Appendix B

Analysis from a Geometric Point of View

B.1. Smooth Functions

We start with the notion of field of view as in Chapter 2, Problems **2.1** and **2.2**.

Definition. A function f from a neighborhood U of a point p in $\mathbf{R}^n$ to $\mathbf{R}^m$ is ***smooth*** if the graph G of f is a smooth n-submanifold of $\mathbf{R}^n \times \mathbf{R}^m = \mathbf{R}^{n+m}$ and the projection of each tangent space of G to $\mathbf{R}^n$ is a one-to-one and onto. An n-***submanifold*** M is a subset of a Euclidean space such that M is ***infinitesimally n-spatial***; that is, for every point p in M, there is an n-hyperplane T_p (called the ***tangent space*** at p) such that, for every tolerance $\tau = (1/N)$, there is a radius $\rho = (1/M)$, such that in any f.o.v. centered at p with radius less than ρ, the projection of M onto T_p is one-to-one and onto and moves each point less than $\tau\rho$ (we describe this by saying that if you zoom in on p, then M and T_p become indistinguishable). The submanifold is said to be ***smooth*** if the zooming is uniform in the sense that (for each tolerance) the same ρ can be used for every point in some neighborhood of p.

Lemma. *The last sentence above is equivalent to saying that the tangent spaces vary continuously over M.*

The proof is essentially the same as Problems **2.2.e** and **3.1.e**.

Definition. If f is a smooth function from a neighborhood U in $\mathbf{R}^n$ to $\mathbf{R}^m$, then for each p in U, the ***differential***, df_p, is the linear function from $\mathbf{R}^n$ to $\mathbf{R}^m$ such that the tangent space T_p is the graph of the affine linear function $t(q) = f(p) + df_p(q - p)$. In the terminology of Appendix **A.1**, we can more accurately say that df_p is a linear transformation from the tangent space $(\mathbf{R}^n)_p$ to the tangent space $(\mathbf{R}^m)_{f(p)}$.

Theorem B.1. *A function, which maps a neighborhood U of p in $\mathbf{R}^n$ to $\mathbf{R}^m$, is smooth* (*in the above geometric sense*) *if and only if it is* C^1 (*in the sense of having for every point p in U a differential df_p that varies continuously with p*).

The proof is essentially the same as the proofs of Problem **2.2.b,c,e** and Problem **3.1.e**.

B.2. Invariance of Domain

In the next section we will need the following result:

Theorem B.2. *Any continuous function that maps an open subset of n-space one-to-one to n-space is open* (*that is, the image of every open set is open*).

This result is commonly known as ***Brouwer's Invariance of Domain***. It was first proved in about 1910 by L.E.J. Brouwer. The proofs of this theorem involve the topological fields of dimension theory or homology theory, and all require a fair amount of machinery. There are proofs in any of the three books listed in the Bibliography in Section **Tp. Topology**. In the context of differentiable functions, there is an easier proof, which involves explicitly constructing a continuous inverse (see [**An:** Strichartz], the proof of Theorem 13.1.1.)

B.3. Inverse Function Theorem

Theorem B.3. *If f is a smooth function from n-space to n-space such that, for the point $p_o = (y_o, f(y_o))$ on the graph of f, the tangent space T_p projects one-to-one onto the range, then there is a neighborhood U of $x_o = f(y_o)$ and a smooth function g from U to n-space such that $f(g(x)) = x$, for every x in U. Furthermore, g maps U one-to-one onto a neighborhood V of y_o and $g(f(y)) = y$, for every y in V.*

Proof: This proof uses the Invariance of Domain but is otherwise shorter and more geometric than the usual proofs in analysis. The only nontrivial things to show are (a) that f is **one-to-one** in a neighborhood of y_o, and (b) that f maps a neighborhood of y_o **onto** a neighborhood of x_o.

(a) Suppose f is not one-to-one in a neighborhood of y_o. Then there is a sequence of point pairs $\{a_n, b_n\}$ such that $f(a_n) = f(b_n)$, for all n, and both sequences $\{a_n\}$ and $\{b_n\}$ converge to y_o. Let l_n be the line segment joining a_n to b_n. Applying the Mean Value Theorem for Space Curves (Problem **4.2.b**), there is a point c_n on l_n between a_n and b_n such that a vector tangent to the graph of $f\,|l_n$ (and, therefore, tangent also to the graph of f) projects to a point on the range n-space. But then the tangent spaces to the graph of f cannot be varying continuously.

(b) The fact that f is onto a neighborhood follows from Invariance of Domain (**B.2**).

For an analytic proof of **B.3**, see [**An:** Strichartz], Theorem 13.1.2.

B.4. Implicit Function Theorem

Theorem B.4.1. *Let $F(x,y)$ be a smooth function defined in a neighborhood of x_0 in $\mathbf{R}^n$ and y_0 in $\mathbf{R}^m$ taking values in $\mathbf{R}^m$, with $F(x_0,y_0) = c$. Then, if the function $f(y) = F(x_0,y)$ is such that, for the point $p_0 = (x_0,y_0,f(y_0))$ on the graph of f, the tangent space T_p projects one-to-one onto the range, then there is a neighborhood U of x_0 and a smooth function h from U to $\mathbf{R}^m$ such that $h(x_0) = y_0$ and $F(x,h(x)) = c$ for every x in U.*

Note that the condition on the graph of f is equivalent to the analytic condition that $F_y(x_0,y_0)$ is invertible, where F_y is the submatrix of dF corresponding to using only the partial derivatives with respect to y.

Proof: We will describe three different proofs:

1. Define $f(x,y) = (x,F(x,y))$. Then it is easy to check that f satisfies the hypotheses of Theorem **B.3**. Thus, if there is a smooth inverse function g defined on a neighborhood of $(x_0,F(x_0,y_0))$, then there is a function $h(x)$ such that $g(x,c) = (x,h(x))$. This h is the desired function.
2. It is possible to construct a direct geometric proof (using Invariance of Domain) along the same lines as the proof of Theorem **B.3**.
3. There is an analysis proof that explicitly constructs the function h. (See [**An:** Strichartz], Theorem 13.1.1.)

Theorem B.4.2. *Let F: $\mathbf{R}^n \to \mathbf{R}^{n-m}$ be a C^1 function, and suppose $dF(x)$ has maximal rank $n-m$ at every point on a level set $M = \{x\,|\,F(x)=c\}$. Then M is a C^1 m-submanifold of $\mathbf{R}^n$.*

We can prove this as a corollary of **B.4.1**, (see [**An:** Strichartz], Theorem 13.2.2.) But there is a more geometric proof. First, change the hypotheses to geometric ones. That dF has maximal rank at p is equivalent to the tangent space $T_{p,F(x)}$ of the graph of F projecting *onto* the range space. Now, if we take the inverse of c under this projection, we get a linear m-dimensional subspace of the tangent space. The projection of this m-subspace onto the domain is a tangent space of the level set M. We have proved:

Theorem B.4.3. *Let F: $\mathbf{R}^{n+m} \to \mathbf{R}^m$ be a smooth function, such that, for every point $p = (x,y,c)$ on the graph of the level set $M = \{x\,|\,F(x,y)=c\}$, the tangent space T_p projects onto the range. Then M is a C^1 m-submanifold of $\mathbf{R}^{n+m}$.*