V. Distributions, 179-211

DOI: 10.3792/euclid/9781429799911-5

from

Advanced Real Analysis

Digital Second Edition
Corrected version 2017

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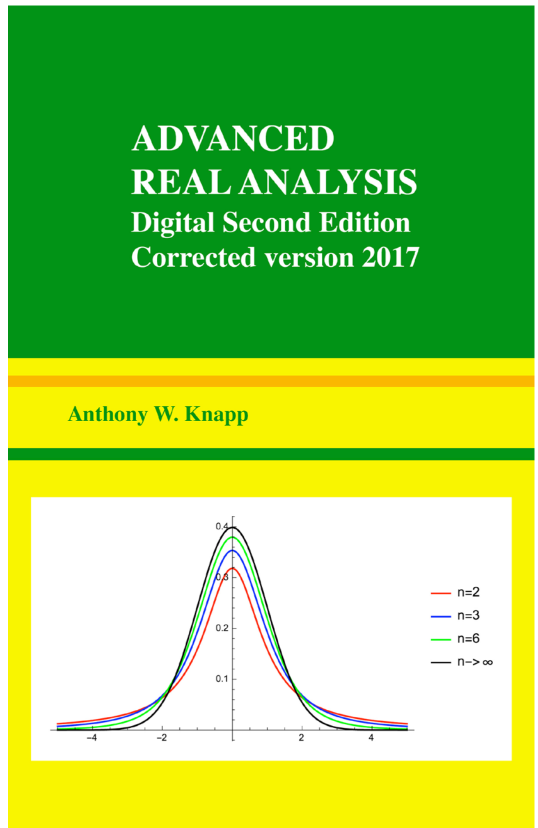

Full Book DOI: 10.3792/euclid/9781429799911
ISBN: 978-1-4297-9991-1

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Title: Advanced Real Analysis
Cover: Normal distribution as a limit of Gosset's t distribution; see page 421.

Mathematics Subject Classification (2010): 46–01, 42–01, 43–01, 35–01, 34–01, 47–01, 58–01, 60A99, 60F05, 28C10, 42C40, 65T60.

First Edition, ISBN-13 978-0-8176-4382-9

Published by Birkhäuser Boston

Digital Second Edition, not to be sold, no ISBN

Published by the Author

CHAPTER V

Distributions

Abstract. This chapter makes a detailed study of distributions, which are continuous linear functionals on vector spaces of smooth scalar-valued functions. The three spaces of smooth functions that are studied are the space $C^\infty_{\text{com}}(U)$ of smooth functions with compact support in an open set U, the space $C^\infty(U)$ of all smooth functions on U, and the space of Schwartz functions $\mathcal{S}(\mathbb{R}^N)$ on $\mathbb{R}^N$. The corresponding spaces of continuous linear functionals are denoted by $\mathcal{D}'(U)$, $\mathcal{E}'(U)$, and $\mathcal{S}'(\mathbb{R}^N)$.

Section 1 examines the inclusions among the spaces of smooth functions and obtains the conclusion that the corresponding restriction mappings on distributions are one-one. It extends from $\mathcal{E}'(U)$ to $\mathcal{D}'(U)$ the definition given earlier for support, it shows that the only distributions of compact support in U are the ones that act continuously on $C^\infty(U)$, it gives a formula for these in terms of derivatives and compactly supported complex Borel measures, and it concludes with a discussion of operations on smooth functions.

Sections 2–3 introduce operations on distributions and study properties of these operations. Section 2 briefly discusses distributions given by functions, and it goes on to work with multiplications by smooth functions, iterated partial derivatives, linear partial differential operators with smooth coefficients, and the operation $(\,\cdot\,)^\vee$ corresponding to $x \mapsto -x$. Section 3 discusses convolution at length. Three techniques are used—the realization of distributions of compact support in terms of derivatives of complex measures, an interchange-of-limits result for differentiation in one variable and integration in another, and a device for localizing general distributions to distributions of compact support.

Section 4 reviews the operation of the Fourier transform on tempered distributions; this was introduced in Chapter III. The two main results are that the Fourier transform of a distribution of compact support is a smooth function whose derivatives have at most polynomial growth and that the convolution of a distribution of compact support and a tempered distribution is a tempered distribution whose Fourier transform is the product of the two Fourier transforms.

Section 5 establishes a fundamental solution for the Laplacian in $\mathbb{R}^N$ for $N > 2$ and concludes with an existence theorem for distribution solutions to $\Delta u = f$ when f is any distribution of compact support.

1. Continuity on Spaces of Smooth Functions

Distributions are continuous linear functionals on vector spaces of smooth functions. Their properties are deceptively simple-looking and enormously helpful. Some of their power is hidden in various interchanges of limits that need to be carried out to establish their basic properties. The result is a theory that is easy to implement and that yields results quickly. In the last section of this chapter, we shall see an example of this phenomenon when we show how it gives information about solutions of partial differential equations involving the Laplacian.

The three vector spaces of scalar-valued smooth functions that we shall consider in the text[1] of this chapter are $C^\infty(U)$, $\mathcal{S}(\mathbb{R}^N)$, and $C^\infty_{\text{com}}(U)$, where U is a nonempty open set in $\mathbb{R}^N$. Topologies for these spaces were introduced in Section IV.2, Section III.1, and Section IV.7, respectively. Let $\{K_p\}$ be an exhausting sequence of compact subsets of U, i.e., a sequence such that $K_p \subseteq K^o_{p+1}$ for all p and such that $U = \bigcup_{p=1}^\infty K_p$.

The vector space $C^\infty(U)$ of all smooth functions on U is given by a separating family of seminorms such that a countable subfamily suffices. The members of the subfamily may be taken to be $\|f\|_{p,\alpha} = \sup_{x\in K_p} |D^\alpha f(x)|$, where $1 \le p < \infty$ and where α varies over all differentiation multi-indices.[2] The space of continuous linear functionals is denoted by $\mathcal{E}'(U)$, and the members of this space are called "distributions of compact support" for reasons that we recall in a moment.

The vector space $\mathcal{S}(\mathbb{R}^N)$ of all Schwartz functions is another space given by a separating family of seminorms such that a countable subfamily suffices. The members of the subfamily may be taken to be $\|f\|_{\alpha,\beta} = \sup_{x\in\mathbb{R}^N} |x^\alpha D^\beta f(x)|$, where α and β vary over all differentiation multi-indices.[3] The space of continuous linear functionals is denoted by $\mathcal{S}'(\mathbb{R}^N)$, and the members of this space are called "tempered distributions."

The vector space $C^\infty_{\text{com}}(U)$ of all smooth functions of compact support on U is given by the inductive limit topology obtained from the vector subspaces $C^\infty_{K_p}$. The space $C^\infty_{K_p}$ consists of the smooth functions with support contained in K_p, the topology on $C^\infty_{K_p}$ being given by the countable family of seminorms $\|f\|_{p,\alpha} = \sup_{x\in K_p} |D^\alpha f(x)|$. The space of continuous linear functionals is traditionally[4] written $\mathcal{D}'(U)$, and the members of this space are called simply "distributions." Since the field of scalars is a locally convex topological vector space, Proposition 4.29 shows that the members of $\mathcal{D}'(U)$ may be viewed as arbitrary sequences of consistently defined continuous linear functionals on the spaces $C^\infty_{K_p}$.

[1] A fourth space, the space of periodic smooth functions on $\mathbb{R}^N$, is considered in Problems 12–19 at the end of the chapter and again in the problems at the end of Chapter VII.

[2] The notation for the seminorms in Chapter IV was chosen for the entire separating subfamily and amounted to $\|f\|_{K_p,D^\alpha}$. The subscripts have been simplified to take into account the nature of the countable subfamily.

[3] The notation for the seminorms in Chapter III was chosen for the entire separating subfamily and amounted to $\|f\|_{x^\alpha,x^\beta}$. The subscripts have been simplified to take into account the nature of the countable subfamily.

[4] The tradition dates back to Laurent Schwartz's work, in which $\mathcal{D}(U)$ was the notation for $C^\infty_{\text{com}}(U)$ and $\mathcal{D}'(U)$ denoted the space of continuous linear functionals.

For the spaces of smooth functions, there are continuous inclusions

$$C^\infty_{\text{com}}(U) \subseteq C^\infty(U) \qquad \text{for all } U,$$
$$C^\infty_{\text{com}}(\mathbb{R}^N) \subseteq \mathcal{S}(\mathbb{R}^N) \subseteq C^\infty(\mathbb{R}^N) \qquad \text{for } U = \mathbb{R}^N.$$

We observed in Section IV.2 that $C^\infty_{\text{com}}(U) \subseteq C^\infty(U)$ has dense image, and it follows that $\mathcal{S}(\mathbb{R}^N) \subseteq C^\infty(\mathbb{R}^N)$ has dense image. Proposition 4.12 showed that $C^\infty_{\text{com}}(\mathbb{R}^N) \subseteq \mathcal{S}(\mathbb{R}^N)$ has dense image.

If $i : A \to B$ denotes one of these inclusions and T is a continuous linear functional on B, then $T \circ i$ is a continuous linear functional on A, and we can regard $T \circ i$ as the restriction of T to A. Since i has dense image, $T \circ i$ cannot be 0 unless T is 0. Thus each restriction map $T \mapsto T \circ i$ as above is one-one. We therefore have *one-one* restriction maps

$$\mathcal{E}'(U) \to \mathcal{D}'(U) \qquad \text{for all } U,$$
$$\mathcal{E}'(\mathbb{R}^N) \to \mathcal{S}'(\mathbb{R}^N) \to \mathcal{D}'(\mathbb{R}^N) \qquad \text{for } U = \mathbb{R}^N.$$

This fact justifies using the term "distribution" for any member of $\mathcal{D}'$ and for using the term "distribution" with an appropriate modifier for members of $\mathcal{E}'$ and $\mathcal{S}'$.

As in Section III.1 it will turn out often to be useful to write the effect of a distribution T on a function φ as $\langle T, \varphi\rangle$, rather than as $T(\varphi)$, and we shall adhere to this convention systematically for the moment.[5]

We introduced in Section IV.2 the notion of "support" for any member of $\mathcal{E}'(U)$, and we now extend that discussion to $\mathcal{D}'(U)$. We saw in Proposition 4.10 that if T is an arbitrary linear functional on $C^\infty_{\text{com}}(U)$ and if U' is the union of all open subsets U_γ of U such that T vanishes on $C^\infty_{\text{com}}(U_\gamma)$, then T vanishes on $C^\infty_{\text{com}}(U')$. We accordingly define the **support** of any distribution to be the complement in U of the union of all open sets U_γ such that T vanishes on $C^\infty_{\text{com}}(U_\gamma)$. If T has empty support, then $T = 0$ because T vanishes on $C^\infty_{\text{com}}(U)$ and because $C^\infty_{\text{com}}(U)$ is dense in the domain of T. Proposition 4.11 showed that the members of $\mathcal{E}'(U)$ have compact support in this sense; we shall see in Theorem 5.1 that no other members of $\mathcal{D}'(U)$ have compact support.

An example of a member of $\mathcal{E}'(U)$ was given in Section IV.2: Take finitely many complex Borel measures ρ_α of compact support within U, the indexing being by multi-indices α with $|\alpha| \le m$, and put $\langle T, \varphi\rangle = \sum_{|\alpha|\le m} \int_U D^\alpha\varphi(x)\, d\rho_\alpha(x)$. Then T is in $\mathcal{E}'(U)$, and the support of T is contained in the union of the supports of the ρ_α's. Theorem 5.1 below gives a converse, but it is necessary in general to allow the ρ_α's to have support a little larger than the support of the given distribution T.

[5]A different convention is to write $\int_U \varphi(x)\, dT(x)$ in place of $\langle T, \varphi\rangle$. This notation emphasizes an analogy between distributions and measures and is especially useful when more than one $\mathbb{R}^N$ variable is in play. This convention will provide helpful motivation in one spot in Section 3.

Theorem 5.1. If T is a member of $\mathcal{D}'(U)$ with support contained in a compact subset K of U, then T is in $\mathcal{E}'(U)$. Moreover, if K' is any compact subset of U whose interior contains K, then there exist a positive integer m and, for each multi-index α with $|\alpha| \leq m$, a complex Borel measure ρ_α supported in K' such that

$$\langle T, \varphi \rangle = \sum_{|\alpha| \leq m} \int_{K'} D^\alpha \varphi \, d\rho_\alpha \qquad \text{for all } \varphi \in C^\infty(U).$$

REMARK. Problems 8–10 at the end of the chapter discuss the question of taking $K' = K$ under additional hypotheses.

PROOF. Let ψ be a member of $C^\infty_{\text{com}}(U)$ with values in $[0, 1]$ that is 1 on a neighborhood of K and is 0 on K'^c; such a function exists by Proposition 3.5f. If φ is in $C^\infty_{\text{com}}(U)$, then we can write $\varphi = \psi\varphi + (1 - \psi)\varphi$ with $\psi\varphi$ in $C^\infty_{K'}$ and with $(1 - \psi)\varphi$ in $C^\infty_{\text{com}}(K^c)$. The assumption about the support of T makes $\langle T, (1 - \psi)\varphi \rangle = 0$, and therefore

$$\langle T, \varphi \rangle = \langle T, \psi\varphi \rangle + \langle T, (1 - \psi)\varphi \rangle = \langle T, \psi\varphi \rangle \quad \text{for all } \varphi \text{ in } C^\infty_{\text{com}}(U). \qquad (*)$$

Since the inclusion $C^\infty_{K'} \to C^\infty_{\text{com}}(U)$ is continuous, we can define a continuous linear functional T_1 on $C^\infty_{K'}$ by $T_1(\phi) = \langle T, \phi \rangle$ for ϕ in $C^\infty_{K'}$. For any φ in $C^\infty_{\text{com}}(U)$, $\phi = \psi\varphi$ is in $C^\infty_{K'}$, and $(*)$ gives $\langle T, \varphi \rangle = \langle T, \psi\varphi \rangle = T_1(\psi\varphi)$. The continuity of T_1 on $C^\infty_{K'}$ means that there exist m and C such that

$$|T_1(\phi)| \leq C \sum_{|\alpha| \leq m} \sup_{x \in K'} |D^\alpha \phi(x)| \qquad \text{for all } \phi \in C^\infty_{K'}. \qquad (**)$$

Let M be the number of multi-indices α with $|\alpha| \leq m$.

We introduce the Banach space X of M-tuples of continuous complex-valued functions on K', the norm for X being the largest of the norms of the components. The Banach-space dual of this space is the space of M-tuples of continuous linear functionals on the components, thus the space of M-tuples of complex Borel measures on K'.

We can embed $C^\infty_{K'}$ as a vector subspace of X by mapping ϕ to the M-tuple with components $D^\alpha \phi$ for $|\alpha| \leq m$. We transfer T_1 from $C^\infty_{K'}$ to its image subspace within X, and the result, which we still call T_1, is a linear functional continuous relative to the norm on X as a consequence of $(**)$. Applying the Hahn–Banach Theorem, we extend T_1 to a continuous linear functional $\widetilde{T}_1$ on all of X without an increase in norm. Then $\widetilde{T}_1$ is given on X by an M-tuple of complex Borel measures ρ'_α on K', i.e., $\widetilde{T}_1(\{f_\alpha\}_{|\alpha| \leq m}) = \sum_{|\alpha| \leq m} \int_{K'} f_\alpha \, d\rho'_\alpha$. Therefore any φ in $C^\infty_{\text{com}}(U)$ has

$$\langle T, \varphi \rangle = T_1(\psi\varphi) = \widetilde{T}_1\big(\{D^\alpha(\psi\varphi)\}_{|\alpha| \leq m}\big) = \sum_{|\alpha| \leq m} \int_{K'} D^\alpha(\psi\varphi) \, d\rho'_\alpha. \qquad (\dagger)$$

The right side of (†) is continuous on $C^\infty(U)$, and therefore T extends to a member of $\mathcal{E}'(U)$. The formula in the theorem follows by expanding out each $D^\alpha(\psi\varphi)$ in (†) by the Leibniz rule for differentiation of products, grouping the derivatives of ψ with the complex measures, and reassembling the expression with new complex measures ρ_α. $\square$

In Chapters VII and VIII we shall be interested also in a notion related to support, namely the notion of "singular support." If f is a locally integrable function on the open set U, then f defines a member T_f of $\mathcal{D}'(U)$ by

$$\langle T_f, \varphi\rangle = \int_U f\varphi\, dx \qquad \text{for } \varphi \in C^\infty_{\text{com}}(U).$$

If U' is an open subset of U and T is a distribution on U, we say that T **equals a locally integrable function** on U' if there is some locally integrable function f on U' such that $\langle T, \varphi\rangle = \langle T_f, \varphi\rangle$ for all φ in $C^\infty_{\text{com}}(U)$. We say that T **equals a smooth function** on U' if this condition is satisfied for some f in $C^\infty(U')$. In the latter case the member of $C^\infty(U')$ is certainly unique.

The **singular support** of a member T of $\mathcal{D}'(U)$ is the complement of the union of all open subsets U' of U such that T equals a smooth function on U'. The uniqueness of the smooth function on such a subset implies that if T equals the smooth function f_1 on U_1' and equals the smooth function f_2 on U_2', then $f_1(x) = f_2(x)$ for x in $U_1' \cap U_2'$. In fact, T equals the smooth function $f_1\big|_{U_1'\cap U_2'}$ on $U_1' \cap U_2'$ and also equals the smooth function $f_2\big|_{U_1'\cap U_2'}$ there. The uniqueness forces $f_1\big|_{U_1'\cap U_2'} = f_2\big|_{U_1'\cap U_2'}$. Taking the union of all the open subsets on which T equals a smooth function, we see that T is a smooth function on the complement of its singular support.

EXAMPLE. Take $U = \mathbb{R}^1$, and define

$$\langle T, \varphi\rangle = \lim_{\varepsilon\downarrow 0} \int_{|x|\geq\varepsilon} \frac{\varphi(x)\, dx}{x} \qquad \text{for } \varphi \in C^\infty_{\text{com}}(\mathbb{R}^1).$$

To see that this is well defined, we choose η in $C^\infty_{\text{com}}(\mathbb{R}^1)$ with η identically 1 on the support of φ and with $\eta(x) = \eta(-x)$ for all x. Taylor's Theorem gives $\varphi(x) = \varphi(0) + xR(x)$ with R in $C^\infty(\mathbb{R}^1)$. Multiplying by $\eta(x)$ and integrating for $|x| \geq \varepsilon$, we obtain

$$\textstyle\int_{|x|\geq\varepsilon} \frac{\varphi(x)\, dx}{x} = \varphi(0) \int_{|x|\geq\varepsilon} \frac{\eta(x)\, dx}{x} + \int_{|x|\geq\varepsilon} R(x)\eta(x)\, dx.$$

The first term on the right side is 0 for every ε, and therefore

$$\textstyle\langle T, \varphi\rangle = \int_{\mathbb{R}^1} R(x)\eta(x)\, dx.$$

It follows that T is in $\mathcal{D}'(\mathbb{R}^1)$. On any function compactly supported in $\mathbb{R}^1 - \{0\}$, the original integral defining T is convergent. Thus T equals the function $1/x$ on $\mathbb{R}^1 - \{0\}$. Since $1/x$ is nowhere zero on $\mathbb{R}^1 - \{0\}$, the (ordinary) support of T has to be a closed subset of $\mathbb{R}^1$ containing $\mathbb{R}^1 - \{0\}$. Therefore T has support $\mathbb{R}^1$. On the other hand, T does not equal a function on all of $\mathbb{R}^1$, and T has $\{0\}$ as its singular support.

Starting in Section 2, we shall examine various operations on distributions. Operations on distributions will be defined by duality from corresponding operations on smooth functions. For that reason it is helpful to know about continuity of various operations on spaces of smooth functions. These we study now.

We begin with multiplication by smooth functions and with differentiation. If ψ is in $C^\infty(U)$, then multiplication $\varphi \mapsto \psi\varphi$ carries $C^\infty_{\text{com}}(U)$ into itself and also $C^\infty(U)$ into itself. The same is true of any iterated partial derivative operator $\varphi \mapsto D^\alpha\varphi$. We shall show that these operations are continuous. A multiplication $\varphi \mapsto \psi\varphi$ need not carry $\mathcal{S}(\mathbb{R}^N)$ into itself, and we put aside $\mathcal{S}(\mathbb{R}^N)$ for further consideration later.

The kind of continuity result for $C^\infty(U)$ that we are studying tends to follow from an easy computation with seminorms, and it is often true that the same argument can be used to handle also $C^\infty_{\text{com}}(U)$. Here is the general fact.

Lemma 5.2. Suppose that $L : C^\infty(U) \to C^\infty(U)$ is a continuous linear map that carries $C^\infty_{\text{com}}(U)$ into $C^\infty_{\text{com}}(U)$ in such a way that for each compact $K \subseteq U$, C^∞_K is carried into $C^\infty_{K'}$ for some compact $K' \supseteq K$. Then L is continuous as a linear map from $C^\infty_{\text{com}}(U)$ into $C^\infty_{\text{com}}(U)$.

PROOF. Proposition 4.29b shows that it is enough to prove for each K that the composition of $L : C^\infty_K \to C^\infty_{K'}$ followed by the inclusion of $C^\infty_{K'}$ into $C^\infty_{\text{com}}(U)$ is continuous, and we know that the inclusion is continuous. Fix K, choose K_p in the exhausting sequence containing the corresponding K', and let α be a multi-index. By the continuity of $L : C^\infty(U) \to C^\infty(U)$, there exist a constant C, some integer q with $q \geq p$, and finitely many multi-indices β_i such that $\|L(\varphi)\|_{p,\alpha} \leq C \sum_i \|\varphi\|_{q,\beta_i}$. Since $L(\varphi)$ has support in $K' \subseteq K_p$ and φ has support in $K \subseteq K' \subseteq K_p \subseteq K_q$, this inequality shows that $\sup_{x\in K'} |D^\alpha(L(\varphi))(x)| \leq C \sum_i \sup_{x\in K} |D^{\beta_i}\varphi(x)|$. Hence $L : C^\infty_K \to C^\infty_{K'}$ is continuous, and the lemma follows. $\square$

Proposition 5.3. If ψ is in $C^\infty(U)$, then $\varphi \mapsto \psi\varphi$ is continuous from $C^\infty(U)$ to $C^\infty(U)$ and from $C^\infty_{\text{com}}(U)$ to $C^\infty_{\text{com}}(U)$. If α is any differentiation multi-index, then $\varphi \mapsto D^\alpha\varphi$ is continuous from $C^\infty(U)$ to $C^\infty(U)$ and from $C^\infty_{\text{com}}(U)$ to $C^\infty_{\text{com}}(U)$.

PROOF. The Leibniz rule for differentiation of products gives $D^\alpha(\psi\varphi) = \sum_{\beta\le\alpha} c_\beta(D^{\beta-\alpha}\psi)(D^\beta\varphi)$ for certain integers c_β. Then

$$\|\psi\varphi\|_{p,\alpha} \le \textstyle\sum_{\beta\le\alpha} c_\beta m_\beta \|\varphi\|_{p,\beta},$$

where $m_\beta = \sup_{x\in K_p} |D^{\beta-\alpha}\psi(x)|$, and it follows that $\varphi \mapsto \psi\varphi$ is continuous from $C^\infty(U)$ into itself. Taking $K' = K$ in Lemma 5.2, we see that $\varphi \mapsto \psi\varphi$ is continuous from $C^\infty_{\text{com}}(U)$ into itself.

Since $\|D^\alpha\varphi\|_{p,\beta} = \|\varphi\|_{p,\alpha+\beta}$, the function $\varphi \mapsto D^\alpha\varphi$ is continuous from $C^\infty(U)$ into itself, and Lemma 5.2 with $K' = K$ shows that $\varphi \mapsto D^\alpha\varphi$ is continuous from $C^\infty_{\text{com}}(U)$ into itself. $\square$

We can combine these two operations into the operation of a **linear partial differential operator**

$$P(x, D) = \sum_{|\alpha|\le m} c_\alpha(x) D^\alpha \qquad \text{with all } c_\alpha \text{ in } C^\infty(U)$$

by means of the formula $P(x, D)\varphi = \sum_{|\alpha|\le m} c_\alpha(x) D^\alpha\varphi$. It is to be understood that the operator has smooth coefficients. It is immediate from Proposition 5.3 that $P(x, D)$ is continuous from $C^\infty(U)$ into itself and from $C^\infty_{\text{com}}(U)$ into itself.

An operator $P(x, D)$ as above is said to be of **order** m if some $c_\alpha(x)$ with $|\alpha| = m$ has c_α not identically 0. The operator reduces to an operator of the form $P(D)$ if the coefficient functions c_α are all constant functions.

We introduce the **transpose operator** $P(x, D)^{\text{tr}}$ by the formula

$$P(x, D)^{\text{tr}}\varphi(x) = \sum_{|\alpha|\le m} (-1)^{|m|} D^\alpha\big(c_\alpha(x)\varphi(x)\big).$$

Expanding out the terms $D^\alpha\big(c_\alpha(x)\varphi(x)\big)$ by means of the Leibniz rule, we see that $P(x, D)^{\text{tr}}$ is some linear partial differential operator of the form $Q(x, D)$. The next proposition gives the crucial property of the transpose operator.

Proposition 5.4. Suppose that $P(x, D)$ is a linear partial differential operator on U. If u and v are in $C^\infty(U)$ and at least one of them is in $C^\infty_{\text{com}}(U)$, then

$$\int_U \big(P(x, D)^{\text{tr}} u(x)\big) v(x)\, dx = \int_U u(x)\big(P(x, D)v(x)\big)\, dx.$$

PROOF. It is enough to prove that the partial derivative operator D_j with respect to x_j satisfies $\int_U (D_j u)v\, dx = -\int_U u(D_j v)\, dx$ since iteration of this formula gives the result of the proposition. Moving everything to one side of the equation and putting $w = uv$, we see that it is enough to prove that $\int_{\mathbb{R}^N} I_U D_j w\, dx = 0$ if w is in $C^\infty_{\text{com}}(U)$, where I_U is the indicator function of U. We can drop the I_U from the integration since $D_j w$ is 0 off U, and thus it is enough to prove that $\int_{\mathbb{R}^N} D_j w\, dx = 0$ for w in $C^\infty_{\text{com}}(\mathbb{R}^N)$. By Fubini's Theorem the integral may be computed as an iterated integral. The integral on the inside extends over the set where x_j is arbitrary in $\mathbb{R}$ and the other variables take on particular values, say $x_i = c_i$ for $i \neq j$. The integral on the outside extends over all choices of the c_i for $i \neq j$. The inside integral is already 0, because for suitable a and b, it is of the form $\int_a^b D_j w\, dx_j = [w]_{x_j=a}^{x_j=b} = 0 - 0 = 0$. □

Next let us consider convolution, taking $U = \mathbb{R}^N$. We shall be interested in the function $\psi * \varphi$ given by

$$\psi * \varphi(x) = \int_{\mathbb{R}^N} \psi(x-y)\varphi(y)\, dy = \int_{\mathbb{R}^N} \psi(y)\varphi(x-y)\, dy,$$

under the assumption that ψ and φ are in $C^\infty(\mathbb{R}^N)$ and that one of them has compact support.

A simple device of localization helps with the analysis of this function: If K is the support of ψ, then the values of $\psi * \varphi(x)$ for x in a bounded open set S depend only on the value of φ on the bounded open set of differences $S - K$. Consequently we can replace φ by $\eta\varphi$, where η is a member of $C^\infty_{\text{com}}(\mathbb{R}^N)$ that is 1 on $S - K$, and the values of $\psi * \varphi(x)$ will match those of $\psi * (\eta\varphi)(x)$ for x in S. The latter function is the convolution of two smooth functions of compact support and is smooth by Proposition 3.5c. Therefore $\psi * \varphi$ is always in $C^\infty(\mathbb{R}^N)$ if ψ is in $C^\infty_{\text{com}}(\mathbb{R}^N)$ and φ is in $C^\infty(\mathbb{R}^N)$. We shall use this same device later in treating convolution of distributions.

Proposition 5.5. If ψ is in $C^\infty_{\text{com}}(\mathbb{R}^N)$ and φ is in $C^\infty(\mathbb{R}^N)$, then

(a) $D^\alpha(\psi * \varphi) = (D^\alpha\psi) * \varphi = \psi * (D^\alpha\varphi)$,

(b) convolution of three functions in $C^\infty(\mathbb{R}^N)$ is associative when at least two of the three functions have compact support,

(c) convolution with ψ is continuous from $C^\infty(\mathbb{R}^N)$ into itself and from $C^\infty_{\text{com}}(\mathbb{R}^N)$ into itself,

(d) convolution with φ is continuous from $C^\infty_{\text{com}}(\mathbb{R}^N)$ into $C^\infty(\mathbb{R}^N)$.

PROOF. For (a), let K be the support of ψ. Concentrating on x's lying in a bounded open set S, choose a function η in $C^\infty_{\text{com}}(\mathbb{R}^N)$ that is 1 on $S - K$, and then $\psi * \varphi(x) = \psi * (\eta\varphi)(x)$ for x in S. Proposition 3.5c says that

$$D^\alpha(\psi * (\eta\varphi))(x) = (D^\alpha\psi) * (\eta\varphi)(x) = \psi * D^\alpha(\eta\varphi)(x)$$

for all x in $\mathbb{R}^N$, and consequently

$$D^\alpha(\psi * \varphi)(x) = (D^\alpha\psi) * \varphi(x) = \psi * D^\alpha\varphi(x)$$

for all x in S. Since S is arbitrary, (a) follows. The proof of (b) is similar.

For (c), again let K be the support of ψ, and apply (a). Then

$$\begin{aligned}\|\psi * \varphi\|_{p,\alpha} &= \sup_{x\in K_p} |D^\alpha(\psi * \varphi)(x)| = \sup_{x\in K_p} |\psi * (D^\alpha\varphi)(x)| \\ &\le \sup_{x\in K_p} \Big|\int_K |\psi(y)||D^\alpha\varphi(x-y)|\,dy\Big| \le \|\psi\|_1 \sup\nolimits_{z\in K_p - K} |D^\alpha\varphi(z)|,\end{aligned}$$

and the right side is $\le \|\psi\|_1\|\varphi\|_{q,\alpha}$ if q is large enough so that $K_p - K \subseteq K_q$. This proves the continuity on $C^\infty(\mathbb{R}^N)$, and the continuity on $C^\infty_{\text{com}}(\mathbb{R}^N)$ then follows from Lemma 5.2.

For (d), Proposition 4.29b shows that it is enough to prove that $\psi \mapsto \psi * \varphi$ is continuous from C^∞_K into $C^\infty(\mathbb{R}^N)$ for each compact set K. The same estimate as for (c) gives

$$\|\psi * \varphi\|_{p,\alpha} \le \|\psi\|_1\|\varphi\|_{q,\alpha} \le |K|\|\varphi\|_{q,\alpha}(\sup_{x\in K} |\psi(x)|)$$

if q is large enough so that $K_p - K \subseteq K_q$. The result follows. □

2. Elementary Operations on Distributions

In this section we take up operations on distributions. If f is a locally integrable function on the open set U, we defined the member T_f of $\mathcal{D}'(U)$ by

$$\langle T_f, \varphi\rangle = \int_U f\varphi\,dx$$

for φ in $C^\infty_{\text{com}}(U)$. If f vanishes outside a compact subset of U, then T_f is in $\mathcal{E}'(U)$, extending to operate on all of $C^\infty(U)$ by the same formula.

Starting from certain continuous operations L on smooth functions, we want to extend these operations to operations on distributions. So that we can regard L as an extension from smooth functions to distributions, we insist on having $L(T_f) = T_{L(f)}$ if f is smooth. To tie the definition of L on distributions T_f to the definition on general distributions T, we insist that L be the "transpose" of *some* continuous operation M on functions, i.e., that $\langle L(T), \varphi\rangle = \langle T, M(\varphi)\rangle$. Taking $T = T_f$ in this equation, we see that we must have $\int_U L(f)\varphi\,dx = \int_U fM(\varphi)\,dx$. On the other hand, once we have found a continuous M on smooth functions with $\int_U L(f)\varphi\,dx = \int_U fM(\varphi)\,dx$, then we can make the definition $\langle L(T), \varphi\rangle = \langle T, M(\varphi)\rangle$ for the effect of L on distributions. In particular the operator M on smooth functions is unique if it exists. We write $L^{\text{tr}} = M$ for it. In summary, our procedure[6] is to find, if we can, a continuous operator L^{tr} on smooth functions such that

$$\int_U L(f)\varphi\,dx = \int_U f L^{\mathrm{tr}}(\varphi)\,dx$$

and then to define

$$\langle L(T), \varphi\rangle = \langle T, L^{\mathrm{tr}}(\varphi)\rangle.$$

We begin with the operations of multiplication, whose continuity is addressed in Proposition 5.3. If L is multiplication by the function ψ in $C^\infty(U)$, then we can take $L^{\mathrm{tr}} = L$ because $\int_U L(f)\varphi\,dx = \int_U(\psi f)\varphi\,dx = \int_U f(\psi\varphi)\,dx = \int_U f L^{\mathrm{tr}}(\varphi)$ if f and φ are in $C^\infty(U)$ and one of them has compact support. Thus our definition of multiplication of a distribution T by ψ in $C^\infty(U)$ is

$$\langle \psi T, \varphi\rangle = \langle T, \psi\varphi\rangle.$$

Here we assume either that T is in $\mathcal{D}'(U)$ and φ is in $C^\infty_{\mathrm{com}}(U)$ or else that T is in $\mathcal{E}'(U)$ and φ is in $C^\infty(U)$. Briefly we say that at least one of T and φ has compact support.

The operation of multiplication by a function can be used to localize the effect of a distribution in a way that is useful in the definition below of convolution of distributions. First observe that if T is in $\mathcal{D}'(U)$ and η is in $C^\infty_{\mathrm{com}}(U)$, then the support of ηT is contained in the support of η; in fact, if φ is any member of $C^\infty_{\mathrm{com}}(U \cap \mathrm{support}(\eta)^c)$, then $\eta\varphi = 0$ and hence $\langle \eta T, \varphi\rangle = \langle T, \eta\varphi\rangle = 0$. In particular, ηT is in $\mathcal{E}'(U)$. On the other hand, we lose no information about T by this operation if we allow all possible η's, because if T is in $\mathcal{D}'(U)$ and if φ is a member of $C^\infty_{\mathrm{com}}(U)$ with support in a compact subset K of U, then $\varphi = \eta\varphi$ and hence $\langle T, \varphi\rangle = \langle T, \eta\varphi\rangle = \langle \eta T, \varphi\rangle$.

Next we consider differentiation, which is a continuous operation by Proposition 5.3. When L gives the iterated derivative D^α of a distribution, we can take the operation L^{tr} on smooth functions to be $(-1)^{|\alpha|}$ times D^α. The definition is then

$$\langle D^\alpha T, \varphi\rangle = (-1)^{|\alpha|}\langle T, D^\alpha\varphi\rangle.$$

Again we assume that at least one of T and φ has compact support.

Putting these definitions together yields the definition of the operation of a linear partial differential operator $P(x, D)$ with smooth coefficients on distributions. The formula is

$$\langle P(x, D)T, \varphi\rangle = \langle T, P(x, D)^{\mathrm{tr}}\varphi\rangle,$$

[6] Another way of proceeding is to use topologies on $\mathcal{E}'(U)$ and $\mathcal{D}'(U)$ such that $C^\infty_{\mathrm{com}}(U)$ is dense in $\mathcal{E}'(U)$ and $C^\infty(U)$ is dense in $\mathcal{D}'(U)$. The approach in the text avoids the use of such topologies on spaces of distributions, and it will not be necessary to consider them.

where $P(x, D)^{\mathrm{tr}}$ is the transpose differential operator defined in Section 1. This definition is forced to satisfy $P(x, D)T = T_{P(x,D)f}$ on smooth f.

For further operations let us specialize to the setting that $U = \mathbb{R}^N$. The first is the operation of acting by -1 in the domain. For a function φ, we define $\varphi^\vee(x) = \varphi(-x)$. It is easy to check that this operation is continuous on $C^\infty(\mathbb{R}^N)$ and on $C^\infty_{\mathrm{com}}(\mathbb{R}^N)$. Since $\int_{\mathbb{R}^N} f^\vee \varphi\, dx = \int_{\mathbb{R}^N} f \varphi^\vee\, dx$ by a change of variables, the operator L^{tr} corresponding to $L(f) = f^\vee$ is just L itself. Thus the corresponding operation $T \mapsto T^\vee$ on distributions is given by

$$\langle T^\vee, \varphi\rangle = \langle T, \varphi^\vee\rangle.$$

The operation $(\,\cdot\,)^\vee$ has the further property that $(\varphi^\vee)^\vee = \varphi$ and $(T^\vee)^\vee = T$.

3. Convolution of Distributions

The next operation, again in the setting of $\mathbb{R}^N$, is the convolution of two distributions. Convolution is considerably more complicated than the operations considered so far because it involves two variables.

The method of Section 2 starts off easily enough. An easy change of variables shows that any three smooth functions, two of which have compact support, satisfy $\int_{\mathbb{R}^N} (\psi * f)\varphi\, dx = \int_{\mathbb{R}^N} (\psi)(f^\vee * \varphi)\, dx$, where $f^\vee(-x) = f(-x)$. This means that $\int_{\mathbb{R}^N} L(\psi)\varphi\, dx = \int_{\mathbb{R}^N} \psi L^{\mathrm{tr}}(\varphi)\, dx$, where $L(\psi) = \psi * f$ and $L^{\mathrm{tr}}(\varphi) = f^\vee * \varphi$. Thus Section 2 says to define $T * f$ by $\langle T * f, \varphi\rangle = \langle T, f^\vee * \varphi\rangle$. To handle the other convolution variable, however, we have to know that $T * f$ is a smooth function and that the passage from f to $T * f$ is continuous, and neither of these facts is immediately apparent. In addition, there are several cases to handle, depending on which two of the functions f, ψ, and φ at the start have compact support.

Sorting out all these matters could be fairly tedious, but there is a model for what happens that will help us anticipate the results. We shall follow the path that the model suggests. Then afterward, if we were to want to do so, it would be possible to go back and see that all the arguments with transposes in the style of Section 2 can be carried through with the tools that we have had to establish anyway.

The model takes a cue from Theorem 5.1, which says that members of $\mathcal{E}'(\mathbb{R}^N)$ are given by integration with compactly supported complex Borel measures and derivatives of them. In particular our definitions ought to specialize to familiar constructions when they are given by compactly supported positive Borel measures. In the case of measures, convolution is discussed in Problem 5 of Chapter VIII of *Basic*. The definition and results are as follows:

(i) $(\mu_1 * \mu_2)(E) = \int_{\mathbb{R}^N} \mu_1(E - x)\, d\mu_2(x)$ by definition,

(ii) $\int_{\mathbb{R}^N} \varphi\, d(\mu_1 * \mu_2) = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \varphi(x + y)\, d\mu_1(x)\, d\mu_2(y)$ for $\varphi \in C_{\mathrm{com}}(\mathbb{R}^N)$,

(iii) $\mu_1 * \mu_2 = \mu_2 * \mu_1$,

(iv) $\varphi\, dx * \mu$ is the continuous function $(\varphi\, dx * \mu)(x) = \int_{\mathbb{R}^N} \varphi(x-y)\, d\mu(y) = \int_{\mathbb{R}^N} (\varphi^\vee)_{-x}\, d\mu$ for $\varphi \in C_{\mathrm{com}}(\mathbb{R}^N)$, where the subscript $-x$ refers to the translate $h_t(y) = h(y+t)$.

The measures and the function φ in these properties are all assumed compactly supported, but some relaxation of this condition is permissible. For example the function φ can be allowed to be any continuous scalar-valued function on $\mathbb{R}^N$.

In defining convolution of distributions and establishing its properties, we shall face three kinds of technical problems: One is akin to Fubini's Theorem and will be handled for $\mathcal{E}'(\mathbb{R}^N)$ by appealing to Theorem 5.1 and using the ordinary form of Fubini's Theorem with measures. A second is a regularity question—showing that certain integrations in one variable of functions of two variables lead to smooth functions of the remaining variable—and will be handled for $\mathcal{E}'(\mathbb{R}^N)$ by Lemma 5.6 below. A third is the need to work with $\mathcal{D}'(\mathbb{R}^N)$, not just $\mathcal{E}'(\mathbb{R}^N)$, and will be handled by the localization device $T \mapsto \eta T$ mentioned in Section 2. We begin with the lemma that addresses the regularity question.

Lemma 5.6. Let K be a compact metric space, and let μ be a Borel measure on K. Suppose that $\Phi = \Phi(x, y)$ is a scalar-valued function on $\mathbb{R}^N \times K$ such that $\Phi(\,\cdot\,, y)$ is smooth for each y in K, and suppose further that every iterated partial derivative $D_x^\alpha \Phi$ in the first variable is continuous on $\mathbb{R}^N \times K$. Then the function

$$F(x) = \int_K \Phi(x, y)\, d\mu(y)$$

is smooth on $\mathbb{R}^N$ and satisfies $D^\alpha F(x) = \int_K D_x^\alpha \Phi(x, y)\, d\mu(y)$ for every multi-index α.

REMARKS. The lemma gives us a new proof of the smoothness shown in Section 1 for $\psi * \varphi$ when ψ is in $C_{\mathrm{com}}^\infty(\mathbb{R}^N)$ and φ is in $C^\infty(\mathbb{R}^N)$. In fact, we write the convolution as $\psi * \varphi(x) = \int_{\mathbb{R}^N} \varphi(x-y)\psi(y)\, dy$ and apply the lemma with μ equal to Lebesgue measure on the compact set support(ψ) and with $F(x) = \psi * \varphi(x)$ and $\Phi(x, y) = \varphi(x-y)\psi(y)$.

PROOF. In the proof we may assume without loss of generality that Φ is real-valued. We begin by showing that F is continuous. If $x_n \to x_0$, then the uniform continuity of Φ on the compact set $\{x_n\}_{n\geq 0} \times K$ implies that $\lim_n \Phi(x_n, y) = \Phi(x_0, y)$ uniformly. Dominated convergence allows us to conclude that $\lim_n \int_K \Phi(x_n, y)\, d\mu(y) = \int_K \Phi(x_0, y)\, d\mu(y)$. Therefore F is continuous.

Let B be a (large) closed ball in $\mathbb{R}^N$, and suppose that x is a member of B that is at distance at least 1 from B^c. If e_j denotes the j^{th} standard basis vector of $\mathbb{R}^N$

and if $|h| < 1$, then the Mean Value Theorem gives

$$\frac{\Phi(x+he_j, y) - \Phi(x, y)}{h} = \frac{\partial \Phi}{\partial x_j}(c, y)$$

for some c on the line segment between x and $x + h$. If $\epsilon > 0$ is given, choose the δ of uniform continuity of $\frac{\partial \Phi}{\partial x_j}$ on the compact set $B \times K$. We may assume that $\delta < 1$. For $|h| < \delta$ and for y in K, we have

$$\Big|\frac{\Phi(x+he_j, y) - \Phi(x, y)}{h} - \frac{\partial \Phi}{\partial x_j}(x, y)\Big| = \Big|\frac{\partial \Phi}{\partial x_j}(c, y) - \frac{\partial \Phi}{\partial x_j}(x, y)\Big| < \epsilon,$$

the inequality holding since (c, y) and (x, y) are both in $B \times K$ and are at distance at most δ from one another. As a consequence, if L is any compact subset of $\mathbb{R}^N$, then

$$\lim_{h\to 0} \frac{\Phi(x+he_j, y) - \Phi(x, y)}{h} = \frac{\partial \Phi}{\partial x_j}(x, y)$$

uniformly for (x, y) in $L \times K$. Because of this uniform convergence we have

$$\lim_{h\to 0} \int_K \frac{\Phi(x+he_j, y) - \Phi(x, y)}{h}\, d\mu(y) = \int_K \frac{\partial \Phi}{\partial x_j}(x, y)\, d\mu(y).$$

The integral on the left side equals $h^{-1}[F(x + he_j, y) - F(x, y)]$, and the limit relation therefore shows that $\frac{\partial}{\partial x_j} \int_K \Phi(x, y)\, d\mu(y)$ exists and equals $\int_K \frac{\partial \Phi}{\partial x_j}(x, y)\, d\mu(y)$.

This establishes the formula $D^\alpha F(x) = \int_K D_x^\alpha \Phi(x, y)\, d\mu(y)$ for α equal to the multi-index that is 1 in the j^{th} place and 0 elsewhere. The remainder of the proof makes the above argument into an induction. If we have established the formula $D^\alpha F(x) = \int_K D_x^\alpha \Phi(x, y)\, d\mu(y)$ for a certain α, then the first paragraph of the proof shows that $D^\alpha F$ is continuous. The second paragraph of the proof shows for each partial derivative operator D_j in one of the x variables that the operator $D^\beta = D_j D^\alpha$ has $D^\beta F(x) = \int_K D_x^\beta \Phi(x, y)\, d\mu(y)$. The lemma follows. $\square$

For our definitions let us begin with the convolution of two members of $\mathcal{E}'(\mathbb{R}^N)$. As indicated at the start of the section, we shall jump right to the final formula. The justification via formulas for transpose operations can be done afterward if desired. If we use notation that treats distributions like measures, the formula (ii) above suggests trying

$$\langle S * T, \varphi\rangle = \textstyle\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \varphi(x + y)\, dT(y)\, dS(x) = \langle S, \langle T, \varphi_x\rangle\rangle = \langle T, \langle S, \varphi_y\rangle\rangle,$$

where the subscript again indicates a translation: $\varphi_x(z) = \varphi(z + x)$. The outside distribution acts on the subscripted variable, and the inside distribution acts on the hidden variable. To make this into a rigorous definition, however, we have to check that $\langle T, \varphi_x\rangle$ and $\langle S, \varphi_y\rangle$ are smooth, that the last equality in the above display is valid, and that the resulting dependence on φ is continuous. We carry out these steps in the next proposition.

Proposition 5.7. Let S and T be in $\mathcal{E}'(\mathbb{R}^N)$, and let φ be in $C^\infty(\mathbb{R}^N)$. Then

(a) the functions $x \mapsto \langle T, \varphi_x \rangle$ and $y \mapsto \langle S, \varphi_y \rangle$ are smooth on $\mathbb{R}^N$,

(b) $D^\alpha(x \mapsto \langle T, \varphi_x \rangle) = \langle T, (D^\alpha \varphi)_x \rangle$,

(c) the function $\varphi \mapsto \langle T, \varphi_x \rangle$ is continuous from $C^\infty(\mathbb{R}^N)$ into itself and from $C^\infty_{\text{com}}(\mathbb{R}^N)$ into itself,

(d) $\langle S, \langle T, \varphi_x \rangle \rangle = \langle T, \langle S, \varphi_y \rangle \rangle$,

(e) the function $\varphi \mapsto \langle S, \langle T, \varphi_x \rangle \rangle$ is continuous from $C^\infty(\mathbb{R}^N)$ into the scalars,

(f) the formula

$$\langle S * T, \varphi \rangle = \langle S, \langle T, \varphi_x \rangle \rangle = \langle T, \langle S, \varphi_y \rangle \rangle$$

determines a well-defined member of $\mathcal{E}'(\mathbb{R}^N)$ such that $S * T = T * S$,

(g) the supports of S, T, and $S * T$ are related by

$$\text{support}(S * T) \subseteq \text{support}(S) + \text{support}(T).$$

PROOF. Let expressions for S and T in Theorem 5.1 be

$$\langle S, \varphi \rangle = \textstyle\sum_\alpha \int_{\mathbb{R}^N} D^\alpha \varphi(x)\, d\rho_\alpha(x) \quad \text{and} \quad \langle T, \varphi \rangle = \textstyle\sum_\beta \int_{\mathbb{R}^N} D^\beta \varphi(y)\, d\sigma_\beta(y),$$

the sums both being over finite sets of multi-indices and the complex measures being supported on some compact subset of $\mathbb{R}^N$. Then

$$\langle T, \varphi_x \rangle = \textstyle\sum_\beta \int_{\mathbb{R}^N} D^\beta \varphi(x + y)\, d\sigma_\beta(y). \tag{$*$}$$

If we apply Lemma 5.6 with $\Phi(x, y) = D^\beta \varphi(x + y)$ and treat y as varying over the union of the compact supports of the σ_β's, then we see that each term in the sum over β is a smooth function of x. Hence $x \mapsto \langle T, \varphi_x \rangle$ is smooth, and symmetrically $y \mapsto \langle S, \varphi_y \rangle$ is smooth. This proves (a).

Applying to $(*)$ the conclusions of Lemma 5.6 about passing the derivative operator D^α under the integral sign, we obtain

$$D^\alpha(x \mapsto \langle T, \varphi_x \rangle) = \textstyle\sum_\beta \int_{\mathbb{R}^N} D^{\alpha+\beta} \varphi(x + y)\, d\sigma_\beta(y) = \langle T, (D^\alpha \varphi)_x \rangle.$$

This proves (b).

If K denotes a subset of $\mathbb{R}^N$ containing the supports of all the σ_β's, then

$$|D^\alpha \langle T, \varphi_x \rangle| \le \sum_\beta \sup_{y \in K} |D^{\alpha+\beta} \varphi(x + y)| \|\sigma_\beta\|,$$

where $\|\sigma_\beta\|$ denotes the total-variation norm of σ_β. Hence

$$\sup_{x\in L}|D^\alpha\langle T,\varphi_x\rangle| \le \sum_\beta \sup_{z\in K+L}|D^{\alpha+\beta}\varphi(z)|\|\sigma_\beta\|.$$

This proves (c) for $C^\infty(\mathbb{R}^N)$. Combining this same inequality with Lemma 5.2, we obtain (c) for $C^\infty_{\text{com}}(\mathbb{R}^N)$.

The formula for $\langle S,\ \cdot\ \rangle$ and the identity $(*)$ together give

$$\begin{aligned}\langle S,\langle T,\varphi_x\rangle\rangle &= \sum_{\alpha,\beta}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} D^\alpha D^\beta\varphi_x(y)\,d\sigma_\beta(y)\,d\rho_\alpha(x)\\ &= \sum_{\alpha,\beta}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} D^{\alpha+\beta}\varphi(x+y)\,d\sigma_\beta(y)\,d\rho_\alpha(x).\end{aligned} \tag{$**$}$$

By Fubini's Theorem the right side is equal to

$$\sum_{\alpha,\beta}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} D^{\alpha+\beta}\varphi(x+y)\,d\rho_\alpha(x)\,d\sigma_\beta(y) = \langle T,\langle S,\varphi_y\rangle\rangle.$$

This proves (d).

Conclusion (e) is immediate from (c) and the continuity of S on $C^\infty(\mathbb{R}^N)$. Thus $S*T$ is in $\mathcal{E}'(\mathbb{R}^N)$. The equality in (d) shows that $S*T = T*S$. This proves (f).

Finally let L be the compact set support(S) + support(T), and suppose that φ is in $C^\infty_{\text{com}}(L^c)$. Let $d>0$ be the distance from support(φ) to L, and let D be the function giving the distance to a set. Define

$$L_S = \{x \mid D(x,\text{support}(S)\} \le \tfrac{1}{3}d$$

and

$$L_T = \{x \mid D(x,\text{support}(T)\} \le \tfrac{1}{3}d.$$

If x_S is in L_S and x_T is in L_T, then $|x_S - s| \le \frac{1}{3}d$ and $|x_T - t| \le \frac{1}{3}d$ for some s in support(S) and t in support(T). Thus $|(x_S+x_T)-(s+t)| \le \frac{2}{3}d$. Hence x_S+x_T is at distance $\le \frac{2}{3}d$ from L. Since every member of support(φ) is at distance $\ge d$ from L, x_S+x_T is not in support(φ). Therefore

$$(L_S+L_T)\cap\text{support}(\varphi)=\varnothing. \tag{$\dagger$}$$

Also, support$(S)\subseteq(L_S)^o$ and support$(T)\subseteq(L_T)^o$. Since L_S contains a neighborhood of support(S), Theorem 5.1 allows us to express S in terms of complex Borel measures ρ_α supported in L_S. Similarly we can express T in terms of complex Borel measures σ_β supported in L_T. By $(\dagger)$ the integrand in $(**)$ is identically 0 on L_S+L_T, and hence $\langle S,\langle T,\varphi_x\rangle\rangle = 0$. Thus $\langle S*T,\varphi\rangle = 0$ for all φ in $C^\infty_{\text{com}}(L^c)$, and we conclude that support$(S*T)\subseteq L=$ support$(S)+$support(T). This proves (g). $\square$

Proposition 5.7 establishes facts about the convolution of two members of $\mathcal{E}'(\mathbb{R}^N)$ as a member of $\mathcal{E}'(\mathbb{R}^N)$. If one of the two members is in fact a smooth function of compact support, then the corresponding results about convolution of measures suggest that the convolution should be a smooth function. The necessary tools for carrying out a proof are already in place in Proposition 5.7 and Theorem 5.1.

Corollary 5.8. If S is in $\mathcal{E}'(\mathbb{R}^N)$, f is in $C^\infty_{\text{com}}(\mathbb{R}^N)$, and φ is in $C^\infty(\mathbb{R}^N)$, then

$$\langle S * T_f, \varphi\rangle = \langle S, f^\vee * \varphi\rangle.$$

Moreover, $S * T_f$ is given by the C^∞ function $y \mapsto \langle S, (f^\vee)_{-y}\rangle$, i.e.,

$$S * T_f = T_F \qquad \text{with } F(y) = \langle S, (f^\vee)_{-y}\rangle.$$

REMARKS. For S in $\mathcal{E}'(\mathbb{R}^N)$ and f in $C^\infty_{\text{com}}(\mathbb{R}^N)$, we write $S * f$ for the $C^\infty_{\text{com}}(\mathbb{R}^N)$ function F of the corollary such that $S * T_f = T_F$. The specific formula that we shall use to simplify notation is

$$S * T_f = T_{S*f},$$

with the right side written as T_{S*f} rather than T_{S*T_f}.

PROOF. Proposition 5.7f gives

$$\begin{aligned}\langle S * T_f, \varphi\rangle &= \langle S, \langle T_f, \varphi_x\rangle\rangle = \big\langle S, \textstyle\int_{\mathbb{R}^N} f(y)\varphi(x+y)\,dy\big\rangle \\ &= \big\langle S, \textstyle\int_{\mathbb{R}^N} f(-y)\varphi(x-y)\,dy\big\rangle = \langle S, f^\vee * \varphi\rangle.\end{aligned} \tag{$*$}$$

This proves the first displayed formula. For the rest let S be written according to Theorem 5.1 as $\langle S, \psi\rangle = \sum_\alpha \int_{\mathbb{R}^N} D^\alpha\psi\,d\rho_\alpha$. Then

$$\begin{aligned}\langle S, f^\vee * \varphi\rangle &= \textstyle\sum_\alpha \int_{\mathbb{R}^N} D^\alpha(f^\vee * \varphi)(x)\,d\rho_\alpha(x) \\ &= \textstyle\sum_\alpha \int_{\mathbb{R}^N} (D^\alpha f^\vee * \varphi)(x)\,d\rho_\alpha(x) \\ &= \textstyle\sum_\alpha \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} D^\alpha f^\vee(x-y)\varphi(y)\,dy\,d\rho_\alpha(x) \\ &= \textstyle\int_{\mathbb{R}^N} \big[\sum_\alpha \int_{\mathbb{R}^N} (D^\alpha f^\vee)_{-y}\,d\rho_\alpha(x)\big]\,\varphi(y)\,dy \\ &= \textstyle\int_{\mathbb{R}^N} \langle S, (f^\vee)_{-y}\rangle\varphi(y)\,dy,\end{aligned}$$

the next-to-last equality following from Fubini's Theorem. Combining this calculation with $(*)$, we see that $S * T_f = T_F$ with $F(y) = \langle S, (f^\vee)_{-y}\rangle$. The function F is smooth by Proposition 5.7a. $\square$

Corollary 5.9. Convolution of members of $\mathcal{E}'(\mathbb{R}^N)$ is consistent with convolution of members of $C^\infty_{\text{com}}(\mathbb{R}^N)$ in the sense that if f and g are in $C^\infty_{\text{com}}(\mathbb{R}^N)$, then $T_g * T_f$ is given by the C^∞ function $T_g * f$, and this function equals $g * f$.

PROOF. The first conclusion is the result of Corollary 5.8 with $S = T_g$. For the second conclusion Corollary 5.8 gives $T_g * T_f = T_F$ with $F(y) = \langle T_g, (f^\vee)_{-y}\rangle = \int_{\mathbb{R}^N} g(x) f^\vee(x-y)\,dx = \int_{\mathbb{R}^N} g(x) f(y-x)\,dy = (g*f)(y)$. Hence $T_{T_g * f} = T_{g*f}$, and the second conclusion follows. $\square$

Corollary 5.10. If T is in $\mathcal{E}'(\mathbb{R}^N)$ and φ is in $C^\infty_{\text{com}}(\mathbb{R}^N)$, then

$$(T^\vee * \varphi)(x) = \langle T, \varphi_x\rangle.$$

PROOF. Corollary 5.8 gives $(T^\vee * \varphi)(x) = \langle T^\vee, (\varphi^\vee)_{-x}\rangle$, and the latter is equal to $\langle T, ((\varphi^\vee)_{-x})^\vee\rangle = \langle T, \varphi_x\rangle$. $\square$

Corollary 5.11. If S and T are in $\mathcal{E}'(\mathbb{R}^N)$ and φ is in $C^\infty_{\text{com}}(\mathbb{R}^N)$, then

$$\langle S * T, \varphi\rangle = \langle S, T^\vee * \varphi\rangle.$$

PROOF. Proposition 5.7f and Corollary 5.10 give $\langle S * T, \varphi\rangle = \langle S, \langle T, \varphi_x\rangle\rangle = \langle S, T^\vee * \varphi\rangle$. $\square$

Corollary 5.12. If T is in $\mathcal{E}'(\mathbb{R}^N)$, then the map $\varphi \mapsto T^\vee * \varphi$ is continuous from $C^\infty_{\text{com}}(\mathbb{R}^N)$ into itself and extends continuously to a map of $C^\infty(\mathbb{R}^N)$ into itself under the definition

$$(T^\vee * \varphi)(x) = \langle T, \varphi_x\rangle.$$

The derivatives of $T^\vee * \varphi$ satisfy $D^\alpha(T^\vee * \varphi) = T^\vee * D^\alpha \varphi$, and also $(T^\vee * \varphi)^\vee = T * \varphi^\vee$.

PROOF. The equality $(T^\vee * \varphi)(x) = \langle T, \varphi_x\rangle$ restates Corollary 5.10, and the statements about continuity follow from Proposition 5.7c. For the derivatives we use Proposition 5.7b to write $D^\alpha(T^\vee * \varphi)(x) = D^\alpha\langle T, \varphi_x\rangle = \langle T, (D^\alpha\varphi)_x\rangle = (T^\vee * D^\alpha\varphi)(x)$. Finally $(T^\vee * \varphi)^\vee(x) = (T^\vee * \varphi)(-x) = \langle T, \varphi_{-x}\rangle = \langle T^\vee, (\varphi_{-x})^\vee\rangle = \langle T^\vee, (\varphi^\vee)_x\rangle = (T * \varphi^\vee)(x)$. $\square$

Since $T^\vee * \varphi$ is now well defined for T in $\mathcal{E}'$ and φ in $C^\infty(\mathbb{R}^N)$, we can use the same formula as in Corollary 5.11 to make a definition of convolution of two arbitrary distributions when only one of the two distributions being convolved has compact support. Specifically if S is in $\mathcal{D}'(\mathbb{R}^N)$ and T is in $\mathcal{E}'(\mathbb{R}^N)$, we define $S * T$ in $\mathcal{D}'(\mathbb{R}^N)$ by the first equality of

$$\langle S * T, \varphi\rangle = \langle S, T^\vee * \varphi\rangle = \langle S, \langle T, \varphi_x\rangle\rangle \qquad \text{for } \varphi \in C^\infty_{\text{com}}(\mathbb{R}^N),$$

the second equality holding by Corollary 5.12. Corollary 5.12 shows also that $S*T$ has the necessary property of being continuous on $C^\infty_{\text{com}}(\mathbb{R}^N)$, and Corollary 5.11 shows that this definition extends the definition of $S * T$ when S and T are in $\mathcal{E}'(\mathbb{R}^N)$.

What is missing with this definition of $S * T$ is any additional relationship that arises for distributions that equal smooth functions. For example:

- Does this new definition make $T_f * T = T_{T*f}$ when T is compactly supported and f does not have compact support?
- Is $S * T_f$ equal to a function when f is compactly supported and S is not?
- If so, are the formulas of Corollaries 5.8, 5.9, and 5.10 valid?
- If so, can we equally well define $S * T$ by $\langle S * T, \varphi\rangle = \langle T, S^\vee * \varphi\rangle = \langle T, \langle S, \varphi_y\rangle\rangle$ when T is compactly supported and S is not?

The answers to these questions are all affirmative. To get at the proofs, we introduce a technique of localization for members of $\mathcal{D}'(\mathbb{R}^N)$. Proposition 5.13 below is a quantitative statement of what we need. We apply the technique to obtain smoothness of functions of the form $\langle S, \varphi_y\rangle$ when S is in $\mathcal{D}'(\mathbb{R}^N)$ and φ is in $C^\infty_{\text{com}}(\mathbb{R}^N)$; this step does not make use of the above enlarged definition of $S * T$. Then we gradually make the connection with the new definition of convolution and establish all the desired properties.

Proposition 5.13. Let N be a bounded open set in $\mathbb{R}^N$. Let S be in $\mathcal{D}'(\mathbb{R}^N)$, and let φ be in $C^\infty_{\text{com}}(\mathbb{R}^N)$. If $\eta \in C^\infty_{\text{com}}(\mathbb{R}^N)$ is identically 1 on the set of differences $\text{support}(\varphi) - N$, then $\langle S, \varphi_y\rangle = \langle \eta S, \varphi_y\rangle$ for y in N. Consequently $y \mapsto \langle S, \varphi_y\rangle$ is in $C^\infty(\mathbb{R}^N)$. Moreover, $D^\alpha(y \mapsto \langle S, \varphi_y\rangle) = \langle S, (D^\alpha\varphi)_y\rangle$, and the linear map $\varphi \mapsto \langle S, \varphi_y\rangle$ of $C^\infty_{\text{com}}(\mathbb{R}^N)$ into $C^\infty(\mathbb{R}^N)$ is continuous.

PROOF. Let y be in N. If $x + y$ is in $\text{support}(\varphi)$, then x is in $\text{support}(\varphi) - N$, and $\eta(x) = 1$. Hence $\eta(x)\varphi(x + y) = \varphi(x + y)$. If $x + y$ is not in $\text{support}(\varphi)$, then $\eta(x)\varphi(x+y) = \varphi(x+y)$ because both sides are 0. Hence $\eta\varphi_y = \varphi_y$ for y in N, and $\langle S, \varphi_y\rangle = \langle S, \eta\varphi_y\rangle = \langle \eta S, \varphi_y\rangle$. The function $y \mapsto \langle \eta S, \varphi_y\rangle$ is smooth by Proposition 5.7a, and hence $y \mapsto \langle S, \varphi_y\rangle$ is smooth on N. Since N is arbitrary, $y \mapsto \langle S, \varphi_y\rangle$ is smooth everywhere.

For the derivative formula Proposition 5.7b gives us $D^\alpha(y \mapsto \langle \eta S, \varphi_y\rangle) = \langle \eta S, (D^\alpha\varphi)_y\rangle$ for y in N. For y in N, $\langle \eta S, \varphi_y\rangle = \langle S, \varphi_y\rangle$ and $\langle \eta S, (D^\alpha\varphi)_y\rangle = \langle S, (D^\alpha\varphi)_y\rangle$. Therefore $D^\alpha(y \mapsto \langle S, \varphi_y\rangle) = \langle S, (D^\alpha\varphi)_y\rangle$ for y in N. Since N is arbitrary, $D^\alpha(y \mapsto \langle S, \varphi_y\rangle) = \langle S, (D^\alpha\varphi)_y\rangle$ everywhere.

For the asserted continuity of $\varphi \mapsto \langle S, \varphi_y\rangle$, it is enough to prove that this map carries C^∞_K continuously into $C^\infty(\mathbb{R}^N)$ for each compact set K. If N is a bounded open set on which we are to make some C^∞ estimates, choose $\eta \in C^\infty_{\text{com}}(\mathbb{R}^N)$ so as to be identically 1 on the set of differences $K - N$. We have just seen that $\langle S, \varphi_y\rangle = \langle \eta S, \varphi_y\rangle$ for all y in N. Proposition 5.7c shows that $\psi \mapsto \langle \eta S, \psi_y\rangle$ is continuous from $C^\infty_{\text{com}}(\mathbb{R}^N)$ into $C^\infty_{\text{com}}(\mathbb{R}^N)$, hence from C^∞_K into $C^\infty_{\text{com}}(\mathbb{R}^N)$, hence from C^∞_K into $C^\infty(\mathbb{R}^N)$. Therefore $\varphi \mapsto \langle S, \varphi_y\rangle$ is continuous from C^∞_K into $C^\infty(\mathbb{R}^N)$. $\square$

Corollary 5.14. Let S be in $\mathcal{D}'(\mathbb{R}^N)$, T be in $\mathcal{E}'(\mathbb{R}^N)$, and φ be in $C^\infty_{\text{com}}(\mathbb{R}^N)$. Then

$$\langle S * T, \varphi\rangle = \langle S, T^\vee * \varphi\rangle = \langle S, \langle T, \varphi_x\rangle\rangle = \langle T, \langle S, \varphi_y\rangle\rangle.$$

Moreover, $D^\alpha(S * T) = (D^\alpha S) * T = S * (D^\alpha T)$ for every multi-index α.

REMARKS. The first two equalities follow by definition of $S * T$ and by application of Corollary 5.12. The new statements in the corollary are the third equality and the derivative formula. The right side $\langle T, \langle S, \varphi_y\rangle\rangle$ of the displayed equation is well defined, since Proposition 5.13 shows that $\langle S, \varphi_y\rangle$ is in $C^\infty(\mathbb{R}^N)$.

PROOF. Let N be a bounded open set containing support(T), and choose a function $\eta \in C^\infty_{\text{com}}(\mathbb{R}^N)$ that is identically 1 on the set of differences support$(\varphi) - N$. Proposition 5.7g shows that

$$\begin{aligned} \text{support}(T^\vee * \varphi) &\subseteq \text{support}(\varphi) + \text{support}(T^\vee) \\ &= \text{support}(\varphi) - \text{support}(T) \\ &\subseteq \text{support}(\varphi) - N, \end{aligned}$$

and the fact that η is identically 1 on support$(\varphi) - N$ implies that

$$(\eta)(T^\vee * \varphi) = T^\vee * \varphi. \tag{$*$}$$

Meanwhile, Proposition 5.13 shows that

$$\langle S, \varphi_y\rangle = \langle \eta S, \varphi_y\rangle \tag{$**$}$$

for all y in N, hence for all y in support(T). Therefore

$$\begin{aligned} \langle T, \langle S, \varphi_y\rangle\rangle &= \langle T, \langle \eta S, \varphi_y\rangle\rangle && \text{by } (**) \\ &= \langle T, (\eta S)^\vee * \varphi\rangle && \text{by Corollary 5.10} \\ &= \langle \eta S * T, \varphi\rangle && \text{by Corollary 5.11} \\ &= \langle \eta S, T^\vee * \varphi\rangle && \text{by Corollary 5.10} \\ &= \langle S, \eta(T^\vee * \varphi)\rangle && \text{by definition} \\ &= \langle S, T^\vee * \varphi\rangle && \text{by } (*). \end{aligned} \tag{$\dagger$}$$

For one of the derivative formulas, we have

$$\langle D^\alpha(S * T), \varphi\rangle = (-1)^{|\alpha|}\langle S * T, D^\alpha\varphi\rangle = (-1)^{|\alpha|}\langle S, \langle T, (D^\alpha\varphi)_x\rangle\rangle.$$

Proposition 5.7b shows that this expression is equal to

$$(-1)^{|\alpha|}\langle S, D^\alpha\langle T, \varphi_x\rangle\rangle = \langle D^\alpha S, \langle T, \varphi_x\rangle\rangle,$$

and the definition of convolution shows that the latter expression is equal to $\langle (D^\alpha S)*T, \varphi\rangle$. Hence $D^\alpha(S*T) = (D^\alpha S)*T$. For the other derivative formula we have

$$\langle D^\alpha(S*T), \varphi\rangle = (-1)^{|\alpha|}\langle S*T, D^\alpha\varphi\rangle = (-1)^{|\alpha|}\langle T, \langle S, (D^\alpha\varphi)_y\rangle\rangle.$$

Proposition 5.13 shows that this expression is equal to

$$(-1)^{|\alpha|}\langle T, D^\alpha\langle S, \varphi_y\rangle\rangle = \langle D^\alpha T, \langle S, \varphi_y\rangle\rangle,$$

and step (†) shows that the latter expression is equal to

$$\langle S, (D^\alpha T)^\vee * \varphi\rangle = \langle S*(D^\alpha T), \varphi\rangle.$$

Hence $D^\alpha(S*T) = S*(D^\alpha T)$. □

For S in $\mathcal{D}'(\mathbb{R}^N)$ and φ in $C^\infty_{\text{com}}(\mathbb{R}^N)$, we now define

$$(S^\vee * \varphi)(y) = \langle S, \varphi_y\rangle.$$

Corollary 5.8 shows that this definition is consistent with our earlier definition when S is in the subset $\mathcal{E}'(\mathbb{R}^N)$ of $\mathcal{D}'(\mathbb{R}^N)$. Proposition 5.13 shows that the linear map $\varphi \mapsto S*\varphi$ is continuous from $C^\infty_{\text{com}}(\mathbb{R}^N)$ into $C^\infty(\mathbb{R}^N)$.

Corollary 5.15. Let S be in $\mathcal{D}'(\mathbb{R}^N)$, T be in $\mathcal{E}'(\mathbb{R}^N)$, and φ be in $C^\infty_{\text{com}}(\mathbb{R}^N)$. Then

$$\langle S*T, \varphi\rangle = \langle S, T^\vee * \varphi\rangle = \langle S, \langle T, \varphi_x\rangle\rangle = \langle T, \langle S, \varphi_y\rangle\rangle = \langle T, S^\vee * \varphi\rangle,$$

and $(S*T)^\vee = S^\vee * T^\vee$.

PROOF. The displayed line just adds the above definition to the conclusion of Corollary 5.14. For the other formula we use Corollary 5.12 to write $\langle (S*T)^\vee, \varphi\rangle = \langle S*T, \varphi^\vee\rangle = \langle S, T^\vee * \varphi^\vee\rangle = \langle S, (T*\varphi)^\vee\rangle = \langle S^\vee, T*\varphi\rangle = \langle S^\vee * T^\vee, \varphi\rangle$. □

With the symmetry that has been established in Corollary 5.15, we allow ourselves to write $T*S$ for $S*T$ when S is in $\mathcal{D}'(\mathbb{R}^N)$ and T is in $\mathcal{E}'(\mathbb{R}^N)$. This notation is consistent with the equality $S*T = T*S$ established in Proposition 5.7f when S and T both have compact support.

Corollary 5.16. Suppose that S is in $\mathcal{D}'(\mathbb{R}^N)$, that f is in $C^\infty(\mathbb{R}^N)$, and that at least one of S and f has compact support. If φ is in $C^\infty_{\text{com}}(\mathbb{R}^N)$, then

$$\langle S * T_f, \varphi \rangle = \langle S, f^\vee * \varphi \rangle.$$

Moreover, $S * T_f$ is given by the C^∞ function $y \mapsto \langle S, (f^\vee)_{-y} \rangle$, i.e.,

$$S * T_f = T_F \qquad \text{with } F(y) = \langle S, (f^\vee)_{-y} \rangle.$$

REMARK. If *both* S and f have compact support, Corollary 5.16 reduces to Corollary 5.8.

PROOF. First suppose that S has compact support. Theorem 5.1 allows us to write S as $\langle S, \psi \rangle = \sum_\alpha \int_{\mathbb{R}^N} D^\alpha \psi \, d\rho_\alpha$, with the sum involving only finitely many terms and with the complex Borel measures ρ_α compactly supported. Applying Corollary 5.15 to $S * T_f$ and using the definition of $S^\vee * \varphi$, we obtain

$$\begin{aligned}
\langle S * T_f, \varphi \rangle &= \textstyle\int_{\mathbb{R}^N} f(y)(S^\vee * \varphi)(y)\, dy \\
&= \textstyle\int_{\mathbb{R}^N} f(y) \sum_\alpha \int_{\mathbb{R}^N} D^\alpha \varphi_y(x)\, d\rho_\alpha(x)\, dy \\
&= \textstyle\int_{\mathbb{R}^N} \sum_\alpha \int_{\mathbb{R}^N} f(y) D^\alpha \varphi(x+y)\, d\rho_\alpha(x)\, dy.
\end{aligned}$$

Since φ and the ρ_α's are compactly supported, we may freely interchange the order of integration to see that the above expression is equal to

$$\begin{aligned}
&\textstyle\sum_\alpha \int_{\mathbb{R}^N} \big[\int_{\mathbb{R}^N} f(y) D^\alpha \varphi(x+y)\, dy \big]\, d\rho_\alpha(x) \\
&\quad= \textstyle\sum_\alpha \int_{\mathbb{R}^N} (f^\vee * D^\alpha \varphi)(x)\, d\rho_\alpha(x) \\
&\quad= \textstyle\sum_\alpha \int_{\mathbb{R}^N} (D^\alpha (f^\vee) * \varphi)(x)\, d\rho_\alpha(x) \\
&\quad= \textstyle\sum_\alpha \int_{\mathbb{R}^N} \big[\int_{\mathbb{R}^N} D^\alpha (f^\vee)(x-y) \varphi(y)\, dy \big]\, d\rho_\alpha(x) \\
&\quad= \textstyle\int_{\mathbb{R}^N} \big[\sum_\alpha \int_{\mathbb{R}^N} D^\alpha (f^\vee)(x-y)\, d\rho_\alpha(x) \big] \varphi(y)\, dy \\
&\quad= \textstyle\int_{\mathbb{R}^N} \langle S, (f^\vee)_{-y} \rangle \varphi(y)\, dy \\
&\quad= \langle T_F, \varphi \rangle,
\end{aligned}$$

as asserted.

Next suppose instead that f has compact support. Then

$$\langle S * T_f, \varphi \rangle = \langle S, (T_f)^\vee * \varphi \rangle = \langle S, T_{f^\vee} * \varphi \rangle = \langle S, f^\vee * \varphi \rangle. \tag{$*$}$$

We are to show that this expression is equal to

$$\langle T_F, \varphi \rangle = \langle T_{\langle S, (f^\vee)_{-y} \rangle}, \varphi \rangle = \textstyle\int_{\mathbb{R}^N} \langle S, (f^\vee)_{-y} \rangle \varphi(y)\, dy. \tag{$**$}$$

We introduce a member η of $C^\infty_{\text{com}}(\mathbb{R}^N)$ that is identically 1 on the set of sums $\text{support}(f^\vee) + \text{support}(\varphi)$. Since ηS is in $\mathcal{E}'(\mathbb{R}^N)$, Corollary 5.8 shows that

$$\langle \eta S, f^\vee * \varphi\rangle = \int_{\mathbb{R}^N} \langle \eta S, (f^\vee)_{-y}\rangle \varphi(y)\,dy = \int_{\mathbb{R}^N} \langle S, \eta (f^\vee)_{-y}\rangle \varphi(y)\,dy.$$

In view of $(*)$ and $(**)$, it is therefore enough to prove the two identities

$$\langle \eta S, f^\vee * \varphi\rangle = \langle S, f^\vee * \varphi\rangle \tag{†}$$

and

$$\int_{\mathbb{R}^N} \langle S, \eta (f^\vee)_{-y}\rangle \varphi(y)\,dy = \int_{\mathbb{R}^N} \langle S, (f^\vee)_{-y}\rangle \varphi(y)\,dy. \tag{††}$$

Since $\text{support}(f^\vee * \varphi) \subseteq \text{support}(f^\vee) + \text{support}(\varphi)$, we have $\eta(f^\vee * \varphi) = f^\vee * \varphi$ and therefore $\langle \eta S, f^\vee * \varphi\rangle = \langle S, \eta(f^\vee * \varphi)\rangle = \langle S, f^\vee * \varphi\rangle$. This proves (†).

To prove (††), it is enough to show that $\eta(f^\vee)_{-y} = (f^\vee)_{-y}$ for every y in $\text{support}(\varphi)$. For a given y in $\text{support}(\varphi)$, there is nothing to prove at points x where $(f^\vee)_{-y}(x) = 0$. If $(f^\vee)_{-y}(x) \neq 0$, then $f^\vee(x - y) \neq 0$ and $x - y$ is in $\text{support}(f^\vee)$. Hence $x = y + (x - y)$ is in $\text{support}(\varphi) + \text{support}(f^\vee)$, and $\eta(x)(f^\vee)_{-y}(x) = (f^\vee)_{-y}(x)$. This proves (††). $\square$

Corollary 5.17. Convolution of two distributions, one of which has compact support, is consistent with convolution of smooth functions, one of which has compact support, in the sense that if f and g are smooth and one of them has compact support, then $T_g * T_f$ is given by the C^∞ function $T_g * f$ and by the C^∞ function $T_f * g$, and these functions equal $g * f$.

PROOF. We apply Corollary 5.16 with $S = T_g$, and we find that $T_g * T_f$ is given by the smooth function that carries y to $\langle T_g, (f^\vee)_{-y}\rangle$. In turn, this latter expression equals $\int_{\mathbb{R}^N} g(x)(f^\vee)_{-y}(x)\,dx = \int_{\mathbb{R}^N} g(x) f^\vee(x - y)\,dx = \int_{\mathbb{R}^N} g(x) f(y - x)\,dx = (g * f)(y)$. Hence $T_g * f = g * f$. Reversing the roles of f and g, we obtain $T_f * g = f * g = g * f$. $\square$

Corollary 5.18. If R, S, and T are distributions and ψ and φ are smooth functions, then

(a) $(T * \psi) * \varphi = T * (\psi * \varphi)$ provided at least two of T, ψ, and φ have compact support,
(b) $(S * T) * \varphi = (S * \varphi) * T$ provided at least two of S, T, and φ have compact support,
(c) $R * (S * T) = (R * S) * T$ provided at least two of R, S, and T have compact support.

PROOF. Let η be in $C^\infty_{\text{com}}(\mathbb{R}^N)$. We make repeated use of Corollaries 5.15 through 5.17 in each part. For (a), we use associativity of convolution of smooth functions (Proposition 5.5b) to write

$$\begin{aligned}\langle T * T_{\psi*\varphi}, \eta\rangle &= \langle T, (\psi * \varphi)^\vee * \eta\rangle = \langle T, (\psi^\vee * \varphi^\vee) * \eta\rangle \\ &= \langle T, \psi^\vee * (\varphi^\vee * \eta)\rangle = \langle T * T_\psi, \varphi^\vee * \eta\rangle \\ &= \langle (T * T_\psi) * T_\varphi, \eta\rangle.\end{aligned}$$

Thus $T * T_{\psi*\varphi} = (T * T_\psi) * T_\varphi$. Since $T * T_{\psi*\varphi} = T_{T*(\psi*\varphi)}$ and $(T * T_\psi) * T_\varphi = T_{T*\psi} * T_\varphi = T_{(T*\psi)*\varphi}$, we obtain $T * (\psi * \varphi) = (T * \psi) * \varphi$. This proves (a).

For (b), we use (a) to write

$$\begin{aligned}\langle (S * T) * T_\varphi, \eta\rangle &= \langle S * T, \varphi^\vee * \eta\rangle = \langle S, T^\vee * (\varphi^\vee * \eta)\rangle \\ &= \langle S, (T^\vee * \varphi^\vee) * \eta\rangle = \langle S, (T * \varphi)^\vee * \eta\rangle \\ &= \langle S, (T * T_\varphi)^\vee * \eta\rangle = \langle S * (T * T_\varphi), \eta\rangle.\end{aligned}$$

Thus $(S*T)*T_\varphi = S*(T*T_\varphi)$. Since $(S*T)*T_\varphi = T_{(S*T)*\varphi}$ and $S*(T*T_\varphi) = S * T_{T*\varphi} = T_{S*(T*\varphi)}$, we obtain $(S * T) * \varphi = S * (T * \varphi)$.

For (c), we use (b) to write

$$\begin{aligned}\langle R * (S * T), \eta\rangle &= \langle R, (S * T)^\vee * \eta\rangle = \langle R, (S^\vee * T^\vee) * \eta\rangle \\ &= \langle R, S^\vee * (T^\vee * \eta)\rangle = \langle R * S, T^\vee * \eta\rangle \\ &= \langle (R * S) * T, \eta\rangle.\end{aligned}$$

Thus $R * (S * T) = (R * S) * T$, and (c) is proved. $\square$

We conclude with a special property of one particular distribution. The **Dirac distribution** at the origin is the member of $\mathcal{E}'(\mathbb{R}^N)$ given by $\langle \delta, \varphi\rangle = \varphi(0)$. It has support $\{0\}$. The proposition below shows that the differentiation operation D^α on distributions equals convolution with the distribution $D^\alpha\delta$.

Proposition 5.19. If T is in $\mathcal{D}'(\mathbb{R}^N)$ and if δ denotes the Dirac distribution at the origin, then $\delta * T = T$. Consequently $D^\alpha\delta * T = D^\alpha T$ for every multi-index α.

PROOF. For φ in $C^\infty_{\text{com}}(\mathbb{R}^N)$, Corollary 5.14 gives $\langle \delta * T, \varphi\rangle = \langle \delta, \langle T, \varphi_x\rangle\rangle = \langle T, \varphi\rangle$, and therefore $\delta * T = T$. Applying D^α and using the second conclusion of Corollary 5.14, we obtain $D^\alpha(\delta * T) = \delta * (D^\alpha T) = D^\alpha T$. $\square$

4. Role of Fourier Transform

The final tool we need in order to make the theory of distributions useful for linear partial differential equations is the Fourier transform. Let us write $\mathcal{F}$ for the Fourier transform on the various places it acts, its initial definition being $\mathcal{F}(f)(\xi) = \int_{\mathbb{R}^N} f(x)e^{-2\pi i x\cdot\xi}\,dx$ on $L^1(\mathbb{R}^N)$. Since the Schwartz space $\mathcal{S}(\mathbb{R}^N)$ is contained in $L^1(\mathbb{R}^N)$, this definition of $\mathcal{F}$ is applicable on $\mathcal{S}(\mathbb{R}^N)$, and it was shown in *Basic* that $\mathcal{F}$ is one-one from $\mathcal{S}(\mathbb{R}^N)$ onto itself. We continue to use the same angular-brackets notation for $\mathcal{S}'(\mathbb{R}^N)$ as for $\mathcal{D}'(\mathbb{R}^N)$ and $\mathcal{E}'(\mathbb{R}^N)$. Then, as a consequence of Corollary 3.3b, the Fourier transform is well defined on elements T of $\mathcal{S}'(\mathbb{R}^N)$ under the definition $\langle \mathcal{F}(T), \varphi\rangle = \langle T, \mathcal{F}(\varphi)\rangle$ for $\varphi \in \mathcal{S}(\mathbb{R}^N)$, and Proposition 3.4 shows that $\mathcal{F}$ is one-one from $\mathcal{S}'(\mathbb{R}^N)$ onto itself. On tempered distributions that are L^1 or L^2 functions, $\mathcal{F}$ agrees with the usual definitions on functions. For f in L^1, the verification comes down to the multiplication formula:

$$\langle \mathcal{F}T_f, \varphi\rangle = \langle T_f, \mathcal{F}\varphi\rangle = \int f(x)(\mathcal{F}\varphi)(x)\,dx = \int (\mathcal{F}f)(x)\varphi(x)\,dx = \langle T_{\mathcal{F}f}, \varphi\rangle.$$

For f in L^2, we choose a sequence $\{f_n\}$ in $L^1 \cap L^2$ tending to f in L^2, obtain $\langle \mathcal{F}T_{f_n}, \varphi\rangle = \langle T_{\mathcal{F}f_n}, \varphi\rangle$ for each n, and then check by continuity that we can pass to the limit.

The formulas that are used to establish the effect of $\mathcal{F}$ on $\mathcal{S}(\mathbb{R}^N)$ come from the behavior of differentiation and multiplication by polynomials on Fourier transforms and are

$$D^\alpha(\mathcal{F}f)(x) = \mathcal{F}((-2\pi i)^{|\alpha|}x^\alpha f)(x)$$

and

$$x^\beta(\mathcal{F}f)(x) = \mathcal{F}((2\pi i)^{-|\beta|}D^\beta f)(x).$$

Let us define the effect of D^α and multiplication by x^β on tempered distributions and then see how the Fourier transform interacts with these operations. If φ is in $\mathcal{S}(\mathbb{R}^N)$, then $D^\alpha\varphi$ is in $\mathcal{S}(\mathbb{R}^N)$, and hence it makes sense to define $D^\alpha T$ for $T \in \mathcal{S}'(\mathbb{R}^N)$ by $\langle D^\alpha T, \varphi\rangle = (-1)^\alpha\langle T, D^\alpha\varphi\rangle$. The product of an arbitrary smooth function on $\mathbb{R}^N$ by a Schwartz function need not be a Schwartz function, and thus the product of an arbitrary smooth function and a tempered distribution need not make sense as a tempered distribution. However, the product of a polynomial and a Schwartz function is a Schwartz function, and thus we can define $x^\beta T$ for $T \in \mathcal{S}'(\mathbb{R}^N)$ by $\langle x^\beta T, \varphi\rangle = \langle T, x^\beta\varphi\rangle$. The formulas for the Fourier transform are then

$$\mathcal{F}(D^\alpha T) = (2\pi i)^{|\alpha|}x^\alpha\mathcal{F}(T)$$

and

$$\mathcal{F}(x^\beta T) = (-2\pi i)^{-|\beta|}D^\beta\mathcal{F}(T).$$

In fact, we compute that $\langle \mathcal{F}(D^\alpha T), \varphi\rangle = \langle D^\alpha T, \mathcal{F}\varphi\rangle = (-1)^{|\alpha|}\langle T, D^\alpha \mathcal{F}\varphi\rangle = (-1)^{|\alpha|}\langle T, \mathcal{F}((-2\pi i)^{|\alpha|}x^\alpha\varphi)\rangle = (2\pi i)^{|\alpha|}\langle \mathcal{F}(T), x^\alpha\varphi\rangle = (2\pi i)^{|\alpha|}\langle x^\alpha\mathcal{F}(T), \varphi\rangle$ and that $\langle \mathcal{F}(x^\beta T), \varphi\rangle = \langle x^\beta T, \mathcal{F}\varphi\rangle = \langle T, x^\beta \mathcal{F}\varphi\rangle = \langle T, \mathcal{F}((2\pi i)^{-|\beta|}D^\beta\varphi)\rangle = (2\pi i)^{-|\beta|}\langle \mathcal{F}(T), D^\beta\varphi\rangle = (-2\pi i)^{-|\beta|}\langle D^\beta\mathcal{F}(T), \varphi\rangle$.

We have seen that the restriction map carries $\mathcal{E}'(\mathbb{R}^N)$ in one-one fashion into $\mathcal{S}'(\mathbb{R}^N)$. Therefore we can identify members of $\mathcal{E}'(\mathbb{R}^N)$ with certain members of $\mathcal{S}'(\mathbb{R}^N)$ when it is convenient to do so, and in particular the Fourier transform becomes a well-defined one-one map of $\mathcal{E}'(\mathbb{R}^N)$ into $\mathcal{S}'(\mathbb{R}^N)$. (The Fourier transform is not usable, however, with $\mathcal{D}'(\mathbb{R}^N)$.) The somewhat surprising fact is that the Fourier transform of a member of $\mathcal{E}'(\mathbb{R}^N)$ is actually a smooth function, not just a distribution. We shall prove this fact as a consequence of Theorem 5.1, which has expressed distributions of compact support in terms of complex measures of compact support.

Theorem 5.20. If T is a member of $\mathcal{E}'(\mathbb{R}^N)$ with support in a compact subset K of $\mathbb{R}^N$, then the tempered distribution $\mathcal{F}(T)$ equals a smooth function that extends to an entire holomorphic function on $\mathbb{C}^N$. The value of this function at $z \in \mathbb{C}^N$ is given by

$$\mathcal{F}(T)(z) = \langle T, e^{-2\pi i z\cdot(\,\cdot\,)}\rangle,$$

and there is a positive integer m such that this function satisfies

$$|D^\beta(\mathcal{F}T)(\xi)| \le C_\beta(1+|\xi|)^m$$

for $\xi \in \mathbb{R}^N$ and for every multi-index β.

REMARK. The estimate shows that the product of $\langle T, e^{-2\pi i z\cdot(\,\cdot\,)}\rangle$ by a Schwartz function is again a Schwartz function, hence that the tempered distribution $\mathcal{F}(T)$ is indeed given by a certain smooth function.

PROOF. Fix a compact set K' whose interior contains K. Theorem 5.1 allows us to write

$$\langle T, \varphi_0\rangle = \textstyle\sum_{|\alpha|\le m}\int_{K'} D^\alpha\varphi_0\, d\rho_\alpha$$

for all $\varphi_0 \in C^\infty(\mathbb{R}^N)$. Replacing φ_0 by $e^{-2\pi i z\cdot(\,\cdot\,)}$ gives

$$\langle T, e^{-2\pi i z\cdot(\,\cdot\,)}\rangle = \textstyle\sum_{|\alpha|\le m}\int_{K'} D_\xi^\alpha e^{-2\pi i z\cdot\xi}\, d\rho_\alpha(\xi),$$

which shows that $z \mapsto \langle T, e^{-2\pi i z\cdot(\,\cdot\,)}\rangle$ is holomorphic in $\mathbb{C}^N$ and gives the estimate

$$|D_x^\beta\langle T, e^{-2\pi i x\cdot(\,\cdot\,)}\rangle| \le \textstyle\sum_{|\alpha|\le m}\int_{\xi\in K'} |D_x^\beta D_\xi^\alpha e^{-2\pi i x\cdot\xi}|\; d|\rho_\alpha|(\xi) \le C_\beta(1+|x|)^m.$$

Replacing φ_0 by $\mathcal{F}\varphi$ with φ in $C_{\text{com}}^\infty(\mathbb{R}^N)$ gives

$$\begin{aligned}\langle \mathcal{F}(T), \varphi\rangle = \langle T, \mathcal{F}\varphi\rangle &= \textstyle\sum_{|\alpha|\le m}\int_{\xi\in K'} D_\xi^\alpha \mathcal{F}\varphi(\xi)\, d\rho_\alpha(\xi)\\ &= \textstyle\sum_{|\alpha|\le m}\int_{\xi\in K'} D_\xi^\alpha \int_{x\in\mathbb{R}^N} e^{-2\pi i x\cdot\xi}\varphi(x)\,dx\, d\rho_\alpha(\xi)\end{aligned}$$

$$= \sum_{|\alpha|\le m} \int_{\xi\in K'} \int_{x\in\mathbb{R}^N} D_\xi^\alpha e^{-2\pi i x\cdot\xi}\varphi(x)\,dx\,d\rho_\alpha(\xi)$$
$$= \int_{x\in\mathbb{R}^N} \Big(\sum_{|\alpha|\le m} \int_{\xi\in K'} D_\xi^\alpha e^{-2\pi i x\cdot\xi}\,d\rho_\alpha(\xi)\Big)\varphi(x)\,dx$$
$$= \int_{x\in\mathbb{R}^N} \langle T, e^{-2\pi i x\cdot(\,\cdot\,)}\rangle\,\varphi(x)\,dx.$$

Both sides are continuous functions of the Schwartz-space variable φ on the dense subset $C^\infty_{\text{com}}(\mathbb{R}^N)$, and hence the formula extends to be valid for φ in $\mathcal{S}(\mathbb{R}^N)$. This proves that $\mathcal{F}(T)$ is given on $\mathcal{S}(\mathbb{R}^N)$ by the function $x \mapsto \langle T, e^{-2\pi i x\cdot(\,\cdot\,)}\rangle$. The estimate on D_x^β of this function has been obtained above, and the theorem follows. $\square$

EXAMPLE. There is an important instance of the formula of the proposition that can be established directly without appealing to the proposition. The Dirac distribution δ at the origin, defined by $\langle\delta,\varphi\rangle = \varphi(0)$, has Fourier transform $\mathcal{F}(\delta)$ equal to the constant function 1 because $\langle\mathcal{F}(\delta),\varphi\rangle = \langle\delta,\mathcal{F}(\varphi)\rangle = \mathcal{F}(\varphi)(0) = \int_{\mathbb{R}^N}\varphi\,dx = \langle T_1,\varphi\rangle$, where T_1 denotes the distribution equal to the smooth function 1. Therefore $\mathcal{F}(D^\alpha\delta) = (2\pi i)^{|\alpha|}x^\alpha T_1$, i.e., $\mathcal{F}(D^\alpha\delta)$ equals the function $x \mapsto (2\pi i)^{|\alpha|}x^\alpha$. The formula of the proposition when $T = D^\alpha\delta$ says that this function equals $(D^\alpha\delta)(e^{-2\pi i x\cdot(\,\cdot\,)})$, and we can see this equality directly because $\langle D^\alpha\delta, e^{-2\pi i x\cdot(\,\cdot\,)}\rangle = (-1)^{|\alpha|}\langle\delta, D^\alpha e^{-2\pi i x\cdot(\,\cdot\,)}\rangle = (-1)^{|\alpha|}(-2\pi i)^{|\alpha|}x^\alpha\langle\delta, e^{-2\pi i x\cdot(\,\cdot\,)}\rangle$ $= (2\pi i)^{|\alpha|}x^\alpha$.

We know that the convolution of two distributions is meaningful if one of them has compact support. Since the (pointwise) product of two general tempered distributions is undefined, we might not at first expect that the Fourier transform could be helpful with understanding this kind of convolution. However, Theorem 5.20 says that there is reason for optimism: the product of the Fourier transform of a distribution of compact support by a tempered distribution is indeed defined. This is the clue that suggests the second theorem of this section.

Theorem 5.21. If S is in $\mathcal{E}'(\mathbb{R}^N)$ and T is in $\mathcal{S}'(\mathbb{R}^N)$, then $S*T$ is in $\mathcal{S}'(\mathbb{R}^N)$, and $\mathcal{F}(S*T) = \mathcal{F}(S)\mathcal{F}(T)$.

PROOF. We know that $S*T$ is in $\mathcal{D}'(\mathbb{R}^N)$, and we shall check that $S*T$ is actually in $\mathcal{S}'(\mathbb{R}^N)$, so that $\mathcal{F}(S*T)$ is defined: We start with φ in $C^\infty_{\text{com}}(\mathbb{R}^N)$ and the identity $\langle S*T,\varphi\rangle = \langle S, T^\vee*\varphi\rangle = \langle S^\vee, T*\varphi^\vee\rangle$. Since S has compact support, there is a compact set K and there are constants C and m such that

$$|\langle S*T,\varphi\rangle \le C\sum_{|\alpha|\le m}\sup_{x\in K}|D^\alpha(T*\varphi^\vee)(x)| = C\sum_{|\alpha|\le m}\sup_{x\in K}|T*D^\alpha(\varphi^\vee)(x)|$$
$$= C\sum_{|\alpha|\le m}\sup_{x\in K}|\langle T, ((D^\alpha(\varphi^\vee))^\vee)_x\rangle| = C\sum_{|\alpha|\le m}\sup_{x\in K}|\langle T, (D^\alpha\varphi)_x\rangle|.$$

Since T is tempered, there exist constants C', m', and k such that the right side is

$$\leq CC' \sum_{\substack{|\alpha|\leq m,\\ |\beta|\leq m'}} \sup_{\substack{x\in K,\\ y\in\mathbb{R}^N}} \big|(1+|y|^2)^k D^\beta (D^\alpha \varphi)_x(y)\big|;$$

in turn, this expression is estimated by Schwartz-space norms for φ, and thus $S * T$ is in $\mathcal{S}'(\mathbb{R}^N)$.

Now let φ and ψ be Schwartz functions with φ and $\mathcal{F}(\psi)$ in $C^\infty_{\text{com}}(\mathbb{R}^N)$. Then

$$\begin{aligned}\langle \mathcal{F}(T_\varphi * T), \psi\rangle &= \langle T_\varphi * T, \mathcal{F}(\psi)\rangle = \langle T, \varphi^\vee * \mathcal{F}(\psi)\rangle\\ &= \langle \mathcal{F}(T), \mathcal{F}^{-1}(\varphi^\vee * \mathcal{F}(\psi))\rangle = \langle \mathcal{F}(T), (\mathcal{F}^{-1}(\varphi^\vee))\mathcal{F}^{-1}(\mathcal{F}(\psi))\rangle\\ &= \langle \mathcal{F}(T), \mathcal{F}^{-1}(\varphi^\vee)\psi\rangle = \langle \mathcal{F}(T), (\mathcal{F}(\varphi))\psi\rangle = \langle \mathcal{F}(\varphi)\mathcal{F}(T), \psi\rangle,\end{aligned}$$

the next-to-last equality following since $\mathcal{F}^{-1}(\varphi^\vee) = \mathcal{F}(\varphi)$ by the Fourier inversion formula. Since the ψ's with $\mathcal{F}(\psi)$ in $C^\infty_{\text{com}}(\mathbb{R}^N)$ are dense in $\mathcal{S}(\mathbb{R}^N)$,

$$\mathcal{F}(T_\varphi * T) = \mathcal{F}(\varphi)\mathcal{F}(T). \tag{$*$}$$

Finally let φ and ψ be in $C^\infty_{\text{com}}(\mathbb{R}^N)$. Corollary 5.18 gives $T_\varphi * (S * T) = (T_\varphi * S) * T$. Taking the Fourier transform of both sides and applying $(*)$ three times, we obtain

$$\begin{aligned}\mathcal{F}(\varphi)\mathcal{F}(S * T) &= \mathcal{F}(T_\varphi * (S * T)) = \mathcal{F}((T_\varphi * S) * T)\\ &= \mathcal{F}(T_\varphi * S)\mathcal{F}(T) = \mathcal{F}(\varphi)\mathcal{F}(S)\mathcal{F}(T).\end{aligned}$$

Hence we have $\langle \mathcal{F}(\varphi)\mathcal{F}(S * T), \psi\rangle = \langle \mathcal{F}(\varphi)\mathcal{F}(S)\mathcal{F}(T), \psi\rangle$ and therefore

$$\langle \mathcal{F}(S * T), \mathcal{F}(\varphi)\psi\rangle = \langle \mathcal{F}(S)\mathcal{F}(T), \mathcal{F}(\varphi)\psi\rangle \qquad \text{for all } \varphi \in C^\infty_{\text{com}}(\mathbb{R}^N).$$

The set of functions $\mathcal{F}(\varphi)$ is dense in $\mathcal{S}(\mathbb{R}^N)$. Moreover, if $\eta_k \to \eta$ in $\mathcal{S}(\mathbb{R}^N)$, then $\eta_k\psi \to \eta\psi$ in $\mathcal{S}(\mathbb{R}^N)$. Choosing a sequence of φ's for which $\mathcal{F}(\varphi)$ tends in $\mathcal{S}(\mathbb{R}^N)$ to a function in $C^\infty_{\text{com}}(\mathbb{R}^N)$ that is 1 on the support of ψ, we obtain

$$\langle \mathcal{F}(S * T), \psi\rangle = \langle \mathcal{F}(S)\mathcal{F}(T), \psi\rangle.$$

Since the set of ψ's is dense in $\mathcal{S}(\mathbb{R}^N)$, we conclude that $\mathcal{F}(S * T) = \mathcal{F}(S)\mathcal{F}(T)$. □

5. Fundamental Solution of Laplacian

The availability of distributions makes it possible to write familiar partial differential equations in a general but convenient notation. For example consider the equation $\Delta u = f$ in $\mathbb{R}^N$, where Δ is the Laplacian. We regard f as known and u as unknown. Ordinarily we might think of f as some function, possibly with some smoothness properties, and we are seeking a solution u that is another function. However, we can regard any locally integrable function f as a distribution T_f and seek a distribution T with $\Delta T = T_f$. In this sense the equation $\Delta u = f$ in the sense of distributions includes the equation in the ordinary sense of functions.

In this section we shall solve this equation when the distribution on the right side has compact support. To handle existence, the technique is to exhibit a **fundamental solution** for the Laplacian, i.e., a solution of the equation $\Delta T = \delta$, where δ is the Dirac distribution at 0, and then to use the rules of Sections 2–3 for manipulating distributions.[7] The argument for this special case will avoid using the full power of Theorem 5.21, but a generalization to other "elliptic" operators with constant coefficients that we consider in Chapter VII will call upon the full theorem.

In this section we shall make use of Green's formula for a ball, as in Proposition 3.14. As we observed in a footnote when applying the proposition in the proof of Theorem 3.16, the result as given in that proposition directly extends from balls to the difference of two balls. The extended result is as follows: If B_R and B_ϵ are closed concentric balls of radii $\epsilon < R$ and if u and v are C^2 functions on a neighborhood of $E = B_R \cap (B_\epsilon^o)^c$, then

$$\int_E (u\Delta v - v\Delta u)\,dx = \int_{\partial E} \Big(u\,\frac{\partial v}{\partial \mathbf{n}} - v\,\frac{\partial u}{\partial \mathbf{n}}\Big)\,d\sigma,$$

where $d\sigma$ is "surface-area" measure on ∂E and the indicated derivatives are directional derivatives pointing outward from E in the direction of a unit normal vector.

Theorem 5.22. In $\mathbb{R}^N$ with $N > 2$, let T be the tempered distribution $-\Omega_{N-1}^{-1}(N-2)^{-1}|x|^{-(N-2)}\,dx$, where Ω_{N-1} is the area of the unit sphere S^{N-1}. Then $\Delta T = \delta$, where δ is the Dirac distribution at 0.

REMARK. The statement uses the name $f(x)\,dx$ for a certain distribution, rather than T_f, for the sake of readability.

[7] Although a fundamental solution for the Laplacian is being shown to exist, it is not unique. One can add to it the distribution T_f for any smooth function f that is harmonic in all of $\mathbb{R}^N$.

PROOF. We are to prove that each φ in $C^\infty_{\text{com}}(\mathbb{R}^N)$ satisfies $\langle \Delta T, \varphi\rangle = \langle \delta, \varphi\rangle$, i.e., that the second equality holds in the chain of equalities

$$\varphi(0) = \langle \delta, \varphi\rangle = \langle \Delta T, \varphi\rangle = \langle T, \Delta\varphi\rangle = -\tfrac{1}{\Omega_{N-1}(N-2)} \int_{\mathbb{R}^N} \tfrac{\Delta\varphi(x)\,dx}{|x|^{N-2}}.$$

We apply Green's formula as above with the closed balls B_R and B_ϵ centered at the origin, with R chosen large enough so that $\operatorname{support}(\varphi) \subseteq B_R^o$, with $u = |x|^{-(N-2)}$, and with $v = \varphi$. Writing r for $|x|$ and observing that $\Delta u = 0$ on $B_R - B_\epsilon$ and that $\frac{\partial\varphi}{\partial \mathbf{n}} = -\nabla\varphi \cdot \frac{x}{r}$ on the boundary of B_ϵ, we obtain

$$\int_{\partial B_\epsilon} \Big(-r^{-(N-2)}\tfrac{x\cdot\nabla\varphi}{r} - \big((\varphi)\big(-\tfrac{d}{dr}\,(r^{-(N-2)})\big)\big)\Big)\,\epsilon^{N-1}\,d\omega = \int_{B_R - B_\epsilon} r^{-(N-2)}\Delta\varphi\,dx.$$

On the left side the first term has $|x\cdot\nabla\varphi|\big/r$ bounded; hence its absolute value is at most a constant times $\int_{\partial B_\epsilon} \epsilon\,d\omega$, which tends to 0 as ϵ decreases to 0. The second term on the left side is $-(N-2)\epsilon^{-(N-1)}\int_{\partial B_\epsilon}\varphi\epsilon^{N-1}\,d\omega$, and it tends, as ϵ decreases to 0, to $-(N-2)\Omega_{N-1}\varphi(0)$. The result in the limit as ϵ decreases to 0 is that

$$-(N-2)\Omega_{N-1}\varphi(0) = \int_{\mathbb{R}^N} r^{-(N-2)}\Delta\varphi\,dx,$$

and the theorem follows. □

Corollary 5.23. In $\mathbb{R}^N$ with $N > 2$, let T be the tempered distribution $-\Omega_{N-1}^{-1}(N-2)^{-1}|x|^{-(N-2)}\,dx$, where Ω_{N-1} is the area of the unit sphere S^{N-1}. If f is in $\mathcal{E}'(\mathbb{R}^N)$, then $u = T * f$ is a tempered distribution and is a solution of $\Delta u = f$.

PROOF. Let δ be the Dirac distribution at 0, so that $\Delta T = \delta$ by Theorem 5.22. Theorem 5.21 shows that $T * f$ is a tempered distribution, and Corollaries 5.14 and 5.19 give $\Delta(T * f) = (\Delta T) * f = \delta * f = f$, as required. □

BIBLIOGRAPHICAL REMARKS. The development in Sections 2–4 is adapted from Hörmander's Volume I of *The Analysis of Linear Partial Differential Equations*.

6. Problems

1. Prove that if U and V are open subsets of $\mathbb{R}^N$ with $U \subseteq V$, then the inclusion $C^\infty_{\text{com}}(U) \to C^\infty_{\text{com}}(V)$ is continuous.
2. Prove that if φ is in $C^\infty_{\text{com}}(U)$, then the map $\psi \mapsto \psi\varphi$ of $C^\infty(U)$ into $C^\infty_{\text{com}}(U)$ is continuous.

3. Let U be a nonempty open set in $\mathbb{R}^N$. Any member T_U of $\mathcal{E}'(U)$ extends to a member T of $\mathcal{E}'(\mathbb{R}^N)$ under the definition $\langle T, \varphi\rangle = \langle T_U, \varphi\big|_U\rangle$ for $\varphi \in C^\infty(\mathbb{R}^N)$. Prove that this is truly an extension in the sense that if φ_1 is in $C^\infty(U)$ and if φ is in $C^\infty(\mathbb{R}^N)$ and agrees with φ_1 in a neighborhood of the support of T_U, then $\langle T, \varphi\rangle = \langle T_U, \varphi\big|_U\rangle = \langle T_U, \varphi_1\rangle$.

4. Prove the following variant of Theorem 5.1: Let K and K' be closed balls of $\mathbb{R}^N$ with K contained in the interior of K'. If T is a member of $\mathcal{E}'(\mathbb{R}^N)$ with support in K, then there exist a positive integer m and members g_α of $L^2(K', dx)$ for each multi-index α with $|\alpha| \le m$ such that
$$\langle T, \varphi\rangle = \textstyle\sum_{|\alpha|\le m} \int_{K'} (D^\alpha \varphi) g_\alpha \, dx \qquad \text{for all } \varphi \in C^\infty(\mathbb{R}^N).$$

5. Let K be a compact metric space, and let μ be a Borel measure on K. Suppose that $\Phi = \Phi(x, y)$ is a scalar-valued function on $\mathbb{R}^N \times K$ such that $\Phi(\,\cdot\,, y)$ is smooth for each y in K, and suppose further that every iterated partial derivative $D_1^\alpha \Phi$ in the first variable is continuous on $\mathbb{R}^N \times K$. Define
$$F(x) = \textstyle\int_K \Phi(x, y)\, d\mu(y).$$
   (a) Prove that any T in $\mathcal{E}'(\mathbb{R}^N)$ satisfies $\langle T, F\rangle = \int_K \langle T, \Phi(\,\cdot\,, y)\rangle\, d\mu(y)$.
   (b) Suppose that Φ has compact support in $\mathbb{R}^N \times K$. Prove that any S in $\mathcal{D}'(\mathbb{R}^N)$ satisfies $\langle S, F\rangle = \int_K \langle S, \Phi(\,\cdot\,, y)\rangle\, d\mu(y)$.

6. Suppose that T is a distribution on an open set U in $\mathbb{R}^N$ such that $\langle T, \varphi\rangle \ge 0$ whenever φ is a member of $C^\infty_{\text{com}}(U)$ that is ≥ 0. Prove that there is a Borel measure $\mu \ge 0$ on U such that $\langle T, \varphi\rangle = \int_U \varphi\, d\mu$ for all φ in $C^\infty_{\text{com}}(U)$.

7. Verify the formula of Theorem 5.22 for $\varphi(x) = e^{-\pi|x|^2}$, namely that
$$\textstyle\int_{\mathbb{R}^N} |x|^{-(N-2)} (\Delta\varphi)(x)\, dx = -\Omega_{N-1}(N-2)\varphi(0)$$
for this φ, by evaluating the integral in spherical coordinates.

Problems 8–11 deal with special situations in which the conclusion of Theorem 5.1 can be improved to say that a distribution with support in a set K is expressible as the sum of iterated partial derivatives of finite complex Borel measures supported in K.

8. This problem classifies distributions on $\mathbb{R}^1$ supported at $\{0\}$. By Proposition 3.5f let η be a member of $C^\infty_{\text{com}}(\mathbb{R}^1)$ with values in $[0, 1]$ that is identically 1 for $|x| \le \frac{1}{2}$ and is 0 for $|x| \ge 1$. Suppose that T is a distribution with support at $\{0\}$. Choose constants C, M, and n such that $|\langle T, \varphi\rangle| \le C \sum_{k=0}^n \sup_{|x|\le M} |D^k \varphi(x)|$ for all φ in $C^\infty(\mathbb{R}^1)$.
   (a) For $\varepsilon > 0$, define $\eta_\varepsilon(x) = \eta(\varepsilon^{-1}x)$. Prove for each $k \ge 0$ that there is a constant C_k independent of ε such that $\sup_x |(\frac{d}{dx})^k\, \eta_\varepsilon(x)| \le C_k \varepsilon^{-k}$.

(b) Using the assumption that T has support at $\{0\}$, prove that $\langle T, \varphi\rangle = \langle T, \eta_\varepsilon\varphi\rangle$ for every φ in $C^\infty(\mathbb{R}^1)$.

(c) Suppose that φ is of the form $\varphi(x) = \psi(x)x^{n+1}$ with ψ in $C^\infty(\mathbb{R}^1)$. By applying (b) and estimating $|\langle T, \eta_\varepsilon\varphi\rangle|$ by means of the Leibniz rule and (a), prove that this special kind of φ has $T(\varphi) = 0$.

(d) Using a Taylor expansion involving derivatives through order n and a remainder term, prove for general φ in $C^\infty(\mathbb{R}^1)$ that $\langle T, \varphi\rangle$ is a linear combination of $\varphi(0), D^1\varphi(0), \dots, D^n\varphi(0)$, hence that T is a linear combination of $\delta, D^1\delta, \dots, D^n\delta$.

9. By suitably adapting the argument in the previous problem, show that every distribution on $\mathbb{R}^N$ that is supported at $\{0\}$ is a finite linear combination of the distributions $D^\alpha\delta$, where δ is the Dirac distribution at 0.

10. Let the members x of $\mathbb{R}^N$ be written as pairs (x', x'') with x' in $\mathbb{R}^L$ and x'' in $\mathbb{R}^{N-L}$. Suppose that T is a compactly supported distribution on $\mathbb{R}^N$ that is supported in $\mathbb{R}^L$. By using a Taylor expansion in the variables x'' with coefficients involving x' and by adapting the argument for the previous two problems, prove that T is a finite sum of the form $\langle T, \varphi\rangle = \sum_{|\alpha|\le n}\langle T_\alpha, (D^\alpha\varphi)\big|_{\mathbb{R}^L}\rangle$, the sum being over multi-indices α involving only x'' variables and each T_α being in $\mathcal{E}'(\mathbb{R}^L)$. (Educational note: The operators D^α of this kind are called **transverse derivatives** to $\mathbb{R}^L$. The result is that T is a finite sum of transverse derivatives of compactly supported distributions on $\mathbb{R}^L$.)

11. Using the result of Problem 9, prove the following uniqueness result to accompany Corollary 5.23: if f is a distribution of compact support in $\mathbb{R}^N$ with $N > 2$, then any two tempered distributions u on $\mathbb{R}^N$ that solve $\Delta u = f$ differ by a polynomial function annihilated by Δ. Is this uniqueness still valid if u is allowed to be *any* distribution that solves $\Delta u = f$?

Problems 12–13 introduce a notion of **periodic distribution** as any continuous linear functional on the space of periodic smooth functions on $\mathbb{R}^N$. Write T for the circle $\mathbb{R}/2\pi\mathbb{Z}$, and let $C^\infty(T^N)$ be the complex vector space of all smooth functions on $\mathbb{R}^N$ that are periodic of period 2π in each variable. Regard $C^\infty(T^N)$ as a vector subspace of $C^\infty((-2\pi, 2\pi)^N)$, and give it the relative topology. Then define $\mathcal{P}'(T^N)$ to be the space of restrictions to $C^\infty(T^N)$ of members of $\mathcal{E}'((-2\pi, 2\pi)^N)$. For S in $\mathcal{P}'(T^N)$, define the **Fourier series** of S to be the trigonometric series $\sum_{k\in\mathbb{Z}^N} c_k e^{ik\cdot x}$ with $c_k = \langle S, e^{-ik\cdot x}\rangle$.

12. Prove that the Fourier coefficients c_k for such an S satisfy $|c_k| \le C(1+|k|^2)^{m/2}$ for some constant C and positive integer m.

13. Prove that any trigonometric series $\sum_{k\in\mathbb{Z}^N} c_k e^{ik\cdot x}$ in which the c_k's satisfy $|c_k| \le C(1+|k|^2)^{m/2}$ for some constant C and positive integer m is the Fourier series of some member S of $\mathcal{P}'(T^N)$.

Problems 14–19 establish the **Schwartz Kernel Theorem** in the setting of periodic functions. We make use of Problems 25–34 in Chapter III concerning Sobolev spaces $L^2_k(T^N)$ of periodic functions. As a result of those problems, the metric on $C^\infty(T^N)$ may be viewed as given by the separating family of seminorms $\|\cdot\|_{L^2_K(T^N)}$, $k \geq 0$, and $C^\infty(T^N)$ is a complete metric space. The Schwartz Kernel Theorem says that any bilinear function $B : C^\infty(T^N) \times C^\infty(T^N) \to \mathbb{C}$ that is separately continuous in the two variables is given by "integration with" a distribution on $T^N \times T^N \cong T^{2N}$. The analogous assertion about signed measures is false.

14. Let $B : C^\infty(T^N) \times C^\infty(T^N) \to \mathbb{C}$ be a function that is bilinear in the sense of being linear in each argument when the other argument is fixed, and suppose that B is continuous in each variable. The continuity in the first variable means that for each $\psi \in C^\infty(T^N)$, there is an integer k and there is some constant $C_{\psi,k}$ such that $|B(\varphi, \psi)| \leq C_{\psi,k}\|\varphi\|_{L^2_k(T^N)}$ for all φ in $C^\infty(T^N)$, and a similar inequality governs the behavior in the ψ variable for each φ. For integers $k \geq 0$ and $M \geq 0$, define
$$E_{k,M} = \big\{\psi \in C^\infty(T^N) \bigm| |B(\varphi, \psi)| \leq M\|\varphi\|_{L^2_k(T^N)} \text{ for all } \varphi \in C^\infty(T^N)\big\}.$$
   (a) Prove that each $E_{k,M}$ is closed and that the union of these sets on k and M is $C^\infty(T^N)$.
   (b) Apply the Baire Category Theorem, and prove as a consequence that there exist an integer $k \geq 0$ and a constant C such that
$$|B(\varphi, \psi)| \leq C\|\varphi\|_{L^2_k(T^N)}\|\psi\|_{L^2_k(T^N)}$$
   for all φ and ψ in $C^\infty(T^N)$.

15. Let B be as in Problem 14, and suppose that k and C are chosen as in Problem 14b. Fix an integer $K > N/2$, and define $k' = k + K$. Prove that
$$|B(D^\alpha\varphi, D^\beta\psi)| \leq C\|\varphi\|_{L^2_{k'}(T^N)}\|\psi\|_{L^2_{k'}(T^N)}$$
for all φ and ψ in $C^\infty(T^N)$ and all multi-indices α and β with $|\alpha| \leq K$ and $|\beta| \leq K$.

16. Let B, C, K, and k' be as in Problem 15. Put $b_{lm} = B(e^{il\cdot(\,\cdot\,)}, e^{im\cdot(\,\cdot\,)})$ for l and m in $\mathbb{Z}^N$, and for each pair of multi-indices (α, β) with $|\alpha| \leq k'$ and $|\beta| \leq k'$, define
$$F_{\alpha,\beta}(x, y) = \sum_{l,m\in\mathbb{Z}^N} \frac{b_{lm}(-i)^{|\alpha|+|\beta|}l^\alpha m^\beta e^{-il\cdot x}e^{-im\cdot y}}{\Big(\sum\limits_{|\alpha'|\leq k'} l^{2\alpha'}\Big)\Big(\sum\limits_{|\beta'|\leq k'} m^{2\beta'}\Big)}$$
for $(x, y) \in T^N \times T^N$. Prove that this series is convergent in $L^2(T^N \times T^N)$.

17. With B, C, K, and k' be as in Problem 15 and with $F_{\alpha,\beta}$ as in Problem 16 for $|\alpha| \le k'$ and $|\beta| \le k'$, define

$$B'(\varphi, \psi) = \sum_{\substack{|\alpha|\le k',\\ |\beta|\le k'}} (2\pi)^{-2N} \int\limits_{[-\pi,\pi]^N\times[-\pi,\pi]^N} F_{\alpha,\beta}(x, y)(D^\alpha\varphi)(x)(D^\beta\psi)(y)\,dx\,dy$$

for φ and ψ in $C^\infty(T^N)$. Prove that B' is well defined for all φ and ψ in $C^\infty(T^N)$ and that $B'(e^{il\cdot(\cdot)}, e^{im\cdot(\cdot)}) = B(e^{il\cdot(\cdot)}, e^{im\cdot(\cdot)})$ for all l and m in $\mathbb{Z}^N$.

18. With B' as in the previous problem, prove that $B'(\varphi, \psi) = B(\varphi, \psi)$ for all φ and ψ in $C^\infty(T^N)$, and conclude that there exists a distribution S in $\mathcal{P}'(T^{2N})$ such that

$$B(\varphi, \psi) = \langle S, \varphi \otimes \psi\rangle$$

for all φ and ψ in $C^\infty(T^N)$ if $\varphi\otimes\psi$ is defined by $(\varphi\otimes\psi)(x, y) = \varphi(x)\psi(y)$.

19. Let η be a function in $C^\infty_{\text{com}}(\mathbb{R}^1)$ with values in [0, 1] that is 1 for $|x| \le \frac{1}{2}$ and is 0 for $|x| \ge 1$. For f continuous on T^1, the Hilbert transform

$$(H(\eta f))(x) = \lim_{\varepsilon\downarrow 0} \frac{1}{\pi} \int_{|y|\ge\varepsilon} \frac{\eta(x-y)f(x-y)\,dy}{y}$$

exists as an $L^2(\mathbb{R}^1)$ limit.

(a) Let $C(T^1)$ be the space of continuous periodic functions on $\mathbb{R}$ of period 2π, and give it the supremum norm. Taking into account that H, as an operator from $L^2(\mathbb{R}^1)$ to itself, has norm 1, prove that

$$B(f, g) = \textstyle\int_{-\pi}^{\pi} (H(\eta f))(x)(\eta g)(x)\,dx$$

is bilinear on $C(T^1)\times C(T^1)$ and is continuous in each variable.

(b) Prove that there is no complex Borel measure $\rho(x, y)$ on $[-\pi, \pi]^2$ such that $B(f, g) = \int_{[-\pi,\pi]^2} f(x)g(y)\,d\rho(x, y)$ for all f and g in $C(T^1)$.