n-1)}u_2^{a_2-1}\\ &\qquad \times (1-u_2)^{a_3+\cdots+a_n-(n-2)+(n-2)}\\ &\qquad \times\cdots\times u_{n-1}^{a_{n-1}-1}(1-u_{n-1})^{a_n-1+1}u_n^{a_n-1}\,du\\ &= \int_0^1 u_1^{a_1-1}(1-u_1)^{a_2+\cdots+a_n}\,du_1\cdot\int_0^1 u_2^{a_2-1}(1-u_2)^{a_3+\cdots+a_n}\,du_2\\ &\qquad \cdot\ldots\cdot\int_0^1 u_{n-1}^{a_{n-1}-1}(1-u_{n-1})^{a_n}\,du_{n-1}\cdot\int_0^1 u_n^{a_n-1}\,du_n.\end{aligned}$$

The right side is the product of 1-dimensional integrals of the kind treated in Problem 11. Substitution of the values from that problem leads to the desired result.

15. The monotonicity makes possible the estimate of uniform convergence, and the continuity then makes the limit continuous. A continuous function is determined by its values on a dense set, and C^c is dense.

16. For each n, $F_n(x) = 1-F_n(1-x)$. Thus $\int_0^1 F_n(x)\,dx = 1-\int_0^1 F_n(1-x)\,dx = 1-\int_0^1 F_n(x)\,dx$ and $\int_0^1 F_n(x)\,dx = \frac{1}{2}$. Passing to the limit and using uniform or dominated convergence, we obtain $\int_0^1 F(x)\,dx = \frac{1}{2}$.

18. Use Proposition 6.47. Then u is harmonic by Problem 14 at the end of Chapter III.

19. Since P_r has L^1 norm 1, the inequality $\|u(r,\,\cdot\,)\|_p \le \|f\|_p$ follows from Minkowski's inequality for integrals. For the limiting behavior as r increases to 1, we extend f periodically and write

$$\begin{aligned}u(r,\theta)-f(\theta) &= \tfrac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\varphi)f(\theta-\varphi)\,d\varphi - f(\theta)\\ &= \tfrac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\varphi)[f(\theta-\varphi)-f(\theta)]\,d\varphi,\end{aligned}$$

the second step following since $\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r\,d\varphi = 1$. Applying Minkowski's inequality for integrals, we obtain

$$\|u(r,\,\cdot\,)-f\|_p \le \tfrac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\varphi)\|f(\theta-\varphi)-f(\theta)\|_{p,\theta}$$

since $P_r \ge 0$. The integration on the right is broken into two sets, $S_1 = (-\delta,\delta)$ and $S_2 = [-\pi,-\delta]\cup[\delta,\pi]$, and the integral is

$$\begin{aligned}&\le \tfrac{1}{2\pi}\int_{S_1} P_r(\varphi)\big(\sup\nolimits_{\varphi\in S_1}\|f(\theta-\varphi)-f(\theta)\|_{p,\theta}\big)\,d\varphi + \tfrac{1}{2\pi}\int_{S_2} P_r(\varphi)2\|f\|_p\,d\varphi\\ &\le \sup_{\varphi\in S_1}\|f(\theta-\varphi)-f(\theta)\|_{p,\theta} + 2\|f\|_p\sup_{\varphi\in S_2} P_r(\varphi).\end{aligned}$$

Let $\epsilon > 0$ be given. If δ is sufficiently small, Proposition 6.16 shows that the first term is $< \epsilon$. With δ fixed, we can then choose r close enough to 1 to make the second term $< \epsilon$.

20. For (a), we argue as in Problem 19, taking S_1 and S_2 to be as in that solution. Then

$$\begin{aligned}|u(r,\theta)-f(\theta)| &\le \tfrac{1}{2\pi}\textstyle\int_{-\pi}^{\pi} P_r(\varphi)|f(\theta-\varphi)-f(\theta)|\,d\varphi\\ &\le \tfrac{1}{2\pi}\textstyle\int_{S_1} P_r(\varphi)|f(\theta-\varphi)-f(\theta)|\,d\varphi\\ &\quad+\tfrac{1}{2\pi}\textstyle\int_{S_2} P_r(\varphi)[\|f\|_\infty+\sup_{\theta\in E}|f(\theta)|]\,d\varphi\\ &\le \sup_{\varphi\in S_1}|f(\theta-\varphi)-f(\theta)|\\ &\quad+\big(\sup_{\varphi\in S_2}P_r(\varphi)\big)[\|f\|_\infty+\sup_{\theta\in E}|f(\theta)|],\end{aligned}$$

and the uniform convergence follows.

For (b), the Poisson integral of f is of the form $\sum_{n=-\infty}^{\infty} c_n r^{|n|}e^{in\theta}$, where the c_n are the Fourier coefficients of f. Any other harmonic function in the disk is of the form $\sum_{n=-\infty}^{\infty} c'_n r^{|n|}e^{in\theta}$. Suppose this tends uniformly to f as r increases to 1. Then the difference is a series $\sum_{n=-\infty}^{\infty} d_n r^{|n|}e^{in\theta}$ that converges uniformly to 0. Then the integral of the product of this series and $e^{-ik\theta}$ tends to 0. Interchanging integral and sum, we see that $d_k r^{|k|}$ tends to 0 for each k. Therefore $d_k=0$ for each k.

In (c) since P_r is even,

$$\begin{aligned}\textstyle\int_{-\pi}^{\pi}(P_r*f)(\theta)g(\theta)\,d\theta &= \textstyle\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} P_r(\theta-\varphi)f(\varphi)g(\theta)\,d\varphi\,d\theta\\ &= \textstyle\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} P_r(\theta-\varphi)f(\varphi)g(\theta)\,d\theta\,d\varphi\\ &= \textstyle\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} P_r(\varphi-\theta)f(\varphi)g(\theta)\,d\theta\,d\varphi,\end{aligned}$$

and thus $\int_{-\pi}^{\pi}(P_r*f)(\theta)g(\theta)\,d\theta=\int_{-\pi}^{\pi}(P_r*g)(\theta)f(\theta)\,d\theta$. Therefore

$$\begin{aligned}\Big|\textstyle\int_{-\pi}^{\pi}(P_r*f)(\theta)g(\theta)d\theta-\int_{-\pi}^{\pi}f(\theta)g(\theta)\,d\theta\Big| &= \Big|\textstyle\int_{-\pi}^{\pi}\big[(P_r*g)(\theta)-g(\theta)\big]f(\theta)\,d\theta\Big|\\ &\le 2\pi\|P_r*g-g\|_1\|f\|_\infty.\end{aligned}$$

By the previous problem the right side tends to 0 as r increases to 1, and the weak-star convergence follows.

21. Let M_f and M_g be upper bounds for $|f|$ and $|g|$ on $[a,b]$. Then

$$\begin{aligned}&\textstyle\sum_i |f(x_i)g(x_i)-f(x_{i-1})g(x_{i-1})|\\ &\quad\le \textstyle\sum_i |f(x_i)g(x_i)-f(x_i)g(x_{i-1})|+\sum_i |f(x_i)g(x_{i-1})-f(x_{i-1})g(x_{i-1})|\\ &\quad\le M_f\textstyle\sum_i |g(x_i)-g(x_{i-1})|+M_g\sum_i |f(x_i)-f(x_{i-1})|\\ &\quad\le M_f\|g\|_{BV}+M_g\|f\|_{BV}.\end{aligned}$$

22. Let us rewrite the given equation $f(x)=f(a)+g_1(x)-g_2(x)$ as $g_2(x)+f(x)-f(a)=g_1(x)$. If $x_i>x_{i-1}$, then subtraction of the values at $x=x_i$ and at $x=x_{i-1}$ gives $g_2(x_i)-g_2(x_{i-1})+f(x_i)-f(x_{i-1})=g_1(x_i)-g_1(x_{i-1})$.

If $f(x_i) - f(x_{i-1}) \geq 0$, then $f(x_i) - f(x_{i-1}) \leq g_1(x_i) - g_1(x_{i-1})$ because g_2 is monotone; if $f(x_i) - f(x_{i-1}) < 0$, then $0 \leq g_1(x_i) - g_1(x_{i-1})$ because g_1 is monotone. Therefore $\big(f(x_i) - f(x_{i-1})\big)^+ \leq g_1(x_i) - g_1(x_{i-1})$. Summing on i for a partition of $[a, x]$ gives $\sum_{i=1}^n \big(f(x_i) - f(x_{i-1})\big)^+ \leq g_1(x) - g_1(a)$. If we take the supremum of the left side and recall that $g_1(a) \geq 0$, we obtain $V^+(f)(x) \leq g_1(x) - g_1(a) \leq g_1(x)$. Starting similarly from $g_1(x) - f(x) + f(a) = g_2(x)$ and arguing in the same way, we obtain $V^-(f)(x) \leq g_2(x) - g_2(a) \leq g_2(x)$.

23. Suppose that $V^+(f)$ and $V^-(f)$ are both discontinuous at some x. Then $V^+(f)(x^-) + \epsilon < V^+(f)(x^+)$ and $V^-(f)(x^-) + \epsilon < V^-(f)(x^+)$ for some $\epsilon > 0$. Define

$$g_1(y) = \begin{cases} V^+(f)(y) & \text{for } y < x, \\ V^+(f)(x^-) & \text{for } y = x, \\ V^+(f)(y) - \epsilon & \text{for } y > x, \end{cases}$$

and define $g_2(y)$ similarly except that V^- replaces V^+. Then g_1 and g_2 are both nonnegative, and $g_1 - g_2 = V^+(f) - V^-(f) = f - f(a)$. If g_1 and g_2 are shown to be monotone, Then Problem 22 leads to the contradiction $g_1(y) < V^+(f)(y)$ for $y > x$, and we conclude that $V^+(f)$ and $V^-(f)$ could not have been discontinuous.

In proving monotonicity for g_1, it is necessary to make comparisons only of x with other points y. Let $h > 0$. For points $y > x$, we have $g_1(x+h) = V^+(f)(x+h) - \epsilon \geq V^+(f)(x^+) - \epsilon \geq V^+(f)(x^-) = g_1(x)$. For points $y < x$, we have $g_1(x-h) = V^+(f)(x-h) \leq V^+(f)(x^-) = g_1(x)$. Monotonicity for g_2 is proved in the same way.

24. The proof is similar in spirit to the proof of Proposition 6.54.

25. For f, let $y_n = (n + \frac{1}{2})^{-1}\pi^{-1}$, so that $f(y_n)$ is $+(n + \frac{1}{2})^{-1}\pi^{-1}$ if n is even and is $-(n + \frac{1}{2})^{-1}\pi^{-1}$ if n is odd. Compute the sum of the absolute values of the difference of values of f at $y_N, y_{N-1}, \ldots, y_1$ and see that this is unbounded as a function of N. The function g has a bounded derivative (even though the derivative is discontinuous), and this is enough to imply bounded variation.

26. Conclusions (a) and (b) can be handled by variants of Lemma B.12 and Corollary B.15. Fix $\sigma_0 > 0$, and let $U = \{\operatorname{Re} s > \sigma_0\} \subseteq \mathbb{C}$. The set $X = [0, +\infty) \cup \{+\infty\}$ is a compact metric space, and $t^{\sigma_0 - 1}e^{-t/2}\,dt$ is a finite measure on it. Also the function $(t, s) \mapsto t^{s-\sigma_0}e^{-t/2}$ is continuous on $U \times X$ and is analytic in the first variable. The argument of Lemma B.12 goes through to prove the continuity of $\Gamma(s)$ for $\operatorname{Re} s > \sigma_0$, and the argument as in Corollary B.15 using Morera's Theorem and an interchange of integrals applies to prove the analyticity of $\Gamma(s)$ for $\operatorname{Re} s > \sigma_0$. Since $\sigma_0 > 0$ is arbitrary, the conclusions first of continuity and then of analyticity apply to $\Gamma(s)$ for $\operatorname{Re} s > 0$.

One can also argue directly with $\Gamma_{\varepsilon,n}(s) = \int_\varepsilon^n t^{s-1}e^{-t}\,dt$ for $\operatorname{Re} s > 0$. Lemma B.12 and then Corollary B.15 apply directly, and then a passage to the limit is needed. For this purpose the relevant tools are Proposition 2.21 for continuity and Problem 55 in Appendix B for analyticity.

27. We enlarge the domain of definition first from $\{\operatorname{Re} s > 0\}$ to $\{\operatorname{Re} s > -1\} - \{0\}$, then to $\{\operatorname{Re} s > -2\} - \{0, -1\}$, then to $\{\operatorname{Re} s > -3\} - \{0, -1, -2\}$, and so on, using the identity $\Gamma(s) = s^{-1}\Gamma(s+1)$ to define the extended function at each stage. The result is analytic except for isolated singularities at the nonpositive integers, and the functional equation $\Gamma(s+1) = s\Gamma(s)$ is valid for the extension. One readily checks that the isolated singularities are all poles of order 1.

Chapter VII

1. If $g(a_k) = g(b_k)$, then a_k would have to be in E. For the second part an example is $g(x) = x$ on $[0, 1]$; there is only one interval (a_k, b_k), and it is $(0, 1)$.

2. No. Corollary 7.4 applied to I_E shows for almost all x that the quotient $m(E \cap (x-h, x+h))/m((x-h, x+h))$ has to tend to 0 or 1 as h decreases to 0.

3. We may work on a bounded interval I. Let $\epsilon > 0$ be given. If x is in E, then $|h^{-1}(F(x+h) - F(x)| \leq \epsilon$ whenever $|h| \leq \delta_x$ for some δ_x depending on x. For each such x, fix a positive number r_x with $r_x \leq \frac{1}{6}\delta_x$. Associate the set $B(r_x; x)$ to x. Then

$$\mu(B(5r_x; x)) \leq \mu((x - 5r_x, x + 5r_x]) = F(x + 5r_x) - F(x - 5r_x) \leq 10 r_x \epsilon.$$

Applying Wiener's Covering Lemma, we can find disjoint sets $B(r_{x_i}; x_i)$ with $E \subseteq \bigcup_{i=1}^\infty B(5r_{x_i}; x_i)$. Then

$$\mu(E) \leq \sum_{i=1}^\infty \mu(B(5r_{x_i}; x_i)) \leq 5\epsilon \sum_{i=1}^\infty 2r_{x_i} = 5\epsilon \sum_{i=1}^\infty m(B(r_{x_i}; x_i)) \leq 5\epsilon m(I).$$

Since I is fixed and ϵ is arbitrary, $\mu(E) = 0$.

4. If F is the function in question, $F - F(0)$ is the distribution function of some Stieltjes measure μ containing no point masses. Proposition 7.8 shows that $\mu(E^c) = 0$ for some countable set E. Since $\mu(\{p\}) = 0$ for each point p, $\mu(E) = 0$ by complete additivity. Thus $\mu = 0$, and F must be constant.

5. For (a), the construction shows that $F'(x) = 0$ for all $x \in C^c$. Then Proposition 7.8 allows us to conclude that μ is singular.

For (b), let F_n be the n^{th} constructed approximation to F (using straight-line interpolations), and let f_n be its derivative (defined except on a finite set and put equal to 0 there). The function f_n is a multiple c_n of the indicator function of the subset C_n of $[0, 1]$ that remains after the first n steps of the construction, and also $m(C_n) = \prod_{k=1}^n (1 - r_k)$. Since $F_n(x) = \int_0^x f_n(t)\,dt$ for all x, we have $1 = F_n(1) = c_n \int_0^1 I_{C_n}(t)\,dt = c_n \prod_{k=1}^n (1 - r_k)$. Therefore $f_n = \big(\prod_{k=1}^n (1 - r_k)\big)^{-1} I_{C_n}$. Put $f = P^{-1} I_C$. The functions f_n converge pointwise to f, and they are uniformly bounded by the constant function P^{-1}. By dominated convergence, $F(x) = \int_0^x f(t)\,dt$ for $0 \leq x \leq 1$. Therefore F is the distribution function of the measure $f(t)\,dt$.

6. Let E be the second described set. The complement of E has measure 0 by Corollary 7.4. Fix x in E, and let $\epsilon > 0$ be given. Choose a rational r such that $|r - f(x)| < \epsilon$. For $h > 0$,

$$h^{-1} \int_x^{x+h} |f(t) - f(x)| \, dt \leq h^{-1} \int_x^{x+h} |f(t) - r| \, dt + h^{-1} \int_x^{x+h} |r - f(x)| \, dt.$$

The second term on the right side equals $|r - f(x)| < \epsilon$, and the first term tends to $|f(x) - r)| < \epsilon$ since x is in E. A similar argument applies if $h < 0$.

7. Part (a) is routine, and part (b) follows by adapting part of the argument for Theorem 6.48. In (c), the assumption that x is in the Lebesgue set implies that $\int_{|t| \leq h} |f(x-t) - f(x)| \, dt \leq hc_x(h)$ for $h > 0$, where $c_x(\,\cdot\,)$ is a function that tends to 0 as h decreases to 0. For each of the described pieces of the integral $\int_{|t| \leq \pi} K_n(t)|f(x-t) - f(x)| \, dt$, we use one of the two estimates in (a), specifically the estimate $K_N(t) \leq N + 1$ for the piece with $|t| \leq 1/N$ and the estimate $K_N(t) \leq c/(Nt^2)$ for all the other pieces. The piece for $1/N$ then contributes $\leq (N+1) \int_{|t| \leq 1/N} |f(x-t) - f(x)| \, dt \leq 2c_x(1/N)$, the piece for $2^{k-1}/N \leq |t| \leq 2^k/N$ contributes $\leq \frac{c}{N} (2^{k-1}/N)^{-2} \int_{2^{k-1}/N \leq |t| \leq 2^k/N} |f(x-t) - f(x)| \, dt \leq \frac{c}{N} (2^{k-1}/N)^{-2} (2^k/N) c_x(2^k/N) = 4 \cdot 2^{-k} c_x(2^k/N)$, and finally the piece for $N^{-1/4} \leq |t| \leq \pi$ contributes $\leq \frac{c}{N} N^{1/2} \int_{N^{-1/4} \leq |t| \leq \pi} |f(x-t) - f(x)| \, dt \leq \frac{c}{N} N^{1/2} 2\pi(\|f\|_1 + |f(x)|)$. The sum of the estimates is

$$\begin{aligned} &\leq 2c_x(1/N) + \sum_{k=1}^{[N^{3/4}]} 4 \cdot 2^{-k} c_x(2^k/N) + 2\pi c N^{-1/2}(\|f\|_1 + |f(x)|) \\ &\leq 4 \sup_{0<h<N^{-1/4}} c_x(h) + c' N^{-1/2}(\|f\|_1 + |f(x)|), \end{aligned}$$

and this tends to 0 as h decreases to 0. (The use of the shells with 2^{-k} is a device that appears frequently in Zygmund's *Trigonometric Series* and may be regarded as a kind of manual integration by parts.)

8. Since μ is singular, find a Borel set E with $\mu(E) = 0$ and $m(E^c) = 0$. Let $\epsilon > 0$ be given. By regularity of $m + \mu$, choose an open set U containing E such that $(m + \mu)(U - E) < \epsilon$. Then $\mu(U) \leq \mu(U - E) + \mu(E) = \mu(U - E) < \epsilon$, and $m(U^c) \leq m(E^c) = 0$.

9. About each x in U, there is some $\delta(x)$ such that $(x - h, x + h) \subseteq U$ for $h \leq \delta(x)$. Then $\nu((x - h, x + h)) = 0$ for $h \leq \delta(x)$, and the limit of this is 0 as h decreases to 0.

11. Since U is open and $\mu_2(U) = 0$, Problem 9 gives

$$\lim_{h \downarrow 0} (2h)^{-1} \mu_2((x - h, x + h)) = 0$$

for all x in U. Since $m(U^c) = 0$, $\lim_{h\downarrow 0}(2h)^{-1}\mu_2((x-h, x+h)) = 0$ for almost every x in $\mathbb{R}^1$. The measure μ_1 has $\mu_1(\mathbb{R}^1) = \mu(U) < \epsilon$, and Problem 10 shows that

$$m\{x \mid \limsup_{h\downarrow 0} \mu_1((x-h, x+h)) > \xi\}$$
$$\leq m\{x \mid \sup_{h>0} \mu_1((x-h, x+h)) > \xi\} \leq 5\mu_1(\mathbb{R}^1)/\xi < 5\epsilon/\xi.$$

12. It is enough to handle the case that μ vanishes outside some interval and hence has $\mu(\mathbb{R}^1)$ finite. Combining the estimates for μ_1 and μ_2 gives

$$m\{x \mid \limsup_{h\downarrow 0} \mu((x-h, x+h)) > \xi\} < 5\epsilon/\xi.$$

Since ϵ is arbitrary, $m\{x \mid \limsup_{h\downarrow 0} \mu((x-h, x+h)) > \xi\} = 0$. Taking the union for $\xi = 1/n$, we conclude that the set where $\limsup_{h\downarrow 0} \mu((x-h, x+h)) > 0$ has measure 0.

To get the better conclusion, the main step is to obtain a bound $10\epsilon/\xi$ for the maximal function formed from the supremum of $\nu((x, x+h))$ or $\nu((x-h, x))$. The proof of Corollary 6.40 shows how to derive this from Problem 10.

Chapter VIII

1. Let $\mathcal{F}$ be the Fourier transform as defined in the text. In each part of the problem, α can be computed by relating matters to the known facts about $\mathcal{F}$, and β can be computed directly from the definitions and Fubini's Theorem.

In (a), we have $\widehat{f}(y) = \int f(x)e^{-ix\cdot y}dy = \int f(x)e^{-2\pi i x\cdot(y/(2\pi))}\,dy = \mathcal{F}f(y/(2\pi))$. To obtain $f(x) = \alpha\int \widehat{f}(y)e^{ix\cdot y}\,dy$, we want $f(x) = \alpha\int \mathcal{F}f(y/(2\pi))e^{ix\cdot y}\,dy = (2\pi)^N\alpha\int \mathcal{F}f(y')e^{ix\cdot(2\pi y')}\,dy' = (2\pi)^N\alpha f(x)$. With $f*g(x) = \beta\int f(x-t)g(t)\,dt$, we have $\widehat{f*g}(y) = \beta\iint f(x-t)g(t)e^{-ix\cdot y}dt\,dx = \beta\iint f(x-t)g(t)e^{-ix\cdot y}dx\,dt = \beta\iint f(x)g(t)e^{-i(x+t)\cdot y}\,dx\,dt = \beta\widehat{f}(y)\widehat{g}(y)$. Thus $\alpha = (2\pi)^{-N}$ and $\beta = 1$.

In (b), we find similarly that $\widehat{f}(y) = (2\pi)^{-N}\mathcal{F}f(y/(2\pi))$, and we are led to $(2\pi)^N(2\pi)^{-N}\alpha = 1$. So $\alpha = 1$. Also, $\beta(2\pi)^N = (2\pi)^{2N}$ and $\beta = (2\pi)^N$.

In (c), we find similarly that $\alpha = (2\pi)^{-N/2}$ and $\beta = (2\pi)^{N/2}$. This normalization has the property that α and β are both 1 if dx is replaced by $dx/(2\pi)^{N/2}$ throughout.

2. This is an operation called "polarization" in linear algebra, and it will be explained further in Chapter XII. Application of the Plancherel formula to $f + cg$, f, and cg gives $\|f+cg\|_2^2 = \|\mathcal{F}(f) + c\mathcal{F}(g)\|_2^2$, $\|f\|_2^2 = \|\mathcal{F}(f)\|_2^2$, and $\|cg\|_2^2 = \|c\mathcal{F}(g)\|_2^2$. We expand the first one in terms of the inner product and subtract the other two to obtain

$$(f, cg)_2 + (cg, f)_2 = (\mathcal{F}(f), c\mathcal{F}(g))_2 + (c\mathcal{F}(g), \mathcal{F}(f))_2.$$

Then $\bar{c}(f,g)_2 + c\overline{(f,g)_2} = \bar{c}(\mathcal{F}(f),\mathcal{F}(g))_2 + c\overline{(\mathcal{F}(f),\mathcal{F}(g))_2}$. Taking $c = 1$ gives $2\operatorname{Re}(f,g)_2 = 2\operatorname{Re}(\mathcal{F}(f),\mathcal{F}(g))_2$, whereas taking $c = i$ gives $2\operatorname{Im}(f,g)_2 = 2\operatorname{Im}(\mathcal{F}(f),\mathcal{F}(g))_2$. The result follows.

3. For any f in L^1, we have $Q_\varepsilon * (Q_{\varepsilon'} * f) = P_{\varepsilon+\varepsilon'} * f$ because the Fourier transforms are equal. Also, $(Q_\varepsilon * Q_{\varepsilon'}) * f = Q_\varepsilon * (Q_{\varepsilon'} * f)$ since we have finiteness when the functions are replaced by their absolute values. Moreover, the functions $Q_\varepsilon * Q_{\varepsilon'}$ and $P_{\varepsilon+\varepsilon'}$ are bounded and continuous. Letting f run through an approximate identity formed with respect to dilations and applying Theorem 6.20c, we see that $Q_\varepsilon * Q_{\varepsilon'}(x) = P_{\varepsilon+\varepsilon'}(x)$ for all x.

4. Since P_t is even, $\int_{\mathbb{R}^N}(P_t*f)(x)g(x)\,dx = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} P_t(x-y)f(y)g(x)\,dy\,dx = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} P_t(x-y)f(y)g(x)\,dx\,dy = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} P_t(y-x)f(y)g(x)\,dx\,dy$, and thus $\int_{\mathbb{R}^N}(P_t*f)(x)g(x)\,dx = \int_{\mathbb{R}^N}(P_t*g)(x)f(x)\,dx$. Therefore

$$\Big|\int_{\mathbb{R}^N}(P_t*f)(x)g(x)\,dx - \int_{\mathbb{R}^N} f(x)g(x)\,dx\Big| = \Big|\int_{\mathbb{R}^N}\big[(P_t*g)(x)-g(x)\big]f(x)\,dx\Big| \\ \le \|P_t*g-g\|_1\|f\|_\infty.$$

By Theorem 8.19c the right side tends to 0 as t decreases to 0, and (a) follows.

For (b), part (a) shows for each g with $\|g\|_1 \le 1$ that $\big|\int_{\mathbb{R}^N} f(x)g(x)\,dx\big| = \lim_{t\downarrow 0}\big|\int_{\mathbb{R}^N}(P_t*f)(x)g(x)\,dx\big|$. Since $\big|\int_{\mathbb{R}^N} P_t*f(x)g(x)\,dx\big| \le \|P_t*f\|_\infty\|g\|_1 \le \|P_t*f\|_\infty$, we have

$$\Big|\int_{\mathbb{R}^N} f(x)g(x)\,dx\Big| \le \liminf_{t\downarrow 0}\|P_t*f\|_\infty$$

whenever $\|g\|_1 \le 1$. For any $\epsilon > 0$ with $\|f\|_\infty - \epsilon > 0$, let S_ϵ be the set where $|f|$ is $\ge \|f\|_\infty - \epsilon$. Then $m(S_\epsilon) > 0$. Take E to be any subset of S_ϵ with $0 < m(E) < +\infty$, and let $g(x)$ be $m(E)^{-1}\overline{f(x)}/|f(x)|$ on E and zero elsewhere. This function has $\|g\|_1 \le 1$. Then $\big|\int_{\mathbb{R}^N} fg\,dx\big| = \int_{\mathbb{R}^N} fg\,dx = m(E)^{-1}\int_E |f|\,dx \ge \|f\|_\infty - \epsilon$. Hence $\|f\|_\infty - \epsilon \le \big|\int fg\,dx\big| \le \liminf_{t\downarrow 0}\|P_t*f\|_\infty$. Since ϵ is arbitrary, $\|f\|_\infty \le \liminf_{t\downarrow 0}\|P_t*f\|_\infty$. On the other hand, Theorem 8.19b shows that $\|P_t*f\|_\infty \le \|f\|_\infty$. So we have $\|f\|_\infty \le \liminf_{t\downarrow 0}\|P_t*f\|_\infty \le \limsup_{t\downarrow 0}\|P_t*f\|_\infty \le \|f\|_\infty$. Equality must hold throughout, and (b) is thereby proved.

5. In (a), the set function is a measure by Corollary 5.27. It has $\mu(\mathbb{R}^N)$ equal to $\mu_1(\mathbb{R}^N)\mu_2(\mathbb{R}^N)$ and is therefore a Borel measure. If $\mu_1 = f\,dx$ and $\mu_2 = \mu$, then

$$\begin{aligned}(f*\mu)(E) &= \int_{\mathbb{R}^N}(f\,dx)(E-x)\,d\mu(x) = \int_{\mathbb{R}^N}\int_{E-x} f(y)\,dy\,d\mu(x) \\ &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} I_{E-x}(y)f(y)\,dy\,d\mu(x) = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} I_E(x+y)f(y)\,dy\,d\mu(x) \\ &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} I_E(y)f(y-x)\,dy\,d\mu(x) = \int_{\mathbb{R}^N}\int_E f(y-x)\,dy\,d\mu(x) \\ &= \int_E\Big[\int_{\mathbb{R}^N} f(y-x)\,d\mu(x)\Big]\,dy.\end{aligned}$$

In (b), we start with an indicator function and compute that

$$\begin{aligned}\textstyle\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} I_E(x+y)\,d\mu_1(x)\,d\mu_2(y) &= \textstyle\int_{\mathbb{R}^N}\big[\int_{\mathbb{R}^N} I_{E-y}(x)\,d\mu_1(x)\big]\,d\mu_2(y)\\ &= \textstyle\int_{\mathbb{R}^N}\mu_1(E-y)\,d\mu_2(y)\\ &= (\mu_1*\mu_2)(E) = \textstyle\int_{\mathbb{R}^N} I_E\,d(\mu_1*\mu_2).\end{aligned}$$

Then we pass to simple functions ≥ 0, use monotone convergence, and finally take linear combinations to get $\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} g(x+y)\,d\mu_1(x)\,d\mu_2(y) = \int_{\mathbb{R}^N} g\,d(\mu_1*\mu_2)$.

In (c), we actually have $\|P_t*\mu\|_1 = \mu(\mathbb{R}^N)$ for every $t>0$ by Fubini's Theorem.

Part (d) is handled in the same way as Problem 4a. First one shows that $\int_{\mathbb{R}^N}(P_t*\mu)(x)g(x)\,dx = \int_{\mathbb{R}^N}(P_t*g)(x)\,d\mu(x)$ for g in $C_{\text{com}}(\mathbb{R}^N)$. The resulting estimate is $\big|\int_{\mathbb{R}^N}[(P_t*g)(x)-g(x)]\,d\mu(x)\big| \le \|P_t*g-g\|_{\sup}\,\mu(\mathbb{R}^N)$, and then (a) follows from Theorem 8.19d.

6. Part (a) follows from the same argument as for Proposition 8.1a. In (b), the measure δ with $\delta(\{0\})=1$ and $\delta(\mathbb{R}^N-\{0\})=0$ has $\widehat{\delta}(y)=1$ for all y. In (c), we use the result of Problem 5b with $g(x)=e^{-2\pi i x\cdot t}$ and get $\int e^{-2\pi i x\cdot t}\,d(\mu_1*\mu_2)(x) = \iint e^{-2\pi i(x+y)\cdot t}\,d\mu_1(x)\,d\mu_2(y) = \widehat{\mu_1}(t)\widehat{\mu_2}(t)$. In (d), let $\varphi(x)=P_1(x)$. Then $\widehat{\mu}=0$ implies $\widehat{\varphi_\varepsilon*\mu}=0$ for every $\varepsilon>0$. Since $\varphi_\varepsilon*\mu$ is a function, Corollary 8.5 gives $\varphi_\varepsilon*\mu=0$ for every $\varepsilon>0$. By Problem 5d, $\varphi_\varepsilon*\mu$ converges weak-star to μ against $C_{\text{com}}(\mathbb{R}^N)$. Therefore $\int_{\mathbb{R}^N} g\,d\mu = 0$ for every g in $C_{\text{com}}(\mathbb{R}^N)$, and Corollary 6.3 shows that $\mu=0$.

7. This is the same kind of approximation argument as was done in Corollary 6.17.

8. We calculate that $\sum_{i,j}\widehat{\mu}(x_i-x_j)\xi_i\overline{\xi_j} = \sum_{i,j}\int e^{-2\pi i t\cdot(x_i-x_j)}\xi_i\overline{\xi_j}\,d\mu(t) = \int\big(\sum_{i,j}(e^{-2\pi i t\cdot x_i}\xi_i)\overline{(e^{-2\pi i t\cdot x_j}\xi_j)}\big)\,d\mu(t) = \int\big|\sum_j e^{-2\pi i t\cdot x_j}\xi_j\big|^2\,d\mu(t)\ge 0$.

9. For the set $\{0\}$, the condition is that $F(0)|\xi_1|^2\ge 0$ for all ξ_1; thus $F(0)\ge 0$. For the set $\{x,0\}$, the condition is that $F(0)|\xi_1|^2+F(x)\xi_1\overline{\xi_2}+F(-x)\xi_2\overline{\xi_1}+F(0)|\xi_2|^2\ge 0$. Taking $\xi_1=\xi_2=1$ shows that $F(x)+F(-x)$ is real; taking $\xi_1=i$ and $\xi_2=1$ shows that $i(F(x)-F(-x))$ is real. Therefore $\overline{F(x)}+\overline{F(-x)} = F(x)+F(-x)$ and $\overline{F(x)}-\overline{F(-x)} = -F(x)+F(-x)$. Adding we obtain $F(-x)=\overline{F(x)}$. Hence $-F(x)\xi_1\overline{\xi_2}-\overline{F(x)}\,\overline{\xi_1}\xi_2 \le F(0)(|\xi_1|^2+|\xi_2|^2)$. If $F(x)\ne 0$, we put $\xi_1=-1$ and $\xi_2=F(x)/|F(x)|$ and obtain $|F(x)|\le F(0)$.

10. $\sum_{i,j}F(x_i-x_j)\Phi(x_i-x_j)\xi_i\overline{\xi_j} = \sum_{i,j}\int F(x_i-x_j)e^{-2\pi i t\cdot(x_i-x_j)}\varphi(t)\xi_i\overline{\xi_j}\,dt = \int\big[\sum_{i,j}F(x_i-x_j)\big(\xi_i e^{-2\pi i t\cdot x_i}\big)\big(\overline{\xi_j}e^{-2\pi i t\cdot x_j}\big)\big]\varphi(t)\,dt\ge 0$.

11. Part (a) follows from the boundedness of F obtained in Problem 9.

In (b), every g in $C_{\text{com}}(\mathbb{R}^N)$ satisfies $0\le\iint F_0(x-y)\overline{g(x)}g(y)\,dx\,dy = \int(F_0*g)(x)\overline{g(x)} = \int\widehat{F_0*g}(y)\overline{\widehat{g}(y)}\,dy = \int\widehat{F_0}(y)\widehat{g}(y)\overline{\widehat{g}(y)}\,dy = \int\widehat{F_0}(y)|\widehat{g}(y)|^2\,dy$.

For (c), if f is in L^2, we can approximate f as closely as we like by a member g of $C_{\text{com}}(\mathbb{R}^N)$. Then $f_0|\widehat{g}|^2 = f_0|\mathcal{F}(f)|^2 + 2f_0\,\mathrm{Re}(\mathcal{F}(f)\overline{(\widehat{g}-\mathcal{F}(f))}) + f_0|\widehat{g}-\mathcal{F}(f)|^2$. We integrate and use the resulting formula to compare $\int f_0|\widehat{g}|^2\,dy$ with $\int f_0|\mathcal{F}(f)|^2\,dy$. By the Schwarz inequality and the Plancherel formula, the absolute value of the difference of these is $\le 2\|f_0\|_{\sup}\|f\|_2\|g-f\|_2 + \|f_0\|_{\sup}\|g-f\|_2^2$. Since $\int f_0|\widehat{g}|^2\,dy$ is ≥ 0, it follows that $\int f_0|\mathcal{F}(f)|^2\,dy \ge 0$ for all f in L^2. Since $\mathcal{F}(f)$ is an arbitrary L^2 function and f_0 is continuous, we conclude that f_0 is ≥ 0.

The integrability in (d) is immediate from Lemma 8.7, and the formula $\int f_0\,dy = F(0)$ follows from the Fourier inversion formula.

12. Let ε_n be a sequence decreasing to 0, let Φ in Problem 11 be the function $e^{-\pi\varepsilon_n^2|x|^2}$, and write F_n for the function $F\Phi$. Then Problem 11d shows that $\mu_n = \widehat{F_n}(y)\,dy$ is a finite Borel measure with $\mu_n(\mathbb{R}^N) = F_n(0) = F(0)$. The Helly–Bray Theorem applies and produces a subsequence of $\{\mu_n\}$ convergent to a finite Borel measure μ weak-star against $C_{\text{com}}(\mathbb{R}^N)$. We shall prove that $F(x) = \int e^{2\pi i x\cdot y}\,d\mu(y)$, i.e., that ν with $\nu(E) = \mu(-E)$ is the desired measure. (The interested reader may wish to compare this argument with the proof of the Portmanteau Lemma (Lemma 9.14) in the companion volume, *Advanced Real Analysis*.)

For each n, the Fourier inversion formula gives $F_n(x) = \int e^{2\pi i x\cdot y}\widehat{F}_n(y)\,dy = \int e^{2\pi i x\cdot y}\,d\mu_n(y)$. Since $F_n(x)$ tends to $F(x)$ pointwise, the result would follow if we could say that the weak-star convergence implies that $\int e^{2\pi i x\cdot y}\,d\mu_n(y)$ tends to $\int e^{2\pi i x\cdot y}\,d\mu(y)$. However, $e^{2\pi i x\cdot y}$ is not compactly supported, and an additional argument is needed.

First we extend the weak-star convergence so that it applies to continuous functions vanishing at infinity. If f is such a function, we can find a sequence $\{f_k\}$ in $C_{\text{com}}(\mathbb{R}^N)$ converging to f uniformly. Then

$$\begin{aligned}
&\left|\int f\,d\mu_n - \int f\,d\mu\right| \\
&\qquad\le \left|\int f\,d\mu_n - \int f_k\,d\mu_n\right| + \left|\int f_k\,d\mu_n - \int f_k\,d\mu\right| + \left|\int f_k\,d\mu - \int f\,d\mu\right| \\
&\qquad\le \|f_k - f\|_{\sup}\,\mu_n(\mathbb{R}^N) + \left|\int f_k\,d\mu_n - \int f_k\,d\mu\right| + \|f_k - f\|_{\sup}\,\mu(\mathbb{R}^N).
\end{aligned}$$

Choose k to make $\|f_k - f\|_{\sup}$ small. With k fixed, choose n to make the middle term small. Then the right side is small since the numbers $\mu_n(\mathbb{R}^N)$ are bounded.

This is not quite good enough by itself because $e^{2\pi i x\cdot y}$ does not vanish at infinity. However, averages of it by L^1 functions (i.e., Fourier transforms of L^1 functions) vanish at infinity, and that will be enough for us.

Define $F^\#(x) = \int e^{2\pi i x\cdot y}\,d\mu(y)$. We prove that $F^\#(x) = F(x)$ for all x. It is enough to prove that $\int F^\#\psi\,dx = \int F\psi\,dx$ for all ψ in L^1. Define $\psi^\vee(y) = \int e^{2\pi i x\cdot y}\psi(x)\,dx$. The multiplication formula (for $(\,\cdot\,)^\vee$ instead of $(\,\cdot\,)\widehat{\ }$) and the Riemann–Lebesgue Lemma give

$$\begin{aligned}
\int F^\#\psi\,dx = \int \psi^\vee\,d\mu(y) &= \lim\nolimits_n \int \psi^\vee\,d\mu_n = \lim\nolimits_n \int \psi^\vee\widehat{F_n}\,dy \\
&= \lim\nolimits_n \int \psi\widehat{F_n}^\vee\,dy = \lim\nolimits_n \int \psi F_n\,dy.
\end{aligned}$$

The right side equals $\int \psi F\,dy$ by dominated convergence since $|F_n(y)| \le |F(y)|$ for all y.

13. Part (a) is easy.

In (b), if χ is a character, then $\sum_x \chi(x) = \sum_x \chi(gx) = \chi(g)\sum_x \chi(x)$. Thus $\sum_x \chi(x) = 0$ if there is some g with $\chi(g) \neq 1$, i.e., if χ is not trivial. If χ and χ' are distinct characters, then $\chi\overline{\chi'}$ is not trivial, and therefore $\sum_x \chi(x)\overline{\chi'(x)} = 0$. The orthogonality implies the linear independence.

In (c), the element 1 of J_m has order m under the group operation of addition. Thus each character χ of J_m must have $\chi(1)$ equal to an m^{th} root of unity. Since 1 generates J_m, $\chi(1)$ determines χ. Thus the listed characters are the only ones.

In (d), any tuple $(n_1, \dots, n_r)$ with $0 \leq n_j < m_j$ for $1 \leq j \leq r$ defines a character by $(k_1, \dots, k_r) \mapsto \prod_{j=1}^r (\zeta_{m_j}^{n_j})^{k_j}$. There are $\prod_{j=1}^r m_j$ distinct characters in this list, and they are linearly independent by (b). Since $\dim L^2(G) = \prod_{j=1}^r m_j$, these characters form a vector-space basis.

14. Since the characters form a basis of $L^2(G)$ as a consequence of Problem 13d, we have $f(t) = \sum_{\chi'} c_{\chi'}\chi'(t)$ for some constants $c_{\chi'}$. Multiply by $\overline{\chi(t)}$ and sum over t to get $\widehat{f}(\chi) = \sum_{\chi'}\sum_t c_{\chi'}\chi'(t)\overline{\chi(t)}$. The orthogonality in Problem 13b shows that this equation simplifies to $\widehat{f}(\chi) = c_\chi \sum_t |\chi(t)|^2 = |G|c_\chi$.

15. $\widehat{f}(\chi) = \sum_{t\in G} f(t)\chi(t) = \sum_{\dot{t}\in G/H}\sum_{h\in H} f(t+h)\dot{\chi}(\dot{t}) = \sum_{\dot{t}\in G/H} F(\dot{t})\dot{\chi}(\dot{t})$ $= \widehat{F}(\dot{\chi})$.

16. The characters of G are the ones with $\chi_n(1) = \zeta_m^n$ for $0 \leq n < m$. Such a character is trivial on H if and only if $\chi_n(q) = 1$, i.e., if and only if $\zeta_m^{nq} = 1$; this means that nq is a multiple of m, hence that n is a multiple of p.

The element 1 of H is the element q of G. Thus the question about the identification of the descended characters asks the value of $\chi_n(1)$ when n is a multiple jp of p. The value is $\chi_n(1) = \zeta_m^n = \zeta_{pq}^{jp} = \zeta_q^j$.

If we have computed F on G/H and want to compute $\widehat{F}$ from the definition of Fourier transform, we have to multiply each of the q values of F by the values of each of the q characters of G/H and then add. The number of multiplications is q^2. The actual computation of F from f involves p additions for each of the q values of $\dot{t}$, hence pq additions.

17. $\widehat{f}(\zeta_m^{jp+k}) = \sum_{i=0}^{m-1} f(i)\zeta_m^{(jp+k)i} = \sum_{i=0}^{m-1}(f(i)\zeta_m^{ki})\zeta_m^{jp}$. The variant of f for the number k is then $i \mapsto f(i)\zeta_m^{ki}$. Handling each value of k involves $m = pq$ steps to compute the variant of f and then the $q^2 + pq$ steps of Problem 16. Thus we have $q^2 + 2pq$ steps for each k, which we regard as on the order of $q^2 + pq$. This means $p(q^2 + pq)$ steps when all k's are counted, hence $pq(p+q)$ steps.

19. For $\operatorname{Re} s > 1$, we have

$$\tfrac{1}{s-1} = \int_1^\infty t^{-s}\,dt = \sum_{n=1}^\infty \int_n^{n+1} t^{-s}\,dt.$$

Thus $\operatorname{Re} s > 1$ implies

$$\zeta(s) = \tfrac{1}{s-1} + \sum_{n=1}^{\infty} \left(\tfrac{1}{n^s} - \int_n^{n+1} t^{-s}\right) dt = \tfrac{1}{s-1} + \sum_{n=1}^{\infty} \int_n^{n+1} (n^{-s} - t^{-s})\, dt.$$

20. Suppose that $\operatorname{Re} s \geq \sigma > 0$, and let $|s| \leq C$. We then have the estimate

$$\left| \int_n^{n+1} (n^{-s} - t^{-s}) \right| \leq \int_n^{n+1} |n^{-s} - t^{-s}|\, dt \leq \tfrac{|s|}{n^{1+\operatorname{Re} s}} \leq \tfrac{C}{n^{1+\sigma}},$$

the next-to-last inequality following from the computation

$$|n^{-s} - t^{-s}| \leq \sup_{n \leq t \leq n+1} \left| \tfrac{d}{dt} t^{-s} \right| \leq \sup_{n \leq t \leq n+1} \tfrac{|s|}{|t^{1+s}|} \leq \tfrac{C}{n^{1+\sigma}}.$$

In combination with the Weierstrass M test, the estimate shows that the series $\sum_{n=1}^{\infty} \int_n^{n+1} (n^{-s} - t^{-s})\, dt$ is uniformly convergent for s in any compact subset of the half plane $\operatorname{Re} s > 0$, and analyticity of $\zeta(s) - \frac{1}{s-1}$ follows from Problem 55 at the end of Appendix B.

21. We have $|e^{in^2\pi\tau}| = e^{-\pi n^2 \sigma}$, and the sum on n of the expression on the right is certainly convergent if $\sigma > 0$. The analyticity follows by using the Weierstrass M test and Problem 55 at the end of Appendix B. The identity $\theta(\tau + 2) = \theta(\tau)$ is clear by inspection.

22. Take $r = \sigma^{1/2}$ and $\sigma > 0$ in the formula of Corollary 8.16. Then $\theta(-1/\tau)$ and $(\tau/i)^{1/2}\theta(\tau)$ are equal on the imaginary axis. Also both are analytic for $\operatorname{Im} \tau > 0$. By the Identity Theorem (Proposition B.23 of Appendix B), they are equal everywhere.

23. The change of variables is $x = n^2 \pi \sigma$.

24. The sum over n of the right side in the previous problem is $\zeta(s)\Gamma(\frac{1}{2}s)\pi^{-\frac{1}{2}s}$. The sum over n of the left side is $\sum_{n=1}^{\infty} \int_0^{\infty} e^{in^2\pi(i\sigma)} \sigma^{\frac{1}{2}s-1}\, d\sigma$ for $\operatorname{Re} s > 1$. If absolute value signs are inserted inside the integral sign then the whole expression is finite. Hence Fubini's Theorem is applicable to interchange sum and integral, and the desired formula results.

25. Put $c(\sigma) = \frac{1}{2}[\theta(i\sigma) - 1]$. Its series is $c(\sigma) = \sum_{n=1}^{\infty} e^{-n^2\pi\sigma}$, and its product with $\sigma^{\frac{1}{2}s-1}$ is a continuous function of the pair (σ, s) that is entire in s for each fixed σ. By Lemma B.12 and Corollary B.15, $\int_1^N c(\sigma)\sigma^{\frac{1}{2}s-1}$ is entire in s for any fixed N. Since $\int_N^{\infty} |c(\sigma)\sigma^{\frac{1}{2}s-1}|\, d\sigma$ tends to 0 uniformly on compact subsets of s values, the entire function $\int_1^N c(\sigma)\sigma^{\frac{1}{2}s-1}$ converges uniformly on compact sets to $\int_1^{\infty} c(\sigma)\sigma^{\frac{1}{2}s-1}$. The limit has to be entire by Problem 55 in Appendix B.

26. Let $\operatorname{Re} s > 1$. In view of Problem 22, we have

$$\begin{aligned}\int_0^1 \tfrac12\theta(i\sigma)\sigma^{\frac12 s-1}\,d\sigma &= \int_0^1 \tfrac12\theta(\tfrac{-1}{i\sigma})(\tfrac{i\sigma}{i})^{-1/2}\sigma^{\frac12 s-1}\,d\sigma\\ &= \int_0^1 \tfrac12\theta(\tfrac{-1}{i\sigma})\sigma^{\frac12 s-\frac32}\,d\sigma\\ &= \int_0^1 \tfrac12[\theta(\tfrac{-1}{i\sigma})-1]\,\sigma^{\frac12 s-\frac32}\,d\sigma + \tfrac{1}{s-1}\end{aligned}$$

The change of variables $\sigma \mapsto 1/\sigma$ shows that the above expression is

$$= \int_1^\infty \tfrac12[\theta(i\sigma)-1]\,\sigma^{\frac12(1-s)-1}\,d\sigma - \tfrac{1}{1-s} = h(1-s) - \tfrac{1}{1-s}.$$

27. The conclusion of Problem 24 gives

$$\begin{aligned}\Lambda(s) &= \int_0^\infty \tfrac12[\theta(i\sigma)-1]\,\sigma^{\frac12 s-1}\,ds\\ &= \int_0^1 \tfrac12\theta(i\sigma)\,\sigma^{\frac12 s-1}\,ds - \tfrac12\int_0^1 \sigma^{\frac12 s-1}d\sigma + \int_1^\infty \tfrac12[\theta(i\sigma)-1]\,\sigma^{\frac12 s-1}\,ds\\ &= \int_0^1 \tfrac12\theta(i\sigma)\,\sigma^{\frac12 s-1}\,ds - \tfrac1s + h(s).\end{aligned}$$

Substituting from Problem 26 shows that

$$\Lambda(s) = h(1-s) - \tfrac{1}{1-s} - \tfrac1s + h(s).$$

Since $\zeta(s)$ extends to be meromorphic in $\operatorname{Re} s > 0$ with its only pole at $s = 1$, $\Lambda(s)$ is meromorphic for $\operatorname{Re} s > 0$. On the other hand, the above expression for $\Lambda(s)$ shows that $\Lambda(s) = \Lambda(1-s)$ for $0 < \operatorname{Re} s < 1$, hence that $\Lambda(s)$ extends to be meromorphic on $\mathbb{C}$. Since h is entire, the only possible poles of $\Lambda(s)$ are at 0 and 1. Since $\zeta(s) = \Lambda(s)\Gamma(\tfrac12 s)^{-1}\pi^{\frac12 s}$ and since $\Gamma(\tfrac12 s)^{-1}$ by assumption has no poles, $\zeta(s)$ can have poles at most at 0 and 1, and any pole is at most simple. Looking at the formula for $\Lambda(s)$ in terms of $\zeta(s)$ shows that $\sigma(s)$ cannot have a pole at $s = 0$.

Chapter IX

1. Let $r = q/p$, and let r' be the dual index. Regard $|f|^p$ as a product $|f|^p \cdot 1$, and apply Hölder's inequality with $|f|^p$ to be raised to the r power and 1 to be raised to the r' power. Compare with Problem 3 below, which is a more complicated version of the same thing.

2. The inequality is routine if any of the indices is ∞. Otherwise, we have

$$\begin{aligned}\int |fgh|\,d\mu &\le \big(\int |fg|^{r'}\,d\mu\big)^{1/r'}\big(\int |h|^r\,d\mu\big)^{1/r}\\ &\le \Big(\big(\int (|f|^{r'})^{p/r'}\,d\mu\big)^{r'/p}\Big)^{1/r'}\Big(\big(\int (|g|^{r'})^{q/r'}\,d\mu\big)^{r'/q}\Big)^{1/r'}\|h\|_r\\ &= \|f\|_p\|g\|_q\|h\|_r.\end{aligned}$$

3. Let us say that $\|f_n\|_p \le C$. Let $\epsilon > 0$ be given. By Egoroff's Theorem, find E with $\mu(E) < \epsilon$ such that f_n tends to f uniformly on E^c. Application of Hölder's inequality with the exponent $r = p/q$ and dual index $r' = p/(p-q)$ to $\int_E |f_n|^q \cdot 1\,d\mu$ gives $\|f_n I_E\|_q \le \big(\int_E |f_n|^{q(p/q)}\,d\mu\big)^{1/p}\big(\int_E 1\,d\mu\big)^{(p-q)/(pq)} \le C\mu(E)^{(p-q)/(pq)} \le C\epsilon^{(p-q)/(pq)}$. Meanwhile, we have

$$\begin{aligned}\|f_n - f\|_q &\le \|f_n - f_n I_{E^c}\|_q + \|f_n I_{E^c} - f I_{E^c}\|_q + \|f I_{E^c} - f\|_q \\ &= \|f_n I_E\|_q + \|(f_n - f) I_{E^c}\|_q + \|f I_E\|_q.\end{aligned}$$

The first term on the right is $\le C\epsilon^{(p-q)/(pq)}$, and so is the third term, by Fatou's Lemma. The middle term tends to 0 as n tends to infinity because of the uniform convergence. Thus $\limsup_n \|f_n - f\|_q \le 2C\epsilon^{(p-q)/(pq)}$. Since ϵ is arbitrary, $\limsup_n \|f_n - f\|_q = 0$.

4. L^1 is 0, and L^∞ consists of the constant functions. All the constant functions give the same linear functional on L^1 because the integral of the product of any constant function and the 0 function is 0.

5. Put $P' = \{f(x) > 0\}$, $N' = \{f(x) < 0\}$, and $Z' = \{f(x) = 0\}$. If E is any measurable subset of Z', then $X = P \cup N$ with $P = P' \cup E$ and $N = N' \cup (Z' - E)$ is a Hahn decomposition. All other Hahn decompositions are obtained by adjusting P and N by taking the symmetric difference of P and of N with any set of μ measure 0.

6. In (a), let X be the positive integers, and let the algebra consist of all finite subsets and their complements; let ν of a finite set be the number of elements in the set, and let ν of the complement of a finite set F be $-\nu(F)$. In (b), use the same X and algebra, define $\nu(\{2k\}) = 2^{-k}$ and $\nu(\{2k-1\}) = -2^{-k}$, and extend ν to be completely additive. In (c), let $X = [0, 1]$, let the σ-algebra consist of the Borel sets, and take ν to be Lebesgue measure and μ to be counting measure.

7. Since P_r has L^1 norm 1, the inequality $\|u(r,\,\cdot\,)\|_p \le \|f\|_p$ follows from Minkowski's inequality for integrals. For the limiting behavior as r increases to 1, we extend f periodically and write

$$\begin{aligned}u(r,\theta) - f(\theta) &= \tfrac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\varphi) f(\theta-\varphi)\,d\varphi - f(\theta) \\ &= \tfrac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\varphi)[f(\theta-\varphi) - f(\theta)]\,d\varphi,\end{aligned}$$

the second step following since $\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r\,d\varphi = 1$. Applying Minkowski's inequality for integrals, we obtain

$$\|u(r,\,\cdot\,) - f\|_p \le \tfrac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\varphi)\|f(\theta-\varphi) - f(\theta)\|_{p,\theta}$$

since $P_r \ge 0$. The integration on the right is broken into two sets, $S_1 = (-\delta, \delta)$ and $S_2 = [-\pi, -\delta] \cup [\delta, \pi]$, and the integral is

$$\begin{aligned}&\le \tfrac{1}{2\pi}\int_{S_1} P_r(\varphi)\big(\sup\nolimits_{\varphi\in S_1}\|f(\theta-\varphi) - f(\theta)\|_{p,\theta}\big)\,d\varphi + \tfrac{1}{2\pi}\int_{S_2} P_r(\varphi) 2\|f\|_p\,d\varphi \\ &\le \sup_{\varphi\in S_1}\|f(\theta-\varphi) - f(\theta)\|_{p,\theta} + 2\|f\|_p \sup_{\varphi\in S_2} P_r(\varphi).\end{aligned}$$

Let $\epsilon > 0$ be given. If δ is sufficiently small, Proposition 9.11 shows that the first term is $< \epsilon$. With δ fixed, we can then choose r close enough to 1 to make the second term $< \epsilon$.

8. Let p be the dual index to p'. Put $r/R = r'$ in Problem 13 at the end of Chapter IV, so that

$$u(r'R, \theta) = \tfrac{1}{2\pi} \int_{-\pi}^{\pi} f_R(\varphi) P_{r'}(\theta - \varphi)\, d\varphi$$

for $r' < 1$. Take a sequence of R's increasing to 1, and let $\{R_n\}$ be a subsequence such that $\{f_{R_n}\}$ converges weak-star in $L^{p'}$ relative to L^p. Let the limit be f. For each θ and r', $P_{r'}(\theta - \cdot\,)$ is in L^p, and the equality $u(r'R_n, \theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f_{R_n}(\varphi) P_{r'}(\theta - \varphi)\, d\varphi$ thus gives $u(r', \theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\varphi) P_{r'}(\theta - \varphi)\, d\varphi$, which is the desired result.

9. If ν is a measure with $0 \leq \nu \leq \mu$, then $\nu(\{n\}) = 0$ for every n, and hence $\nu(\{\text{integers}\}) = 0$. So $\nu = 0$.

10. Let μ be given on the space X, and consider the set S of all completely additive ν with $0 \leq \nu \leq \mu$. This contains 0 and hence is nonempty. Order S by saying that $\nu_1 \leq \nu_2$ if $\nu_1(E) \leq \nu_2(E)$ for all E. If we are given a chain $\{\nu_\alpha\}$, let $C = \sup_\alpha \nu_\alpha(X)$. This is $\leq \mu(X)$ and hence is finite. Choose a sequence $\{\nu_{\alpha_k}\}$ from the chain with $\nu_{\alpha_k}(X)$ monotone increasing with limit C.

If $m < n$, let us see that $\nu_{\alpha_m} \leq \nu_{\alpha_n}$. Since the ν_α's form a chain, the only way this can fail is to have $\nu_{\alpha_m}(E) > \nu_{\alpha_n}(E)$ for some E and also $\nu_{\alpha_m}(E^c) \geq \nu_{\alpha_n}(E^c)$. But then $\nu_{\alpha_m}(X) > \nu_{\alpha_n}(X)$ by additivity, and this contradicts the fact that $\nu_{\alpha_k}(X)$ is monotone increasing. So $m < n$ implies $\nu_{\alpha_m} \leq \nu_{\alpha_n}$.

Define $\nu_0(E) = \lim_k \nu_{\alpha_k}(E)$. Corollary 1.14 shows that ν_0 is completely additive, and certainly $\nu_0 \leq \mu$. So ν_0 is an upper bound for the chain. Zorn's Lemma therefore shows that S has a maximal element ν.

Write $\sigma = \mu - \nu$. This is bounded nonnegative additive as a result of the construction. If there were a completely additive λ such that $0 \leq \lambda \leq \sigma$, then $\nu + \lambda$ would contradict the construction of ν from Zorn's Lemma. Thus σ is purely finitely additive.

11. It is enough to prove that μ is completely additive. If the contrary is the case, then there exists an increasing sequence of sets E_n with union E in the algebra such that the monotone increasing sequence $\{\mu(E_n)\}$ does not have limit $\mu(E)$. Since μ is nonnegative additive, $\mu(E_n) \leq \mu(E)$ for all n. Thus $\lim_n \mu(E_n) < \mu(E)$. Since $\nu - \mu$ is nonnegative additive, $\nu - \mu$ similarly has $\lim_n(\nu - \mu)(E_n) \leq (\nu - \mu)(E)$. Adding, we obtain $\lim_n \nu(E_n) < \nu(E)$, in contradiction to the complete additivity of ν.

12. Suppose μ is nonnegative bounded additive. Let $\mu = \nu_1 + \rho_1 = \nu_2 + \rho_2$ with ν_1 and ν_2 nonnegative completely additive and with ρ_1 and ρ_2 nonnegative purely finitely additive. Then $\nu_1 - \nu_2 = \rho_2 - \rho_1$. Let $\nu^+ - \nu^-$ be the Jordan decomposition of $\nu_1 - \nu_2$. Since $\nu_1 - \nu_2$ is completely additive, so are ν^+ and ν^-. The equality $\nu^+ - \nu^- = \rho_2 - \rho_1$ and the minimality of the Jordan decomposition together imply

that $0 \le \nu^+ \le \rho_2$ and $0 \le \nu^- \le \rho_1$. Problem 11 then shows that $\nu^+ = \nu^- = 0$. Hence $\nu_1 - \nu_2 = 0$, $\nu_1 = \nu_2$, and $\rho_1 = \rho_2$.

13. Let $R = I \times J$ be centered at (x, y). Then $\frac{1}{m(R)} \iint_R |f(u, v)|\,dv\,du = \frac{1}{m(I)} \int_I \big[\frac{1}{m(J)} \int_J |f(u, v)|\,dv\big]\,du \le \frac{1}{m(I)} \int_I f_1(u, y)\,du = f_2(x, y)$. Taking the supremum over R gives $f^{**}(x, y) \le f_2(x, y)$.

14. $\iint |f^{**}(x, y)|^p\,dx\,dy \le \iint |f_2(x, y)|^p\,dx\,dy = \int \big[\int |f_2(x, y)|^p\,dx\big]\,dy \le A_p^p \int \big[\int |f_1(x, y)|^p\,dx\big]\,dy$ by Corollary 9.21. If we interchange integrals and apply Corollary 9.21 a second time, we see that this is $\le A_p^{2p} \int \big[\int |f(x, y)|^p\,dy\big]\,dx = A_p^{2p} \|f\|_p^p$.

15. This is done in the style of Corollary 6.39.

16. Let $\Phi_1 \ge 0$ be a decreasing C^1 function on [0, 1] with $\Phi_1'(0) = 0$, $\Phi_1(1) = 1$, and $\Phi_1'(1) = -1$. Define $\Phi_0(x)$ on [0, 1] to be $\Phi_1(x)\big/(\pi(1 + x^2))$ on [0, 1] and to be $1\big/(\pi x(1 + x^2))$ on $[1, +\infty)$. Then $\Phi(x) = \Phi_0(|x|)$ has the required property.

17. $\sup_{\varepsilon>0} |(\psi_\varepsilon * f)(x)| \le \sup_{\varepsilon>0}(|\psi_\varepsilon| * |f|)(x) \le \sup_{\varepsilon>0}(\Phi_\varepsilon * |f|)(x)$, and then $\sup_{\varepsilon>0} |(\psi_\varepsilon * f)(x)| \le Cf^*(x)$ by Corollary 6.42. Since $\int_{\mathbb{R}^1} \psi(x)\,dx = 0$, the last part of the proof of Corollary 6.42 shows that $\lim_{\varepsilon>0}(\psi_\varepsilon * f)(x) = 0$ a.e. for f in $L^1(\mathbb{R}^1)$. If f is in $L^\infty(\mathbb{R}^1)$ and a bounded interval is specified, we can write f as the sum of an L^1 function carried on that interval and an L^∞ function vanishing on that interval. The L^1 part is handled by the previous case, and the L^∞ part is handled on that bounded interval by Theorem 6.20c.

18. We use the fact that $Q_\varepsilon = h_\varepsilon + \psi_\varepsilon$, where ψ is integrable with integral 0. Since $h_\varepsilon * f$ and $\psi_\varepsilon * f$ are in L^p, so is $Q_\varepsilon * f$. Convolution by an L^1 function such as P_ε is continuous on L^p by Proposition 9.10. With all limits being taken in L^p as $\varepsilon' \downarrow 0$, we have $P_\varepsilon * (Hf) = P_\varepsilon * (\lim(h_{\varepsilon'} * f)) = \lim P_\varepsilon * (h_{\varepsilon'} * f) = \lim P_\varepsilon * (Q_{\varepsilon'} * f - \psi_{\varepsilon'} * f) = \lim P_\varepsilon * (Q_{\varepsilon'} * f) - (\lim P_\varepsilon * \psi_{\varepsilon'}) * f$. The second term on the right side is 0. If we think of P_ε as in L^1 and $Q_{\varepsilon'}$ as in $L^{p'}$, then we have $P_\varepsilon * (Q_{\varepsilon'} * f) = (P_\varepsilon * Q_{\varepsilon'}) * f = Q_{\varepsilon'+\varepsilon} * f = (P_{\varepsilon'} * Q_\varepsilon) * f = P_{\varepsilon'} * (Q_\varepsilon * f)$. Thus $\lim P_\varepsilon * (Q_{\varepsilon'} * f) = \lim P_{\varepsilon'} * (Q_\varepsilon * f) = Q_\varepsilon * f$, and we conclude that $P_\varepsilon * (Hf) = Q_\varepsilon * f$.

19. $\sup_{\varepsilon>0} |(h_\varepsilon * f)(x)| \le \sup_{\varepsilon>0} |(Q_\varepsilon * f)(x)| + \sup_{\varepsilon>0} |(\psi_\varepsilon * f)(x)| \le \sup_{\varepsilon>0} |(P_\varepsilon * Hf)(x)| + Cf^*(x) \le C'(Hf)^*(x) + Cf^*(x)$, the last inequality following from Corollary 6.42 for P_ε. Let $1 < p < \infty$. Then it follows from Corollary 9.21 that $\|\sup_{\varepsilon>0} |h_\varepsilon * f|\,\|_p \le C_p(\|Hf\|_p + \|f\|_p)$, and we conclude from Theorem 9.23c that $\|\sup_{\varepsilon>0} |h_\varepsilon * f|\,\|_p \le D_p\|f\|_p$. Lemma 9.24 shows that $\lim_{\varepsilon\downarrow 0}(h_\varepsilon * f)(x) = f(x)$ everywhere if f is in a certain dense subspace of L^p, and it follows as in Problem 15 that $\lim_{\varepsilon\downarrow 0}(h_\varepsilon * f)(x) = f(x)$ almost everywhere if f is arbitrary in L^p.

20. Imitating the proof of parts (a) and (b) of Fejér's Theorem (Theorem 6.48), we readily prove that $K_n * f \to f$ in L^p, where K_n is the Fejér kernel. Therefore finite linear combinations of the exponentials are dense in $L^p([-\pi, \pi])$. For each such

linear combination f of exponentials, we have $S_n f = f$ for all sufficiently large n, and hence $S_n f \to f$ in L^p for a dense subset of L^p. Using the given estimate on $\|S_n f\|_p$ and the convergence of $S_n f$ on the dense set, we argue as in the proof of Theorem 9.23b to deduce convergence for all f in L^p.

21. Let $F_n(t) = \frac{2\sin(n+\frac{1}{2})t}{t}$ for $0 < |t| \le \pi$, and extend F_n periodically. Then $\frac{t}{2} F_n(t) = \sin(n+\frac{1}{2})t = (\sin\frac{1}{2}t)D_n(t)$. Since $(t/2)\big/\sin\frac{1}{2}t = 1 + t\psi(t)$ with $\psi(t)$ bounded above and below by positive constants on $[-\pi, \pi]$, we see that $D_n(t) - F_n(t) = \big[\frac{\frac{t}{2}}{\sin\frac{1}{2}t} - 1\big]F_n(t) = 2\psi(t)\sin(n+\frac{1}{2})t$. Then the functions $\psi_n(t) = 2\psi(t)\sin(n+\frac{1}{2})t$ have $D_n - F_n = \psi_n$ and $\|\psi_n\|_1$ bounded. By inspection, $F_n - E_n$ equals the function that is $\frac{2\sin(n+\frac{1}{2})t}{t}$ for $|t| < \frac{1}{2n+1}$ and is 0 for $\frac{1}{2n+1} \le |t| \le \pi$. These functions are $\le 2(n+\frac{1}{2})$ for $|t| < \frac{1}{2n+1}$ and are 0 otherwise; so their L^1 norms are bounded. This proves that $D_n - E_n = \varphi_n$ with $\|\varphi_n\|_1 \le C$ for some C.

If $\|T_n f\|_p \le B_p\|f\|_p$, then we have $\|S_n f\|_p = \|D_n * f\|_p = \|E_n * f + \varphi_n * f\|_p \le \|E_n * f\|_p + \|\varphi_n * f\|_p \le B_p\|f\|_p + \|\varphi_n\|_1\|f\|_p$, and we can take $A_p = B_p + C$.

22. We have $2i\sin(n+\frac{1}{2})t = e^{i(n+\frac{1}{2})t} - e^{-(n+\frac{1}{2})t}$. Thus the effect of the operator T_n on f is the sum of two terms $T_n^{(1)}f + T_n^{(2)}f$, one of which is

$$T_n^{(1)}f(x) = \int_{\frac{1}{2n+1}\le|t|\le\pi} \frac{-if(x-t)e^{-i(n+\frac{1}{2})(x-t)}e^{i(n+\frac{1}{2})x}}{t}\,dt.$$

If we regard f as continued periodically to the interval $[-3\pi, 3\pi]$ and we put f equal to 0 outside that interval, then

$$T_n^{(1)}f(x) = e^{i(n+\frac{1}{2})x}((H_\pi - H_{1/(2n+1)})g)(x) \quad \text{for } x \in [-\pi, \pi],$$

where $g(y) = -i\pi f(y)e^{-i(n+\frac{1}{2})(y)}$ on $[-3\pi, 3\pi]$. With A_p as the constant from Theorem 9.23, Theorem 9.23 gives

$$\begin{aligned}\big(\textstyle\int_{-\pi}^{\pi}|T_n^{(1)}f(x)|^p\,dx\big)^{1/p} &\le \big(\textstyle\int_{\mathbb{R}}|T_n^{(1)}f(x)|^p\,dx\big)^{1/p}\\ &\le \big(\textstyle\int_{\mathbb{R}}|H_\pi g|^p\,dx\big)^{1/p} + \big(\textstyle\int_{\mathbb{R}}|H_{1/(2n+1)}g|^p\,dx\big)^{1/p}\\ &\le 2A_p\big(\textstyle\int_{\mathbb{R}}|g|^p\,dx\big)^{1/p} \le 2\pi A_p\big(3\textstyle\int_{-\pi}^{\pi}|f|^p\,dx\big)^{1/p}.\end{aligned}$$

We get a similar estimate for $T_n^{(2)}f$, and the desired estimate for $T_n f$ follows.

23. Define a signed measure ν on $\mathcal{B}$ by $\nu(B) = \int f\,d\mu$. Then ν is absolutely continuous with respect to the restriction of μ to $\mathcal{B}$, and the Radon–Nikodym Theorem yields a function g measurable with respect to $\mathcal{B}$ such that $\nu(B) = \int_B g\,d\mu$ for all B in $\mathcal{B}$. This function g is $E[f|\mathcal{B}]$. Uniqueness is built into the uniqueness aspect of the Radon–Nikodym Theorem.

24. For those n's such that $\mu(X_n) \neq 0$, $E[f|\mathcal{B}]$ may be defined to be equal everywhere on X_n to the constant $\mu(X_n)^{-1}\int_{X_n} f\,d\mu$. For definiteness, $E[f|\mathcal{B}]$ may be defined to be 0 on each X_n with $\mu(X_n) = 0$.

25. The function f satisfies the defining properties (i) and (ii) of $E[f|\mathcal{A}]$.

26. In (a), we identify $E[E[f|\mathcal{B}] \mid \mathcal{C}]$ as $E[f|\mathcal{C}]$. It is measurable with respect to $\mathcal{C}$ and hence satisfies (i) toward being $E[f|\mathcal{C}]$. Any $C \in \mathcal{C}$ has $\int_C E[E[f|\mathcal{B}] \mid \mathcal{C}]\,d\mu = \int_C E[f|\mathcal{B}]\,d\mu$. In turn this equals $\int_C f\,d\mu$ since C is in $\mathcal{B}$. Hence $E[E[f|\mathcal{B}] \mid \mathcal{C}]$ satisfies (ii) toward being $E[f|\mathcal{C}]$.

In (b), we identify $E[f|\mathcal{B}]+E[g|\mathcal{B}]$ as $E[f+g \mid \mathcal{B}]$. It is measurable with respect to $\mathcal{B}$ and hence satisfies (i). For (ii), each B in $\mathcal{B}$ has $\int_B (E[f|\mathcal{B}] + E[g|\mathcal{B}])\,d\mu = \int_B E[f|\mathcal{B}]\,d\mu + \int_B E[g|\mathcal{B}]\,d\mu = \int_B f\,d\mu + \int_B g\,d\mu = \int_B (f+g)\,d\mu$.

In (c), it is enough to handle $f \geq 0$, and then it is enough to handle $g \geq 0$. If $g = I_B$ with $B \in \mathcal{B}$, then we shall identify $I_B E[f|\mathcal{B}]$ as $E[f I_B \mid \mathcal{B}]$. Certainly $I_B E[f|\mathcal{B}]$ satisfies (i). For (ii), each B' in $\mathcal{B}$ has $\int_{B'} I_B E[f|\mathcal{B}]\,d\mu = \int_{B'\cap B} E[f|\mathcal{B}]\,d\mu = \int_{B'\cap B} f\,d\mu = \int_{B'} I_B f\,d\mu$. This handles g equal to an indicator function. Part (b) allows us to handle g equal to a simple function, and monotone convergence allows us to handle g equal to any nonnegative integrable function. (For this last conclusion one needs to use that $f \geq 0$ implies $E[f|\mathcal{B}] \geq 0$, but this is built into the construction via the Radon–Nikodym Theorem.)

In (d), the important thing is that X is a set in $\mathcal{B}$. Then (ii) and (c) successively give $\int_X f E[g|\mathcal{B}]\,d\mu = \int_X E[fE[g|\mathcal{B}] \mid \mathcal{B}]\,d\mu = \int_X E[f|\mathcal{B}]E[g|\mathcal{B}]\,d\mu$. The right side is symmetric in f and g, and hence the left side is also.

27. For f in $L^1 \cap L^2$, we compute from the definition of $\mathcal{F}$ that $\mathcal{F}(\delta_r f)(y) = r\delta_r^{-1}(\mathcal{F}f)(y)$. It follows for all L^2 functions f that $\mathcal{F}(\delta_r f) = r\delta_r^{-1}(\mathcal{F}f)$ as an equality of L^2 functions. Let $A : L^2 \to L^2$ be bounded linear commuting with translations and dilations. Theorem 8.14 produces an L^∞ function m such that $\mathcal{F}(Af) = m(\mathcal{F}f)$ for all f in L^2. Using the commutativity of A with dilations, we have

$$\begin{aligned}(m)(\mathcal{F}f) &= \mathcal{F}(Af) = \mathcal{F}(\delta_r^{-1}A\delta_r f) = r^{-1}\delta_r(\mathcal{F}(A\delta_r f)) = r^{-1}\delta_r(m\mathcal{F}(\delta_r f))\\ &= r^{-1}(\delta_r m)(\delta_r(\mathcal{F}(\delta_r f))) = r^{-1}(\delta_r m)(\delta_r(r\delta_r^{-1}(\mathcal{F}f))) = (\delta_r m)(\mathcal{F}f).\end{aligned}$$

Consequently $\delta_r m = m$ for all $r > 0$. It follows that m is constant a.e. on each half line. The result follows.

28. Lemma 8.13 relies on Proposition 6.16 and Corollary 6.17. Proposition 9.11 extends Proposition 6.16 to $1 \leq p < \infty$ and is to be quoted in place of Proposition 6.16.

To generalize Corollary 6.17 appropriately, one can use any number p with $1 \leq p < \infty$, and it is important to allow the p associated to g_k to depend on k. In other words the statement of the corollary concerns functions g_k in L^{p_k}, and the norm on the expression involving g_k is to be $\|\cdot\|_{p_k}$. The same kinds of adjustments are needed in the proof of the corollary, and then the proof goes through.

The statement of Lemma 8.13 remains valid for any bounded linear operator $A : L^p \to L^q$ commuting with translations, provided $1 \le p < \infty$ and $1 \le q < \infty$. Corollary 6.17 is to be applied with $g_1 = g$, $p_1 = p$, $g_2 = Ag$, and $p_2 = q$, and then the argument goes through.

29. In (a), the simple functions f and g are in L^1, L^p, and L^q, and also $L^{p'}$ and $L^{q'}$ for the dual indices p' and q'. Problem 28 gives $(Af)*g = A(f*g)$ as an equality of L^q functions, and it similarly gives $A(f*g) = f*(Ag)$. Thus $(Af)*g = f*(Ag)$ as L^q functions. On the other hand, $(Af*g)$ is a bounded continuous function by Proposition 9.12 because Af is in L^q and g is in $L^{q'}$. Similarly $f*(Ag)$ is a bounded continuous function. Then we must have $(Af*g) = f*(Ag)$ pointwise. Evaluating both sides at 0 yields (a).

In (b), we take the supremum of the absolute value of both sides of (a) over all simple f with $\|f\|_p \le 1$. The right side becomes $\|Ag\|_{p'}$, and the left side, by Hölder's inequality, is $\le \|Af\|_p\|g\|_{p'} \le \|A\|_{p,p}\|g\|_{p'}$, where $\|A\|_{p,p}$ is the norm of $A : L^p \to L^p$. Thus each simple g has $\|Ag\|_{p'} \le \|A\|_{p,p}\|g\|_{p'}$. Since the space of simple functions is dense, A extends to a bounded linear operator from $L^{p'}$ into itself with $\|A\|_{p',p'} \le \|A\|_{p,p}$. The extension commutes with translations by a continuity argument. Reversing roles of p and p', we see that $\|A\|_{p,p} \le \|A\|_{p',p'}$. Thus $\|A\|_{p',p'} = \|A\|_{p,p}$.

In (c), the bounded operator obtained by the dual construction is from $L^{q'}$ to $L^{p'}$.

30. Problem 29 shows that A is also bounded from $L^{p'}$ to itself. By the Riesz Convexity Theorem (Theorem 9.19A), it is bounded also from L^2 to itself, since 2 is between p and p'. Being bounded from L^2 into itself and commuting with translations, it is given, according to Theorem 8.14, by multiplication on the Fourier transform side by an L^∞ function m. Thus $\mathcal{F}(Af) = m\mathcal{F}(f)$ for that same m on a dense subspace of L^p. Since both sides are continuous linear operators, this equality extends to all of L^p.

31. In (a), the real and imaginary parts of ρ are treated separately and come from the L^1 functions $A\varphi_\varepsilon$; let us ignore the imaginary parts, which are handled in the same way as the real parts. Since A is bounded from L^1 to itself, $\|A\varphi_\varepsilon\|_1 \le \|A\|\|\varphi_\varepsilon\|_1 = \|A\|$. Take a sequence of ε's tending to 0 and apply the Helly–Bray Theorem to extract a subsequence $\{\varepsilon_k\}$ such that $\{(A\varphi_{\varepsilon_k})^+\,dx\}$ and $\{(A\varphi_{\varepsilon_k})^-\,dx\}$ both converge weak-star against $C_{\text{com}}(\mathbb{R}^N)$. Let ρ be the difference of the limits of these sequences. This is a signed measure on the Borel sets $\mathbb{R}^N$, and its positive and negative parts ρ^+ and ρ^- in the Jordan decomposition (Theorem 9.14) have $\rho^+(\mathbb{R}^N) + \rho^-(\mathbb{R}^N) \le \|A\|$.

In (b), g is uniformly continuous and ρ is finite. The continuity of $g*\rho$ is immediate.

In (c), we have

$$\begin{aligned}(Ah^\# * \varphi_{\varepsilon_k})(y) &= A(h^\# * \varphi_{\varepsilon_k})(y) = (h^\# * A\varphi_{\varepsilon_k})(y)\\ &= \int_{\mathbb{R}^N} h^\#(y-x)(A\varphi_{\varepsilon_k})(x)\,dx\end{aligned}$$

$$= \int_{\mathbb{R}^N} h(x-y)(A\varphi_{\varepsilon_k})(x)\,dx$$
$$= \int_{\mathbb{R}^N} (\tau_y h)(x)(A\varphi_{\varepsilon_k})(x)\,dx,$$

and this tends by (a) to

$$\int_{\mathbb{R}^N} (\tau_y h)(x)\,d\rho(x) = \int_{\mathbb{R}^N} h(x-y)\,d\rho(x) = \int_{\mathbb{R}^N} h^\#(y-x)\,d\rho(x) = (h^\# * \rho)(y).$$

In (d), we observe that the equality in (c) is a pointwise equality. Since $\{\varphi_\varepsilon\}$ is an approximate identity, $Ah^\# * \varphi_\varepsilon \to Ah^\#$ n L^1. Thus we have $Ah^\# = h^\# * \rho$ as an equality of L^1 functions whenever h is in $C_{\text{com}}(\mathbb{R}^N)$. The operators on the two sides, A and $(\,\cdot\,) * \rho$, are continuous on L^1; this fact is given in the case of A and is easily checked in the case of $(\,\cdot\,) * \rho$. By continuity the equality $Ah^\# = h^\# * \rho$ valid on $C_{\text{com}}(\mathbb{R}^N)$ extends to an equality $Af = f * \rho$ valid on all of L^1.

32. Define r by $1 - \frac{1}{r} = \frac{1}{p} - \frac{1}{q}$. Young's inequality (Corollary 9.19D) shows that convolution with an L^r function h is bounded from L^p to L^q, and it commutes with translations. To obtain a nonzero convolution operator of this kind, we take h to be nonzero, simple, and real-valued. Putting $h^\#(x) = h(-x)$, we observe that $h * h^\#$ is a bounded continuous function and has $h * h^\#(0) = \int_{\mathbb{R}^N} h(x)^2\,dx > 0$.

33. For (a), if f is in $C_{\text{com}}(\mathbb{R}^N)$, then $\|\tau_h f + f\|_p^p = \int_{\mathbb{R}^N} |f(x-h) + f(x)|^p\,dx$, and for h sufficiently large, this equals $\int_{\mathbb{R}^N} |f(x-h)|^p\,dx + \int_{\mathbb{R}^N} |f(x)|^p\,dx = 2\|f\|_p^p$. Thus (a) is proved in this special case. The general case follows from a 3ϵ argument, $C_{\text{com}}(\mathbb{R}^N)$ being dense in L^p.

For (b), we have $\|\tau_h(Af) + Af\|_q = \|A(\tau_h f) + Af\|_q \le M\|\tau_h f + f\|_p$. Letting h tend to infinity and applying (a) to both sides, we obtain $2^{1/q}\|Af\|_q \le 2^{1/p}M\|f\|_p$ and thus $\|Af\|_q \le 2^{1/p-1/q}M\|f\|_p$. Since M is the norm of $\|A\|$ and since $2^{1/p-1/q}M < M$, we can find an $f \ne 0$ with $\|Af\|_q > M\|f\|_p$, and then we have a contradiction.

Chapter X

1. For (a), the diagonal $\Delta = \{(y, y) \in Y \times Y\}$ is a closed subset of $Y \times Y$ since Y is Hausdorff, and the function $F : X \to Y \times Y$ given by $F(x) = (f(x), g(x))$ is continuous. Therefore $F^{-1}(\Delta)$ is closed.

2. The argument is the same as for Problem 18 in Chapter II.

3. We argue as in the proof of Theorem 2.53. Taking complements, we see that it is enough to prove that the intersection of countably many open dense sets is nonempty. Suppose that U_n is open and dense for $n \ge 1$. Let x_1 be in U_1. Since U_1 is open, local compactness and regularity together allow us to find an open neighborhood B_1 of x_1 with B_1^{cl} compact and $B_1^{\text{cl}} \subseteq U_1$. We construct inductively points x_n and open neighborhoods B_n of them such that $B_n \subseteq U_1 \cap \cdots \cap U_n$ and $B_n^{\text{cl}} \subseteq B_{n-1}$. Suppose B_n

with $n \geq 1$ has been constructed. Since U_{n+1} is dense and B_n is nonempty and open, $U_{n+1} \cap B_n$ is not empty. Let x_{n+1} be a point in $U_{n+1} \cap B_n$. Since $U_{n+1} \cap B_n$ is open, we can find an open neighborhood B_{n+1} of x_{n+1} in U_{n+1} such that $B_{n+1}^{\text{cl}} \subseteq U_{n+1} \cap B_n$. Then B_{n+1} has the required properties, and the inductive construction is complete. The sets B_n^{cl} have the finite-intersection property, and they are closed subsets of B_1^{cl}, which is compact. By Proposition 10.11 their intersection is nonempty. Let x be in the intersection. For any integer N, the inequality $n > N$ implies that x_n is in B_{N+1}. Thus x is in $B_{N+1}^{\text{cl}} \subseteq B_N \subseteq U_1 \cap \cdots \cap U_N$. Since N is arbitrary, x is in $\bigcap_{n=1}^{\infty} U_n$.

4. Let Y be a locally compact dense subset of the Hausdorff space X. If y is in Y, let N be a relatively open neighborhood of y such that $N \subseteq K$ with K compact in Y. Since N is relatively open, $N = U \cap Y$ for some open U in X. It will be proved that $N = U$, so that each point of Y has an X open neighborhood, and then Y will be open. The set K is compact in X and must be closed since X is Hausdorff. The points of $U \cap K$ are in Y since $K \subseteq Y$, and hence $U \cap K \subseteq U \cap Y = N$. Consider a point x of the open set $U - K$. Suppose x is not in Y. Then x is a limit point of Y since Y is dense. Hence the open neighborhood $U - K$ of y contains a point y' of Y. Then y' is in $U \cap Y = N \subseteq K$ and cannot be in $U - K$, contradiction. We conclude that x is in Y. Then x is in $U \cap Y = N$, and $U = N$.

5. First consider any continuous function $f : Y^* \to [0, 1]$ with $f(y_\infty) = 0$. The set of y's with $f(y) > 1/k$ is open and contains y_∞, thus is a compact subset of Y and must be finite. Hence the set of y's with $f(y) = 0$ has a countable complement.

If Z is normal, apply Urysohn's Lemma to A and B, obtaining a continuous $F : Z \to [0, 1]$ with $f(A) = 1$ and $f(B) = 0$. Enumerate the members of X as $x_1, x_2, \ldots$. For fixed n, $f(y) = F(x_n, y)$ is continuous from Y^* to $[0, 1]$ and is 0 at y_∞. Thus $F(x_n, y) > 0$ only on a countable set S_n of y's, and $F(x_n, y) > 0$ for some n at most on the countable set $S = \bigcup_{n=1}^{\infty} S_n$. If y_0 is not in S, then $x \mapsto F(x, y_0)$ is continuous from X^* to $[0, 1]$, is 0 for every x other than x_∞, and is 1 at x_∞. This contradicts the continuity, and we conclude that Z is not normal.

6. If E is an infinite set with no limit point, then E is closed and each x in E is relatively open. Hence each x has an open set U_x in X with $U_x \cap E = \{x\}$. These open sets and E^c cover X, and there is no finite subcover. Thus X compact implies that each infinite subset has a limit point.

8. Part (a) follows from Problem 7b and Proposition 10.34. For (b), $f^{-1}(-\infty, a)$ is $\varnothing$ if $a < 0$, is $\mathbb{R} - \{0\}$ if $0 \leq a < 1$, and is $\mathbb{R}$ if $a \geq 1$; hence it is open in every case. Part (d) follows from (a). For (e), there exists an upper semicontinuous function $\geq f(x)$, namely the constant function everywhere equal to $\sup |f(x)|$. Then (d) shows that the pointwise infimum over all upper semicontinuous functions $\geq f(x)$ meets the conditions on f^-.

9. For (a), we have $Q_f(x) = f^-(x) + (-f)^-(x)$. Both terms on the right are upper semicontinuous, and the sum is upper semicontinuous by Problem 8c. For (c), $f_-(x) \leq f(x) \leq f^-(x) = Q_f(x) + f_-(x)$. If $Q_f = 0$, then $f_- = f = f^-$ shows that f is continuous with respect to all sets $\{x < b\}$ and all sets $\{x > a\}$. Hence

$f^{-1}(a, b)$ is open for every a and b, and f is continuous with respect to the metric topology. Conversely if f is continuous, then the definition makes $f^- = f$ and $(-f)^- = -f$. Therefore $f^- = f_- = f$ and $Q_f = f^- - f_- = 0$.

10. In (a), that subset of pairs is $(A \times A) \cup (B \times B) \cup \{(x, x) \mid x \in X\}$, which is the union of three closed sets and hence is closed. In (b), let X be a Hausdorff space that is not normal, and take A and B to be disjoint closed sets that cannot be separated by open sets.

11. In (a), $q^{-1}q(x) = p_2((\{x\} \times X) \cap R)$, where p_2 is the projection to the second coordinate of $X \times X$. Since $\{x\}$ is closed and X is compact and R is closed, $(\{x\} \times X) \cap R$ is compact. Then $q^{-1}q(x)$ is compact, hence closed, being the continuous image of a compact set.

In (b), we have $p_2((U^c \times X) \cap R) = \{y \in X \mid (x, y) \in R \text{ for some } x \in U^c\} = \{y \in X \mid q^{-1}q(y) \cap U^c \neq \varnothing\} = \{y \in X \mid q^{-1}q(y) \subseteq U\}^c = V^c$. Since U is open, the left side is closed, by the same considerations as in (a). Thus V^c is closed, and V is open.

In (c), let $q(x)$ and $q(y)$ be distinct points of $X/\sim$. By (a), the disjoint subsets $q^{-1}q(x)$ and $q^{-1}q(y)$ are closed. Since X is normal, find disjoint open sets U_1 and U_2 containing $q^{-1}q(x)$ and $q^{-1}q(y)$, respectively. Let $V_1 = \{z \in X \mid q^{-1}q(z) \subseteq U_1\}$ and $V_2 = \{z \in X \mid q^{-1}q(z) \subseteq U_2\}$. These are disjoint sets, and they are open by (b). Then $q(V_1)$ is open in $X/\sim$ because $q^{-1}q(V_1) = V_1$ is open, and similarly $q(V_2)$ is open. The sets $q(V_1)$ and $q(V_2)$ are disjoint because $q^{-1}q(V_1) = V_1$ and $q^{-1}q(V_2) = V_2$ are disjoint. Thus $q(V_1)$ and $q(V_2)$ are the required open sets separating $q(x)$ and $q(y)$.

For (d), part (c) shows that $X/\sim$ is Hausdorff, and therefore its compact subsets are closed. The image of any closed set is X is the image of a compact set, hence is compact and must be closed. For (e), the answer is "no," and part (f) supplies a counterexample. For (f), the function $p : X \to S^1$ is continuous, and Proposition 10.38a produces a continuous function $p_0 : X/\sim \to S^1$ such that $p = p_0 \circ q$, where q is the quotient map. Then p_0 is continuous and one-one from a compact space onto a Hausdorff space and must be a homeomorphism.

12–13. The proofs are the same as in Section II.8.

14. This is proved in the same way as in Problems 13 and 11 in Chapter II.

15. For (a), call the relation $\sim$. This is certainly reflexive and symmetric. For transitivity let $x \sim y$ and $y \sim z$. Then x and y lie in a connected set E, and y and z lie in a connected set F. The sets E and F have y in common, and Problem 13a shows that $E \cup F$ is connected. Thus $x \sim z$. Part (b) is immediate from Problem 13b. For (c), let x be given, and let U be a connected neighborhood of x. Then U lies in the component of x. Thus the component of x is a neighborhood of each of its points and is therefore open.

16. Form the class $\mathcal{C}$ of all functions F as described, including the empty function, and order the class by inclusion; for the purposes of the ordering, each function is

to be regarded as a set of ordered pairs. The class $\mathcal{C}$ is nonempty since the empty function is in it. If we have a chain in $\mathcal{C}$, we form the union F of the functions in the chain. We show that F is an upper bound for the chain. To do so, we need to see that the indicated sets cover X. Thus let $x \in X$ be given. Only finitely many sets U in $\mathcal{U}$ contain x, by assumption. Say these are $U_1, \dots, U_n$. If one of these fails to be in the domain of F, then x lies in $\bigcup_{V \in \mathcal{U},\, V \notin \text{domain}(F)} V$, and x is covered. Thus all of $U_1, \dots, U_n$ may be assumed to be in the domain of F. Each U_j is in the domain of some function F_j in the chain, and all of them are in the domain of the largest of the F_j's, say F_0. Since x is not in $\bigcup_{V \in \mathcal{U},\, V \notin \text{domain}(F)} V$, it is not in the larger union $\bigcup_{V \in \mathcal{U},\, V \notin \text{domain}(F_0)} V$. Thus it must be in $\bigcup_{U \in \text{domain} F_0}(U)$. Since $F_0(U)^{\text{cl}} \subseteq U$ for each U, x must lie in some $F_0(U_j)$. Then x lies in $F(U_j)$, and F is an upper bound for the chain.

By Zorn's Lemma let F be a maximal element in $\mathcal{C}$. To complete the argument, we show that every set in $\mathcal{U}$ lies in domain(F). Suppose that U_0 is a set in $\mathcal{U}$ that is not in domain(F). Let U' be the union of all $F(U)$ for U in domain(F) and all V other than U_0 that are not in domain(F). Since F is in $\mathcal{C}$, $U' \cup U_0 = X$. Hence U'^c is a closed subset of the open set U_0. Since X is normal, we can find an open set W such that $U'^c \subseteq W \subseteq W^{\text{cl}} \subseteq U$. If we define $F(U) = W$, then we succeed in enlarging the domain of F, in contradiction to the maximality of F. Hence every member of $\mathcal{U}$ lies in domain(F), as asserted.

17. Form the open sets V_U as in the previous problem. For each U in $\mathcal{U}$, apply Urysohn's Lemma to find a continuous function $g_U : X \to [0, 1]$ with g_U equal to 1 on V_U and equal to 0 on U^c. The open cover $\{V_U\}$ is locally finite since $\mathcal{U}$ is locally finite. Therefore $g = \sum_{U \in \mathcal{U}} g_U$ is a continuous function on X. Since g_U is positive on V_U and the sets V_U cover X, g is everywhere positive. Therefore the functions $f_U = g_U/g$ have the required properties.

18. If $c_0 = 0$, take $F_0 = 0$. If $c_0 \neq 0$, apply Urysohn's Lemma to obtain a continuous function h with values in $[0, 1]$ that is 1 on P_0 and is 0 on N_0, and then put $F_0 = \frac{2}{3}c_0 h - \frac{1}{3}c_0$.

19. On $P_0 \cap C$, g_0 is $\geq c_0/3$ and F_0 is $c_0/3$. Therefore $g_0 - F_0$ is ≥ 0 and $\leq 2c_0/3$. Similarly on $N_0 \cap C$, $g_0 - F_0$ is ≤ 0 and $\geq -2c_0/3$. Elsewhere on C, g_0 and F_0 are both between $-c_0/3$ and $c_0/3$, and hence $|g_0 - F_0| \leq 2c_0/3$. Thus $|g_0 - F_0| \leq 2c_0/3$ everywhere on C. The function F_1 is continuous from X into $\mathbb{R}$, has $|F_1| \leq \frac{2}{3}(\frac{1}{3}c_0)$, and takes a value $c_1 \leq \frac{2}{3}(\frac{1}{3}c_0)$ on $\{x \in C \mid g_1(x) \geq c_1/3\}$ and the value $-c_1$ on $\{x \in C \mid g_1(x) \leq -c_1/3\}$.

20. Iteration produces continuous functions $F_n : X \to \mathbb{R}$ with $|F_n(x)| \leq \frac{1}{3}\big(\frac{2}{3}\big)^n c_0$ for all x in X and $\big|f(x) - \sum_{i=0}^{n-1} F_i(x)\big| \leq \big(\frac{2}{3}\big)^n c_0$ for all x in C. Let $F(x) = \sum_{n=0}^{\infty} F_n(x)$. The series converges uniformly on X by the estimate on $F_n(x)$ and the Weierstrass M test, and Proposition 10.30 shows that F is continuous on X. If we let n tend to infinity in the estimate on $f(x) - \sum_{i=0}^{n-1} F_i(x)$, we see that F and f agree on C. Finally for x in X,

$$|F(x)| \le \sum_{n=0}^{\infty} |F_n(x)| \le \sum_{n=0}^{\infty} \tfrac{1}{3}\big(\tfrac{2}{3}\big)^n c_0 = c_0 = \sup_{y\in C} |f(y)|.$$

Thus $|F|$ and $|f|$ have the same supremum.

21. Every open interval is in the base and hence is open. The closed interval $\{a \le x \le b\}$ is the complement of the open set $\{x < a\} \cup \{b < x\}$ and is therefore closed.

22. Let $a < b$ be given. If there exists a c with $a < c < b$, then the open sets $\{x < c\}$ and $\{c < x\}$ separate a and b; otherwise the open sets $\{x < b\}$ and $\{a < x\}$ separate them. Hence X is Hausdorff.

Let a and a closed set F be given with a not in F. Since F^c is a neighborhood of a, there exists a basic open set B containing a that is disjoint from F. If B has some element larger than a, let d be such an element; otherwise let d be undefined. If B has some element smaller than a, let c be such an element; otherwise let c be undefined. If c and d are both defined, then $F \subseteq \{x < c\} \cup \{d < x\}$, while a is in $\{c < x < d\}$. If c is not defined but d is defined, then $F \subseteq \{x < a\} \cup \{d < x\}$, while a is in $B \cap \{x < d\}$. If d is not defined but c is defined, we argue symmetrically. If neither c nor d is defined, then $B = \{a\}$ is open and closed; hence B^c and B are the required open sets separating F and a.

23. Suppose that any nonempty set with an upper bound has a least upper bound, and let E be a set with a lower bound. We are to produce a greatest lower bound. Let F be the set of all lower bounds for E. This is nonempty, and all elements of F are $\le e$, where e is an element of E. So F has an upper bound. Let c be a least upper bound. We show that c is a greatest lower bound for E.

If c is not a lower bound for E, then E has some e with $e \le c$, $e \ne c$, i.e., with $e < c$. All f in F have $f \le e < c$. So e is a smaller upper bound for F, contradiction. Thus c is a lower bound for E. If there is some greater lower bound, say d, then $c < d \le e$ for all e in E. This implies that d is in F, and hence c is not an upper bound for F.

24. In (a), suppose that Y is nonempty closed and has an upper bound and a lower bound. We are to prove that Y is compact. It is enough to handle a set $Y = [a, b]$. Let an open cover $\mathcal{U}$ of Y be given, and suppose there is no finite subcover. Let E be the set of all x in $[a, b]$ such that some finite subcollection from $\mathcal{U}$ covers $[a, x]$. Then a is in E. Since E is nonempty and has b as an upper bound, the order completeness shows that E has a least upper bound c. Since we are assuming that $\mathcal{U}$ has no finite subcover of $[a, b]$, $E^c \cap [a, b]$ is nonempty. This set has a lower bound, namely a, and therefore it has a greatest lower bound d.

If e is in E and f is in $E^c \cap [a, b]$, then $e \le f$. So $e \le d$, and then $c \le d$. Suppose $c < d$. Then c must be in E. Any x with $c < x < d$ cannot be in E or E^c, and hence there is no such x. Then a finite subclass of $\mathcal{U}$ that covers $[a, c]$, together with a member of $\mathcal{U}$ that contains d, is a finite open subcover for $[a, d]$ and contradicts the fact that d is not in E. Thus $c = d$.

Now suppose that c is in $E^c \cap [a, b]$. Since $c = d$, E has no largest element. Choose a member U of $\mathcal{U}$ containing c, and find a basic open neighborhood B of c contained in U. Then $B \cap E$ must contain some c' with $c' < c$. A finite subclass of $\mathcal{U}$ covers $[a, c']$, and U covers $[c', c]$. Thus c is in E, and we have a contradiction.

We conclude that c is in E. Since $c = d$, $E^c \cap [a, b]$ has no smallest element. Choose a member U of $\mathcal{U}$ containing c, and find a basic open neighborhood B of c contained in U. Then $B \cap (E^c \cap [a, b])$ must contain an element c' with $c < c'$, and then there must be some c'' with $c < c'' < c'$. A finite subclass of $\mathcal{U}$ that covers $[a, c]$, together with the set U, then covers $[a, c'']$ and shows that c'' is in E. This contradicts the fact that c is an upper bound of E.

In (b), let x be given in X. If $a < x < b$ for some a and b, then $[a, b]$ is the required compact neighborhood of x. If x is a lower bound for X and there exists b with $x < b$, then $[x, b]$ is the required compact neighborhood. If x is an upper bound for X and there exists a with $a < x$, then $[a, x]$ is the required compact neighborhood. Since X has at least two members, there are no other possibilities. So X is locally compact.

25. In (a), the sets $\{x < b\}$ and $\{a < x\}$ are open and disjoint, contain a and b respectively, and have union X. Thus X is disconnected.

In (b), suppose that X is order complete and has no gaps. Assume, on the contrary, that U and V are disjoint nonempty open sets with union X. Say that $u < v$ for some u in U and v in V. It will be convenient to assume that u is not the smallest element in X and v is not the largest; when this assumption is not in place, the same line of proof works except that one may below have to use basic open sets of the form $\{r < x\}$ and $\{x < s\}$, as well as $\{r < x < s\}$.

Form the set S of all $x \in X$ with $x \le v$ and $(x, v] \subseteq V$. This set has u as a lower bound, and we let b be the greatest lower bound. Then $u \le b \le v$. First suppose that b is in V. Choose a basic open set $(r, s) \subseteq V$ with $r < b < s$; this is possible by our temporary assumption because V is open. Then $(\max\{u, r\}, v] \subseteq V$. If $\max\{u, r\} < b$, then $\max\{u, r\}$ is in S and b is not a lower bound for S; thus $b \le \max\{u, r\}$, i.e., $b = u$. This is impossible since b is assumed to be in V. We conclude that b is in U. Choose a basic open set $(r, s) \subseteq U$ with $r < b < s$; again this is possible by our temporary assumption because U is open. Since there are no gaps, we can find s' with $b < s' < s$. Then $\min\{v, s'\}$ is a lower bound for S, and b cannot be the greatest lower bound unless $\min\{v, s'\} \le b$, i.e., $b = v$. This is impossible since b is assumed to be in U, and we have arrived at a contradiction.

26. As an ordered set, X is the same as $\mathbb{R}$, and hence its order topology is the same as for $\mathbb{R}$, which is connected. In its relative topology, X is disconnected, being the disjoint union of the open sets $[0, 1)$ and $[2, 3)$.

27. The subset $[0, 1)$ is closed, being the intersection of all sets $\{x \mid x \le y\}$ for $y \in (1, 2]$. Similarly $(1, 2]$ is closed. Hence they are both open, and X is disconnected. It follows immediately from the definition that there are no gaps.

28. If a nonempty subset of points (x, y) is given, let x_0 be the least upper bound

of the x's. If no (x_0, y) is in the set, then $(x_0, 0)$ is the least upper bound for the set. If some (x_0, y) is in the set, let y_0 be the least upper bound of the y's. Then (x_0, y_0) is the least upper bound of the set. We conclude that X is order complete. Problem 24a then shows that X is compact. This proves the compactness in (a). There are no gaps, and Problem 25b thus proves the connectedness. For each $x \in [0, 1]$, the set $\{(x, y) \mid 0 < y < 1\}$ is open. Thus we have an uncountable disjoint union of open sets, and X cannot be separable. Part (b) is handled in the same way.

Chapter XI

1. In (a), every compact subset of X is compact when viewed as in X^*, and this gives inclusion in one direction. In the reverse direction it is enough to show that when U is open in X^*, then $U - \{\infty\}$ is a Borel set in X. Since X is σ-compact, we can choose an increasing sequence of compact sets K_n with $K_n \subseteq K_{n+1}^o$ and $\bigcup_{n=1}^\infty K_n = X$. Then $U \cap K_{n+1}^o$ is open and bounded, hence is a Borel subset of X. The countable union of these sets is U, and hence U is a Borel set. In (b), the Borel sets of X are the countable sets and their complements. However, every subset U of X is open in X and therefore open in X^*. Its complement in X^* is compact and is a Borel set in X^*. Thus U is a Borel set in X^*.

2. Part (a) of the previous problem shows that every open subset of X is a Borel set, and hence every continuous function is a Borel function.

3. Use the regularity to show that the conclusion holds for indicator functions and hence simple functions. Then pass to the limit.

4. Let I_E be an indicator function. Given $\epsilon > 0$, find by regularity a compact set L and an open set U with $L \subseteq E \subseteq U$ and $\mu(U - L) < \epsilon$. The compact set K will be $K = (U - L)^c = L \cap U^c$. Thus consider the restriction of I_E to the compact set K. Let x be in K. If x is in E, then x is in L. The set $U \cap K = L$ is a relatively open neighborhood of x, and I_E is identically 1 on this. Hence the restriction of I_E to K is continuous at the points of E. Similarly if x is in E^c, then x is in U^c. The set $L^c \cap K = U^c$ is a relatively open neighborhood of x, and we argue similarly. This handles indicator functions, and the result for simple functions follows immediately.

Next suppose that f is a real-valued Borel function ≥ 0. Choose an increasing sequence of simple functions $s_n \geq 0$ with limit f. Let $\epsilon > 0$ be given, and find, by Egoroff's Theorem, a Borel set E with $\mu(E^c) < \epsilon$ such that $\lim s_n(x) = f(x)$ uniformly for x in E. Next find, for each n, a compact subset K_n of X with $\mu(K_n^c) < \epsilon/2^n$ such that $s_n\big|_{K_n}$ is continuous. The set $F = E \cap \big(\bigcap_{n=1}^\infty K_n\big)$ has complement of measure $< 2\epsilon$, and the restriction of every s_n to F is continuous. Since $\{s_n\}$ converges to f uniformly on E, the restriction of f to F is continuous. Using regularity once more, we can find a compact subset K_0 of F such that $\mu(F - K_0) < \epsilon$. Then $\mu(K_0^c) < 3\epsilon$, and the restriction of f to K_0 is continuous.

5. In (a), any rotation preserves Euclidean distances and fixes the origin. Since S_{ab} is exactly the set of points whose distance d from the origin has $a < d < b$, S_{ab} is mapped to itself. Part (b) follows from the change-of-variables formula (Theorem 6.32). The determinant that enters the formula is the determinant of the matrix of the rotation and is 1. The first conclusion of (c) is what the change-of-variables formula gives for the transformation to spherical coordinates when applied to the set S_{ab} if we take Fubini's Theorem into account. It yields $\int_{S_{ab}} LF\,dx = \big(\int_a^b r^2\,dr\big)\big(\int_{S^2} Lf\,d\omega\big) = \big(\int_a^b r^2\,dr\big)\big(\int_{S^2} Lf\,d\omega\big)$. Since $\int_a^b r^2\,dr$ is not zero, we can divide by it and obtain the second conclusion of (c). Part (d) is proved by setting it up to be a special case of the uniqueness in Theorem 11.1.

6. In (a), monotonicity of μ gives $\mu(K) \le \inf_\alpha \mu(K_\alpha)$. Suppose that $<$ holds. Choose by regularity an open set U containing K such that $\mu(U) < \inf_\alpha \mu(K_\alpha)$. The sets K_α^c form an open cover of the compact set U^c, and there is a finite subcover. The intersection of the complements is one of the sets K_{α_0}, and it has the property that $K_{\alpha_0} \subseteq U$. Monotonicity then gives $\mu(K_{\alpha_0}) \le \mu(U)$, and thus $\inf_\alpha \mu(K_\alpha) \le \mu(U)$, contradiction.

For (b), consider all compact subsets K of X for which $\mu(K) = 1$. The intersection of any two of these is again one by Lemma 11.9. If K_0 is the intersection of all of them, then K_0 is compact, and (a) shows that $\mu(K_0) = 1$. If K_0 contains two distinct points x and y, find disjoint open neighborhoods U_x and U_y. Then $K_0 = (K_0 - U_x) \cup (K_0 - U_y)$ exhibits K_0 as the union of two proper compact sets. At least one of them must have measure 1, and then K_0 is shown *not* to be the intersection of all compact subsets of measure 1.

In (c) let K_0 be any compact G_δ, and choose a decreasing sequence $\{f_n\}$ in $C(X)$ with limit I_{K_0}. Passing to the limit from the formula $\int_X f_n^2\,d\mu = \big(\int_X f_n\,d\mu\big)^2$, we obtain $\mu(K_0) = \mu(K_0)^2$. Thus $\mu(K_0)$ is 0 or 1. By regularity, μ takes only the values 0 and 1, and (b) shows that μ is a point mass.

For (d), apply Theorem 11.1 and obtain the regular Borel measure μ corresponding to ℓ. Then μ has the property in (c) and must be a point mass.

7. The statement for (a) is that $u(r, \theta)$ is the Poisson integral of a signed or complex Borel measure on the circle if and only if $\sup_{0<r<1} \|u(r, \theta)\|_{1,\theta}$ is finite. The necessity is proved in the same way as in Problem 7 at the end of Chapter IX. The sufficiency is proved in the same way as in Problem 8 in that group, except that the weak-star convergence is in $M(\text{circle})$ relative to $C(\text{circle})$. For (b), expand $u(r, \theta)$ in series as in Problem 13 at the end of Chapter IV. Since u is nonnegative, the L^1 norm over any circle centered at the origin is just the integral, and the result of integrating in θ is that the $n = 0$ term is picked out. Thus $\|u(r, \theta)\|_{1,\theta} = c_0$ for every r. The condition in (a) is satisfied, and u is therefore the Poisson integral of a Borel complex measure. Examination of the proof of (a) shows that the complex measure is a measure.

8. Order topologies are always Hausdorff. Since Ω^* has a smallest element and a largest element, Problems 23 and 24 of Chapter X show that Ω^* is compact if every nonempty subset has a least upper bound. Since the ordering for Ω^* has the property

that every nonempty subset has a least element, the existence of least upper bounds is satisfied.

9. First we prove that the intersection of any two uncountable relatively closed sets C and D is uncountable. Assume the contrary. Since $C \cap D$ is countable and the countable union of countable sets is countable, there is some countable ordinal ω greater than all members of $C \cap D$. Since C and D are uncountable, we can find a sequence $\omega < \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \cdots$ such that each α_j is in C and each α_j is in D. The least ordinal γ greater or equal to all members of the sequence is a countable ordinal and has to be a limit point of both C and D. Since C and D are closed, γ is in $C \cap D$. But $C \cap D$ was supposed to have no ordinals greater than ω. This contradiction shows that $C \cap D$ is uncountable, and of course it is relatively closed also.

Now let a sequence of uncountable relatively closed sets C_n be given. By the previous step we may assume that they are decreasing with n. If $\bigcap_{n=1}^{\infty} C_n = C$ is countable, then there is some countable ordinal ω greater than all members of C. Replacing C_n by $C_n \cap \{x \geq \omega\}$ we may assume that the C_n have empty intersection. Let α_n be the least member of C_n. The result is a monotone increasing sequence since the C_n are decreasing. If α is the least ordinal $\geq$ all α_n, then α is a countable ordinal. It is a limit point of each C_n, hence lies in each C_n. The existence of α contradicts the fact that the C_n have been adjusted to have empty intersection. This contradiction shows that $\bigcap_{n=1}^{\infty} C_n$ is uncountable.

10. For additivity the question is whether the union of two sets that fail to meet the condition of the previous problem can meet the condition. The answer is no because the previous problem shows that the intersection of any two sets meeting the condition again meets the condition. The complete additivity is then a consequence of Corollary 5.3 and the result of the previous problem. The measure μ takes on only the values 0 and 1 and yet is not a point mass because one-point sets do not satisfy the defining property for measure 1. Problem 6b therefore allows us to conclude that μ is not regular.

11. Let μ be a Borel measure on X, and let S be the set of all regular Borel measures ν with $\nu \leq \mu$. This contains 0 and hence is nonempty. Order S by saying that $\nu_1 \leq \nu_2$ if $\nu_1(E) \leq \nu_2(E)$ for all E. If we are given a chain $\{\nu_\alpha\}$, let $C = \sup_\alpha \nu_\alpha(X)$. This is $\leq \mu(X)$ and hence is finite. Choose a sequence $\{\nu_{\alpha_k}\}$ from the chain with $\nu_{\alpha_k}(X)$ monotone increasing with limit C. Then $\nu_{\alpha_k}(E)$ is monotone increasing for every Borel set E, and we define $\nu(E)$ to be its limit. The complete additivity of ν follows from Corollary 1.14, and it is easy to check that $\nu_\alpha \leq \nu \leq \mu$ for all α. We have to check that ν is regular. Let $\epsilon > 0$ be given, and choose ν_{α_k} with $\nu_{\alpha_k}(X) \geq \nu(X) - \epsilon$. If E is given, find K and U with $K \subseteq E \subseteq U$, K compact, U open, and $\nu_{\alpha_k}(U - K) < \epsilon$. Then

$$\begin{aligned}\nu_{\alpha_k}(U - K) + \nu((U - K)^c) + \epsilon &\geq \nu_{\alpha_k}(U - K) + \nu_{\alpha_k}((U - K)^c) + \epsilon \\ &= \nu_{\alpha_k}(X) + \epsilon \geq \nu(X) = \nu(U - K) + \nu((U - K)^c),\end{aligned}$$

and hence $\nu_{\alpha_k}(U-K)+\epsilon \geq \nu(U-K)$. Since $\nu_{\alpha_k}(U-K) < \epsilon$, we obtain $\nu(U-K) < 2\epsilon$. Thus ν is regular. The decomposition readily follows.

12. This follows immediately from Proposition 11.20.

13. Let $\mu = \mu_r + \mu_p = \nu_r + \nu_p$ with μ_r and ν_r regular and with μ_p and ν_p purely irregular. Write $\sigma = \mu_r - \nu_r = \nu_p - \mu_p$ in terms of its Jordan decomposition as $\sigma = \sigma^+ - \sigma^-$. Then $\sigma^+ \leq \mu_r$ and $\sigma^- \leq \nu_r$, and hence σ^+ and σ^- are regular by Proposition 11.20. Also, $\sigma^+ \leq \nu_p$ and $\sigma^- \leq \mu_p$, and the definition of "purely irregular" forces σ^+ and σ^- to be 0. Then $\mu_r = \nu_r$ and $\mu_p = \nu_p$.

14. Let μ be as in Problem 10, and suppose that ν is a regular Borel measure with $\nu \leq \mu$. Since $\nu(\{\infty\}) = 0$, Problem 6a shows that $\lim_{\alpha\uparrow\infty} \nu(\{x \geq \alpha\}) = 0$. For each n, let α_n be the least ordinal such that $\nu(\{x \geq \alpha_n\}) \leq 1/n$. The least ordinal $\geq$ all α_n is a countable ordinal β, and $\nu(\{x \geq \beta\}) = 0$. Since $\{x < \beta\}$ is a countable set, $\mu(\{x < \beta\}) = 0$. Therefore $\nu(\{x < \beta\}) = 0$, and we conclude that $\nu = 0$.

16. For the regularity any set in $\mathcal{F}$ is in some $\mathcal{F}_n$. The sets in $\mathcal{F}_n$ are of the form $\widetilde{E} = E \times \big(\bigtimes_{n=N+1}^{\infty} X_n\big)$ with $E \subseteq \Omega^{(n)}$ and $\nu(\widetilde{E}) = \nu_n(E)$. Given $\epsilon > 0$, choose K compact and U open in $\Omega^{(n)}$ with $K \subseteq E \subseteq U$ and $\nu_n(U-K) < \epsilon$. In Ω, $\widetilde{K}$ is compact, $\widetilde{U}$ is open, $\widetilde{K} \subseteq \widetilde{E} \subseteq \widetilde{U}$, and $\nu(\widetilde{U} - \widetilde{K}) < \epsilon$.

17. Let $E = \bigcup_{n=1}^{\infty} E_n$ disjointly in $\mathcal{F}$. Since ν is nonnegative additive, we have $\sum_{n=1}^{\infty} \nu(E_n) \leq \nu(E)$. For the reverse inequality let $\epsilon > 0$ be given. Choose K compact and U_n open with $K \subseteq E$, $E_n \subseteq U_n$, $\nu(U_n - E_n) < \epsilon/2^n$, and $\nu(E-K) < \epsilon$. Then $K \subseteq \bigcup_{n=1}^{\infty} U_n$, and the compactness of K forces $K \subseteq \bigcup_{n=1}^{N} U_n$ for some N. Then $\nu(E) \leq \nu(K) + \epsilon \leq \nu\big(\bigcup_{n=1}^{N} U_n\big) + \epsilon \leq \sum_{n=1}^{N} \nu(U_n) + \epsilon \leq \sum_{n=1}^{N} \nu(E_n) + 2\epsilon \leq \sum_{n=1}^{\infty} \nu(E_n) + 2\epsilon$. Since ϵ is arbitrary, $\nu(E) \leq \sum_{n=1}^{\infty} \nu(E_n)$.

18. The key is that Ω is a separable metric space. Every open set is therefore the countable union of basic open sets, which are in the various $\mathcal{F}_n$'s.

Chapter XII

1. In (a), the closed ball is closed and contains the open ball; also every point of the closed ball is a limit point of the open ball since $\|x_1 - x_0\| = r$ implies that $\|[(1-\frac{1}{n})(x_1-x_0)+x_0]-x_0\| = (1-\frac{1}{n})\|x_1-x_0\| < r$ and $\lim_n[(1-\frac{1}{n})(x_1-x_0)+x_0] = x_1$.

For (b), let the closed balls be $B(r_n; x_n)^{\text{cl}}$. If $m \geq n$, then $\|x_m - x_n\| \leq r_n$ since $B(r_m; x_m)^{\text{cl}} \subseteq B(r_n; x_n)^{\text{cl}}$. Let $r = \lim_n r_n$. If $r = 0$, then $\{x_n\}$ is Cauchy and hence is convergent. In this case if $x = \lim x_n$, then $\|x - x_n\| \leq r_n$ for all n, and hence x is in $B(r_n; x_n)^{\text{cl}}$ for all n. If $r > 0$, fix n_0 large enough so that $r_{n_0} \leq 3r/2$. It is enough to show that x_{n_0} is in $B(r_n; x_n)^{\text{cl}}$ for $n \geq n_0$. We may assume that $x_{n_0} \neq x_n$. The members of $B(r_n; x_n)$ are the vectors of the form $x_n + v$ with $\|v\| \leq r_n$, and these are assumed to lie in $B(r_{n_0}; x_{n_0})$. Therefore $\|x_n - x_{n_0} + v\| \leq r_{n_0}$ for all such v. Take $v = r_{n_0}^{-1} r_n (x_n - x_{n_0})$. Then $r_{n_0} \geq \|x_n - x_{n_0} + v\| = \|(1 + r_{n_0}^{-1} r_n)(x_n - x_{n_0})\| =$

$(1+r_{n_0}^{-1}r_n)\|x_n-x_{n_0}\|$. Here $r_{n_0}^{-1}r_n \geq (\frac{3}{2}r)^{-1}r = \frac{2}{3}$. So $\|x_n-x_{n_0}\| \leq (1+\frac{2}{3})^{-1}r_{n_0} = \frac{3}{5}r_{n_0} \leq \frac{3}{5}\frac{3}{2}r < r \leq r_n$, as required.

2. Reduce to the real-valued case, and there use Theorem 1.23 and the remarks at the end of Section A3 of Appendix A.

3. Convergence in either case is uniform convergence. For $H^\infty(D)$, suppose therefore that $\big\{\sum_{k=0}^\infty c_k^{(n)}z_k\big\}$ is a Cauchy sequence in $H^\infty(D)$ indexed by n. Write $z = re^{i\theta}$, multiply by $e^{-im\theta}$, and integrate in θ from $-\pi$ to π. The result is that $\big\{c_m^{(n)}r^m\big\}$ is Cauchy in n for each $r < 1$ and each m. Then $\lim_n c_m^{(n)}r^m = c_m r^m$ exists for each r and m. Taking $r = 1/2$, we see that $\lim_n c_m^{(n)} = c_m$ exists for each m. Arguing as in the proof of Theorem 1.37, we see that $f(z) = \sum_{k=0}^\infty c_k z_k$ is convergent for $|z| < 1$ and that the sequence of functions $f_n(z) = \sum_{k=0}^\infty c_k^{(n)} z_k$ converges to it pointwise. Since $\{f_n\}$ is uniformly Cauchy and pointwise convergent to f, it converges uniformly to f. For the vector subspace $A(D)$, we have $A(D) = H^\infty(D)\cap C(D^{\mathrm{cl}})$. Hence $A(D)$ is a closed subspace of $H^\infty(D)$.

4. In (a), let us check the triangle inequality. For $y \in Y$, we have $\|a+b+y\| \leq \|a+y'\| + \|b+(y-y')\|$ for all $y' \in Y$. Comparing the definition of $\|a+b+Y\|$ with the left side, we obtain $\|a+b+Y\| \leq \|a+y'\| + \|b+(y-y')\|$ for all y and y' in Y. Thus $\|a+b+Y\| \leq \|a+y'\| + \|b+y''\|$ for all y' and y'' in Y. Taking the infimum over y' and y'' gives the desired conclusion.

In (b), let a Cauchy sequence in X/Y be given. It is enough to prove that some subsequence in convergent. Thus it is enough to prove that if $\{x_n\}$ is a sequence in X with $\|x_n - x_{n+1} + Y\| \leq 2^{-n}$, then $\{x_n + Y\}$ is convergent in X/Y. We define a sequence $\{\widetilde{x}_n\}$ in X with $\widetilde{x}_n = x_n - y_n$ and y_n in Y such that $\|\widetilde{x}_n - \widetilde{x}_{n+1}\| \leq 2\cdot 2^{-n}$. It is then easy to check that $\{\widetilde{x}_n\}$ is Cauchy in X and that if x' is its limit, then $\{x_n+Y\}$ tends to $x'+Y$. To define the y_n's, we proceed inductively, starting with $y_1 = 0$. If $y_1,\dots,y_n$ have been defined such that $\|\widetilde{x}_k - \widetilde{x}_{k+1}\| \leq 2\cdot 2^{-k}$ for $k < n$, choose y_{n+1} in Y such that $\|\widetilde{x}_n - x_{n+1} + y_{n+1}\| \leq \|x_n - x_{n+1} + Y\| + 2^{-n} \leq 2\cdot 2^{-n}$. Then $\widetilde{x}_{n+1} = x_{n+1} - y_{n+1}$ has $\|\widetilde{x}_n - \widetilde{x}_{n+1}\| \leq 2\cdot 2^{-n}$, and the induction is complete.

5. In (a), we have $c^{\mathrm{tr}}G(v_1,\dots,v_n)\bar{c} = \sum_{i,j} c_i(v_i,v_j)\bar{c}_j = \sum_{i,j}(c_iv_i, c_jv_j) = \big(\sum_i c_iv_i, \sum_j c_jv_j\big) = \big\|\sum_i c_iv_i\big\|^2$. In (b), $G(v_1,\dots,v_n)$ is Hermitian, and thus the finite-dimensional Spectral Theorem says that there exists a unitary matrix $u = [u_{ij}]$ with $u^{-1}G(v_1,\dots,v_n)u$ diagonal, say $= \mathrm{diag}(d_1,\dots,d_n)$. Then $d_j = e_j^{\mathrm{tr}}u^{-1}G(v_1,\dots,v_n)ue_j$, and this, by (a), equals $\big\|\sum_i c_iv_i\big\|^2$ with $\bar{c} = ue_j$. Hence $d_j \geq 0$. In (c), we have $\det G(v_1,\dots,v_n) = \det(u^{-1}G(v_1,\dots,v_n)u) = d_1d_2\cdots d_n \geq 0$ with equality if and only if some d_j is 0. If $d_j = 0$, then $\sum_i c_iv_i = 0$ for $\bar{c} = ue_j$, and hence $v_1,\dots,v_n$ is dependent. Conversely if $v_1,\dots,v_n$ is dependent, then $\sum_i c_iv_i = 0$ for some nonzero tuple $(c_1,\dots,c_n)$, and therefore $0 = \big(\sum_i c_iv_i, v_j\big) = \sum_i c_i(v_i,v_j)$ for all j; this equality shows that a nontrivial linear combination of the rows of $G(v_1,\dots,v_n)$ is 0, and hence $\det G(v_1,\dots,v_n) = 0$.

6. A single induction immediately shows the following: $\mathrm{span}\{v_1',\dots,v_k'\} =$

span$\{u_1, \ldots, u_k\}$, v'_k is $\neq 0$, and v_k is defined. Then each v_k has norm 1. If $k < l$, then $(v'_l, v_k) = \big(u_l - \sum_{j=1}^{l-1} (u_l, v_j)u_j, v_k\big) = (u_l, v_k) - (u_l, v_k) = 0$. This proves the orthogonality.

7. Define F on each u_α to be the vector v_β given in the statement of the problem, and extend F linearly to a mapping defined on the linear span V of $\{u_\alpha\}$. Corollary 12.8c shows that $\|F(u)\|_{H_2} = \|u\|_{H_1}$ for u in V. Corollary 12.8b shows that V is dense. Proposition 2.47 shows that F extends to a bounded linear operator from H_1 into H_2 satisfying $\|F(u)\|_{H_2} = \|u\|_{H_1}$ for u in H_1. Arguing in the same way with F^{-1} proves that F is onto H_2. The second conclusion follows by using Proposition 12.11.

8. In (a), the boundedness is elementary, and the operator norm is $\|f\|_\infty$. In (b), the adjoint is multiplication by the complex conjugate of f.

9. The linear span V of $\{x_n\}$ is a separable vector subspace. Suppose that it is not dense. Choose by Corollary 12.15 a member $x^* \neq 0$ of X^* with $x^*(V) = 0$. Since $\{x_n^*\}$ is dense, choose a subsequence $\{x_{n_k}^*\}$ with $x_{n_k}^* \to x^*$. Then

$$\|x^* - x_{n_k}^*\| \geq |(x^* - x_{n_k}^*)(x_{n_k})| = |x_{n_k}^*(x_{n_k})| \geq \tfrac{1}{2}\|x_{n_k}^*\|.$$

Since the left side tends to 0, so does the right side. Thus $x_{n_k}^*$ tends to 0, and $x^* = 0$, contradiction.

10. The dual of $C(X)$ is $M(X)$. Define a linear functional x^* on $M(X)$ by $x^*(\rho) = \rho(\{s_0\})$. Then $\|x^*\| = 1$, so that x^* is in $M(S)^*$. Let δ_s denote a point mass at s. If x^* were given by integration with a continuous function f, then we would have $I_{\{s_0\}}(s) = \delta_s(\{s_0\}) = x^*(\delta_s) = \int_S f\, d\delta_s = f(s)$. Thus the only possibility would be $f = I_{\{s_0\}}$, and this is discontinuous.

11. Let X and Y be normed linear with X complete, and let $\{L_n\}$ be a family of bounded linear operators $L_n : X \to Y$ such that $\|L_n(x)\| \leq C_x$ for each x in X. For each y^* in Y^* with $\|y^*\| \leq 1$, the linear functional $y^* \circ L_n$ on X is bounded and has $|y^*(L_n(x))| \leq C_x$. Since X is complete, the Uniform Boundedness Theorem for linear functionals shows that $|y^*(L_n(x))| \leq C\|x\|$ for all x. Taking the supremum over y^* and applying Corollary 12.17, we obtain $\|L_n(x)\| \leq C\|x\|$, as required.

12. For x in X and y in Y, we have

$$\begin{aligned}\|L_n(x) - L_m(x)\| &\leq \|L_n(x - y)\| + \|L_n(y) - L_m(y)\| + \|L_m(y - x)\| \\ &\leq 2C\|x - y\| + \|L_n(y) - L_m(y)\|.\end{aligned}$$

Given $x \in X$ and $\epsilon > 0$, choose y in Y to make the first term $< \epsilon$, and then choose n and m large enough to make the second term $< \epsilon$. It follows that $\{L_n(x)\}$ is Cauchy for each x. Since X' is complete, $L(x) = \lim_n L_n(x)$ exists for all x. Continuity of addition and scalar multiplication implies that L is linear. Then $\|L(x)\| = \lim \|L_n(x)\| \leq \liminf_n \|L_n\|\|x\| \leq C\|x\|$. Hence $\|L\| \leq C$.

13. Proposition 12.1 shows that X^* is a Banach space. We identify the elements x_α in X with their images $\iota(x_\alpha)$ under the canonical map $\iota : X \to X^{**}$. Corollary 12.18 shows that the element $\iota(x_\alpha)$ of X^{**} has $\|\iota(x_\alpha)\| = \|x_\alpha\|$. The hypothesis shows for each x^* that $|(\iota(x_\alpha))(x^*)| = |x^*(x_\alpha)| \le C_{x^*}$ for a constant C_{x^*} independent of α. Since X^* is complete, the Uniform Boundedness Theorem (Theorem 12.22) shows that $\|\iota(x_\alpha)\| \le C$ for a constant C independent of α. Applying Corollary 12.18 a second time, we conclude that $\|x_\alpha\| \le C$ independently of α.

14. For (a), let u and v have $\|u - x\| \le r$ and $\|v - x\| \le r$. Then the estimate $\|(1-t)u+tv-x\| = \|(1-t)(u-x)+t(v-x)\| \le \|(1-t)(u-x)\|+\|t(v-x)\| = (1-t)\|u-x\| + t\|v-x\| \le (1-t)r + tr = r$ proves the convexity.

For (b), let X be the space of sequences $s = \{s_n\}$ with $\|s\| = \sum_n |s_n|$. Let E_k be the set of sequences with all $s_n \ge 0$, with $\|s\| = 1$, and with $s_j = 0$ for $j \le k$. If s and t are two sequences with terms ≥ 0, then $\|s + t\| = \|s\| + \|t\|$. The convexity follows, and everything else is easy.

15. Denote open balls in X by B_X and open balls in Y by B_Y. The Interior Mapping Theorem says that $L(B_X(1; 0))$ is open. Hence it contains a ball $B_Y(\epsilon; 0)$. Put $C = \epsilon^{-1}$. By linearity, $L(B_X(Cr; 0)) \supseteq B_Y(r; 0)$ for every $r \ge 0$. Since L is onto Y, we can choose x_0 in X with $L(x_0) = y_0$. Linearity gives $L(B_X(Cr; x_0)) \supseteq B_Y(r; y_0)$. For each y_n, we can take $r = 2\|y_n - y_0\|$ and choose x_n in $B_X(C2\|y_n - y_0\|; x_0)$ with $L(x_n) = y_n$. Since $y_n \to y_0$, $x_n \to x_0$. Also, we have $\|x_n - x_0\| \le 2C\|y_n - y_0\|$.

In this construction if $y_0 = 0$, we could choose $x_0 = 0$, and then the result follows with $M = 2C$.

If $y_0 \ne 0$, then $\|y_n\| \to \|y_0\| \ne 0$ says that $\|y_n\| \le \frac{1}{2}\|y_0\|$ only finitely often. For these exceptional n's, we can adjust x_n when $y_n = 0$ so that $x_n = 0$, and then we have $\|x_n\| \le M\|y_n\|$ for a suitable M and the exceptional n's. For the remaining n's, an inequality $\|x_n\| \le M\|y_n\|$ is valid as soon as $\{x_n\}$ is bounded, and $\{x_n\}$ has to be bounded since it is convergent.

16. It will be proved that the distance from e to X_0 is ≥ 1. The set X_{00} of all sequences $s_1, s_2 - s_2, s_3 - s_2, \ldots$ such that $\{s_n\}$ is in X is closed under addition and scalar multiplication. Hence it is a dense vector subspace of X_0, and it is enough to prove that $\|e - s\| \ge 1$ for all s in X_{00}. Let s be in X_{00}, and let $c = e - s$. Adding the first n entries gives $c_1 + \cdots + c_n = n - s_n$. Hence $|c_1 + \cdots + c_n| \ge n - \|s\|$. If, by way of contradiction, $\|c\| = 1 - \epsilon$ with $\epsilon > 0$, then $|c_j| \le 1 - \epsilon$ for all j, and we have $|c_1 + \cdots + c_n| \le n - n\epsilon$. Thus $n - \|s\| \le n - n\epsilon$, and we get $n\epsilon \le \|s\|$, in contradiction to the finiteness of $\|s\|$.

17. This is immediate from Corollary 12.15 and the previous problem.

18. For (a), let $s \ge 0$ have $\|s\| = 1$. Then $\|e - s\| \le 1$, and so $|x^*(e - s)| \le 1$. Since $x^*(e) = 1$, this says that $|1 - x^*(s)| \le 1$. On the other hand, $|x^*(s)| \le 1$ since $\|s\| \le 1$. Thus $0 \le x^*(s) \le 1$. We can scale this inequality to handle general s.

For (b), the two sequences differ by a member of X_0, on which the Banach limit vanishes identically; then (c) follows by iterated application of (b) since the Banach limit of the 0 sequence is 0.

In (d), let $\epsilon > 0$ be given. By applying (c), we see that we may adjust the sequence so that $\sup_n s_n - \inf_n s_n \le \epsilon$ and so that the Banach limit is unchanged. By (a), Banach limits preserve order. Since $(\inf s_n)e \le s \le (\sup s_n)e$, we have $\inf s_n \le \text{LIM}_{n\to\infty} s_n \le \sup s_n$. Since $\sup s_n = (\sup s_n - \limsup s_n) + \limsup s_n \le (\sup_n - \inf_n) + \limsup_n \le \limsup_n + \epsilon$, we obtain $\text{LIM}_{n\to\infty} s_n \le \limsup_n + \epsilon$. Since ϵ is arbitrary, $\text{LIM}_{n\to\infty} s_n \le \limsup_n$. Similarly $\liminf s_n \le \text{LIM}_{n\to\infty} s_n$. Conclusion (e) is immediate from (d).

20. The parallelogram law gives

$$2(\|x+y+z\|^2 + \|y-z\|^2) = \|x+2y\|^2 + \|x+2z\|^2.$$

If we set $z = 0$ in this identity and then set $y = 0$ in it, we get two relations, one involving an expression for $\|x+2y\|^2$ and the other involving an expression for $\|x+2z\|^2$. If we substitute these relations into the displayed equation for the terms $\|x+2y\|^2$ and $\|x+2z\|^2$, we obtain the formula $\|x+y+z\|^2 + \|y-z\|^2 = \|x+y\|^2 + \|x+z\|^2 - \|x\|^2 + \|y\|^2 + \|z\|^2$. Substitution of $2\|y\|^2 + 2\|z\|^2 - \|y+z\|^2$ for $\|y-z\|^2$ in this formula gives the desired identity.

21. We have

$$\begin{aligned}(x_1+x_2, y) &= \sum_k \tfrac{i^k}{4}\|x_1+x_2+i^k y\|^2 \\ &= \sum_k \tfrac{i^k}{4}(\|x_1+x_2\|^2 - \|x_1\|^2 - \|x_2\|^2 - \|y\|^2) \\ &\qquad + \sum_k \tfrac{i^k}{4}\|x_1+i^k y\|^2 + \textstyle\sum_k \tfrac{i^k}{4}\|x_2+i^k y\|^2.\end{aligned}$$

Each term of the first line on the right is 0 because $\sum_k i^k/4 = 0$, and thus the right side simplifies to $(x_1, y) + (x_2, y)$, as required.

22. Induction with the result of the previous problem gives $(nx, y) = n(x, y)$ for every integer $n \ge 0$. Replacing nx by z, we obtain $\frac{1}{n}(z, y) = (\frac{1}{n}z, y)$. Hence $(rx, y) = r(x, y)$ for every rational $r \ge 0$. It follows from the definition of $(\,\cdot\,,\,\cdot\,)$ that $(-x, y) = -(x, y)$ and that if the scalars are complex, $(ix, y) = i(x, y)$. Consequently $(rx, y) = r(x, y)$ if r is in the set D.

23. We are to prove that $|(x, y)| \le \|x\|\|y\|$, and we may assume that $y \ne 0$. If r is in D, we have

$$0 \le \|x - ry\|^2 = (x - ry, x - ry) = \|x\|^2 - r(y, x) - \bar{r}(x, y) + |r|^2\|y\|^2.$$

Letting r tend to $(x, y)\big/\|y\|^2$ through members of D, we obtain

$$0 \le \|x\|^2 - 2|(x, y)|^2\big/\|y\|^2 + |(x, y)|^2\|y\|^2\big/\|y\|^4 = \|x\|^2 - |(x, y)|^2\big/\|y\|^2,$$

and it follows that $|(x, y)| \le \|x\|\|y\|$.

24. The Schwarz inequality gives

$$|r(x, y) - (cx, y)| = |(rx - cx, y)| \leq \|(r - c)x\|\|y\| = |r - c|\|x\|\|y\|.$$

As r tends to c through D, the right side tends to 0, and the left side tends to $|c(x, y) - (cx, y)|$. Hence $c(x, y) = (cx, y)$.

25. If $L_n \to L$ in $B(X, Y)$ and $x_n \to x$ in X, then the triangle inequality gives $|L_n(x_n) - L(x)| \leq |L_n(x_n) - L(x_n)| + |L(x_n) - L(x)| \leq \|L_n - L\||x_n| + \|L\||x_n - x|$. The first term on the right side tends to 0 because $|x_n|$ is bounded (being convergent to $|x|$) and $\lim_n \|L_n - L\| = 0$, and the second term tends to 0 because $\lim_n |x_n - x| = 0$.

26. Since $|\cdot|$ is a continuous function on Y, the equality $L(x) = \lim_n L_n(x)$ implies $|L(x)| = \limsup_n |L_n(x)| \leq \limsup_n (\|L_n\||x|) = (\limsup_n \|L_n\|)|x|$. Taking the supremum of this inequality for $|x| \leq 1$ yields $\|L\| \leq \limsup_n \|L_n\|$. The inequality $\sup_n \|L_n\| < \infty$ follows from the Uniform Boundedness Theorem (Theorem 12.22).

For an example with strict inequality, let $X = Y = L^1(\mathbb{R})$, and let L_n be multiplication by the indicator function of $[n, \infty)$. Then the limit operator is $L = 0$ but $\|L_n\| = 1$ for every n.

27. We have $|L_n(u_n) - L(u)| \leq |L_n(u_n) - L_n(u)| + |L_n(u) - L(u)|$. The first term on the right side is $\leq \|L_n\||u_n - u\|$, and this tends to 0, since $\|L_n\|$ is bounded (according to Problem 26) and $u_n \to u$. The second term on the right side tends to 0 because $L_n(u) \to L(u)$ by hypothesis.

Appendix B

1. For (a), the answer is yes. An example is $f(z) = |z|^2 = x^2 + y^2$. It is a differentiable function on all of $\mathbb{R}^2$, and its first partial derivatives are both 0 at $z = 0$. So it has a complex derivative at 0 by Proposition B.1. At a general point (x, y), $f(z) = u(x, y)$ with $v(x, y) = 0$. Thus the first partial derivatives of v are 0 everywhere, but the first partial derivatives of u vanish together only at $z = 0$; so the Cauchy–Riemann equations are satisfied only when $x = y = 0$.

For (b), the answer is yes. An example is $f(z) = y^2$. The argument is similar to the argument for (a).

2. We can parametrize γ as $t \mapsto t(1 + i)$ for $0 \leq t \leq 1$. Then the integral equals $\int_0^1 t(1 + i)\,dt = \frac{1}{2}(1 + i)$.

3. Let R be given by $a \leq x \leq b$ and $c \leq y \leq d$, and write $f(z) = u(x, y) + iv(x, y)$. Making use of the continuity of the first partial derivatives of u and v, we have

$$\begin{aligned}\int_{\partial R} f(z)\,dz &= \int_a^b (u(x,c)+iv(x,c))\,dx + \int_c^d (u(b,y)+iv(b,y))i\,dy \\ &\quad - \int_a^b (u(x,d)+iv(x,d))\,dx - \int_c^d (u(a,y)+iv(a,y))i\,dy \\ &= -\int_a^b \int_c^d \Big(\tfrac{\partial u}{\partial y} + i\tfrac{\partial v}{\partial y}\Big)\,dy\,dx + \int_c^d \int_a^b i\Big(\tfrac{\partial u}{\partial x} + i\tfrac{\partial v}{\partial x}\Big)\,dx\,dy \\ &= \iint_R \Big(i\Big(\tfrac{\partial u}{\partial x} - \tfrac{\partial v}{\partial y}\Big) - \Big(\tfrac{\partial u}{\partial y} + \tfrac{\partial v}{\partial x}\Big)\Big)\,dx\,dy.\end{aligned}$$

with the last equality following from Fubini's Theorem (Corollary 3.33). In the double integral on the right side, the two terms within the inner parentheses are 0 by the Cauchy–Riemann equations. Thus the integrand is identically 0, and the double integral is 0.

4. For (a), write $z = x + iy$ with x and y given by the column vectors $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$, and identify the column vector $z = (z_1, \dots, z_n)$ with $x = (x_1, \dots, x_n, y_1, \dots, y_n)$. Also write $f(z) = u(x, y) + iv(x, y)$. A candidate for $f'(z_0)$ is a certain n-dimensional row vector with n complex entries, write we write as $a + ib$, the sum of its real and imaginary parts. Temporarily we put $z - z_0 = h + ik$. We calculate exactly as in the proof of Proposition B.1 except that z, z_0, h, and k are now column vectors rather than numbers. The expression that is to tend to 0 in the definition of $f'(z_0)$ is

$$\begin{aligned}&|z|^{-1}\big(f(z) - f(z_0) - f'(z_0)(z - z_0)\big) \\ &= |z|^{-1}\big(f(z) - f(z_0) - (a+ib)(h+ik)\big) \\ &= |x+iy|^{-1}\big(u(x,y) - u(x_0,y_0) + iv(x,y) - iv(x_0,y_0) - (a+ib)(h+ik)\big) \\ &= |x+iy|^{-1}\big(u(x,y) - u(x_0,y_0) - \begin{pmatrix} a & -b \end{pmatrix}\begin{pmatrix} x-x_0 \\ y-y_0 \end{pmatrix}\big) \\ &\quad + |x+iy|^{-1} i\big(v(x,y) - v(x_0,y_0) - \begin{pmatrix} b & a \end{pmatrix}\begin{pmatrix} x-x_0 \\ y-y_0 \end{pmatrix}\big),\end{aligned}$$

and this tends to 0 in $\mathbb{C}$ if and only if

$$|(x,y)|^{-1}\Big(\begin{pmatrix} u(x,y)-u(x_0,y_0) \\ v(x,y)-v(x_0,y_0) \end{pmatrix} - \begin{pmatrix} a & -b \\ b & a \end{pmatrix}\begin{pmatrix} x-x_0 \\ y-y_0 \end{pmatrix}\Big)$$

tends to 0 in $\mathbb{R}^2$. Here $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ is a real 2-by-$2n$ matrix, and the fact that the above expression tends to 0 says exactly that the function $(x, y) \mapsto (u(x, y), v(x, y))$ is differentiable at (x_0, y_0) with Jacobian matrix $J = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$. The condition that J be of this form is exactly the condition that J satisfy the matrix equation in the statement of (a).

For the two equivalences in (b), first suppose that f is complex differentiable at every point of an open set. Then (a) shows that f is real differentiable at every point of the open set and that the Cauchy–Riemann equations hold in each variable. Therefore f is analytic in each variable and by definition is holomorphic on the open set. Next if f is holomorphic on the open set, then $f_{\mathbb{R}}$ is C^∞ by Theorem B.50. Since

f is analytic in each complex variable, the Cauchy–Riemann equations hold in each variable. The matrix equation in (a) follows, and then (a) shows that f is complex differentiable at every point of the open set. Finally if $f_{\mathbb{R}}$ is C^∞ and its Jacobian matrix satisfies the equality in (a), then (a) shows that f is complex differentiable at every point of the open set.

5. We have $|z|^2 = (x_1^2+x_2^2)/(1-x_3)^2 = (1-x_3^2)/(1-x_3)^2 = (1+x_3)/(1-x_3)$. Then the formulas for x_1, x_2, x_3 are routine.

6. The line through $(0, 0, 1)$ and $(x, y, 0)$ can be parametrized as $t \mapsto (x, y, 0) + t(-x, -y, 1)$. For the value $t = x_3$, this line passes through the point $(x(1 - x_3), y(1 - x_3), x_3) = (x_1, x_2, x_3)$, and hence the three points in question are collinear.

7. Stereographic projection and the coordinate function of its inverse are manifestly continuous.

8. A plane in $\mathbb{R}^3$ is of the form $\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 = \alpha_0$ with $\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1$ and $0 \le \alpha_0$. Suppose it meets S. Specializing the equation to (x_1, x_2, x_3) of the form $\varphi^{-1}(z)$ gives

$$\alpha_1(z + \bar{z}) - \alpha_2 i(z - \bar{z}) + \alpha_3(|z|^2 - 1) = \alpha_0(|z|^2 + 1)$$

and thus

$$(\alpha_0 - \alpha_3)(x^2 + y^2) - 2\alpha_1 x - 2\alpha_2 y + \alpha_0 + \alpha_3 = 0,$$

which is trying to be the equation of a circle in the z plane if $\alpha_0 \ne \alpha_3$. However, a little computation shows that the circle degenerates if and only if $(\alpha_0 + \alpha_3)(\alpha_0 - \alpha_3) \le \alpha_1^2 + \alpha_2^2$, i.e., if and only if $\alpha_1^2 + \alpha_2^2 + \alpha_3^2 \ge \alpha_0^2$. So we must have $\alpha_0 < 1$. In this case we do have a circle in the z plane. In the case that $\alpha_0 = \alpha_3$, we obtain $2\alpha_1 x + 2\alpha_2 y = \alpha_0 + \alpha_3$, which is the equation of a line in the z plane. Conversely if we have a line or a circle in the z plane, we can choose parameters as above and see that it comes from a the intersection of S with a plane in $\mathbb{R}^3$.

9. By the Cauchy Integral Formula this is $\int_{|z|=1} \frac{e^z}{z}\, dz = e^z\big|_{z=0} = 1$.

10. For (a), the function $f(z) = \sin(2\pi z)$ is a counterexample.

For (b), by the Identity Theorem, $f(z + 1) = f(z)$ for all $z \in \mathbb{C}$ and $f(iz + i) = f(iz)$ for all $z \in \mathbb{C}$. The latter implies that $f(z + i) = f(z)$ for all z. If M denotes the supremum of $f(z)$ for $0 \le \operatorname{Re} z \le 1$ and $0 \le \operatorname{Im} z \le 1$, then it follows that $|f(z)| \le M$ everywhere. Liouville's Theorem implies that f is a constant function.

11. False, false, false, true, as follows:

(a) $f(z) = e^z$ with $z_n = -n$.

(b) $f(z) = e^z$ with $\theta = \pi$.

(c) $f(z) = e^{-z^4}$.

(d) The limit relation forces f to be bounded, Liouville's Theorem says that f is constant, and the limit relation says that the constant must be zero.

12. Apply Theorem B.11 to the interior point -1 with $n = 1$ and $f(z) = e^z$. Then the integral equals $2\pi i f'(-1) = 2\pi i e^{-1}$.

13. The points in question have a limit point at 0. For $z = 1/n$, we have $n = z^{-1}$, $n - 1 = z^{-1} - 1 = \frac{1-z}{z}$. Thus $\frac{1}{n(n-1)} = f(z) = \frac{z^2}{1-z}$ for those values of z. By the Identity Theorem, $f(z) = \frac{z^2}{1-z}$ everywhere. But the result is not an entire function. So f does not exist.

14. No. As an even entire function, $f(z)$ satisfies $f(-z) = f(z)$. The power series expansion $\sum a_n z^n$ of $f(z)$ must then have $\sum_n a_n(-z)^n = \sum_n a_n z^n$, and uniqueness of coefficients forces $a_n = 0$ for n odd. Since $f'''(0)$ equals 6 times the coefficient of z^3, $f'''(0)$ must be 0.

15. First solution: For $m = n = 1$, we have $\frac{1}{(z-a)(z-b)} = \frac{1}{a-b}\big(\frac{1}{z-a} - \frac{1}{z-b}\big)$. Only the term with $1/(z-a)$ contributes to the integral, and the result is that $\int_{|z|=1} \frac{dz}{(z-a)(z-b)} = 2\pi i/(a-b)$. For general m and n, we can differentiate this result $m-1$ times in a and $n-1$ times in b, using Corollary B.15. We obtain $\int_{|z|=1} \frac{dz}{(z-a)^m(z-b)^n} = 2\pi i(-1)^{m-1}(m-1)!(n-1)!/(a-b)^{m+n-1}$.

Second solution. Use Theorem B.11 for a function $f(\zeta)$ of the form $\frac{1}{(\zeta-b)^{\text{power}}}$ and the point $z = a$.

16. For (a), let $f(z) = u(x, y) + iv(x, y)$ and $\overline{f(\bar{z})} = u^\#(x, y) + iv^\#(x, y)$ be the decompositions of $f(z)$ and $\overline{f(\bar{z})}$ into real and imaginary parts, and denote by subscripts 1 and 2 the first partial derivatives of these functions in the first and second variables. The formulas for $u^\#$ and $v^\#$ are $u^\#(x, y) = u(x, -y)$ and $v^\#(x, y) = -v(x, -y)$. Then $u^\#_1(x, y) = u_1(x, -y)$, $u^\#_2(x, y) = -u_2(x, -y)$, $v^\#_1(x, y) = -v_1(x, -y)$, and $v^\#_2(x, y) = v_2(x, -y)$, and the equations $u_1 = v_2$ and $u_2 = -v_1$ imply $u^\#_1 = v^\#_2$ and $u^\#_2 = -v^\#_1$. Since analytic functions have continuous first partial derivatives, the result follows from Corollary B.2.

For (b), if $f(z)$ has a Taylor series expansion $f(z) = \sum a_n(z - z_0)^n$ about z_0, then $g(z)$ near $\bar{z}_0$ is given by $g(z) = \sum \bar{a}_n(z - \bar{z}_0)^n$ and hence is analytic near $\bar{z}_0$.

17. Apply Problem 16. The entire functions $f(z)$ and $\overline{f(\bar{z})}$ are equal on the real axis and hence are equal everywhere, by the Identity Theorem. Also the entire functions $f(z)$ and $\overline{f(-\bar{z})}$ are equal on the imaginary axis and hence are equal everywhere, by the Identity Theorem. Putting these conclusions together gives $\overline{f(\bar{z})} = \overline{f(-\bar{z})}$, from which we see that $f(\bar{z}) = f(-\bar{z})$ everywhere and $f(z) = f(-z)$ everywhere.

18. Apply Problem 16. The condition for real z says that $\overline{f(\bar{z})} = f(z)$ for real z and therefore for all z, while the condition for imaginary z says that $\overline{f(\bar{z})} = -f(-z)$ for imaginary z and therefore for all z. Putting these results together gives $f(z) = -f(-z)$ for all z.

19. This would be immediate from Corollary B.42 except that the stated version of the corollary assumes the domain of F to be bounded. Nevertheless, the same proof works: Any line integral $\int_\gamma F'(\zeta)/F(\zeta)\, d\zeta$ over a piecewise smooth C^1 closed

curve γ is equal to 0 by the Cauchy Integral Theorem, and Corollary B.6 shows that $F'(z)/F(z) = g'(z)$ for some analytic function $g(z)$. Hence

$$\tfrac{d}{dz}\big(F(z)e^{-g(z)}\big) = F'(z)e^{-g(z)} + F(z)e^{-g(z)}(-g'(z)) = 0,$$

and it follows that $F(z)e^{-g(z)}$ is a constant, say c. Then $F(z) = ce^{g(z)}$. If we write $c = e^k$ for some constant k, then we obtain $F(z) = e^{f(z)}$ with $f(z) = k + g(z)$.

20. For any $R < 2$, we have $|f'(z)| \le R^{-1}$ for $|z| = R$ and therefore also for $|z| \le R$ by the Maximum Modulus Theorem. Consequently $|f'(z)| \le \frac{1}{2}$ for $|z| < 2$. If γ is a straight line segment from 0 to 1, then $f(1) - f(0) = \int_\gamma f'(z)\,dz$. Taking absolute values gives $|f(1) - f(0)| \le \max_{\text{image}(\gamma)} |f'(z)|\ell(\gamma) \le 1/2$.

21. The function $1/(zf(z))$ is analytic for $|z| < 1$ and has $|1/(zf(z))| \le 1$ everywhere and $1/(\frac{1}{2}f(\frac{1}{2})) = 1$. By the Maximum Modulus Theorem $1/(zf(z)) = 1$ everywhere. Thus $f(z) = 1/z$ everywhere.

22. For any positive integer K, the given estimate implies that $|f(z)| \le A(KR)^\alpha$ for $|z| = KR$. Thus we can take $C = A(KR)^\alpha$ in Cauchy's estimate (Corollary B.16) and get $|f^{(n)}(0)| \le A(KR)^\alpha n!\,r^{-n}$ as long as $r \le KR$. Thus for $r = \frac{1}{2}(KR)$, we have $|f^{(n)}(0)| \le 2^n n!\,A(KR)^{\alpha-n}$. Letting K tend to infinity shows that $f^{(n)}(0) = 0$ for $n > \alpha$. Since f is given by a convergent power series, all the terms are 0 except at most the terms $c_j z^j$ with $j \le \alpha$, and $f(z)$ is a polynomial of degree at most the integer part of α.

23. If $f(z)$ is analytic in a region containing the closed disk of center 0 and radius r, then Cauchy's estimate (Corollary B.16) gives $|f^{(n)}(0)| \le Kn!r^{-n}$, where $K = \sup_{|z|=r} |f(z)|$. Thus all that is required is that $Kr^{-n} \le M^n$, and this happens if $M = r^{-1}\max\{1, K\}$.

24. (a) Essential singularity, just as with $-\sin(1/w)$ at $w = 0$.

(b) Pole of order 1, just as with $\frac{1}{1-e^z}$ at $z = 0$.

(c) Pole of order 1. Write $w = z - \pi/4$, so that $\sin z = \sin(w + \pi/4) = \sin w\cos(\pi/4) + \cos w\cos(\pi/4) = \frac{1}{2}\sqrt{2}(\sin w + \cos w))$. Still with $z = w + \pi/4$, we have $\cos z = \cos(w + \pi/4) = \cos w\cos(\pi/4) - \sin w\sin(\pi/4) = \frac{1}{2}\sqrt{2}(\cos w - \sin w)$. Thus $\sin z - \cos z = \sqrt{2}\sin w$. This has a simple zero at $w = 0$, and thus $\sin z - \cos z$ has a simple zero at $z = \pi/4$.

25. We investigate the isolated singularity of $f(z)$ at infinity, i.e., the isolated singularity of $f(1/z)$ at $z = 0$. If the singularity is removable, then f is constant (by Liouville's Theorem) and is not one-one.

If the singularity is essential, then the Weierstrass result (Proposition B.25) shows that there exists a sequence $\{z_n\}$ tending to ∞ with $w_n = f(z_n)$ tending to 0. If f^{-1} exists, then $f^{-1}(w_n) = z_n$, and continuity of f^{-1} at 0 forces $f^{-1}(0) = \lim z_n = \infty$, so that F^{-1} has a singularity at 0, contradiction.

So the singularity must be a pole. Then Cauchy's estimate shows that f is a polynomial, and the Fundamental Theorem of Algebra shows f has degree at most one.

26. For $j = 1, \dots, r$, let $P_j(\frac{1}{z-r_j})$ be the singular part of $P(z)/Q(z)$ about $z = r_j$. Then $P(z)/Q(z) - \sum_{j=1}^{k} P_j(\frac{1}{z-r_j})$ has no pole at any of $r_1, \dots, r_k$, and there are no other possibilities for a pole. Hence it is an entire function $g(z)$. It is also the quotient of polynomials. Its denominator can have no root, and the Fundamental Theorem of Algebra shows that the denominator is constant. Therefore $g(z)$ is a polynomial.

28. The right side is the sum of the singular parts at each of the poles of $P(z)/Q(z)$. Thus the difference of the two sides is an entire function that vanishes at infinity. Hence it is 0.

29. Put $Q(z) = (z - r_1) \cdot \dots \cdot (z - r_n)$, and define $P(z) = Q(z) \sum_{k=1}^{n} \frac{c_k}{Q'(r_k)(z-r_k)}$. In view of the previous problem, $\sum_{k=1}^{n} \frac{P(r_k)-c_k}{Q'(r_k)(z-r_k)} = 0$. The singular parts at r_k for the two sides must match, and thus $P(r_k) = c_k$ for $1 \le k \le n$.

30. Use Proposition B.34, or argue as follows: We may assume that f is not the 0 function. Since f has isolated zeros, we can choose $r > 0$ so that $f(z) \ne 0$ for $0 < |z| \le r$. Define $c > 0$ to be the minimum value of $|f(z)|$ for $|z| = r$. For each t with $0 \le t \le 1$, $|f(z) - tc/2| \ne 0$ for $|z| = r$. By the Argument Principle the integral $\frac{1}{2\pi i} \int_{|z|=r} \frac{f'(z)\,dz}{f(z)-tc/2}$ is a nonnegative integer that varies continuously with t for $0 \le t \le 1$. It is ≥ 2 for $t = 0$, and thus it is ≥ 2 for $t = 1$. Then it follows that there are two points z with $|z| < r$ such that $f(z) = c/2$.

31. Near z_0, where $f'_\lambda(z_0) = 0$, f'_λ is not one-one. Since $f'_\lambda(z) = 1 + 2\lambda z$, $f'_\lambda(z_0) = 0$ for some $|z| < 1$ if $z_0 = -(2\lambda)^{-1}$ has $|2\lambda|^{-1} < 1$, i.e., $|\lambda| > \frac{1}{2}$. Thus a necessary condition for f_λ to be one-one is that $|\lambda| \le \frac{1}{2}$. Conversely we show that the condition $|\lambda| \le \frac{1}{2}$ is also sufficient. Arguing by contradiction, suppose $f_\lambda(z) = f_\lambda(z')$ with $z \ne z'$. Then $z + \lambda z^2 = z' + \lambda z'^2$. So $(z - z') + \lambda(z - z')(z + z') = 0$, $1 + \lambda(z + z') = 0$, and $2\lambda\frac{1}{2}(z + z') = -1$. Taking the absolute value of both sides shows that $1 = 2|\lambda|\frac{1}{2}|z + z'| \le \frac{1}{2}|z + z'| \le \frac{1}{2}(|z| + |z'|) < \frac{1}{2}(1 + 1) = 1$, contradiction.

32. The condition on f implies that $f'(z)$ is real for all z. By the open mapping property of analytic functions (Corollary B.35), f' is constant. Thus $f'(z) = az + b$.

33. Arguing by contradiction, suppose f is not constant. Let ∂ denote boundary and $(\,\cdot\,)^{\mathrm{o}}$ denote interior. Since f is continuous, $f(E)$ is compact in $\mathbb{C}$. Since a nonconstant analytic function is an open mapping, $f(E^{\mathrm{o}})$ is open in $\mathbb{C}$. By continuity, $\partial(f(E^{\mathrm{o}}) \subseteq f(\partial E) \subseteq i\mathbb{R}$. Let H be the open right half plane. Then it follows that $\partial(f(E^{\mathrm{o}}) \cap H) = \varnothing$, and hence the open set $f(E^{\mathrm{o}}) \cap H$ is closed in H. Since H is connected, $f(E^{\mathrm{o}}) \cap H$ is empty or equals H. It cannot equal H, being contained in the compact set $f(E)$. Thus $f(E^{\mathrm{o}}) \cap H$ is empty. Arguing similarly with H replaced by the open left half plane, we conclude that $f(E^{\mathrm{o}}) \subseteq i\mathbb{R}$. This shows that $f(E^{\mathrm{o}})$ is not open, contradiction.

34. The tangent function has $\tan z = -i(e^{iz} - e^{-iz})/(e^{iz} + e^{-iz})$ Solving $w = -i(e^{iz} - e^{-iz})/(e^{iz} + e^{-iz})$ for z in terms of w yields $z = \frac{1}{2i}\log\big(\frac{-w+i}{w+i}\big)$ for some branch of the logarithm. We readily check that the derivative of this expression with respect to w is $1/(w^2+1)$, consistently with the case that w is a real number in $(-\infty, \infty)$, known from Corollary 1.46b. As with the case of arcsine let us try for the principal branch of the logarithm. Then the argument of the logarithm must not be real and ≤ 0. The exceptional case is that $\frac{-w+i}{w+i} = r \le 0$. If we write $w = u + iv$, this equation says that $u + iv = i(1-r)/(1+r)$, hence that $u = 0$ and $v = (1-r)/(1+r)$. For $r \le 0$, this has $|v| \ge 1$. Hence we can use the principal branch Log as long as we cut out from the plane the pieces of the imaginary axis corresponding to $|v| \ge 1$. In other words, the branch of arctangent that we seek is given by $\arctan w = \frac{1}{2i}\operatorname{Log}\big(\frac{-w+i}{w+i}\big)$ on $\mathbb{C} - \{w \bigm| |\operatorname{Im} w| \ge 1\}$.

35. For (a), set $a_0 = b_0 = 1$. For $n > 0$, the coefficient of z^n in the power series expansion of $f(z)g(z) = 1$ is

$$b_n + b_{n-1}a_1 + \cdots + b_1 a_{n-1} + 1 = 0.$$

Thus the desired recursive formula is $b_n = -b_{n-1}a_1 - \cdots - b_1 a_{n-1} - 1$.

For (b), the series $\sum_{n=1}^{\infty} a_n z^n$ is absolutely convergent for $|z| < r_0$, and therefore $c(r) = \sum_{n=1}^{\infty} |a_n| r^n$ is finite-valued for $r < r_0$. As the sum of a power series, $c(r)$ is continuous as a function of r. Under the assumption that $f(z)$ is not a constant function, it is strictly increasing with $c(0) = 0$. Thus there exists a positive number ρ such that $c(\rho) < 1$. For any such ρ, $f(z)$ is nonvanishing for $|z| < \rho$, and therefore $1 + \sum_{n=1}^{\infty} b_n z^n$ is convergent for $|z| < \rho$.

36. The given conditions and the Maximum Modulus Theorem imply that the function $f(z)/z$ is analytic for $|z| < 1$ and has for each $r < 1$, $|f(z)/z| \le \sup_{|\zeta|=r} |f(\zeta)|/r$ whenever $|z| \le r$. The condition $|f(z)| \le 1$ implies that $\sup_{|\zeta|\le r} |f(\zeta)| \le 1$, and thus $|f(z)/z| \le 1$ for $|z| < 1$. Since $\lim_{z\to 0} f(z)/z = f'(0)$, this inequality forces $|f'(0)| \le 1$.

If equality holds, i.e., if either $|f(z)| = |z|$ somewhere or $|f'(0)| = 1$, then $|f(z)/z)|$ attains its maximum somewhere in the interior of the unit disk, and $f(z)/z$ must be constant. Thus $f(z) = cz$. Taking absolute values shows that $|c| = 1$.

37. The Maximum Modulus Theorem shows that $|f(z)| \le |e^z|$ everywhere for $|z| \le 1$. Schwarz's Lemma therefore applies to $e^{-z} f(z)$ on the open unit disk and shows that $|e^{-z} f(z)| \le |z|$ for $|z| < 1$. Hence $|f(z)| \le |z||e^z|$, and $|f(\log 2)| \le (\log 2)|e^{\log 2}| = 2\log 2$.

38. Arguing by contradiction, suppose that $\alpha > 1$. Since f carries open sets to open sets, f^{-1} is an analytic function from $f(D)$ onto D with $f^{-1}(0) = 0$ and $(f^{-1})'(0) = 1$. By assumption the domain of f^{-1} contains $\{|z| < \alpha\}$. Thus the

domain of the analytic function $g(z) = f^{-1}(\alpha z)$ contains D, and we have $g(0) = f^{-1}(0) = 0$ and $|g(z)| \le 1$ for $|z| < 1$. By Schwarz's Lemma $|g'(0)| \le 1$. However, direct computation gives $g'(0) = \alpha(f^{-1})'(0) = \alpha > 1$, contradiction.

39. From the condition $|f(e^{i\theta})| \le M$ for $0 \le \theta \le 2\pi$ and the Maximum Modulus Theorem, $|f(z)| \le M$ for $|z| \le 1$. Since $f(0) = f'(0) = 0$, $z^{-2}f(z)$ is analytic. This function has $|z^{-2}f(z)| \le M$ for $|z| = 1$. By the Maximum Modulus Theorem, $|z^{-2}f(z)| \le M$ for $|z| \le 1$, and $|f(z)| \le M|z|^2$.

40. For (a), the inequality follows by dividing $|f(z) - g(z)| < |f(z)|$ through by $|f(z)|$. Then $|F(z) - 1| < 1$ for z in the image of γ, and (a) is proved. From (a), it follows that 0 lies in the unbounded component of the complement of the image of Γ, and $n(\Gamma, 0) = 0$ by Proposition B.29. For (c), the Argument Principle says precisely that $n(\Gamma, 0) = \sum_j h_j n(\gamma, a_j) - \sum_l k_l n(\gamma, b_l)$. Since the left side is 0, so is the right side.

41. The Argument Principle shows that the integral equals $2\pi i$ times the number of zeros of $g(z)$ inside $|z| = 1$. To compute the number of zeros, we can use Rouché's Theorem. For $|z| = 1$, the term $f(z) = 10z^8$ has $|f(z) - g(z)| < |f(z)|$, and both $f(z)$ and $g(z)$ are nonvanishing for $|z| = 1$. Thus $f(z)$ and $g(z)$ have the same number of zeros for $|z| < 1$, counting multiplicities. For $f(z)$, this number is 8, and thus it is 8 for $g(z)$ also. Hence the given integral equals $16\pi i$.

42. For $|z| = 1$, the term $4z^5$ dominates the sum of the others. Thus $|f(z) - g(z)| < |f(z)|$ for $|z| = 1$ if $f(z) = 4z^5$. Neither $f(z)$ nor $g(z)$ vanishes anywhere with $|z| = 1$. The conditions of Rouché's Theorem are satisfied, and $f(z)$ and $g(z)$ have the same number of zeros inside $|z| = 1$. Since $f(z)$ has 5 zeros inside $|z| = 1$, counting multiplicities, so does $g(z)$.

43. When $|z| = 2$, the term $2z^5$ dominates the sum of the others in absolute value. Thus $f(z) = 2z^5$ and $g(z) = 2z^5 - 6z^2 + z + 1$ have $|f(z) - g(z)| < |f(z)|$ for $|z| = 2$. In addition, neither $f(z)$ nor $g(z)$ vanishes anywhere for $|z| = 2$. The conditions of Rouché's Theorem are satisfied, and $f(z)$ and $g(z)$ have the same number of zeros inside $|z| = 2$. Since $f(z)$ has 5 zeros inside $|z| = 2$, counting multiplicities, so does $g(z)$. For $|z| = 1$, we argue similarly, using $f_1(z) = -6z^2$. Again we have $|f_1(z) - g(z)| < |f_1(z)|$ for $|z| = 1$ with neither f_1 nor g vanishing anywhere for $|z| = 1$. Since f_1 has 2 zeros inside $|z| = 1$, so does g. Thus the number of zeros for $g(z)$ with $1 < |z| < |2|$ is $5 - 2 = 3$.

44. Let $P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$ be a polynomial of degree $n > 0$, and write $P(z)$ as $z^n + Q(z)$. Since $\lim_{|z|\to\infty} Q(z)/|z|^n = 0$, there exists $R > 0$ such that $|Q(z)|/|z^n| < 1$ for $|z|| \ge R$. Then $|z^n - P(z)| < |z^n|$ for $|z| \ge R$. Applying Rouché's Theorem to the standard circle about 0 of radius R and taking $f(z) = z^n$ and $g(z) = P(z)$, we see that $P(z)$ and z^n have the same number of zeros, counting multiplicities, inside the circle $|z| = R$. That is, $P(z)$ has n zeros inside the circle.

45. By the residue theorem, $\int_{|z|=2} \frac{dz}{z^2+1} = 2\pi i\big(\mathrm{Res}_{z=i}\big(\frac{1}{z^2+1}\big) + \mathrm{Res}_{z=-i}\big(\frac{1}{z^2+1}\big)\big) = 2\pi i\big(\frac{1}{z+i}\big|_{z=i} + \frac{1}{z-i}\big|_{z=-i}\big) = 2\pi i\big(\frac{1}{2i} + \frac{1}{-2i}\big) = 0$.

46. The factorization $2z^2+3z-2=(2z-1)(z+2)$ shows that the only pole inside C is at $z=\frac{1}{2}$. The Residue Theorem gives $\int_C \frac{dz}{2z^2+3z-2} = 2\pi i \operatorname{Res}_{z=\frac{1}{2}} \frac{1}{(2z-1)(z+2)} = 2\pi i \lim_{z\to\frac{1}{2}} \frac{z-\frac{1}{2}}{(2z-1)(z+2)} = 2\pi i \frac{1}{2(\frac{5}{2})} = 2\pi i/5$. This problem can also be done more directly by using the Cauchy Integral Formula.

47. This integral is of the type of Example 1 in Section B11, and the answer is $2\pi i$ times the sum of the residues in the open upper half plane. The roots of z^4+3z^2+2 are $\pm i\sqrt{2}$ and $\pm i$. We need to compute the residues at $i\sqrt{2}$ and i. These are

$$\begin{aligned}\operatorname{Res}_{i\sqrt{2}}\left(\tfrac{1}{z^4+3z^2+2}\right) &= \lim_{z\to i\sqrt{2}} \frac{1}{(z+i\sqrt{2})(z^2+1)} = \frac{1}{(2i\sqrt{2})(-2+1)} = i\sqrt{2}/4,\\ \operatorname{Res}_{i}\left(\tfrac{1}{z^4+3z^2+2}\right) &= \lim_{z\to i} \frac{1}{(z+i)(z^2+2)} = \frac{1}{(2i)(-1+2)} = -i/2.\end{aligned}$$

Thus the integral equals $(2\pi i)(i\sqrt{2}/4 - i/2) = 2\pi(1/2-\sqrt{2}/4) = \frac{1}{2}\pi(2-\sqrt{2})$.

48. The denominator factors as $(x^2+9)(x^2+1)$, and its roots in the upper half plane are $3i$ and i. The degree of the denominator is 2 greater than the degree of the numerator. This is of the type of Example 1 in Section B11. Thus the integral equals $2\pi i$ times the sum of the residues at $3i$ and i. These residues are respectively $\frac{z^2-z+2}{(z+3i)(z^2+1)}\Big|_{z=3i}$ and $\frac{z^2-z+2}{(z+i)(z^2+9)}\Big|_{z=i}$, which equal $\frac{-9-3i+2}{6i(-8)} = \frac{-7-3i}{-48i}$ and $\frac{-1-i+2}{2i(8)} = \frac{1-i}{16i}$. The integral is $2\pi i$ times the sum of these two complex numbers, namely $5\pi/12$.

49. This is similar to Examples 2 and 3 in Section B11, and the qualitative conclusion there applies here. The polynomial z^2-2z+2 has roots $1\pm i\sqrt{2}$, with $z=1+i\sqrt{2}$ as the only root in the upper half plane. The results of those examples show that the integral equals $\operatorname{Im}\big(\operatorname{Res}_{1+i\sqrt{2}}\big(\frac{(1+z)e^{iz}}{z^2-2z+2}\big)\big) = \operatorname{Im}\big(\frac{(1+z)e^{iz}}{z-(1-i\sqrt{2})}\big)_{z=1+i\sqrt{2}} = \operatorname{Im}\big(\frac{(2-i\sqrt{2})e^{i(1-i\sqrt{2})}}{2i\sqrt{2}}\big)$.

50. This is $\frac{1}{2}\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta}$, which is of the form in Example 5 in Section B11. If C denotes the standard unit circle, the substitution $z=e^{i\theta}$ and $dz=ize^{i\theta}$, in which $d\theta = \frac{-i\,dz}{z}$, changes it into $\frac{-i}{2}\int_C \frac{dz}{z(a+\frac{1}{2}b(z+z^{-1}))} = \frac{-i}{2}\int_C \frac{2\,dz}{bz^2+2az+b}$. The roots of the denominator in the integrand are $z=\frac{-a\pm\sqrt{a^2-b^2}}{b}$, and the one and only root in the unit disk is $\frac{-a+\sqrt{a^2-b^2}}{b}$. Thus

$$\begin{aligned}\int_0^{\pi} \frac{dx}{a+b\cos x} &= -i(2\pi i)\operatorname{Res}_{z=b^{-1}(-a+\sqrt{a^2-b^2})}\left(\frac{1}{bz^2+2az+b}\right)\\ &= 2\pi\left(\frac{1}{b(z-b^{-1}(-a-\sqrt{a^2-b^2}))}\right)\Big|_{z=b^{-1}(-a+\sqrt{a^2-b^2})}\\ &= 2\pi\frac{1}{2\sqrt{a^2-b^2}} = \frac{\pi}{\sqrt{a^2-b^2}}\end{aligned}$$

51. (a) $\frac{1}{z-1} = -(1 + z + z^2 + z^3 + \dots)$. So $f(z) = -\frac{1}{z} - \sum_{n=0}^{\infty} z^n$.

(b) $\frac{1}{1-z} = z^{-1}\frac{1}{z^{-1}-1} = -z^{-1}\frac{1}{1-z^{-1}} = z^{-1}(1 + z^{-1} + z^{-2} + z^{-3} + \dots)$. So $f(z) = \sum_{n=2}^{\infty} z^{-n}$.

52. For (a), there are three such expansions, valid in the disk $|z| < 1$, the annulus $1 < |z| < 3$, and the annulus $3 < |z|$.

For (b) we treat the expansion in the annulus $|z| < 1$, writing $\frac{1}{1-z^2} = 1 + z^2 + z^4 + z^6 + \dots$ and $\frac{1}{3-z} = \frac{1}{3}(1 + \frac{z}{3} + (\frac{z}{3})^2 + (\frac{z}{3})^3 + \dots)$. Then the series has

$$c_n = \begin{cases} 0 & \text{for } n < 0, \\ 3^{-(n+1)} & \text{for } n > 0 \text{ and odd,} \\ 1 + 3^{-(n+1)} & \text{for } n \geq 0 \text{ and even.} \end{cases}$$

53. The function

$$\frac{z}{e^z-1} - 1 + \frac{1}{2}z = \frac{z-e^z+1+\frac{1}{2}ze^z-\frac{1}{2}z}{e^{z/2}(e^{z/2}-e^{-z/2})} = \frac{\frac{1}{2}z-e^z+1+\frac{1}{2}ze^z}{e^{z/2}(e^{z/2}-e^{-z/2})}$$
$$= \frac{\frac{1}{2}ze^{-z/2}-e^{z/2}+e^{-z/2}+\frac{1}{2}ze^{z/2}}{e^{z/2}-e^{-z/2}} = \frac{\frac{1}{2}z(e^{z/2}+e^{-z/2})-(e^{z/2}-e^{-z/2})}{e^{z/2}-e^{-z/2}}$$

is the quotient of two odd functions and hence is an even function. Also it is analytic is a disk about 0. Therefore $\frac{z}{e^z-1} - 1 + \frac{1}{2}z = \sum_{n=0}^{\infty} b_n z^{2n}$, and the result follows if we set $b_k = (-1)^{k-1}\frac{B_k}{(2k)!}$.

54. The solution of Problem 53 shows that $\frac{iz}{e^{iz}-1} + \frac{1}{2}iz = \frac{1}{2}\frac{iz(e^{iz/2}+e^{-iz/2})}{e^{iz/2}-e^{-iz/2}} = \frac{1}{2}z\cot(z/2)$ and hence $z\cot z = \frac{2iz}{e^{2iz}-1} + iz$. From the result of Problem 53,

$$\frac{2iz}{e^{2iz}-1} = 2iz\Big(\frac{1}{2iz} - \frac{1}{2} + \sum_{k=1}^{\infty}(-1)^{k-1}\frac{B_k}{(2k)!}(2iz)^{2k-1}\Big) = 1 - iz + \sum_{k=1}^{\infty}(-1)^{k-1}\frac{B_k}{(2k)!}(2iz)^{2k}$$

and

$$z\cot z = 1 - \sum_{k=1}^{\infty}\frac{B_k}{(2k)!}(2z)^{2k}.$$

The desired Laurent series is therefore

$$\cot z = \frac{2}{e^{2iz}-1} + z = \frac{1}{z} - \sum_{k=1}^{\infty}\frac{B_k 2^{2k}}{(2k)!}z^{2k-1}.$$

55. The function $f(z)$ is continuous on each compact subset of U by Proposition 2.21. Hence $f(z)$ is continuous on U. Fix attention on an open disk D in U. If γ is any piecewise C^1 closed curve in D, then $\int_\gamma f_n(z)\,dz = 0$ by the Cauchy Integral Theorem. Since the image of γ is compact and the convergence of integrands is uniform on compact sets, we can pass to the limit by Theorem 1.31 and obtain $\int_\gamma f(z) = 0$. Since $f(z)$ is known to be continuous, Morera's Theorem shows that $f(z)$ is analytic in D. Since D is an arbitrary open disk in U, $f(z)$ is analytic in U.

56. Let $K \subseteq U$ be compact, and let d be the distance from K to U^c, i.e., the positive minimum of the distance from x to U^c for x in the compact set K. Let K' be the larger compact set $\{z \in U \mid \text{distance from } z \text{ to } K \text{ is } \le \frac{1}{2}d\}$. By assumption $\lim_n f_n(z) = 0$ uniformly for $z \in K'$. Let $\epsilon > 0$ be given, and choose N so that $n \ge N$ implies $|f_n(\zeta)| \le \epsilon$ for all $\zeta \in K'$. If z is in K, let γ be a standard circle of radius $\frac{1}{2}d$ about z. The complex derivative $f_n'(z)$ is given by $f_n'(z) = (2\pi i)^{-1} \int_C \frac{f_n(\zeta)}{(\zeta - z)^2}\, d\zeta$, according to Theorem B.11. Since γ has radius $\frac{1}{2}d$, each point ζ in the integration lies in K'. Thus $n \ge N$ implies $|f_n'(z)| \le \frac{1}{2\pi} \frac{\epsilon}{(\frac{1}{2}d)^2} 2\pi(\frac{1}{2}d) = 2\epsilon/d$, and $\{f_n'(z)\}$ indeed tends uniformly to 0 for $z \in K$.

57. Arguing by contradiction, suppose that f is not identically 0 and that $f(z_0) = 0$. Choose $r > 0$ small enough so that $\{|z - z_0| \le r\}$ is contained in U and so that f vanishes for $|z - z_0| \le r$ only when $z = z_0$. Let γ be the standard circle about z_0 of radius r. For each n, $\frac{1}{2\pi i} \int_\gamma \frac{f_n'(z)}{f_n(z)}\, dz = 0$ by the Argument Principle, since each f_n is nowhere vanishing. Since $\{f_n(z)\}$ converges uniformly to $f(z)$ on the compact set image(γ) and since $f(z)$ is nowhere 0 on image(γ), $\{1/f_n(z)\}$ converges to $1/f(z)$ uniformly on image(γ). Also Problem 56 shows that $\{f_n'(z)\}$ converges uniformly to $f'(z)$ on image(γ). Thus $\left\{\frac{f_n'(z)}{f_n(z)}\right\}$ converges uniformly to $\frac{f'(z)}{f(z)}$, and

$$\lim_n \frac{1}{2\pi i} \int_\gamma \frac{f_n'(z)}{f_n(z)}\, dz = \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)}\, dz.$$

We have seen that the left side is 0, and the right side is positive by the Argument Principle, since $f(z_0)$ has been assumed to be 0. This contradiction shows that $f(z)$ is indeed either nowhere 0 or identically 0.

58. Let $K \subseteq U$ be compact, and let d be the distance from K to U^c, i.e., the positive minimum of the distance from x to U^c for x in the compact set K. Let K' be the larger compact set $\{z \in U \mid \text{distance from } z \text{ to } K \text{ is } \le \frac{1}{2}d\}$. By assumption there is some constant $c_{K'}$ such that $|f(z)| \le c_{K'}$ for all $z \in K'$. If z is in K, let γ be a standard circle of radius $\frac{1}{2}d$ about z. For f in E, the complex derivative $f'(z)$ is given by $f'(z) = (2\pi i)^{-1} \int_C \frac{f(\zeta)}{(\zeta - z)^2}\, d\zeta$, according to Theorem B.11. Since γ has radius $\frac{1}{2}d$, each point ζ in the integration lies in K'. Thus $|f'(z)| \le \frac{1}{2\pi} \frac{c_{K'}}{(\frac{1}{2}d)^2} 2\pi(\frac{1}{2}d) = 2c_{K'}/d$, and the derivative $f'(z)$ of each member $f(z)$ of E is bounded by $2c_{K'}/d$ for $z \in K$.

59.

(a) K' is certainly bounded, and it is closed by Proposition 2.16. Hence it is compact. If z_0 is in K and $|z - z_0| \le r$, then z' is in K' and hence is in U.

(b) From

$$f(z) - f(z_0) = \frac{1}{2\pi i} \int_{|\zeta - z_0| = r} \left(\frac{1}{\zeta - z} - \frac{1}{\zeta - z'}\right) f(\zeta)\, dz = \frac{z - z}{2\pi i} \int_{|\zeta - z_0| = r} \frac{f(\zeta)\, d\zeta}{(\zeta - z)(\zeta - z')},$$

we obtain $|f(z) - f(z')| \le \frac{1}{2\pi} |z' - z|\, 2\pi r \frac{M}{(r/2)(r/2)} = (4M/r)|z - z'|$.

(c) We can apply (b) with $z_0 = z' = z_1$ and $z = z_2$ because $|z - z'| = |z_2 - z_1| \le \delta \le r/2$. Then we obtain $|f(z_1) - f(z_2)| \le 4M|z_2 - z_1|/r \le 4M\delta/r \le 4M(\epsilon r/4M) = \epsilon$.

60. We shall combine Ascoli's Theorem with a diagonal process. We can choose an increasing sequence $\{K_n\}$ of compact sets with union U such that K_n is contained in the interior of K_{n+1} for each n; namely for each n, we let K_n be the intersection of the closed disk of radius n about 0 with the set of points at distance $\ge 1/n$ from U^c. Let a sequence $\{f_k\}$ of members of E be given. Problem 59 shows that $\{f_k\}$ is uniformly equicontinuous on K_1, and $\{f_k\}$ is by assumption uniformly bounded on K_1. By Ascoli's Theorem it has a subsequence that is uniformly convergent on K_1. Repeating this process with K_2, we can find a further subsequence that is uniformly convergent on K_2 as well. Continuing in this way, we can find successive subsequences that are uniformly convergent on K_n for each n. Then the sequence whose n^{th} term is the n^{th} member of the n^{th} subsequence converges uniformly on each K_n. This subsequence in fact converges uniformly on every compact subset of U because each compact subset of U lies in some K_n. Indeed, the construction was arranged so that the interiors of the K_n's form an open cover of U, hence of any given compact subset K of U; a finite subcover suffices to cover K, and since the K_n's are nested, one single such interior covers K.

65. Conclusion (a) is a routine computation. For the first part of (b), take $L(z) = \frac{z-z_3}{z-z_4}\frac{z_2-z_4}{z_2-z_3}$.

66. ST^{-1} carries Tz_2, Tz_3, Tz_4 into $(1, 0, \infty)$. Then

$$(Tz_1, Tz_2, Tz_3, Tz_4) = (ST^{-1})(Tz_1) = Sz_1 = (z_1, z_2, z_3, z_4).$$

67. For (a), we compute $\operatorname{Im}\left(\frac{az+b}{cz+d}\right) = \operatorname{Im}\big((az+b)(c\bar{z}+d)\big) = \operatorname{Im}(azd + bc\bar{z}) = (ad - bc)\operatorname{Im} z$.

For (b) let the transformation be given by the complex matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. This transformation carries $\mathbb{R} \cup \{\infty\}$ into $\mathbb{R} \cup \{\infty\}$, sending 0 to b/d and ∞ to a/c. Also the real derivative with respect to r of $r \mapsto \frac{ar+b}{cr+d}$, which is $\frac{ad-bc}{(cr+d)^2}$, has to be real for real r. Therefore the polynomial function $r \mapsto (ad - bc)^{-1}(c^2r^2 + 2cdr + d^2)$, which is the reciprocal of the derivative, has real coefficients.

Suppose for the moment that $d \ne 0$. Adjusting the given matrix by a scalar, we may assume that $d > 0$. Then $(ad - bc)^{-1}d$ real implies $ad - bc$ real, $(ad - bc)^{-1}2cd$ real implies c real, b/d real implies b real, and $ad - bc$ real implies a real. Also the computation in (a) shows that $ad - bc > 0$. This completes the argument when $d \ne 0$.

Now suppose that $d = 0$. Adjusting the given matrix by a scalar, we may assume that $c > 0$. Then $(ad - bc)^{-1}c^2$ real implies $-bc$ real and therefore also b real. Also a/c real implies a real. Again the computation in (a) shows that $ad - bc > 0$. This completes the argument when $d = 0$.

68. This problem can be reduced to Problem 67 by making use of the unique linear fractional transformation that sends $0, -1, 1$ into $i, 0, \infty$ in this order, namely $z \mapsto \frac{z+1}{iz-i}$, verifying that it carries the unit disk onto the upper half plane.

71. For the last part of (b), the property of being C^∞ in a region U is local, and it holds in any open set where the harmonic function is the real part of an analytic function. Every point of U has a filled disk about it that lies in U that satisfies this condition, and hence the harmonic function is C^∞ everywhere.

72. The idea is that although v is unknown, its first partial derivatives are known because of the Cauchy–Riemann equations. Therefore the first partial derivatives are known for the unknown analytic function $F(z)$ whose real part is $u(x, y)$. Write u_1 for $\frac{\partial u}{\partial x}$ and u_2 for $\frac{\partial u}{\partial y}$. Along any horizontal segment that lies in U, we must have

$$F(x_2, y) - F(x_1, y) = \int_{x_1}^{x_2} (u_1 - iu_2)(s, y)\, ds,$$

and along any vertical segment that lies in U, we must have

$$F(x, y_2) - F(x, y_1) = \int_{y_1}^{y_2} (u_2 - iu_1)(x, t)\, dt.$$

Fix the base point $z_0 = (x_0, y_0)$, define $F(z_0) = u(x_0, y_0)$, let σ be any polygonal path from z_0 to z in U with sides parallel to the axes, and define F along σ one segment at a time, using one or the other of the above two formulas. The main step is to prove that $F(z)$ is well defined. Once this step is done, we find as in the proof of Theorem B.40 that F is continuous and has $\frac{\partial F}{\partial x} = u_1 - iu_2$ and $\frac{\partial F}{\partial y} = u_2 - iu_1$. These partial derivatives are continuous and satisfy $\frac{\partial F}{\partial x} = -i\frac{\partial F}{\partial y}$. By Corollary B.2$'$, F has a complex derivative at each point and is therefore analytic. (The value of the complex derivative is $f(z) = \frac{\partial F}{\partial z} = u_1 - iu_2$.) The real part of F has first partial derivatives u_1 and u_2 and therefore equals u except for an additive constant. The imaginary part of F is a well defined conjugate harmonic function throughout U.

Thus we are to prove that $F(z)$ is well defined. The combinatorial part of the proof of Theorem B.40 works without change: We take two polygonal paths σ_1 and σ_2 in U from z_0 to z with sides parallel to the axes and work with $\gamma = \sigma_1 - \sigma_2$. We are then able to show that γ has a decomposition

$$\gamma = \sum_i n(\gamma, a_i)\partial R_i.$$

Using that U is simply connected, we argue exactly as in the last paragraph of the proof of Theorem B.40 to show that the interior of each R_i for which $n(\gamma, a_i) \neq 0$ lies completely in U. From this fact we can see as follows that $\int_{\partial R_i} f(z)\, dz = 0$, where $f(z) = \frac{\partial F}{\partial x} = u_1 - iu_2$: we simply write out $\int_{\partial R_i} f(z)\, dz$ as the sum of the complex line integrals over each side and proceed as in the solution to Problem 3. The equality $\int_{\partial R_i} f(z)\, dz = 0$ follows. Summing over i the product of $n(\gamma, a_i)$ by this equality, we obtain $\int_\gamma f(z)\, dz = 0$. Thus $\int_{\sigma_1} f(z)\, dz = \int_{\sigma_2} f(z)\, dz$, and $F(z)$ is well defined.

73. Problem 71b shows that u has a conjugate harmonic function v defined on $\mathbb{R}^2$. Then $(u + iv) \circ A$ is analytic as the composition of two analytic functions, and its real part, namely $u \circ A$, is harmonic.

74. Since U is assumed connected, the image of U is connected. Let $u(x_0, y_0) = c$, and let D be an open disk about (x_0, y_0) lying in D. On D, u is the real part of an analytic function f, by Problem 71b. If $f(z)$ is not constant on D, then $f(z)$ is an open mapping, by Corollary B.35. The intersection of $f(D)$ with the real axis is therefore an open subset of $\mathbb{R}$ containing c.

75. If u has a local maximum at (x_0, y_0), then on some open disk D about (x_0, y_0), u has an absolute maximum at (x_0, y_0). By the previous problem, u is constant on D. Thus the interior E of the subset of U where $u(x, y) = c$ is nonempty, as well as open. Let (x_0, y_0) be a limit point of E in U, and choose an open disk D' about (x_0, y_0) that lies in U. Since (x_0, y_0) is a limit point of E, there exists a member (x_1, y_1) of E in D'. Since (x_1, y_1) is in the open set E, there is a disk D'' about (x_1, y_1) contained in E and D'. On this disk, $u(x, y) = c$. Thus the analytic function on D' of which u is the real part is constant on D'' and necessarily also on D'. In other words, (x_0, y_0) is in E, and E is closed within U. Since U is connected, $E = U$.

76. By Problem 71b, $u(x, y)$ is the real part of an analytic function $f(z)$ on all of $\mathbb{C}$. Then $e^{-f(z)}$ is an entire function that takes values in the unit disk. By Liouville's Theorem, $e^{-f(z)}$ is constant. Therefore $f(z)$ is constant, and so is its real part $u(x, y)$.

77. Problem 71b shows that $u(x, y)$ is the real part of an analytic function $f(z)$ for $|z| < 1$. For $r < 1$, the Cauchy Integral Formula gives

$$\begin{aligned} f(0) &= \tfrac{1}{2\pi i} \int_{|z|=r} z^{-1} f(z)\, dz = \tfrac{1}{2\pi i} \int_0^{2\pi} (re^{i\theta})^{-1} f(re^{i\theta}) ire^{i\theta}\, d\theta \\ &= \tfrac{1}{2\pi} \int_0^{2\pi} (u(r\cos\theta, r\sin\theta) + iv(r\cos\theta, r\sin\theta))\, d\theta. \end{aligned}$$

Taking the real part of both sides gives $u(0,0) = \frac{1}{2\pi}\int_0^{2\pi} u(r\cos\theta, r\sin\theta)\, d\theta$. We apply the operation $\lim_{r\uparrow 1}$ to both sides. Since u is continuous as a function of two variables, the convergence of $u(r\cos\theta, r\sin\theta)$ to $u(\cos\theta, \sin\theta)$ is uniform in θ. Thus we can put the limit $\lim_{r\uparrow 1}$ under the integral sign and obtain $u(0,0) = \frac{1}{2\pi}\int_0^{2\pi} u(\cos\theta, \sin\theta)\, d\theta$, as required.

78. In (a), the matrix equation follows by applying the matrix equation of Problem 4a to each component function f_k and lining up the results.

In (b), the $(k, \ell)^{\text{th}}$ entry of $J_{\mathbb{C}}$ is $\frac{\partial f_k}{\partial z_\ell}$. This equals $\frac{\partial f_k}{\partial x_\ell} = \frac{\partial \operatorname{Re} f_k}{\partial x_\ell} + i\frac{\operatorname{Im} \partial f_k}{\partial x_\ell}$, which is the sum of the $(k, \ell)^{\text{th}}$ entry of J and i times the $(k+m, \ell)^{\text{th}}$ entry. Thus in block form the first column of J is $\begin{pmatrix} \operatorname{Re} J_{\mathbb{C}} \\ \operatorname{Im} J_{\mathbb{C}} \end{pmatrix}$. If we write $J = \begin{pmatrix} \operatorname{Re} J_{\mathbb{C}} & \gamma \\ \operatorname{Im} J_{\mathbb{C}} & \delta \end{pmatrix}$, apply (a), and multiply out the block matrices, we find that $\gamma = -\operatorname{Im} J_{\mathbb{C}}$ and $\delta = \operatorname{Re} J_{\mathbb{C}}$.

79. This is a matter of combining Problems 4 and 79 with the chain rule (Theorem 3.10) in the real-variable theory. The functions $f_{\mathbb{R}}$ and $g_{\mathbb{R}}$ are C^∞ by Problem 4, and $(g \circ f)_{\mathbb{R}} = g_{\mathbb{R}} \circ f_{\mathbb{R}}$ from the definitions. Since $g_{\mathbb{R}} \circ f_{\mathbb{R}}$ is C^∞ with Jacobian matrix

the product of the Jacobian matrices for $g_{\mathbb{R}}$ and $f_{\mathbb{R}}$, $(g \circ f)_{\mathbb{R}}$ is C^∞, and we have a formula for its Jacobian matrix. Applying Problem 78, we see that the Jacobian matrix of $g \circ f$ satisfies the equation in Problem 78a. Then it follows from Problem 4 that each entry of $g \circ f$ is holomorphic; by definition $g \circ f$ is holomorphic. Combining the formula for $(g \circ f)_{\mathbb{R}}$ with Problem 78b, we see that the complex Jacobian matrix of $g \circ f$ is the product of the complex Jacobian matrices.

80. Statement: Suppose that f is a holomorphic function from an open set E of $\mathbb{C}^n$ into $\mathbb{C}^n$, and suppose that the complex derivative of f is invertible for some a in E. Put $b = f(a)$. Then

(a) there exist open sets $U \subseteq E \subseteq \mathbb{C}^n$ and $V \subseteq \mathbb{C}^n$ such that a is in U, b is in V, f is one-one from U onto V, and
(b) the inverse function $g : V \to U$ is holomorphic.

Consequently, the complex Jacobian matrix of g at $f(z)$ is the inverse of the complex Jacobian matrix of f at z for $z \in U$.

The proof consists in reducing matters to Theorem 3.17 by using Problems 4, 78, and 79.

81. The statement is just the analog of Theorem 3.16 with complex variables replacing real variables. The proof comes by imitating the proof that Theorem 3.17 implies Theorem 3.16.