CHAPTER XVIII.

TRANSFORMATION OF PERIODS, ESPECIALLY LINEAR TRANSFORMATION.

318. IN the foregoing portion* of the present volume, the fundamental algebraic equation has been studied with the help of a Riemann surface. Much of the definiteness of the theory depends upon the adoption of a specific mode of dissecting the surface by means of period loops; for instance this is the case for the normal integrals, and their periods, and consequently also for the theta functions, which were defined in terms of the periods $\tau_{i,j}$ of the normal integrals of the first kind; it is also the case for the places $m_1, \dots, m_p$ of § 179 (Chap. X.), upon which the theory of the vanishing of the theta functions depends. The question then arises; if we adopt a different set of period loops as fundamental, how is the theory modified, and, in particular, what is the relation between the new theta functions obtained, and the original functions? We have given a geometrical method (§ 183, Chap. X.) of determining the places $m_1, \dots, m_p$ from the place m, from which it appears that they cannot have more than a finite number of positions when m is given, and coresidual places are reckoned equivalent; the enquiry then suggests itself; can they take all these possible positions by a suitable choice of period loops, or is one of these essentially different from the others? The answers to such questions as these are to be sought from the theory of the present chapter.

There is another enquiry, not directly related to the Riemann surface, but arising in connexion with the analytical theory of the theta functions. Taking p independent variables $u_1, \dots, u_p$, and associating with them, in accordance with the suggestion of §§ 138—140 (cf. § 284), the matrices 2ω, $2\omega'$, 2η, $2\eta'$, we are thence able, with the help of the resulting equations

$$2h\omega = \pi i, \quad 2h\omega' = b, \quad \eta = 2a\omega, \quad \eta' = 2a\omega' - \bar{h},$$

to formulate a theta function. But it is manifest that this procedure makes an unsymmetrical use of the columns of periods arising respectively from the matrices ω and ω'; and it becomes a problem to enquire whether this

* References to the literature dealing with transformation are given at the beginning of Chap. XX.

want of symmetry can be removed; and more generally to enquire what general linear functions of the original $2p$ columns of periods, with integral coefficients, can be formed to replace the original columns of periods; and, if theta functions be formed with the new periods, as with the original ones, to investigate the expression of the new theta functions in terms of the original ones.

So far as the theta functions are concerned, it will appear that the theory of the transformation of periods, and of characteristics, includes the consideration of the effect of a modification of the period loops of a Riemann surface; for that reason we give in this chapter the fundamental equations for the transformation of the periods and characteristic of a theta function, when the coefficients of transformation are integers; but the main object of this chapter is to deal with the transformation of the period loops on a Riemann surface. The analytical theory of the expression of the transformed theta functions in terms of the original functions is considered in the two following chapters.

In virtue of the algebraical representation which is possible for quotients of Riemann theta functions (as exemplified in Chap. XI.), the theory of the expression of the transformed theta functions in terms of the original functions, includes a theory of the algebraical transformation of the fundamental algebraical equation associated with a Riemann surface; it is known what success was achieved by Jacobi, from this point of view, in the case of elliptic functions; and some of the earliest contributions to the general theory of transformation of theta functions approach the matter from that side*. We deal briefly with particular results of this algebraical theory in Chap. XXII.

319. Take any undissected Riemann surface associated with a fundamental algebraic equation of deficiency p. The most general set of $2p$ period loops may be constructed as follows:

Draw on the surface any closed curve whatever, not intersecting itself, which is such that if the surface were cut along this curve it would not be divided into two pieces; of the two possible directions in which this curve can be described, choose either, and call it the positive direction; call the side of the curve which is on the left hand when the curve is described positively, the left side; this curve is the period loop (A_1); starting now from any point on the left side of (A_1), a curve can be drawn on the surface, which, without cutting itself, or the curve (A_1), and without dividing the surface, ends at the point of the curve (A_1) at which it began, but on the right side of (A_1); this is the loop (B_1), and the direction in which it has

* See, in particular, Richelot, *Crelle*, XVI. (1837), De transformatione...integralium Abelianorum primi ordinis; in the papers of Königsberger, *Crelle*, LXIV., LXV., LXVII., some of the algebraical results of Richelot are obtained by means of the transformation of theta functions.

been described is its positive direction; its left side is that on the left hand in the positive description of it. The period associated with the loop (A_1), of any Abelian integral, is the constant whereby the value of the integral on the left side of (A_1) exceeds the value on the right side, and is equal to the value obtained by taking the integral along the loop (B_1) in the negative direction, from the end of the loop (B_1) to its beginning. The period associated with the loop (B_1) is similarly the excess of the value of the integral on the left side of the loop (B_1) over its value on the right side, and may be obtained by taking the integral round the loop (A_1) in the positive direction, from the right side of the loop (B_1) to the left side. These periods may be denoted respectively by Ω_1 and Ω_1'.

320. It is useful further to remark that there is no essential reason why what we have called the loops (A_1), (B_1) should not be called respectively the loops $[B_1]$ and $[A_1]$. If this be done, and the positive direction of the (original) loop (B_1) be preserved, the convention as to the relation of the directions of the loops $[A_1]$, $[B_1]$ will necessitate a reversal of the convention as to the positive direction of the (original) loop (A_1). If the periods associated with the (new) loops $[A_1]$, $[B_1]$ be respectively denoted by $[\Omega]$ and $[\Omega']$, we have, therefore, the equations

$$[\Omega]=\Omega', \quad [\Omega']=-\Omega.$$

These equations represent a process—of interchange of the loops (A_1), (B_1), with retention of the direction of (B_1)—which may be repeated. The repetition gives equations which we may denote by

$$\{\Omega\}=[\Omega']=-\Omega, \quad \{\Omega'\}=-[\Omega]=-\Omega',$$

and the two processes are together equivalent to reversing the direction of loop (A_1), and (therefore) of the loop (B_1). The convention that the loop (B_1) shall begin from the left side of the loop (A_1) is not necessary for the purpose of the dissection of the surface into a simply connected surface; but it affords a convenient way of specifying the necessary condition for the convergence of the series defining the theta functions.

321. The pair of loops (A_1), (B_1) being drawn, the successive pairs (A_2), (B_2), ..., (A_p), (B_p) are then to be drawn in accordance with precisely similar conventions—the additional convention being made that neither loop of any pair is to cross any one of the previously drawn loops. If the Riemann surface be cut along these $2p$ loops it will become a p-ply connected surface, with p closed boundary curves. It may be further dissected into a simply connected surface by means of $(p-1)$ further cuts (C_1), ..., (C_{p-1}), taken so as to reduce the boundary to one continuous closed curve.

Upon the p-ply connected surface formed by cutting the original surface along the loops (A_1), (B_1), ..., (A_p), (B_p), the Riemann integrals of the first and second kind are single-valued. In particular if W_1, ..., W_p be a set of linearly independent integrals of the first kind defined by the conditions that the periods of W_r at the loops (A_1), ..., (A_p) are all zero, except that at

(A_r), which is 1, and if $\tau_{r,s}$ be the period of W_r at the loop (B_s), the imaginary part of the quadratic form

$$\tau_{11}n_1^2 + \ldots\ldots + 2\tau_{12}n_1n_2 + \ldots\ldots + \tau_{pp}n_p^2$$

is necessarily positive* for real values of $n_1, \ldots, n_p$. This statement remains true when, for each of the p pairs, the loops (A_r), (B_r) are interchanged, with *e.g.* the retention of the direction of (B_r) and a consequent change in the sign of the period associated with (A_r), as explained above (§ 320); if the loops (A_r), (B_r) be interchanged without the change in the sign of the period associated with (A_r), the imaginary part of the corresponding quadratic form is negative†.

322. In addition now to such a general system of period loops as has been described, imagine another system of loops, which for distinctness we shall call the original system; the loops of the original system may be denoted by (a_r), (b_r) and the periods of any integral, u_i, associated therewith, by $2\omega_{i,r}$, $2\omega'_{i,r}$; the general system of period loops is denoted by (A_r), (B_r), and the periods associated therewith by $[2\omega_{i,r}]$, $[2\omega'_{i,r}]$. For the values of the integral u_i, the circuit of the loop (B_r), in the negative direction, from the right to the left side of the loop (A_r), is equivalent to a certain number, say‡ to $\alpha_{j,r}$, of circuits of the loop (b_j) in the negative direction, together with a certain number, say $\alpha'_{j,r}$, of circuits of the loop (a_j) in the positive direction $(r, j = 1, 2, \ldots, p)$; hence we have

$$[\omega_{i,r}] = \sum_{j=1}^{p} (\omega_{i,j}\alpha_{j,r} + \omega'_{i,j}\alpha'_{j,r}), \qquad (r = 1, 2, \ldots, p);$$

similarly we have equations which we write in the form

$$[\omega'_{i,r}] = \sum_{j=1}^{p} (\omega_{i,j}\beta_{j,r} + \omega'_{i,j}\beta'_{j,r}), \qquad (r = 1, 2, \ldots, p),$$

the interpretation of the integers $\beta_{j,r}$, $\beta'_{j,r}$ being similar to that of the integers $\alpha_{j,r}$, $\alpha'_{j,r}$.

Thus, if $u_1, \ldots, u_p$ denote p linearly independent integrals of the first kind, and the matrices of their periods for the original system of period loops be denoted by 2ω, $2\omega'$, and for the general system of period loops by $[2\omega]$, $[2\omega']$, we have

$$[\omega] = \omega\alpha + \omega'\alpha', \quad [\omega'] = \omega\beta + \omega'\beta',$$

where α, α', β, β' denote matrices whose elements are integers.

* And not zero, since $n_1W_1 + \ldots + n_pW_p$ cannot be a constant. Cf. for instance, Neumann, *Riemann's Theorie der Abel'schen Integrale* (Leipzig, 1884), p. 247, or Forsyth, *Theory of Functions* (1893), p. 447. (Riemann, *Werke*, 1876, p. 124.)

† As previously remarked, p. 247, note.

‡ A circuit of (b_j) in the positive direction furnishing a contribution of -1 to $\alpha_{j,r}$.

If $L_1, \ldots, L_p$ be a set of p integrals of the second kind *associated* with $u_1, \ldots, u_p$, as in § 138, Chap. VII., and satisfying, therefore, the condition

$$\sum_{i=1}^{p} [D_x u_i^{x, a} D_z L_i^{z, c} - D_z u_i^{z, c} D_x L_i^{x, a}] = D_x \left[(z, x) \frac{dz}{dt}\right] - D_z \left[(x, z) \frac{dx}{dt}\right],$$

and the period matrices of $L_1, \ldots, L_p$ at the original and general period loops be denoted respectively by -2η, $-2\eta'$ and $-[2\eta]$, $-[2\eta']$, we have, similarly, for the same values of α, α', β, β',

$$[\eta] = \eta\alpha + \eta'\alpha', \quad [\eta'] = \eta\beta + \eta'\beta'.$$

We have used the notation Ω_P for the row of P quantities $2\omega P + 2\omega' P'$, where P, P' each denotes a row of p quantities; we extend this notation to the matrix $2\omega\alpha + 2\omega'\alpha'$, where α, α' each denotes a matrix of p rows and columns, and denote this matrix by Ω_α; similarly we denote the matrix $2\eta\alpha + 2\eta'\alpha'$ by H_α; then the four equations just obtained may be written

$$[2\omega] = \Omega_\alpha, \quad [2\omega'] = \Omega_\beta, \quad [2\eta] = H_\alpha, \quad [2\eta'] = H_\beta. \qquad \text{(I.)}$$

Noticing now that the matrices $[2\omega]$, $[2\omega']$, $[2\eta]$, $[2\eta']$ must satisfy the relations obtained in § 140, we have

$$\begin{aligned}
\tfrac{1}{2}\pi i &= [\bar{\eta}]\,[\omega'] - [\bar{\omega}]\,[\eta'] = \tfrac{1}{4}(\bar{H}_\alpha \Omega_\beta - \bar{\Omega}_\alpha H_\beta) \\
&= (\bar{\alpha}\bar{\eta} + \bar{\alpha}'\bar{\eta}')(\omega\beta + \omega'\beta') - (\bar{\alpha}\bar{\omega} + \bar{\alpha}'\bar{\omega}')(\eta\beta + \eta'\beta') \\
&= \bar{\alpha}(\bar{\eta}\omega - \bar{\omega}\eta)\beta + \bar{\alpha}'(\bar{\eta}'\omega - \bar{\omega}'\eta)\beta + \bar{\alpha}(\bar{\eta}\omega' - \bar{\omega}\eta')\beta' + \bar{\alpha}'(\bar{\eta}'\omega' - \bar{\omega}'\eta')\beta' \\
&= (\bar{\alpha}\beta' - \bar{\alpha}'\beta)\tfrac{1}{2}\pi i,
\end{aligned}$$

in virtue of the relations satisfied by the matrices 2ω, $2\omega'$, 2η, $2\eta'$; and similarly

$$0 = [\bar{\eta}][\omega] - [\bar{\omega}][\eta] = \tfrac{1}{4}(\bar{H}_\alpha \Omega_\alpha - \bar{\Omega}_\alpha H_\alpha) = (\bar{\alpha}\alpha' - \bar{\alpha}'\alpha)\tfrac{1}{2}\pi i,$$

and

$$0 = [\bar{\eta}'][\omega'] - [\bar{\omega}'][\eta'] = \tfrac{1}{4}(\bar{H}_\beta \Omega_\beta - \bar{\Omega}_\beta H_\beta) = (\bar{\beta}\beta' - \bar{\beta}'\beta)\tfrac{1}{2}\pi i;$$

thus we have

$$\bar{\alpha}\beta' - \bar{\alpha}'\beta = 1 = \bar{\beta}'\alpha - \bar{\beta}\alpha', \quad \bar{\alpha}\alpha' - \bar{\alpha}'\alpha = 0, \quad \bar{\beta}\beta' - \bar{\beta}'\beta = 0, \qquad \text{(II.)}$$

namely, the matrices α, β, α', β' satisfy relations precisely similar to those respectively satisfied by the matrices ω, ω', η, η', the $\frac{1}{2}\pi i$ which occurs for the latter case being, in the case of the matrices α, β, α', β', replaced by -1; therefore also, as in § 141, the relations satisfied by α, β, α', β' can be given in the form

$$\alpha\bar{\beta}' - \beta\bar{\alpha}' = 1 = \beta'\bar{\alpha} - \alpha'\bar{\beta}, \quad \alpha\bar{\beta} - \beta\bar{\alpha} = 0, \quad \alpha'\bar{\beta}' - \beta'\bar{\alpha}' = 0. \qquad \text{(III.)}$$

In virtue of these equations, if

$$J = \begin{pmatrix} \alpha, & \beta \\ \alpha', & \beta' \end{pmatrix}$$

denote the matrix of $2p$ rows and columns formed with the elements of the matrices α, β, α', β', we have (cf., for notation, Appendix ii.)

$$\begin{pmatrix} \alpha, & \beta \\ \alpha', & \beta' \end{pmatrix} \begin{pmatrix} \bar{\beta}', & -\bar{\beta} \\ -\bar{\alpha}', & \bar{\alpha} \end{pmatrix} = \begin{pmatrix} \alpha\bar{\beta}' - \beta\bar{\alpha}', & \beta\bar{\alpha} - \alpha\bar{\beta} \\ \alpha'\bar{\beta}' - \beta'\bar{\alpha}', & \beta'\bar{\alpha} - \alpha'\bar{\beta} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and therefore

$$J^{-1}=\begin{pmatrix} \bar{\beta}', & -\bar{\beta} \\ -\bar{a}', & \bar{a} \end{pmatrix},$$

and the original periods can be expressed in terms of the general periods in the form

$$\omega=[\omega]\bar{\beta}'-[\omega']\bar{a}', \quad \omega'=-[\omega]\bar{\beta}+[\omega']\bar{a},$$
$$\eta=[\eta]\bar{\beta}'-[\eta']\bar{a}', \quad \eta'=-[\eta]\bar{\beta}+[\eta']\bar{a}.$$

If 0 denote the matrix of p rows and columns whereof every element is zero, and 1 denote the matrix of p rows and columns whereof every element is zero except those in the diagonal, which are all equal to 1, and if ϵ denote the matrix of $2p$ rows and columns given by

$$\epsilon=\begin{pmatrix} 0, & -1 \\ 1, & 0 \end{pmatrix}, \text{ so that } \epsilon^2=\begin{pmatrix} -1, & 0 \\ 0, & -1 \end{pmatrix}=-1,$$

then it is immediately proved that the relations (II.), (III.) are respectively equivalent to the two equations

$$\bar{J}\epsilon J=\epsilon, \quad J\epsilon\bar{J}=\epsilon,$$

where

$$\bar{J}=\begin{pmatrix} \bar{a}, & \bar{a}' \\ \bar{\beta}, & \bar{\beta}' \end{pmatrix};$$

and it will be noticed that the equations (III.) are obtained from the equations (II.) by changing the elements of J into the corresponding elements of $\bar{J}$.

It follows* from the equation $\bar{J}\epsilon J=\epsilon$ that the determinant of the matrix J is equal to $+1$ or to -1. It will subsequently (§ 333) appear that the determinant is equal to $+1$.

Ex. Verify, for the case $p=2$, that the matrices

$$a=\begin{pmatrix} 4, & -20 \\ 4, & 1 \end{pmatrix}, \quad \beta=\begin{pmatrix} -29, & 124 \\ -28, & -6 \end{pmatrix},$$
$$a'=\begin{pmatrix} -3, & 20 \\ -8, & -7 \end{pmatrix}, \quad \beta'=\begin{pmatrix} 22, & -124 \\ 56, & 43 \end{pmatrix}$$

satisfy the conditions (III.) (Weber, *Crelle*, LXXIV. (1872), p. 72).

323. It is often convenient, simultaneously with the change of period loops which has been described, to make a linear transformation of the fundamental integrals of the first kind, $u_1, \ldots, u_p$. Suppose that we introduce, in place of $u_1, \ldots, u_p$, other p integrals $w_1, \ldots, w_p$, such that

$$u_i=M_{i,1}w_1+\ldots\ldots+M_{i,p}w_p, \qquad (i=1, 2, \ldots, p),$$

or, as we shall write it, $u=Mw$, M being a matrix whose elements are constants and of which the determinant is not zero. We enquire then what are the integrals of the second kind associated with $w_1, \ldots, w_p$. We have (§ 138) denoted $Du_i^{x,a}$ by $\mu_i(x)$, and the matrix of the quantities $\mu_i(c_j)$ by μ;

* For another proof of the relations (II.), (III.) of the text, the reader may compare Thomae, *Crelle*, LXXV. (1873), p. 224. A proof directly on the lines followed here may of course be constructed with the employment only of Riemann's normal elementary integrals of the first and second kind. Cf. § 142.

denote now, also, $Dw_i^{x,a}$ by $\rho_i(x)$, and the matrix of the quantities $\rho_i(c_j)$ by ρ; then we immediately find $\mu = \rho\bar{M}$, and the equation (§ 138)

$$L^{x,a} = \mu^{-1}H^{x,a} - 2au^{x,a}$$

gives

$$\bar{M}L^{x,a} = \rho^{-1}H^{x,a} - 2\bar{M}aMw^{x,a};$$

thus the integrals of the second kind associated with $w_1, \ldots, w_p$ are the p integrals given by $\bar{M}L^{x,a}$, and, corresponding to the matrix a for the integrals $L_1^{x,a}, \ldots, L_p^{x,a}$, we have, for the integrals $\bar{M}L^{x,a}$, the matrix $\mathrm{a} = \bar{M}aM$. If $2v$, $2v'$ denote the matrices of the periods of the integrals w, and -2ζ, $-2\zeta'$ denote the matrices of the periods of the integrals $\bar{M}L^{x,a}$, so that (§ 139)

$$\zeta = 2\mathrm{a}v, \quad \zeta' = 2\mathrm{a}v' - \tfrac{1}{2}\rho^{-1}\Delta,$$

we therefore have $\omega = Mv$, $\omega' = Mv'$ and

$$\zeta = 2\bar{M}aMv = \bar{M}\eta, \quad \zeta' = 2\bar{M}aMv' - \tfrac{1}{2}\bar{M}\mu^{-1}\Delta = \bar{M}\eta'; \qquad \text{(IV.)}$$

it is immediately apparent from these equations that the matrices v, v', ζ, ζ' satisfy the equations of § 140,

$$v\bar{v}' - v'\bar{v} = 0, \ \ \zeta\bar{\zeta}' - \zeta'\bar{\zeta} = 0, \ \ v'\bar{\zeta} - v\bar{\zeta}' = \tfrac{1}{2}\pi i = \zeta\bar{v}' - \zeta'\bar{v}.$$

324. The preceding Articles have sufficiently shewn how the equations of transformation of the periods arise by the consideration of the Abelian integrals. It is of importance to see that equations of the same character, but of more general significance, arise in connexion with the analytical theory of the theta functions.

Let ω, ω', η, η' be any four matrices of p rows and columns satisfying the conditions (i) that the determinant of ω does not vanish, (ii) that $\omega^{-1}\omega'$ is a symmetrical matrix, (iii) that the quadratic form $\omega^{-1}\omega' n^2$ has its imaginary part positive when $n_1, \ldots, n_p$ are real, (iv) that $\eta\omega^{-1}$ is a symmetrical matrix, (v) that $\eta' = \eta\omega^{-1}\omega' - \frac{1}{2}\pi i\bar{\omega}^{-1}$. The conditions (i), (ii), (iv), (v) are equivalent to equations of the form of (B) and (C), § 140, and, taking matrices a, b, h such that $a = \frac{1}{2}\eta\omega^{-1}$, $h = \frac{1}{2}\pi i\omega^{-1}$, $b = \pi i\omega^{-1}\omega'$, or $2h\omega = \pi i$, $2h\omega' = b$, $\eta = 2a\omega$, $\eta' = 2a\omega' - \bar{h}$, the condition (iii) ensures the existence of the function defined by

$$\vartheta\left(u\,;\, \begin{smallmatrix} Q' \\ Q \end{smallmatrix}\right) = \Sigma e^{au^2 + 2hu(n+Q') + b(n+Q')^2 + 2\pi i Q(n+Q')},$$

wherein Q, Q' are any constants (cf. § 174).

Introduce now two other matrices $[\omega]$, $[\omega']$, also of p rows and columns, defined by the equations

$$[\omega] = \omega\alpha + \omega'\alpha', = \tfrac{1}{2}\Omega_\alpha, \text{ say}, \quad [\omega'] = \omega\beta + \omega'\beta', = \tfrac{1}{2}\Omega_\beta, \text{ say},$$

where α, α', β, β', are matrices of p rows and columns whose elements are integers*, it being supposed† that the determinant of the matrix $[\omega]$ does not vanish; and introduce p other variables $w_1, \ldots, w_p$ defined by

$$u_i = M_{i,1}w_1 + \ldots\ldots + M_{i,p}w_p, \qquad (i = 1, 2, \ldots, p)$$

or $u = Mw$, where M is a matrix of constants, whose determinant does not vanish; let the simultaneous increments of $w_1, \ldots, w_p$ when $u_1, \ldots, u_p$ are simultaneously increased by the constituents of the j-th column of $[\omega]$ be denoted by $v_{1,j}, \ldots, v_{p,j}$, and the simultaneous increments of $w_1, \ldots, w_p$ when $u_1, \ldots, u_p$ are simultaneously increased by the elements of the j-th column of $[\omega']$ be denoted by $v'_{1,j}, \ldots, v'_{p,j}$; then we have the equations $2Mv = 2[\omega] = \Omega_\alpha$, $2Mv' = 2[\omega'] = \Omega_\beta$, where v, v' denote the matrices of which respectively the (i, j) elements are $v_{i,j}$ and $v'_{i,j}$.

The function $\vartheta\left(u;\, \substack{Q'\\Q}\right)$ is a function of $w_1, \ldots, w_p$; we proceed now to investigate whether it is possible to choose the matrices $\alpha, \alpha', \beta, \beta'$ and the matrix M, so that the function may be regarded as a theta function in $w_1, \ldots, w_p$ of order r (cf. Chap. XV. § 284).

Let the arguments $w_1, \ldots, w_p$ be simultaneously increased by the constituents of the j-th column of the matrix $2v$; thereby $u_1, \ldots, u_p$ will be increased by the constituents of the j-th column of the matrix $[2\omega]$, and, since $\alpha, \alpha', \beta, \beta'$ consist of integers, the function $\vartheta\left(u;\, \substack{Q'\\Q}\right)$ will (Chap. X. § 190) be multiplied by a factor e^{L_j} where

$$L_j = (H_\alpha)^{(j)}\left[u + \tfrac{1}{2}(\Omega_\alpha)^{(j)}\right] - \pi i\,(\alpha)^{(j)}\,(\alpha')^{(j)} + 2\pi i\left[(\alpha)^{(j)}\,Q' - (\alpha')^{(j)}\,Q\right],$$

$(\alpha)^{(j)}$ denoting the row of p elements forming the j-th column of the matrix α, and $(\Omega_\alpha)^{(j)}$, $(H_\alpha)^{(j)}$ denoting, similarly, the j-th columns of the matrices $2\omega\alpha + 2\omega'\alpha'$, $2\eta\alpha + 2\eta'\alpha'$ respectively; this expression L_j, is linear in $w_1, \ldots, w_p$, and can be put into the form

$$L_j = r\,(2\zeta_{1,j}, \ldots, 2\zeta_{p,j})\left[(w_1, \ldots, w_p) + (v_{1,j}, \ldots, v_{p,j})\right] + 2\pi i K_j',$$

where $(w_1, \ldots, w_p)$ denotes the row letter whose elements are $w_1, \ldots, w_p$, and similarly $(v_{1,j}, \ldots, v_{p,j})$ is the row letter formed by the elements of the j-th column of the matrix v, r is a positive integer which is provisionally arbitrary, K_j' and $2\zeta_{1,j}, \ldots, 2\zeta_{p,j}$ are properly chosen constants, and $(2\zeta_{1,j}, \ldots, 2\zeta_{p,j})$ is the row letter formed of the last of these. Similarly, if the arguments $w_1, \ldots, w_p$ be simultaneously increased by $2v'_{1,j}, \ldots, 2v'_{p,j}$, the function $\vartheta\left(u;\, \substack{Q'\\Q}\right)$ takes a factor $e^{L'_j}$, where

$$L_j' = (H_\beta)^{(j)}\left[u + \tfrac{1}{2}(\Omega_\beta)^{(j)}\right] - \pi i\,(\beta)^{(j)}(\beta')^{(j)} + 2\pi i\left[(\beta)^{(j)}\,Q' - (\beta')^{(j)}\,Q\right],$$

and, with the same value of r, this can be put into the form

$$L_j' = r\,(2\zeta'_{1,j}, \ldots\ldots, 2\zeta'_{p,j})\left[(w_1, \ldots\ldots, w_p) + (v'_{1,j}, \ldots\ldots, v'_{p,j})\right] - 2\pi i K_j,$$

* The case when $\alpha, \alpha', \beta, \beta'$ are not integers is briefly considered in chapter XX.

† We have $\pi i\omega^{-1}[\omega] = \pi i a + b a'$; we suppose that the determinant of $\pi i a + b a'$ does not vanish.

where K_j, $\zeta'_{1,j}, \ldots, \zeta'_{p,j}$ are properly chosen constants. In these equations we suppose j to be taken in turn equal to $1, 2, \ldots, p$.

Comparing the two forms of L_j we have

$$(H_\alpha)^{(j)} Mw, \text{ or } \bar{M}(H_\alpha)^{(j)} w, = r(2\zeta_{1,j}, \ldots, 2\zeta_{p,j})(w_1, \ldots, w_p),$$

so that the (i, j)th element of the matrix $\bar{M}H_\alpha$ is $2r\zeta_{i,j}$; hence if ζ, ζ' denote respectively the matrices of the quantities $\zeta_{i,j}$ and $\zeta'_{i,j}$, we have

$$\bar{M}H_\alpha = 2r\zeta, \; \bar{M}H_\beta = 2r\zeta'; \qquad \text{(V.)}$$

from these we deduce, in virtue of the equations $2Mv = \Omega_\alpha$, $2Mv' = \Omega_\beta$,

$$\tfrac{1}{2}\bar{H}_\alpha\Omega_\alpha = \tfrac{1}{2}\bar{H}_\alpha \,.\, 2Mv = 2r\bar{\zeta}v, \quad \tfrac{1}{2}\bar{H}_\beta\Omega_\beta = \tfrac{1}{2}\bar{H}_\beta \,.\, 2Mv' = 2r\bar{\zeta}'v',$$

and therefore, in particular, comparing the (j, j)th elements on the two sides of these equations,

$$\tfrac{1}{2}(H_\alpha)^{(j)}(\Omega_\alpha)^{(j)} = 2r(\zeta)^{(j)}(v)^{(j)}, \quad \tfrac{1}{2}(H_\beta)^{(j)}(\Omega_\beta)^{(j)} = 2r(\zeta')^{(j)}(v')^{(j)},$$

where, as before, $(v)^{(j)}$ is the row letter formed by the elements of the j-th column of the matrix v, etc.; therefore the only remaining conditions necessary for the identification of the two forms of L_j and L_j', are

$$K_j' = (\alpha)^{(j)}Q' - (\alpha')^{(j)}Q - \tfrac{1}{2}(\alpha)^{(j)}(\alpha')^{(j)}, \; -K_j = (\beta)^{(j)}Q' - (\beta')^{(j)}Q - \tfrac{1}{2}(\beta)^{(j)}(\beta')^{(j)},$$

and the p pairs of equations of this form are included in the two

$$K' = \bar{\alpha}Q' - \bar{\alpha}'Q - \tfrac{1}{2}d(\bar{\alpha}\alpha'), \; -K = \bar{\beta}Q' - \bar{\beta}'Q - \tfrac{1}{2}d(\bar{\beta}\beta'), \qquad \text{(VI.)}$$

where K', K are row letters of p elements and $d(\bar{\alpha}\alpha')$, $d(\bar{\beta}\beta')$ are respectively the row letters of p elements constituted by the diagonal elements of the matrices $\bar{\alpha}\alpha'$, $\bar{\beta}\beta'$.

The equations (VI.) arise by identifying the two forms of L_j and L_j'; it is effectively sufficient to identify the two forms of e^{L_j} and $e^{L_j'}$; thus it is sufficient to regard the equations (VI.) as *congruences*, to the modulus 1.

We now impose upon the matrices v, v', ζ, ζ' the conditions

$$\bar{\zeta}v - \bar{v}\zeta = 0 = \bar{\zeta}'v' - \bar{v}'\zeta', \; \bar{\zeta}v' - \bar{v}\zeta' = \tfrac{1}{2}\pi i, \qquad \text{(VII.)}$$

which, as will be proved immediately, are equivalent to certain conditions for the matrices α, β, α', β'; then, denoting $\vartheta\left(u;\, {Q' \atop Q}\right)$ by $\phi(w_1, \ldots, w_p)$ or $\phi(w)$, it can be verified* that the $2p$ equations

$$\phi(\ldots, w_r + 2v_{r,j}, \ldots) = e^{L_j}\phi(w), \; \phi(\ldots, w_r + 2v'_{r,j}, \ldots) = e^{L_j'}\phi(w), \; (j = 1, \ldots, p),$$

where L_j, L_j' have the specified forms, lead to the equation

$$\phi(w + 2vm + 2v'm') = e^{r(2\zeta m + 2\zeta' m')(w + vm + v'm') - r\pi i mm' + 2\pi i(mK' - m'K)}\phi(w),$$

wherein m, m' are row letters consisting of any p integers; and this is the

* The verification is included in a more general piece of work which occurs in Chap. XIX.

characteristic equation for a theta function of order r with the associated constants $2v$, $2v'$, 2ζ, $2\zeta'$ (§ 284, p. 448).

The equations (VII.) are equivalent to conditions for the matrices v, v', ζ, ζ', entirely analogous to the conditions (ii), (iv), (v) of § 324 for the matrices ω, ω', η, η'. The condition analogous to (i) of § 324, namely that the determinant of the matrix v do not vanish, is involved in the hypothesis that the determinant of $\pi i a + b a'$ do not vanish. It will be proved below (§ 325) that the remaining condition involved in the definition of a theta function, viz. that the quadratic form $v^{-1}v'n^2$ has its imaginary part positive for real values of $n_1, \dots, n_p$, is a consequence of the corresponding condition for the matrices ω, ω'. We consider first the conditions for the equations (VII.).

In virtue of equations (V.), the equations (VII.) require

$$\bar{H}_\alpha \Omega_\beta - \bar{\Omega}_\alpha H_\beta = 2\bar{H}_\alpha M v' - 2\bar{v}\bar{M}H_\beta = 4r(\bar{\zeta}v' - \bar{v}\zeta') = 2r\pi i,$$

and, similarly,

$$\bar{H}_\alpha \Omega_\alpha - \bar{\Omega}_\alpha H_\alpha = 0,\ \ \bar{H}_\beta \Omega_\beta - \bar{\Omega}_\beta H_\beta = 0;$$

but

$$\tfrac{1}{4}(\bar{H}_\alpha \Omega_\beta - \bar{\Omega}_\alpha H_\beta),\ = (\bar{\alpha}\bar{\eta} + \bar{\alpha}'\bar{\eta}')(\omega\beta + \omega'\beta') - (\bar{\alpha}\bar{\omega} + \bar{\alpha}'\bar{\omega}')(\eta\beta + \eta'\beta'),$$
$$= \bar{\alpha}(\bar{\eta}\omega - \bar{\omega}\eta)\beta + \bar{\alpha}(\bar{\eta}\omega' - \bar{\omega}\eta')\beta' + \bar{\alpha}'(\bar{\eta}'\omega - \bar{\omega}'\eta)\beta + \bar{\alpha}'(\bar{\eta}'\omega' - \bar{\omega}'\eta')\beta',$$

and this, by the equations (B), § 140, is equal to

$$\tfrac{1}{2}\pi i(\bar{\alpha}\beta' - \bar{\alpha}'\beta);$$

thus

$$\bar{\alpha}\beta' - \bar{\alpha}'\beta = \bar{\beta}'\alpha - \bar{\beta}\alpha' = r, \qquad \text{(VIII.)}$$

and, similarly,

$$\bar{\alpha}\alpha' - \bar{\alpha}'\alpha = 0,\ \ \bar{\beta}\beta' - \bar{\beta}'\beta = 0;$$

and as before (§ 322) these three equations can be replaced by the three

$$\alpha\bar{\beta} = \beta\bar{\alpha},\ \ \alpha'\bar{\beta}' = \beta'\bar{\alpha}',\ \ \alpha\bar{\beta}' - \beta\bar{\alpha}' = r = \beta'\bar{\alpha} - \alpha'\bar{\beta}, \qquad \text{(IX.)}$$

the relations satisfied by the matrices α, β, α', β' respectively being *similar to those satisfied by* ω, ω', η, η', *with the change of the* $\frac{1}{2}\pi i$, *which occurs in the latter case, into* $-r$.

The number r which occurs in these equations is called the order of the transformation; when it is equal to 1 the transformation is called a linear transformation.

Ex. i. Prove that, with matrices of $2p$ rows and $2p$ columns,

$$\begin{pmatrix} \alpha & \beta \\ \alpha' & \beta' \end{pmatrix}\begin{pmatrix} \bar{\beta}' & -\bar{\beta} \\ -\bar{\alpha}' & \bar{\alpha} \end{pmatrix} = r\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \bar{\alpha} & \bar{\alpha}' \\ \bar{\beta} & \bar{\beta}' \end{pmatrix}\begin{pmatrix} \beta' & -\alpha' \\ -\beta & \alpha \end{pmatrix},$$

and

$$\begin{pmatrix} \alpha & \beta \\ \alpha' & \beta' \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \bar{\alpha} & \bar{\alpha}' \\ \bar{\beta} & \bar{\beta}' \end{pmatrix} = r\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The determinant of the matrix will be subsequently proved to be $+r^p$.

Ex. ii. Prove that the equations (V.) of § 324 are equivalent to

$$\begin{pmatrix} M & 0 \\ 0 & r\bar{M}^{-1} \end{pmatrix} \begin{pmatrix} 2\upsilon & 2\upsilon' \\ 2\zeta & 2\zeta' \end{pmatrix} = \begin{pmatrix} 2\omega & 2\omega' \\ 2\eta & 2\eta' \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \alpha' & \beta' \end{pmatrix}.$$

Ex. iii. If x, y, x_1, y_1 be any row letters of p elements, and X, Y, X_1, Y_1 be other such row letters, such that

$$(X, Y) = \begin{pmatrix} \alpha & \beta \\ \alpha' & \beta' \end{pmatrix} (x, y), \text{ or } \begin{matrix} X = \alpha x + \beta y, & X_1 = \alpha x_1 + \beta y_1, \\ Y = \alpha' x + \beta' y, & Y_1 = \alpha' x_1 + \beta' y_1, \end{matrix}$$

then the equations (VIII.) are the conditions for the self-transformation of the bilinear form $xy_1 - x_1 y$, which is expressed by the equation

$$XY_1 - X_1 Y = r(xy_1 - x_1 y).$$

325. Conversely when the matrices α, α', β, β' satisfy the equations (VIII.), the function $\vartheta\left(u;\, \begin{smallmatrix} Q' \\ Q \end{smallmatrix}\right)$ satisfies the determining equation for a theta function in $w_1, \ldots, w_p$, of order r, with the characteristic (K, K'), and with the associated constants 2υ, $2\upsilon'$, 2ζ, $2\zeta'$; and in virtue of the equations (VII.), the determinant of υ not vanishing, matrices a, b, h, of which the first two are symmetrical, can be taken such that

$$\mathrm{a} = \tfrac{1}{2}\zeta\upsilon^{-1}, \quad \mathrm{h} = \tfrac{1}{2}\pi i \upsilon^{-1}, \quad \mathrm{b} = \pi i \upsilon^{-1}\upsilon';$$

we proceed now to shew* that the real part of the quadratic form $\mathrm{b}n^2$ is negative for real values of $n_1, \ldots, n_p$, r being positive, as was supposed.

The quantity, or matrix, obtainable from any complex quantity, or matrix of complex quantities, by changing the sign of the imaginary part of that quantity, or of the imaginary parts of every constituent of that matrix, will be denoted by the suffix 0; and a similar notation will be used for row letters; further the symmetrical matrices $\omega^{-1}\omega'$, $\upsilon^{-1}\upsilon'$ will be denoted respectively by τ and τ', so that $b = \pi i \tau$, $\mathrm{b} = \pi i \tau'$; also τ, τ' will be written, respectively, in the forms $\tau_1 + i\tau_2$, $\tau_1' + i\tau_2'$, where τ_1, τ_2, τ_1', τ_2' are matrices of real quantities. Then, putting

$$x' = \bar{\upsilon}\bar{M}\bar{\omega}^{-1}x, \text{ and therefore } x_0' = \bar{\upsilon}_0\bar{M}_0\bar{\omega}_0^{-1}x_0,$$

where x', x denote rows of p complex quantities, and x_0', x_0 the rows of the corresponding conjugate complex quantities, and recalling that

$$\tau' = \bar{\tau}' = \bar{\upsilon}'\bar{\upsilon}^{-1}, \quad \omega^{-1}M\upsilon = \alpha + \tau\alpha', \quad \omega^{-1}M\upsilon' = \beta + \tau\beta',$$

we have

$$\tau' x' x_0' = \tau'\bar{\upsilon}\bar{M}\bar{\omega}^{-1}x \,.\, \bar{\upsilon}_0\bar{M}_0\bar{\omega}_0^{-1}x_0 = \bar{\upsilon}'\bar{M}\bar{\omega}^{-1}x \,.\, \bar{\upsilon}_0\bar{M}_0\bar{\omega}_0^{-1}x_0$$
$$= (\bar{\beta} + \bar{\beta}'\tau)\,x \,.\, (\bar{\alpha} + \bar{\alpha}'\tau_0)\,x_0;$$

and, if $x = x_1 + ix_2$, $x_0 = x_1 - ix_2$, where x_1, x_2 are real, this is equal to

$$(\bar{\beta} + \bar{\beta}'\tau_1 + i\bar{\beta}'\tau_2)(x_1 + ix_2) \,.\, (\bar{\alpha} + \bar{\alpha}'\tau_1 - i\bar{\alpha}'\tau_2)(x_1 - ix_2)$$

or

$$[\bar{\beta}P + \bar{\beta}'P' + i(\bar{\beta}Q + \bar{\beta}'Q')]\,[\bar{\alpha}P + \bar{\alpha}'P' - i(\bar{\alpha}Q + \bar{\alpha}'Q')],$$

* Hermite, *Compt. Rendus*, XL. (1855), Weber, *Ann. d. Mat.*, Ser. 2, t. ix. (1878—9).

where P, P', Q, Q' are row letters of p real quantities given by

$$P = x_1,\ P' = \tau_1 x_1 - \tau_2 x_2,\ Q = x_2,\ Q' = \tau_1 x_2 + \tau_2 x_1,$$

so that

$$PQ' - P'Q = \tau_2 (x_1^2 + x_2^2);$$

thus the coefficient of i in $\tau' x' x_0'$ is

$$(\bar{\alpha}P + \bar{\alpha}'P')(\bar{\beta}Q + \bar{\beta}'Q') - (\bar{\beta}P + \bar{\beta}'P')(\bar{\alpha}Q + \bar{\alpha}'Q'),$$

which, in virtue of the equations (IX.), is equal to $r(PQ' - P'Q)$ or $r\tau_2 (x_1^2 + x_2^2)$; thus the coefficient of i in $\tau' x' x_0'$ is equal to the coefficient of i in $r\tau x x_0$. Since x' may be regarded as arbitrarily assigned this proves that the imaginary part of $\tau' x' x_0$ is necessarily positive; and this includes the proposition we desired to establish.

Ex. Prove that the equation obtained is equivalent to

$$M_0 v_0 \tau_2' \bar{v} \bar{M} = r\omega_0 \tau_2 \bar{\omega}.$$

326. Of the general formulae thus obtained for the transformation of theta functions, the case of a linear transformation, for which $r = 1$, is of great importance; and we limit ourselves mainly to that case in the following parts of this chapter. We have shewn that a theta function of the first order, with assigned characteristic and associated constants, is unique, save for a factor independent of the argument; we have therefore, for $r = 1$, as a result of the theory here given, the equation

$$\vartheta\left(u;\ 2\omega,\ 2\omega',\ 2\eta,\ 2\eta';\ {Q' \atop Q}\right) = A\vartheta\left(w;\ 2v,\ 2v',\ 2\zeta,\ 2\zeta';\ {K' \atop K}\right).$$

We suppose α, β, α', β' to be any arbitrarily assigned matrices of integers satisfying the equations (VIII.) or (IX.); then there remains a certain redundancy of disposable quantities; we may for instance suppose ω, ω', η, η' and M to be given, and choose v, v', ζ, ζ' in accordance with these equations; or we may suppose ω, ω', v, ζ and ζ' to be prescribed and use these equations to determine M, v', η and η'. It is convenient to specify the results in two cases. We replace u, w respectively by U, W.

(i) $2\omega = 1,\ 2\omega' = \tau,\ \eta = a,\ \eta' = a\tau - \pi i,\ h = \pi i,\ b = \pi i\tau,$
$2v = 1,\ 2v' = \tau',\ \zeta = 0,\ \zeta' = -\pi i\ ,\ \mathrm{a} = 0\ ,\ \mathrm{h} = \pi i\ ,\ \mathrm{b} = \pi i\tau',$
$U = MW,\ M = \alpha + \tau\alpha',\ (\alpha + \tau\alpha')\tau' = \beta + \tau\beta',$

so that, as immediately follows from equations (IX.),

$$(\alpha + \tau\alpha')(\bar{\beta}' - \tau'\bar{\alpha}') = r = (\beta' - \alpha'\tau')(\bar{\alpha} + \bar{\alpha}'\tau),\ U = (\alpha + \tau\alpha')W,\ W = \frac{1}{r}(\bar{\beta}' - \tau'\bar{\alpha}')U,$$

and, because $\eta' = \eta\tau - \pi i$ and $\zeta = 0$,

$$a = \eta = \pi i\alpha'(\alpha + \tau\alpha')^{-1} = \frac{\pi i}{r}\alpha'(\bar{\beta}' - \tau'\bar{\alpha}'),$$

from which we get

$$aU^2 = \frac{\pi i}{r}\alpha'(\bar{\beta}' - \tau'\bar{\alpha}')\,U^2 = \pi i\alpha' W U = \pi i\bar{\alpha}'(\alpha + \tau\alpha')\,W^2.$$

These equations satisfy the necessary conditions, and lead, when $r = 1$, to

$$e^{\pi i\bar{\alpha}'(\alpha+\tau\alpha')W^2}\,\Theta\left(U;\,\tau;\,{Q' \atop Q}\right) = A\Theta\left(W;\,\tau';\,{K' \atop K}\right), \qquad \text{(X.)}$$

where A is independent of $U_1, \ldots, U_p$, and the characteristic (K, K') is determined from (Q, Q') by the equations (§ 324)

$$K' = \bar{\alpha}Q' - \bar{\alpha}'Q - \tfrac{1}{2}d(\bar{\alpha}\alpha'), \quad -K = \bar{\beta}Q' - \bar{\beta}'Q - \tfrac{1}{2}d(\bar{\beta}\beta').$$

The appearance of the exponential factor outside the Θ-function, in equation (X.), would of itself be sufficient reason for using, as we have done, the ϑ-function, in place of the Θ-function, in all general algebraic investigations*.

If in § 324 we put

$$u = 2\omega U, \quad \tau = \omega^{-1}\omega', \quad w = 2v W, \quad \tau' = v^{-1}v'$$

we easily find

$$\pi i\bar{\alpha}'(a + \tau a')\,W^2 = \tfrac{1}{2}\eta\omega^{-1}u^2 - \tfrac{1}{2}r\zeta v^{-1}w^2;$$

thus (§ 189, p. 283) equation (X.) includes the initial equation of this Article.

In general the function occurring on the left side of equation (X.) is a theta function in W of order r with associated constants $2v = 1$, $2v' = \tau'$, $2\zeta = 0$, $2\zeta' = -2\pi i$, and characteristic (K, K').

(ii) A particular case of (i), when the matrix α' consists of zeros, is given by the formulae

$$2\omega = 1,\ 2\omega' = \tau,\ \eta = 0,\ \eta' = -\pi i,\ a = 0,\ h = \pi i,\ b = \pi i\tau,$$
$$2v = 1,\ 2v' = \tau',\ \zeta = 0,\ \zeta' = -\pi i,\ \mathrm{a} = 0,\ \mathrm{h} = \pi i,\ \mathrm{b} = \pi i\tau',$$

$$U = \alpha W, \quad \tau' = \alpha^{-1}(\beta + \tau\beta'), \quad \tau = \frac{1}{r}(\alpha\tau' - \beta)\,\bar{\alpha},$$

$$\begin{pmatrix}\alpha & \beta \\ \alpha' & \beta'\end{pmatrix} = \begin{pmatrix}\alpha & \beta \\ 0 & r\bar{\alpha}^{-1}\end{pmatrix}, \quad \text{where } \alpha\bar{\beta} = \beta\bar{\alpha}.$$

Then the function $\Theta\left(U;\,\tau;\,{Q' \atop Q}\right)$ or $\Theta\left[\alpha W;\,\frac{1}{r}(\alpha\tau' - \beta)\,\bar{\alpha};\,{Q' \atop Q}\right]$ is a theta function in W, of order r, with associated constants $2v = 1$, $2v' = \tau'$, $2\zeta = 0$, $2\zeta' = -2\pi i$, and characteristic (K, K') given by

$$K' = \bar{\alpha}Q', \quad -K = \bar{\beta}Q' - r\alpha^{-1}Q - \tfrac{1}{2}d(r\bar{\beta}\bar{\alpha}^{-1}),$$

and, in particular, when $r = 1$ we have

$$\Theta\left(U;\,\tau;\,{Q' \atop Q}\right) = A\Theta\left(W;\,\tau';\,{K' \atop K}\right), \qquad \text{(XI.)}$$

where A is independent of $U_1, \ldots, U_p$.

* Cf. § 189 (Chap. X.); and for the case $p = 1$, Cayley, *Liouville*, x. (1845), or *Collected Works*, Vol. I., p. 156 (1889).

327. It is clear that the results just obtained, for the linear transformation of theta functions, contain the answer to the enquiry as to the changes in the Riemann theta functions which arise in virtue of a change in the fundamental system of period loops. Before considering the results in further detail, it is desirable to be in possession of certain results as to the transformation of the characteristics of the theta function, which we now give; the reader who desires may omit the demonstrations, noticing only the results, and proceed at once to § 332. We retain the general value r for the order of the transformation, though the applications of greatest importance are those for which $r = 1$.

As before let $d(\gamma)$ denote the row of p quantities constituted by the diagonal elements of any matrix γ of p rows and columns; in all cases here arising γ is a symmetrical matrix; then we have

$$\left.\begin{array}{ll} \alpha\, d(\bar\beta\beta') + \beta\, d(\bar\alpha\alpha') \equiv r d(\alpha\bar\beta), & \bar\beta' d(\alpha\bar\beta) + \bar\beta d(\alpha'\bar\beta') \equiv r d(\bar\beta\beta') \\ \alpha' d(\bar\beta\beta') + \beta' d(\bar\alpha\alpha') \equiv r d(\alpha'\bar\beta'), & \bar\alpha' d(\alpha\bar\beta) + \bar\alpha d(\alpha'\bar\beta') \equiv r d(\bar\alpha\alpha') \end{array}\right\} \pmod 2$$

and

$$\left.\begin{array}{l} d(\bar\alpha\alpha')\, d(\bar\beta\beta') \equiv (r+1)\,\Sigma d(\bar\beta\alpha') \equiv (r+1)\,\Sigma d(\bar\beta'\alpha) \\ d(\alpha\bar\beta)\, d(\alpha'\bar\beta') \equiv (r+1)\,\Sigma d(\alpha\bar\beta') \equiv (r+1)\,\Sigma d(\beta\bar\alpha') \end{array}\right\} \pmod 2,$$

so that, when $r = 1$ or is any odd integer,

$$d(\bar\alpha\alpha') \,.\, d(\bar\beta\beta') \equiv d(\alpha\bar\beta) \,.\, d(\alpha'\bar\beta') \equiv 0 \pmod 2.$$

The last result contains the statement that the linear transformation of the zero theta-characteristic is always an even characteristic.

For the equations

$$\beta'\bar\alpha - \alpha'\bar\beta = r, \quad \alpha\bar\beta = \beta\bar\alpha,$$

give

$$\alpha\bar\beta\beta'\bar\alpha - \beta\bar\alpha\alpha'\bar\beta = r\alpha\bar\beta,$$

and therefore

$$\bar\beta\beta' z^2 - \bar\alpha\alpha' y^2 = r\alpha\bar\beta x^2,$$

where x is any row letter of p integers, and $z = \bar\alpha x$, $y = \bar\beta x$; but if γ be a symmetrical matrix of integers and t be any row letter of p integers γt^2, $= \gamma_{11} t_1^2 + \ldots + 2\gamma_{12} t_1 t_2 + \ldots$, is $\equiv \gamma_{11} t_1^2 + \ldots + \gamma_{pp} t_p^2$, and therefore $\equiv \gamma_{11} t_1 + \ldots + \gamma_{pp} t_p$, or $\equiv d(\gamma) \,.\, t$, for modulus 2; hence

$$d(\bar\beta\beta') z - d(\bar\alpha\alpha') y \equiv r d(\alpha\bar\beta) x \pmod 2$$

or

$$[\alpha d(\bar\beta\beta') + \beta d(\bar\alpha\alpha') - r d(\alpha\bar\beta)]\, x \equiv 0 \pmod 2;$$

and as this is true for any row letter of integers, x, the first of the given equations follows at once. The second of the equations also follows from $\beta'\bar\alpha - \alpha'\bar\beta = r$, in the same way, and the third and fourth follow similarly from $\bar\beta'\alpha - \bar\beta\alpha' = r$.

To prove the fifth equation, we have, since $\beta'\bar\alpha - \alpha'\bar\beta = r$,

$$\bar\beta\beta'\bar\alpha\alpha' = \bar\beta\alpha'\bar\beta\alpha' + r\bar\beta\alpha'$$

or

$$ba = c^2 + rc,$$

where $b=\bar{\beta}\beta'$, $a=\bar{a}a'$, $c=\bar{\beta}a'$; hence, equating the sums of the diagonal elements on the two sides of the equation, we have

$$\sum_{j=1}^{p}\sum_{i=1}^{p} b_{i,j}a_{j,i}=\sum_{j=1}^{p}\sum_{i=1}^{p} c_{i,j}c_{j,i}+r\sum_{i=1}^{p} c_{i,i};$$

therefore, as, unless $i=j$, $b_{i,j}\,a_{j,i}=b_{j,i}\,a_{i,j}$, because a, b are symmetrical matrices, and as

$$c_{i,j}\,c_{j,i}=c_{j,i}\,c_{i,j},$$

we obtain

$$\sum_{i=1}^{p} a_{i,i}\,b_{i,i}\equiv\sum_{i=1}^{p}(c^2_{i,i}+rc_{i,i})\equiv(r+1)\sum_{i=1}^{p} c_{i,i}.$$

The sixth equation is obtained in a similar way, starting from $\bar{\beta}'a-\bar{\beta}a'=r$.

Of the results thus derived we make, now, application to the case when r is odd, limiting ourselves to the case when the characteristic (Q, Q') consists of half-integers; we put then $Q=\frac{1}{2}q$, $Q'=\frac{1}{2}q'$, so that q, q' each consist of p integers; then K, K' are also half-integers, respectively equal to $\frac{1}{2}k$, $\frac{1}{2}k'$, say, where

$$k'=\bar{a}q'-\bar{a}'q-d(\bar{a}a'),\quad -k=\bar{\beta}q'-\bar{\beta}'q-d(\bar{\beta}\beta').$$

In most cases of these formulae, it is convenient to regard them as congruences, to modulus 2. This is equivalent to neglecting additive *integral* characteristics.

From these equations we derive immediately, in virtue of the equations of the present Article

$$q\equiv ak+\beta k'+d(a\bar{\beta}),\quad q'\equiv a'k+\beta'k'+d(a'\bar{\beta}')\quad(\text{mod. } 2)$$

and

$$qq'\equiv kk'\quad(\text{mod. } 2).$$

Further if μ, μ' be row letters of p integers, and

$$\nu'=\bar{a}\mu'-\bar{a}'\mu-d(\bar{a}a'),\quad -\nu=\bar{\beta}\mu'-\bar{\beta}'\mu-d(\bar{\beta}\beta'),$$

we find, also in virtue of the equations of the present Article,

$$k\nu'-k'\nu\equiv q\mu'-q'\mu+(\mu'+q')\,d(a\bar{\beta})+(\mu+q)\,d(a'\bar{\beta}'),\quad(\text{mod. } 2);$$

therefore, if also

$$\sigma'=\bar{a}\rho'-\bar{a}'\rho-d(\bar{a}a'),\quad -\sigma=\bar{\beta}\rho'-\bar{\beta}'\rho-d(\bar{\beta}\beta'),$$

we have

$$k\nu'-k'\nu+\nu\sigma'-\nu'\sigma+\sigma k'-\sigma'k\equiv q\mu'-q'\mu+\mu\rho'-\mu'\rho+\rho q'-\rho'q\quad(\text{mod. } 2).$$

Denoting the half-integer characteristics $\frac{1}{2}\begin{pmatrix}q'\\q\end{pmatrix}$, $\frac{1}{2}\begin{pmatrix}\mu'\\\mu\end{pmatrix}$, $\frac{1}{2}\begin{pmatrix}\rho'\\\rho\end{pmatrix}$ by A, B, C, and the characteristics $\frac{1}{2}\begin{pmatrix}k'\\k\end{pmatrix}$, $\frac{1}{2}\begin{pmatrix}\nu'\\\nu\end{pmatrix}$, $\frac{1}{2}\begin{pmatrix}\sigma'\\\sigma\end{pmatrix}$, which we call *the transformed characteristics*, by A', B', C', we have therefore the results (§ 294)

$$|A|\equiv|A'|,\quad |A, B, C|\equiv|A', B', C'|,\quad(\text{mod. } 2)$$

or, in words, *in a linear transformation of a theta function with half-integer characteristic, and in any transformation of odd order, an odd (or even) characteristic transforms into an odd (or even) characteristic, and three syzygetic (or azygetic) characteristics transform into three syzygetic (or azygetic) characteristics.*

Of these the first result is immediately obvious when $r=1$ from the equation of transformation (§ 326), by changing w into $-w$.

Hence also it is obvious that if A be an even characteristic for which $\vartheta(0; A)$ vanishes, then the transformed characteristic A' is also an even characteristic for which the transformed function $\underline{\vartheta}(0; A')$ vanishes.

328. If in the formula of linear transformation of theta functions with half-integer characteristic, which we may write

$$\vartheta\left[u;\ \tfrac{1}{2}\binom{q'}{q}\right] = A\underline{\vartheta}\left[w;\ \tfrac{1}{2}\binom{k'}{k}\right],$$

we replace u by $u + \frac{1}{2}\Omega_m = u + \omega m + \omega' m'$, where m, m' denote rows of integers, and, therefore, since $\omega = M(v\bar{\beta}' - v'\bar{\alpha}')$, $\omega' = M(-v\bar{\beta} + v'\bar{\alpha})$, (cf. Ex. i., § 324), replace w by $w + vn + v'n'$, where

$$n' = \bar{\alpha}m' - \bar{\alpha}'m, \quad -n = \bar{\beta}m' - \bar{\beta}'m,$$

we obtain (§ 189, formula (L))

$$\vartheta\left[u;\ \tfrac{1}{2}\binom{q'+m'}{q+m}\right] = A'\underline{\vartheta}\left[w;\ \tfrac{1}{2}\binom{k'+n'}{k+n}\right],$$

where A' is independent of $u_1, \ldots, u_p$, and $k'+n'$, $k+n$ are obtainable from $q'+m'$, $q+m$ by the same formulae whereby k', k are obtained from q', q, namely

$$k' + m' = \bar{\alpha}(q' + m') - \bar{\alpha}'(q+m) - d(\bar{\alpha}\alpha'),$$

$$-(k+m) = \bar{\beta}(q' + m') - \bar{\beta}'(q+m) - d(\bar{\beta}\beta');$$

these formulae are different from those whereby n', n are obtained from m', m; for this reason it is sometimes convenient to speak of $\frac{1}{2}\binom{q'}{q}$ as a *theta characteristic*, and of $\frac{1}{2}\binom{m'}{m}$ as a *period characteristic*; as it arises here the difference lies in the formulae of transformation; but other differences will appear subsequently; these differences are mainly consequences of the obvious fact that, when half-integer characteristics which differ by integer characteristics are regarded as identical, the sum of any odd number of theta characteristics is transformed as a theta characteristic, while the sum of any even number of theta characteristics is transformed as a period characteristic. In other words, a period characteristic is to be regarded as the (sum or) difference of two theta characteristics.

It will appear for instance that the characteristics associated in §§ 244, 245, Chap. XIII. with radical functions of the form $\sqrt{X^{(2\nu+1)}}$ are to be regarded as theta characteristics—and the characteristics associated in § 245 with radical functions of the form $\sqrt{X^{(2\mu)}}$, which are defined as sums of characteristics associated with functions $\sqrt{X^{(2\nu+1)}}$, are to be regarded as period characteristics.

We may regard the distinction* thus explained somewhat differently, by taking as the fundamental formula of linear transformation that which expresses $\vartheta\left[u\,;\ \tfrac{1}{2}\binom{q'}{q}\right]$ in terms of $\vartheta\left[w+\tfrac{1}{2}\Omega_r\,;\ \tfrac{1}{2}\binom{l'}{l}\right]$, where

$$r'=d\,(\bar{a}a'),\quad r=d\,(\bar{\beta}\beta'),$$

and

$$l'=k'+d\,(\bar{a}a')=\bar{a}q'-\bar{a}'q,\quad -l=-k+d\,(\bar{\beta}\beta')=\bar{\beta}q'-\bar{\beta}'q.$$

In the following pages we shall always understand by 'characteristic,' a theta characteristic; when it is necessary to call attention to the fact that a characteristic is a period characteristic this will be done.

329. It is clear that the formula of linear transformation of a theta function with any half-integer characteristic is obtainable from the particular case

$$\vartheta\,(u)=A\vartheta\left[w\,;\ \tfrac{1}{2}\binom{r'}{r}\right],$$

where $r'=d\,(\bar{\alpha}\alpha')$, $r=d\,(\bar{\beta}\beta')$, by the addition of half periods to the arguments. It is therefore of interest to shew that matrices α, β, α', β' can be chosen, satisfying the equations

$$\alpha\bar{\beta}=\beta\bar{\alpha},\quad \alpha'\bar{\beta}'=\beta'\bar{\alpha}',\quad \alpha\bar{\beta}'-\beta\bar{\alpha}'=1,$$

which will make the characteristic $\tfrac{1}{2}\binom{r'}{r}$ equal to any even half-integer characteristic.

Any even half-integer characteristic, being denoted by

$$\tfrac{1}{2}\begin{pmatrix}k_1'\ldots k_p'\\ k_1\ldots k_p\end{pmatrix},$$

we may, momentarily, call $\binom{k_i'}{k_i}$ the i-th column of the characteristic; then the columns may be of four sorts,

$$\binom{0}{0},\ \binom{1}{0},\ \binom{0}{1},\ \binom{1}{1},$$

but the number of columns of the last sort must be even; we build now a matrix

$$\begin{pmatrix}\alpha & \beta\\ \alpha' & \beta'\end{pmatrix}$$

* Theta characteristics have also been named eigentliche Charakteristiken and Primcharakteristiken; they consist of $2^{p-1}(2^p-1)$ odd and $2^{p-1}(2^p+1)$ even characteristics. The period characteristics have been called Gruppencharakteristiken and Elementarcharakteristiken or sometimes relative Charakteristiken. For them the distinction of odd and even is unimportant—while the distinction between the zero characteristic—which cannot be written as the sum of two different theta characteristics—and the remaining $2^{2p}-1$ characteristics, is of great importance. The distinction between theta characteristics and period characteristics has been insisted on by Noether, in connection with the theory of radical forms—Cf. Noether, *Math. Annal.* XXVIII. (1887), p. 373, Klein, *Math. Annal.* XXXVI. (1890), p. 36, Schottky, *Crelle*, CII. (1888), p. 308. The distinction is in fact observed in the *Abel'sche Functionen* of Clebsch and Gordan, in the manner indicated in the text.

of $2p$ rows and columns by the following rule*—Corresponding to a column of the characteristic of the first sort, say the i-th column, we take $\alpha_{i,i}=\beta'_{i,i}=1$, but take every other element of the i-th row and i-th column of α and β', and every element of the i-th row and i-th column of β and α' to be zero; corresponding to a column of the characteristic of the second sort, say the j-th column, we take $\alpha_{j,j}=\beta'_{j,j}=\alpha'_{j,j}=1$, but take every other element of the j-th row and j-th column of α, β', α', and every element of the j-th row and column of β, to be zero; corresponding to a column of the characteristic of the third sort, say the m-th column, we take $\alpha_{m,m}=\beta_{m,m}=\beta'_{m,m}=1$, but take every other element of the m-th row and column of α, β, β' and every element of the m-th row and column of α' to be zero; corresponding to a pair of columns of the characteristic of the fourth sort, say the ρ-th and σ-th, we take $\alpha_{\rho,\rho}=\beta_{\rho,\rho}=\beta'_{\rho,\rho}=1$, $\alpha_{\sigma,\sigma}=\alpha'_{\sigma,\sigma}=\beta'_{\sigma,\sigma}=1$, $\alpha_{\sigma,\rho}=1$, $\beta_{\rho,\sigma}=-1$, $\alpha'_{\sigma,\rho}=1$, $\beta'_{\rho,\sigma}=-1$, and take every other element of the ρ-th row and column and of the σ-th row and column, of each of the four matrices α, α', β, β', to be zero. Then it can be shewn that the matrix thus obtained satisfies all the necessary conditions and gives $k'=d(\bar{\alpha}\alpha')$, $k=d(\bar{\beta}\beta')$.

Consider for instance the case $p=5$, and the characteristic

$$\tfrac{1}{2}\begin{pmatrix}0&1&0&1&1\\0&0&1&1&1\end{pmatrix};$$

the matrix formed by the rules from this characteristic is

$$\left(\begin{array}{ccccc|ccccc}
1&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&1&0&0\\
0&0&0&1&0&0&0&0&1&-1\\
0&0&0&1&1&0&0&0&0&0\\
\hline
0&0&0&0&0&1&0&0&0&0\\
0&1&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&1&-1\\
0&0&0&1&1&0&0&0&0&1
\end{array}\right)$$

and it is immediately verified that this satisfies the equations for a linear transformation (§ 324 (IX.), for $r=1$), and gives, for the diagonal elements of $\bar{\alpha}\alpha'$, $\bar{\beta}\beta'$, respectively, the elements 01011 and 00111.

Since we can transform the zero characteristic into any even characteristic, we can of course transform any even characteristic into the zero characteristic; for instance, when there is an even theta function which vanishes for zero values of the arguments, we can, by making a linear transformation, take for this function the theta function with zero characteristic.

* Clebsch and Gordan, *Abel. Fctnen* (Leipzig, 1866), p. 318.

Ex. For the hyperelliptic case, when $p=3$, the period loops being taken as in § 200, the theta-function whose characteristic is $\frac{1}{2}\begin{pmatrix}1&0&1\\1&1&1\end{pmatrix}$ vanishes for zero arguments (§ 203); prove that the transformation given by

$$\alpha=\begin{vmatrix}1&0&0\\0&1&0\\-1&0&1\end{vmatrix},\quad \beta=\begin{vmatrix}-1&0&0\\0&-1&0\\1&0&0\end{vmatrix},\quad \alpha'=\begin{vmatrix}0&0&-1\\0&0&0\\0&0&-1\end{vmatrix},\quad \beta'=\begin{vmatrix}1&0&1\\0&1&0\\0&0&1\end{vmatrix}$$

is a linear transformation and gives an equation of the form

$$\vartheta\left[u;\ \tfrac{1}{2}\begin{pmatrix}1&0&1\\1&1&1\end{pmatrix}\right]=A\vartheta[w;\ 0],$$

where A is independent of $u_1, \ldots, u_p$.

330. We have proved (§ 327) that if three half-integer theta characteristics be syzygetic (or azygetic) the characteristics arising from them by any linear transformation are also syzygetic (or azygetic). It follows therefore that a Göpel system of 2^r characteristics, syzygetic in threes (§ 297, Chap. XVII.), transforms into such a Göpel system. Also the $2^{2\sigma}$ Göpel systems of § 298, having a definite character, that of being all odd or all even, transform into systems having the same character. And the $2\sigma+1$ fundamental Göpel systems (§ 300), which satisfy the condition that any three characteristics chosen from different systems of these are azygetic, transform into such systems; moreover since the linear transformation of a characteristic which is the sum of an odd number of other characteristics is the sum of the transformations of these characteristics, the transformations of these $2\sigma+1$ systems possess the property belonging to the original systems, that all the $2^{2\sigma}$ Göpel systems having a definite character are representable by the combinations of an odd number of them. It follows therefore that the theta relations obtained in Chap. XVII., based on the properties of the Göpel systems, persist after any linear transformation.

331. But questions are then immediately suggested, such as these: What are the simplest Göpel systems from which all others are obtainable* by linear transformation? Is it possible to derive the $2^{2\sigma}$ Göpel systems of § 298, having a definite character, by linear transformation, from systems based upon the $2^{2\sigma}$ characteristics obtainable by taking all possible half-integer characteristics in which $p-\sigma$ columns consist of zeros? Are the fundamental sets of $2p+1$ three-wise azygetic characteristics, by the odd combinations of which all the 2^{2p} half-integer characteristics can be represented (§ 300), all derivable by linear transformation from one such set?

We deal here only with the answer to the last question—and prove the following result: *Let $D, D_1, \ldots, D_{2p+1}$ be any $2p+2$ half-integer characteristics, such that, for $i<j$,*

* An obvious Göpel group of 2^p characteristics is formed by all the characteristics in which the upper row of elements are all zeros, and the lower row of elements each $=0$ or $\frac{1}{2}$.

$i=1, \ldots, 2p$, $j=2, \ldots, 2p+1$, *we have* $|D, D_i, D_j|=1$; *then it is possible to choose a half-integer characteristic* E, *and a linear transformation, such that the characteristics*

$$ED,\ ED_1,\ \ldots,\ ED_{2p+1}$$

transform into

$$0,\ \lambda_1,\ \ldots,\ \lambda_{2p+1},$$

where $\lambda_1, \ldots, \lambda_{2p+1}$ *are certain characteristics to be specified, of which (by § 327) every two are azygetic.* It will follow that if $D', D_1', \ldots, D'_{2p+1}$ be any other set of $2p+2$ characteristics of which every three are azygetic, a characteristic E', and a linear transformation, can be found such that, with a proper characteristic E, the set $ED, ED_1, \ldots, ED_{2p+1}$ transforms into $E'D', E'D_1', \ldots, E'D'_{2p+1}$. It will be shewn that the characteristics $\lambda_1, \ldots, \lambda_{2p+1}$ can be written down by means of the hyperelliptic half-periods denoted (§ 200) by u^{a, c_1}, u^{a, a_1}, $u^{a, c_2}, \ldots, u^{a, a_p}$, $u^{a, c}$; it has already been remarked (§ 294, Ex.) that the characteristics associated with these half-periods are azygetic in pairs. The proof which is to be given establishes an interesting connexion between the conditions for a linear transformation and the investigation of § 300, Chap. XVII.

Taking an Abelian matrix,

$$\begin{pmatrix} a & \beta \\ a' & \beta' \end{pmatrix},$$

for which

$$\bar{a}a' - \bar{a}'a = 0, \quad \bar{\beta}\beta' - \bar{\beta}'\beta = 0, \quad \bar{a}\beta' - \bar{a}'\beta = 1,$$

define characteristics of integers by means of the equations

$$a_r = \begin{pmatrix} a'_{1,r}, & a'_{2,r}, & \ldots, & a'_{p,r} \\ a_{1,r}, & a_{2,r}, & \ldots, & a_{p,r} \end{pmatrix}, \quad b_r = \begin{pmatrix} \beta'_{1,r}, & \beta'_{2,r}, & \ldots, & \beta'_{p,r} \\ \beta_{1,r}, & \beta_{2,r}, & \ldots, & \beta_{p,r} \end{pmatrix}, \quad a_r' = -a_r,$$

where $a'_{s,r}$ is the r-th element of the s-th row of the matrix a', etc. and $r=1, 2, \ldots, p$; then the symbol which, in accordance with the notation of § 294, Chap. XVII., we define by the equation

$$|A_r, B_s| = a_{1,r}\beta'_{1,s} + \ldots + a_{p,r}\beta'_{p,s} - a'_{1,r}\beta_{1,s} - \ldots - a'_{p,r}\beta_{p,s},$$

is the (r, s)-th element of the matrix $\bar{a}\beta' - \bar{a}'\beta$, and may be denoted by $(\bar{a}\beta' - \bar{a}'\beta)_{r,s}$; thus the conditions for the matrices a, a', β, β' are equivalent to the $p(2p-1)$ equations

$$|A_r, B_r|=1, \quad |A_r, B_s|=0, \quad |A_r, A_s|=0, \quad |B_r, B_s|=0, \quad (r \neq s,\ r, s=1, 2, \ldots, p),$$

whereof the first gives p conditions, the second $p(p-1)$ conditions, and the third and fourth each $\frac{1}{2}p(p-1)$ conditions. It is convenient also to notice, what are corollaries from these, the equations

$$|B_s, A_r| = -|A_r, B_s| = 0, \quad |B_r, A_r| = -|A_r, B_r| = -1, \quad |B_r, A_r'| = -|A_r', B_r| = |A_r, B_r| = 1.$$

Consider now the $2p+1$ characteristics, of integers, given by

$$a_1,\ b_1,\ a_1'b_1a_2,\ a_1'b_1b_2,\ a_1'b_1a_2'b_2a_3,\ a_1'b_1a_2'b_2b_3,\ \ldots,\ a_1'b_1\ldots b_{p-1}b_p,\ a_1'b_1\ldots a_p'b_p,$$

whereof the first $2p$ are pairs of the type

$$a_1'b_1\ldots a'_{r-1}b_{r-1}a_r, \quad a_1'b_1\ldots a'_{r-1}b_{r-1}b_r,$$

for $r=1, 2, \ldots, p$, and $a_1'b_1a_2$ means the sum, without reduction, of the characteristics a_1', b_1, a_2, and so in general. The sum of these characteristics is a characteristic consisting wholly of even integers. If these characteristics be denoted, in order, by $c_1, c_2, \ldots, c_{2p+1}$, it immediately follows, from the fundamental equations connecting $a_1, \ldots, b_p$, that

$$c_{i,1}c'_{j,1} + \ldots - c'_{i,1}c_{j,1} - \ldots = 1, \quad \left(i<j,\ \begin{matrix} i=1, 2, \ldots, 2p \\ j=2, 3, \ldots, 2p+1 \end{matrix}\right).$$

Thus the $(2p+1)$ half-integer characteristics derivable from $c_1, c_2, \ldots, c_{2p+1}$, namely $C_1=\frac{1}{2}c_1, \ldots, C_{2p+1}=\frac{1}{2}c_{2p+1}$, are azygetic in pairs.

Conversely let $D, D_1, \ldots, D_{2p+1}$ be any half-integer characteristics such that, for $i<j$, $i=1, \ldots, 2p$, $j=2, \ldots, 2p+1$, we have $|D, D_i, D_j|=1$, so that (§ 300, p. 496) there exist connecting them only two relations (i) that their sum is a characteristic of integers, and (ii) a relation connecting an odd number of them; putting $C_i=D'D_i\,(i=1, \ldots, 2p)$, where $D'=-D$, we obtain a set of independent characteristics $C_1, \ldots, C_{2p}$, such that for $i<j$,

$$|C_i, C_j|=1, \quad \begin{pmatrix} i=1, 2, \ldots, 2p-1 \\ j=2, 3, \ldots, 2p \end{pmatrix};$$

taking $C_{2p+1}=C_1'C_2C_3'C_4\ldots C'_{2p-1}C_{2p}$, where $C'_{2r-1}=-C_{2r-1}$, we have also the $2p$ equations

$$|C_m, C_{2p+1}|=1, \quad (m=1, 2, \ldots, 2p).$$

Thus putting $C_1=\frac{1}{2}c_1, \ldots, C_{2p+1}=\frac{1}{2}c_{2p+1}$, we can obtain an Abelian matrix by means of the equations, previously given,

$$c_{2r-1}=a_1'b_1\ldots a'_{r-1}b_{r-1}a_r, \quad c_{2r}=a_1'b_1\ldots a'_{r-1}b_{r-1}b_r, \quad c_{2p+1}=a_1'b_1\ldots a_p'b_p,$$

the i-th column of this matrix consisting of the elements of the lower and upper rows of the integer characteristic a_i or b_i, according as $i<p+1$ or $i>p$. We proceed now to find the result of applying the linear transformation, given by this Abelian matrix, to the half-integer characteristics $C_1, \ldots, C_{2p+1}$.

The equations for the transformation of the characteristic $\frac{1}{2}\begin{pmatrix} q' \\ q \end{pmatrix}$ to the characteristic $\frac{1}{2}\begin{pmatrix} k' \\ k \end{pmatrix}$, which are (§ 324, VI.),

$$k'=\bar{a}q'-\bar{a}'q-d(\bar{a}a'), \quad -k=\bar{\beta}q'-\bar{\beta}'q-d(\bar{\beta}\beta'),$$

are equivalent, in the notation here employed, to

$$k_i'=|A_i, Q|-[d(\bar{a}a')]_i, \quad -k_i=|B_i, Q|-[d(\bar{\beta}\beta')]_i, \quad (i=1, 2, \ldots, p),$$

where $A_i=\frac{1}{2}a_i$, $Q=\frac{1}{2}q$; taking

$$Q=\tfrac{1}{2}a_1'b_1\ldots a'_{r-1}b_{r-1}a_r, \ =\tfrac{1}{2}a_1'b_1\ldots a'_{r-1}b_{r-1}b_r, \text{ and } =\tfrac{1}{2}a_1'b_1\ldots a_p'b_p,$$

in turn, we immediately find that the transformations of the characteristics C_{2r-1}, C_{2r}, C_{2p+1}, are given, omitting integer characteristics, by

$$\tfrac{1}{2}\begin{pmatrix} d(\bar{a}a') \\ d(\bar{\beta}\beta') \end{pmatrix}+\tfrac{1}{2}\begin{pmatrix} 1\,1\ldots1\,0\,0\ldots0 \\ 1\,1\ldots1\,1\,0\ldots0 \end{pmatrix}, \ \tfrac{1}{2}\begin{pmatrix} d(\bar{a}a') \\ d(\bar{\beta}\beta') \end{pmatrix}+\tfrac{1}{2}\begin{pmatrix} 1\,1\ldots1\,1\,0\ldots0 \\ 1\,1\ldots1\,0\,0\ldots0 \end{pmatrix}, \ \tfrac{1}{2}\begin{pmatrix} d(\bar{a}a') \\ d(\bar{\beta}\beta') \end{pmatrix}+\tfrac{1}{2}\begin{pmatrix} 1\,1\ldots1 \\ 1\,1\ldots1 \end{pmatrix},$$

or, say, by

$$\tfrac{1}{2}\begin{pmatrix} d(\bar{a}a') \\ d(\bar{\beta}\beta') \end{pmatrix}+\tfrac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}^{r-1}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix}^{p-r}, \ \tfrac{1}{2}\begin{pmatrix} d(\bar{a}a') \\ d(\bar{\beta}\beta') \end{pmatrix}+\tfrac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}^{r-1}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix}^{p-r}, \ \tfrac{1}{2}\begin{pmatrix} d(\bar{a}a') \\ d(\bar{\beta}\beta') \end{pmatrix}+\tfrac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}^{p},$$

respectively.

Now let the characteristics

$$\tfrac{1}{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix}^{p-1}, \ \tfrac{1}{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix}^{p-1}, \ldots, \tfrac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}^{r-1}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix}^{p-r}, \ \tfrac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}^{r-1}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix}^{p-r}, \ldots, \tfrac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}^{p},$$

be respectively denoted by

$$\lambda_1, \lambda_2, \ldots, \lambda_{2r-1}, \lambda_{2r}, \ldots, \lambda_{2p+1};$$

then we have proved that the half-integer characteristic DD_i transforms, save for an integer characteristic, into $\lambda_i+\frac{1}{2}\begin{pmatrix} r' \\ r \end{pmatrix}$, where $r=d(\bar{\beta}\beta')$, $r'=d(\bar{a}a')$; since the transformation of the sum of two characteristics is the sum of their transformations added to $\frac{1}{2}\begin{pmatrix} r' \\ r \end{pmatrix}$, and since the characteristic $\frac{1}{2}\begin{pmatrix} s' \\ s \end{pmatrix}$, where $s'=d(\alpha'\bar{\beta}')$, $s=d(\alpha\bar{\beta})$, transforms into the zero characteristic (§ 327), it follows that the transformation of the characteristic $\frac{1}{2}\begin{pmatrix} s' \\ s \end{pmatrix}+DD_i$ is the characteristic λ_i; hence, putting $E=\frac{1}{2}\begin{pmatrix} s' \\ s \end{pmatrix}+D$, and omitting integer characteristics, the characteristics

$$ED,\ ED_1,\ \dots,\ ED_{2p+1}$$

transform, respectively, into

$$0,\ \lambda_1,\ \dots,\ \lambda_{2p+1};$$

and this is the result we desired to prove.

The number of matrices of integers, of the form

$$\begin{pmatrix} \alpha & \beta \\ \alpha' & \beta' \end{pmatrix},$$

in which $\bar{\alpha}\alpha'-\bar{\alpha}'\alpha=0$, $\bar{\beta}\beta'-\bar{\beta}'\beta=0$, $\bar{\alpha}\beta'-\bar{\alpha}'\beta=1$, is infinite; but it follows from the investigation just given that if all the elements of these matrices be replaced by their smallest positive residues for modulus 2, the number of different matrices then arising is finite, being equal to the number of sets of $2p+1$ half-integer characteristics, with integral sum, of which every two characteristics are azygetic. As in § 300, Chap. XVII., this number is

$$(2^{2p}-1)\,2^{2p-1}\,(2^{2p-2}-1)\,2^{2p-3}\ldots\ldots(2^2-1)\,2\,;$$

we may call this the number of incongruent Abelian matrices, for modulus 2. Similarly the number* of incongruent Abelian matrices for modulus n is

$$(n^{2p}-1)\,n^{2p-1}\,(n^{2p-2}-1)\,n^{2p-3}\ldots\ldots(n^2-1)\,n.$$

Ex. By adding suitable integers to the characteristics denoted by 1, 2, 3, 4, 5, 6, 7 in the table of § 205, for $p=3$, we obtain respectively

$$\tfrac{1}{2}\begin{pmatrix} -1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix},\ \tfrac{1}{2}\begin{pmatrix} -1 & -1 & 0 \\ 0 & -1 & 0 \end{pmatrix},\ \tfrac{1}{2}\begin{pmatrix} -1 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix},\ \tfrac{1}{2}\begin{pmatrix} -1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix},$$

$$\tfrac{1}{2}\begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & -1 \end{pmatrix},\ \tfrac{1}{2}\begin{pmatrix} 0 & 1 & -1 \\ 1 & 1 & 0 \end{pmatrix},\ \tfrac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix};$$

denoting these respectively by $C_1, C_2, \dots, C_7$, we find, for $i<j$, that

$$|C_i, C_j|=1, \qquad (i=1, \dots, 6;\ j=2, \dots, 7).$$

The equations of the text

$$c_{2r-1}=a_1'b_1\ldots\ldots a'_{r-1}b_{r-1}a_r, \quad c_{2r}=a_1'b_1\ldots\ldots a'_{r-1}b_{r-1}b_r,$$

give

$$a_r=c_1c_2'\ldots\ldots c_{2r-3}c'_{2r-2}c_{2r-1}, \quad b_r=c_1c_2'\ldots\ldots c_{2r-3}c'_{2r-2}c_{2r},$$

and therefore, in this case, we find

$$a_1=\begin{pmatrix} -1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix},\quad a_2=\begin{pmatrix} -1 & 0 & 1 \\ -1 & 1 & 1 \end{pmatrix},\quad a_3=\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & -1 \end{pmatrix},$$

$$b_1=\begin{pmatrix} -1 & -1 & 0 \\ 0 & -1 & 0 \end{pmatrix},\quad b_2=\begin{pmatrix} -1 & 1 & 1 \\ -1 & 2 & 1 \end{pmatrix},\quad b_3=\begin{pmatrix} 0 & 1 & -1 \\ 0 & 1 & 0 \end{pmatrix};$$

* Another proof is given by Jordan, *Traité des Substitutions* (Paris, 1870), p. 176.

hence the linear substitution, of the text, for transforming the fundamental set of characteristics $C_1, \ldots, C_7$ is

$$\left(\begin{array}{ccc|ccc} -1 & -1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 2 & 1 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ \hline -1 & -1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 \\ 0 & 1 & -1 & 0 & 1 & -1 \end{array}\right)$$

From this we find $\frac{1}{2}\begin{pmatrix} s' \\ s \end{pmatrix} = \frac{1}{2}\begin{pmatrix} d(a'\bar{\beta}') \\ d(a\bar{\beta}) \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2 & 0 & 2 \\ 1 & 2 & 1 \end{pmatrix}$; since the sum of $C_1, \ldots, C_7$ is an integral characteristic, it follows by the general theorem, that *if the characteristic* $\frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$ *be added to each of* $C_1, \ldots, C_7$, *and then the linear transformation given by the matrix be applied, they will be transformed respectively into the characteristics* $\lambda_1, \ldots, \lambda_7$.

A further result should be mentioned. On the hyperelliptic Riemann surface suppose the period loops drawn as in the figure (12);

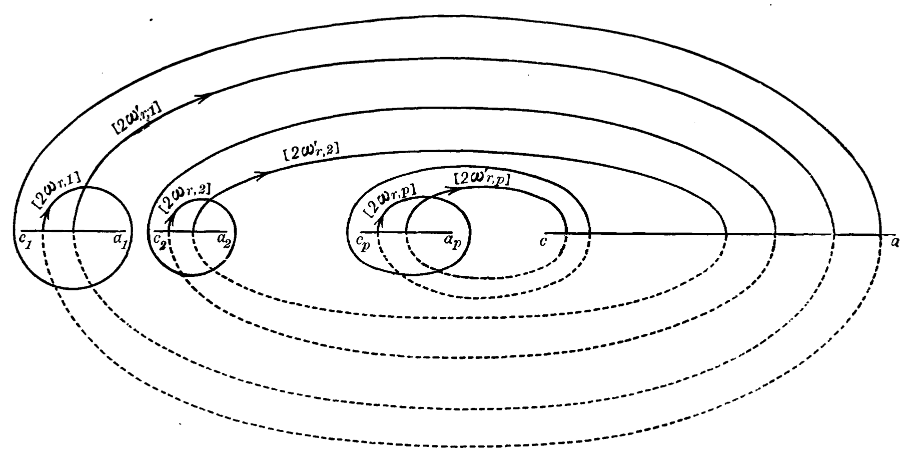

FIG. 12.

then the characteristics associated with the half-periods $u^{a,\,c_1}, u^{a,\,a_1}, \ldots, u^{a,\,c_p}, u^{a,\,a_p}, u^{a,\,c}$ will be, save for integer characteristics, respectively $\lambda_1, \lambda_2, \ldots, \lambda_{2p}, \lambda_{2p+1}$; this the reader can immediately verify by means of the rule given at the bottom of page 297 of the present volume.

Ex. Prove that if the characteristics $0, \lambda_1, \ldots, \lambda_{2p+1}$ be subjected to the transformation given by the Abelian matrix of $2p$ rows and columns which is denoted by $\begin{pmatrix} 1, & -1 \\ 0, & 1 \end{pmatrix}$, then, save for integer characteristics, λ_i is changed to $\Sigma_i + \frac{1}{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix}^p$, where

$$\Sigma_{2r-1} = \frac{1}{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix}^{r-1}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix}^{p-r}, \quad \Sigma_{2r} = \frac{1}{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix}^{r-1}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix}^{p-r}, \quad \Sigma_{2p+1} = \frac{1}{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix}^{p}, \quad (r = 1, 2, \ldots, p),$$

are the characteristics which arise in § 200, Chap. XI. as associated with the half-periods $u^{a,\,c_r}$, $u^{a,\,a_r}$, $u^{a,\,c}$ respectively. The characteristics $\Sigma_1, \dots, \Sigma_{2p+1}$ satisfy the $p(2p-1)$ conditions $|\Sigma_i, \Sigma_j| = 1$, for $i < j$.

332. We proceed now to shew how any linear transformation may be regarded as the result of certain very simple linear transformations performed in succession. As a corollary from the investigation we shall be able to infer that every linear transformation may be associated with a change in the method of taking the period loops on a Riemann surface; we have already proved the converse result, that every change in the period loops is associated with matrices, α, α', β, β', belonging to a linear substitution (§ 322).

It is convenient to give first the fundamental equations for a composition of two transformations of any order. It has been shewn (§ 324) that the equations for the transformation of a theta function of the first order, in the arguments u, with characteristic (Q, Q') and associated constants 2ω, $2\omega'$, 2η, $2\eta'$, to a theta function of order r, in the arguments w, where $u = Mw$, with characteristic (K, K') and associated constants $2v$, $2v'$, 2ζ, $2\zeta'$, are

$$K' = \bar{\alpha}Q' - \bar{\alpha}'Q - \tfrac{1}{2}d(\bar{\alpha}\alpha'), \quad -K = \bar{\beta}Q' - \bar{\beta}'Q - \tfrac{1}{2}d(\bar{\beta}\beta'),$$

$$\begin{pmatrix} M, & 0 \\ 0, & r\bar{M}^{-1} \end{pmatrix} \begin{pmatrix} 2v, & 2v' \\ 2\zeta, & 2\zeta' \end{pmatrix} = \begin{pmatrix} 2\omega, & 2\omega' \\ 2\eta, & 2\eta' \end{pmatrix} \begin{pmatrix} \alpha, & \beta \\ \alpha', & \beta' \end{pmatrix};$$

and from the last equation, writing it in the form $\mu\mathbf{U} = \Omega\Delta$, it follows, in virtue of the equations $\Omega\epsilon\bar{\Omega} = -\tfrac{1}{2}\pi i\epsilon$, $\mathbf{U}\epsilon\bar{\mathbf{U}}' = -\tfrac{1}{2}\pi i\epsilon$ (§ 140, Chap. VII.), and the easily verifiable equation $\bar{\mu}\epsilon\mu = r\epsilon$, where the matrix ϵ is given by

$$\epsilon = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

that also $\bar{\Delta}\epsilon\Delta = r\epsilon$, as in Ex. i., § 324. And, just as in § 324, it can be proved that equations for the transformation of a theta function of order r in the arguments w, with characteristic (K, K'), and associated constants $2v$, $2v'$, 2ζ, $2\zeta'$, to a theta-function of order rs, in the arguments u_1, given by $w = Nu_1$, with characteristic (Q_1, Q_1'), and associated constants $2\omega_1$, $2\omega_1'$, $2\eta_1$, $2\eta_1'$, are

$$Q_1' = \bar{\gamma}K' - \bar{\gamma}'K - \tfrac{1}{2}rd(\bar{\gamma}\gamma'), \quad -Q_1 = \bar{\delta}K' - \bar{\delta}'K - \tfrac{1}{2}rd(\bar{\delta}\delta'),$$

$$\begin{pmatrix} N, & 0 \\ 0, & s\bar{N}^{-1} \end{pmatrix} \begin{pmatrix} 2\omega_1, & 2\omega_1' \\ 2\eta_1, & 2\eta_1' \end{pmatrix} = \begin{pmatrix} 2v, & 2v' \\ 2\zeta, & 2\zeta' \end{pmatrix} \begin{pmatrix} \gamma, & \delta \\ \gamma', & \delta' \end{pmatrix};$$

and writing the last equation in the form $\nu\Omega_1 = \mathbf{U}\nabla$, we infer as before that $\bar{\nabla}\epsilon\nabla = s\epsilon$.

Now from the equations $\mu\mathbf{U} = \Omega\Delta$, $\nu\Omega_1 = \mathbf{U}\nabla$, we obtain $\mu\nu\Omega_1 = \mu\mathbf{U}\nabla = \Omega\Delta\nabla$, or, if $\Delta_1 = \Delta\nabla$,

$$\begin{pmatrix} MN, & 0 \\ 0, & rs\overline{MN}^{-1} \end{pmatrix} \begin{pmatrix} 2\omega_1, & 2\omega_1' \\ 2\eta_1, & 2\eta_1' \end{pmatrix} = \begin{pmatrix} 2\omega, & 2\omega' \\ 2\eta, & 2\eta' \end{pmatrix} \Delta_1;$$

from this equation we find as before that the matrix Δ_1, given by

$$\Delta_1 = \Delta\nabla = \begin{pmatrix} \alpha\gamma + \beta\gamma', & \alpha\delta + \beta\delta' \\ \alpha'\gamma + \beta'\gamma', & \alpha'\delta + \beta'\delta' \end{pmatrix} = \begin{pmatrix} \alpha_1, & \beta_1 \\ \alpha_1', & \beta_1' \end{pmatrix}, \text{ say,}$$

satisfies the equation $\bar{\Delta}_1 \epsilon \Delta_1 = rs\epsilon$. Similarly from the two sets of equations transforming the characteristics, by making use of the equations

$$d(\bar{\alpha}_1\alpha_1') \equiv \bar{\gamma} d(\bar{\alpha}\alpha') + \bar{\gamma}' d(\bar{\beta}\beta') + rd(\bar{\gamma}\gamma'),$$
$$d(\bar{\beta}_1\beta_1') \equiv \bar{\delta} d(\bar{\alpha}\alpha') + \bar{\delta}' d(\bar{\beta}\beta') + rd(\bar{\delta}\delta'), \quad (\text{mod. } 2),$$

which can be proved by the methods of § 327, we immediately find

$$Q_1' \equiv \bar{\alpha}_1 Q' - \bar{\alpha}_1' Q - \tfrac{1}{2} d(\bar{\alpha}_1\alpha_1'), \quad -Q_1 \equiv \bar{\beta}_1 Q' - \bar{\beta}_1' Q - \tfrac{1}{2} d(\bar{\beta}_1\beta_1'), \quad (\text{mod. } 2).$$

Hence any transformation of order rs may be regarded as compounded of two transformations, of which the first transforms a theta-function of the first order into a theta function of the r-th order, and the second transforms it further into a theta function of order rs.

It follows therefore that the most general transformation may be considered as the result of successive transformations of prime order. It is convenient to remember that the matrix of integers, Δ_1, associated with the compound transformation, is equal to $\Delta\nabla$, the matrix Δ, associated with the transformation which is first carried out, being the left-hand factor.

One important case should be referred to. The matrix

$$r\Delta^{-1} = \begin{pmatrix} \bar{\beta}' & -\bar{\beta} \\ -\bar{\alpha}' & \bar{\alpha} \end{pmatrix}$$

is easily seen to be that of a transformation of order r; putting it in place of ∇, the final equations for the compound transformation ∇_1 may be taken to be

$$u_1 = ru, \quad 2\omega_1 = 2\omega, \quad 2\omega_1' = 2\omega', \quad 2\eta_1 = 2\eta, \quad 2\eta_1' = 2\eta'.$$

The transformation $r\Delta^{-1}$ is called *supplementary* to Δ (cf. Chap. XVII., § 317, Ex. vii.).

333. Limiting ourselves now to the case of linear transformation, let A_k $(k = 2, 3, \dots, p)$ denote the matrix of $2p$ rows and columns indicated by

$$A_k = \begin{pmatrix} \mu_k, & 0 \\ 0, & \mu_k \end{pmatrix},$$

where μ_k has unities in the diagonal except in the first and k-th places, in which there are zeros, and has elsewhere zeros, except in the k-th place of the first row, and the k-th place of the first column, where there are unities; let B denote the matrix of $2p$ rows and columns indicated by

$$B = \begin{pmatrix} 0 & & & & -1 & & & \\ & 1 & & & & 0 & & \\ & & 1 & & & & 0 & \\ & & & 1 & & & & 0 \\ 1 & & & & 0 & & & \\ & 0 & & & & 1 & & \\ & & 0 & & & & 1 & \\ & & & 0 & & & & 1 \end{pmatrix},$$

which has unities in the diagonal, except in the first and $(p+1)$-th places, where there are zeros, and has elsewhere zeros except in the $(p+1)$-th place of the first row, where there is -1, and the $(p+1)$-th place of the first column, where there is $+1$; let C denote the matrix of $2p$ rows and columns indicated by

$$C = \begin{pmatrix} 1 & & & & -1 & & & \\ & 1 & & & & 0 & & \\ & & 1 & & & & 0 & \\ & & & 1 & & & & 0 \\ 0 & & & & 1 & & & \\ & 0 & & & & 1 & & \\ & & 0 & & & & 1 & \\ & & & 0 & & & & 1 \end{pmatrix},$$

which has unities everywhere in the diagonal and has elsewhere zeros, except in the $(p+1)$-th place of the first row, where it has -1; let D denote the matrix of $2p$ rows and columns indicated by

$$D = \begin{pmatrix} 1 & & & & 0 & -1 & & \\ & 1 & & & -1 & 0 & & \\ & & 1 & & & & 0 & \\ & & & 1 & & & & 0 \\ 0 & & & & 1 & & & \\ & 0 & & & & 1 & & \\ & & 0 & & & & 1 & \\ & & & 0 & & & & 1 \end{pmatrix},$$

which has unities everywhere in the diagonal and has elsewhere zeros, except in the $(p+2)$-th place of the first row and the $(p+1)$-th place of the second row, in each of which there is -1. It is easy to see that each of these matrices satisfies the conditions (IX.) of § 324, for $r = 1$.

Then it can be proved that every matrix of $2p$ rows and columns of integers,

$$\begin{pmatrix} \alpha, & \beta \\ \alpha', & \beta' \end{pmatrix},$$

for which $\alpha\bar{\beta} = \beta\bar{\alpha}$, $\alpha'\bar{\beta}' = \beta'\bar{\alpha}'$, $\alpha\bar{\beta}' - \beta\bar{\alpha}' = 1$, can be written* as a product of positive integral powers of the $(p+2)$ matrices $A_2, \dots, A_p, B, C, D$. The proof of this statement is given in the Appendix (II) to this volume.

We shall therefore obtain a better understanding of the changes effected by a linear transformation by considering these transformations in turn. We have seen that any linear transformation may be considered as made up of two processes, (i) the change of the fundamental system of periods, effected by the equations

$$[\omega] = \omega\alpha + \omega'\alpha', \quad [\omega'] = \omega\beta + \omega'\beta',$$
$$[\eta] = \eta\alpha + \eta'\alpha', \quad [\eta'] = \eta\beta + \eta'\beta',$$

(ii) the change of the arguments, effected by the equation $u = Mw$, and leading to

$$[\omega] = Mv, \quad [\omega'] = Mv', \quad \zeta = \bar{M}[\eta], \quad \zeta' = \bar{M}[\eta'];$$

of these we consider here the first process. Applying the equations†

$$[\omega] = \omega\alpha + \omega'\alpha', \quad [\omega'] = \omega\beta + \omega'\beta',$$

respectively for the transformations A_k, B, C, D, we obtain the following results:

For the matrix (A_k) we have

$$[\omega_{r,1}] = \omega_{r,k}, \quad [\omega_{r,k}] = \omega_{r,1}, \quad [\omega'_{r,1}] = \omega'_{r,k}, \quad [\omega'_{r,k}] = \omega'_{r,1}, \quad (r = 1, 2, \dots, p);$$

or, in words, if $2\omega_{r,i}$, $2\omega'_{r,i}$ be called the i-th pair of periods for the argument u_r, the change effected by the substitution A_k is an interchange of the first and k-th pairs of periods—no other change whatever being made.

When we are dealing with p quantities, the interchange of the first and k-th of these quantities can be effected by a composition of the two processes (i) an interchange of the first and second, (ii) a cyclical change whereby the second becomes the first, the third becomes the second, ..., the p-th becomes the $(p-1)$-th, and the first becomes the p-th. Such a cyclical change is easily seen to be effected by the matrix

$$E = \left(\begin{array}{cccc|cccc} 0 & & & 1 & & & & \\ 1 & 0 & & & & & & \\ & 1 & 0 & & & 0 & & \\ & & \ddots & \vdots & & & & \\ & & 1 & 0 & & & & \\ \hline & & & & 0 & & & 1 \\ & & & & 1 & 0 & & \\ & 0 & & & & 1 & 0 & \\ & & & & & & \ddots & \vdots \\ & & & & & & 1 & 0 \end{array}\right)$$

* Other sets of elementary matrices, by the multiplication of which any Abelian matrix can be formed, can easily be chosen. One other obvious set consists of the matrices obtained by interchanging the rows and columns of the matrices A_k, B, C, D.

† We may state the meaning of the matrices A_k, B, C, D somewhat differently in accordance with the property remarked in Ex. iii., § 324.

which verifies the equations (IX.) § 324, for $r=1$. Hence the matrices $A_3, \ldots, A_p$ can each be represented by a product of positive powers of the matrices E and A_2. Thereby the $(p+2)$ elementary matrices $A_2, \ldots, A_p, B, C, D$ can be replaced by only 5 matrices E, A_2, B, C, D*.

Considering next the matrix B we obtain

$$[\omega_{r,1}]=\omega'_{r,1},\quad [\omega'_{r,1}]=-\omega_{r,1},\quad [\omega_{r,i}]=\omega_{r,i},\quad [\omega'_{r,i}]=\omega'_{r,i},\quad \begin{pmatrix} r=1,\,2,\,\ldots,\,p \\ i=\quad 2,\,\ldots,\,p \end{pmatrix},$$

so that this transformation has the effect of interchanging $\omega_{r,1}$ and $\omega'_{r,1}$, changing the sign of one of them; no other change is introduced.

The matrix C gives the equation

$$[\omega'_{r,1}]=\omega'_{r,1}-\omega_{r,1},\quad (r=1,\,2,\,\ldots,\,p),$$

but makes no other change.

The matrix D makes only the changes expressed by the equations

$$[\omega'_{r,1}]=\omega'_{r,1}-\omega_{r,2},\quad [\omega'_{r,2}]=\omega'_{r,2}-\omega_{r,1}.$$

In applying these transformations to the case of the theta functions we notice immediately that A_k, C and D all belong to the case considered in § 326 (ii), in which the matrix $\alpha'=0$.

Thus in the case of the transformation A_k we have

$$\Theta\left(u;\ \tau \,\middle|\, {}^{Q'}_{Q}\right)=A\Theta\left(w;\ \tau' \,\middle|\, {}^{K'}_{K}\right),$$

where w differs from u only in the interchange of u_1 and u_k, τ' differs from τ only in the interchange of the suffixes 1 and k in the constituents $\tau_{r,s}$ of the matrix τ, and K, K' differ from Q, Q' only in the interchange of the first and k-th elements both in Q and Q'. Thus in this case the constant A is equal to 1.

In the case of the matrix (C), the equations of § 326 (2) give

$$\Theta\left(u;\ \tau \,\middle|\, {}^{Q'}_{Q}\right)=A\Theta\left(w;\ \tau' \,\middle|\, {}^{K'}_{K}\right),$$

where

$u=w$, $\tau'=\tau$ save that $\tau'_{1,1}=\tau_{1,1}-1$, and $K'=Q'$, $K=Q$ save that $K_1=Q_1+Q_1'-\frac{1}{2}$; now the general term of the left-hand side, or

$$e^{2\pi iu(n+Q')+i\pi\tau(n+Q')^2+2\pi iQ(n+Q')}$$

is equal to

$$e^{2\pi iw(n+K')+i\pi\tau'(n+K')^2+i\pi(n_1+Q_1')^2+2\pi iK(n+K')-2i\pi(Q_1'-\frac{1}{2})(n_1+Q_1')}$$

$$=e^{-i\pi(Q_1'^2-Q_1')}\,e^{2\pi iw(n+K')+i\pi\tau'(n+K')^2+2\pi iK(n+K')};$$

thus in the case of the transformation (C) the constant A is equal to $e^{-i\pi(Q_1'^2-Q_1')}$; when Q_1' is a half-integer, this is an eighth root of unity.

* See Krazer, *Ann. d. Mat.*, Ser. II., t. xii. (1884). The number of elementary matrices is stated by Burkhardt to be further reducible to 3, or, in case $p=2$, to 2; *Götting. Nachrichten*, 1890, p. 381.

In the case of the matrix (D), the equations of § 326 (ii) lead to

$$\Theta\left(u;\ \tau \,|\, {}^{Q'}_{Q}\right) = A\Theta\left(w;\ \tau' \,|\, {}^{K'}_{K}\right),$$

where $u = w$, $\tau' = \tau$ save that $\tau'_{1,2} = \tau_{1,2} - 1$, $\tau'_{2,1} = \tau_{2,1} - 1$, and $K' = Q'$, $K = Q$ save that $K_1 = Q_1 + Q_2'$, $K_2 = Q_2 + Q_1'$; now we have

$$e^{2\pi i u (n+Q') + i\pi\tau (n+Q')^2 + 2\pi i Q (n+Q')} = e^{2i\pi (n_1 n_2 - Q_1' Q_2')}\, e^{2\pi i w (n+K') + i\pi\tau' (n+K')^2 + 2\pi i K (n+K')};$$

thus, in the case of the matrix (D) the constant A is equal to $e^{-2\pi i Q_1' Q_2'}$.

We consider now the transformation (B)—which falls under that considered in (i) § 326. In this case $\pi i \bar{\alpha}' (\alpha + \tau\alpha') w^2$ is equal to $\pi i \tau_{1,1} w_1^2$, and the equation $(\alpha + \tau\alpha')\tau' = \beta + \tau\beta'$ leads to the equations

$$\tau'_{1,1} = -1/\tau_{1,1}, \quad \tau'_{1,r} = \tau_{1,r}/\tau_{1,1}, \quad \tau'_{r,s} = \tau_{r,s} - \tau_{1,r}\tau_{1,s}/\tau_{1,1},$$

or, the equivalent equations $(r, s = 2, 3, \ldots, p)$,

$$\tau_{1,1} = -1/\tau'_{1,1}, \quad \tau_{1,r} = -\tau'_{1,r}/\tau'_{1,1}, \quad \tau_{r,s} = \tau'_{r,s} - \tau'_{1,r}\tau'_{1,s}/\tau'_{1,1};$$

also $u_1 = \tau_{1,1} w_1$, $u_r = \tau_{1,r} w_1 + w_r$, so that $w_1 = -\tau'_{1,1} u_1$, $w_r = u_r - \tau'_{1,r} u_1$, and $\tau_{1,1} w_1^2 = -\tau'_{1,1} u_1^2$; further we find

$K' = Q'$ save that $K_1' = -Q_1$, and $K = Q$ save that $K_1 = Q_1'$;

with these values we have the equation

$$e^{\pi i \tau_{1,1} w_1^2}\,\Theta\left(u;\ \tau \,|\, {}^{Q'}_{Q}\right) = A\Theta\left(w;\ \tau' \,|\, {}^{K'}_{K}\right).$$

334. To determine the constant A in the final equation of the last Article we proceed as follows*:—We have

(i) $$\int_0^1 e^{2\pi i m w}\, dw = 0 \text{ or } 1,$$

according as m is an integer other than zero, or is zero;

(ii) if α be a positive real quantity other than zero, and β, γ, δ be real quantities,

$$\int_{-\infty}^{\infty} e^{(-\alpha + i\beta)(x + \gamma + i\delta)^2}\, dx = \sqrt{\frac{\pi}{\alpha - i\beta}},$$

where for the square root is to be taken that value of which the real part is positive†;

* For indications of another method consult Clebsch u. Gordan, *Abel. Funct.*, § 90; Thomae, *Crelle*, LXXV. (1873), p. 224.

† By the symbol $\sqrt{\mu}$, where μ is any constant quantity, is to be understood that square root whose real part is positive, or, if the real part be zero, that square root whose imaginary part is positive.

(iii) with the relations connecting u, w and τ, τ' given in the previous Article,

$$un = (wn)_1 + (\tau_{1,1} n_1 + \ldots\ldots + \tau_{1,p} n_p) w_1,$$

where $(wn)_1$ denotes $w_2 n_2 + \ldots\ldots + w_p n_p$;

(iv) the series representing the function $\Theta(w, \tau')$ is uniformly convergent for all finite values of $w_1, \ldots, w_p$, and therefore, between finite limits, the integral of the function is the sum of the integrals of its terms.

Therefore, taking the case when $\binom{Q'}{Q}$ and therefore $\binom{K'}{K}$ are $\binom{0}{0}$, and integrating the equation

$$e^{\pi i \tau_{1,1} w_1^2}\Theta(u;\ \tau) = A_0 \Theta(w;\ \tau'),$$

in regard to $w_1, \ldots, w_p$, each from 0 to 1, we have

$$A_0 = \sum_{n_1=-\infty}^{\infty} \sum_{n_2, \ldots, n_p}^{-\infty, \infty} \int_0^1 \ldots \int_0^1 e^{\pi i \tau_{1,1} w_1^2 + 2\pi i (wn)_1 + 2\pi i (\tau_{1,1} n_1 + \ldots + \tau_{1,p} n_p) w_1 + i\pi \tau n^2} dw_1 \ldots dw_p,$$

where, on the right hand, the integral is zero except for $n_2 = 0, \ldots, n_p = 0$; thus

$$A_0 = \sum_{n_1=-\infty}^{\infty} \int_0^1 e^{\pi i \tau_{1,1} w^2 + 2\pi i \tau_{1,1} n_1 w_1 + i\pi \tau_{1,1} n_1^2} dw_1$$

$$= \sum_{n_1=-\infty}^{\infty} \int_0^1 e^{\pi i \tau_{1,1} (w_1 + n_1)^2} dw_1$$

$$= \int_{-\infty}^{\infty} e^{\pi i \tau_{1,1} x^2} dx;$$

hence since the real part of $\pi i \tau_{1,1}$ is negative (§ 174), we have

$$A_0 = \sqrt{\frac{\pi}{-\pi i \tau_{1,1}}} = \sqrt{\frac{i}{\tau_{1,1}}},$$

where the square root is to be taken of which the real part is positive.

Hence

$$e^{\pi i \tau_{1,1} w_1^2}\Theta(u;\ \tau) = \sqrt{\frac{i}{\tau_{1,1}}}\,\Theta(w;\ \tau'),$$

and from this equation, by increasing w by $K + \tau' K'$, we deduce that

$$e^{\pi i \tau_{1,1} w_1^2}\Theta\left(u;\ \tau \,\middle|\, \begin{matrix} Q' \\ Q \end{matrix}\right) = \sqrt{\frac{i}{\tau_{1,1}}} \cdot e^{2\pi i Q_1 Q_1'}\,\Theta\left(w;\ \tau' \,\middle|\, \begin{matrix} K' \\ K \end{matrix}\right).$$

Hence, when the decomposition of any linear transformation into transformations of the form A_k, B, C, D is known, the value of the constant factor, A, can be determined.

335. But, save for an eighth root of unity, we can immediately specify the value in the general case; for when Q, Q' are zero, the value of the constant A has been found to be unity for each of the transformations A_k, C, D, and for the transformation B to have a

value which is in fact equal to $\sqrt{i/|M|}$, $|M|$ denoting the determinant of the matrix M. Hence for a transformation which can be put into the form

$$\begin{pmatrix} \alpha & \beta \\ \alpha' & \beta' \end{pmatrix} = \dots B^{r_2} \dots A_k^{\rho} \dots D^{\nu} \dots B^{r_1} \dots C^{\mu} \dots A_k^{\lambda} \dots,$$

if the values of the matrix M for these component transformations be respectively

$$\dots M_2^{r_2} \dots 1 \dots 1 \dots M_1^{r_1} \dots 1 \dots 1 \dots,$$

the value of the constant A, when Q, Q' are zero, for the complete transformation, will be

$$\dots \left(\sqrt{\frac{i}{|M_2|}}\right)^{r_2} \dots\dots \left(\sqrt{\frac{i}{|M_1|}}\right)^{r_1} \dots\dots;$$

but if the complete transformation give $u = Mw$, we have $M = \dots M_2 M_1 \dots$; thus, for any transformation we have the formula

$$e^{\pi i \bar{a}' (a + \tau a') w^2} \Theta(u, \tau) = \frac{\epsilon}{\sqrt{|M|}} \Theta\left[w, \tau' \,|\, \tfrac{1}{2}\begin{pmatrix} r' \\ r \end{pmatrix}\right],$$

where $M = a + \tau a'$, $u = Mw$, and ϵ is an eighth root of unity, r, r' being as in § 328, p. 544.

Putting $2\omega u$, $2vw$ for u, w, as in § 326, this equation is the same as

$$\frac{1}{\sqrt{|\omega|}} \vartheta(u;\, 2\omega, 2\omega', 2\eta, 2\eta') = \frac{\epsilon}{\sqrt{|M|\,|v|}} \vartheta\left[w;\, 2v, 2v', 2\zeta, 2\zeta' \,|\, \tfrac{1}{2}\begin{pmatrix} r' \\ r \end{pmatrix}\right]$$

where $|\omega|$ is the determinant of the matrix ω, etc.

Of such composite transformations there is one which is of some importance, that, namely, for which

$$\begin{pmatrix} \alpha & \beta \\ \alpha' & \beta' \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

so that

$$[\omega_{r,i}] = \omega'_{r,i}, \quad [\omega'_{r,i}] = -\omega_{r,i}; \qquad (r, i = 1, 2, \dots, p).$$

Then

$$M = \tau, \quad \tau\tau' = -1, \quad u = \tau w, \quad \pi i \bar{a}'(a + \tau a') w^2 = \pi i \tau w^2 = \pi i u w = -\pi i \tau' u^2.$$

We may suppose this transformation obtained from the formula given above for the simple transformation B—thus—Apply first the transformation B which interchanges $\omega_{r,1}$, $\omega'_{r,1}$ with a certain change of sign of one of them; then apply the transformation $A_2 B A_2$ which effects a similar change for the pair $\omega_{r,2}$, $\omega'_{r,2}$; then the transformation $A_3 B A_3$, and so on. Thence we eventually obtain the formula

$$e^{\pi i \tau w^2} \Theta\left(u;\, \tau \,\Big|\, {Q' \atop Q}\right) = \sqrt{\frac{i}{\tau_{1,1}}} \sqrt{\frac{i}{\tau'_{2,2}}} \sqrt{\frac{i}{\tau''_{3,3}}} \dots\dots e^{2\pi i (Q_1 Q_1' + \dots + Q_p Q_p')} \Theta\left(w;\, \tau' \,\Big|\, {-Q \atop Q'}\right),$$

where

$$\tau'_{2,2} = \tau_{2,2} - \frac{\tau^2_{1,2}}{\tau_{1,1}}, \quad \tau''_{3,3} = \tau'_{3,3} - \frac{\tau'^2_{2,3}}{\tau'_{2,2}}, \dots,$$

and, save for an eighth root of unity,

$$\sqrt{\frac{i}{\tau_{1,1}}} \sqrt{\frac{i}{\tau'_{2,2}}} \sqrt{\frac{i}{\tau''_{3,3}}} \dots = \frac{1}{\sqrt{|\tau|}},$$

where $|\tau|$ is the determinant of the matrix τ.

The result can also be obtained immediately, and the constant obtained by an integration as in the simple case of the transformation B; we thus find, for the value of the constant here denoted by $\sqrt{\frac{i}{\tau_{1,1}}}\sqrt{\frac{i}{\tau'_{2,2}}}\ldots$, the integral*

$$\int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty} e^{\pi i\tau x^2}\,dx_1\ldots dx_p.$$

Ex. i. Prove that another way of expressing the value of this integral is

$$e^{\frac{1}{2}i\sum_{r=1}^{p}\tan^{-1}\lambda_r}\Big/\sqrt[4]{|\tau\tau_0|},$$

where, if the matrix τ be written $\rho+i\sigma$, $|\tau\tau_0|$ is the determinant of the matrix $\rho^2+\sigma^2$, which is equal to the square of the modulus of the determinant of the matrix τ, also $\lambda_1, \ldots, \lambda_p$ are the (real) roots of the determinantal equation $|\rho-\lambda\sigma|=0$, and $\tan^{-1}\lambda_r$ lies between $-\pi/2$ and $\pi/2$. Of the fourth root the positive real value is to be taken.

Ex. ii. For the case $p=1$, the constant for any linear transformation is given by

$$e^{\pi i a'(a+\tau a')w^2}\Theta\left[u;\ \tau\,\middle|\,\tfrac{1}{2}\binom{s'}{s}\right]\div\Theta(w;\ \tau')=L\sum_{\mu=0}^{a'-1}e^{-\frac{\pi i a}{a'}(\mu+\frac{1}{2}a')^2}$$

$$=L\sqrt{a'}\left(\frac{a'}{a}\right)e^{-\frac{\pi i a}{4}}\text{ or }L\sqrt{a'}\left(\frac{a}{a'}\right)e^{-\frac{\pi i}{4}[a+(a-1)(a'-1)]}$$

according as a or a' is odd; where a' is positive, and

$$\begin{array}{l} as'-a's=aa', \\ \beta s'-\beta' s=\beta\beta', \end{array}\quad L=e^{\frac{\pi i a'}{4a}s^2}\sqrt{\frac{i}{a'(a+\tau a')}}.$$

336. Returning now to consider the theory more particularly in connexion with the Riemann surface, we prove first that every linear transformation of periods such as

$$[\omega]=\omega\alpha+\omega'\alpha',\quad [\omega']=\omega\beta+\omega'\beta',$$

where

$$\alpha\bar{\beta}-\beta\bar{\alpha}=0,\quad \alpha'\bar{\beta}'-\beta'\bar{\alpha}'=0,\quad \alpha\bar{\beta}'-\beta\bar{\alpha}'=1,$$

can be effected by a change in the manner in which the period loops are taken. For this it is sufficient to prove that each of the four elementary types of transformation, A_k, B, C, D, from which, as we have seen, every such transformation can be constructed, can itself be effected by a change in the period loops.

The change of periods due to substitutions A_k can clearly be effected without drawing the period loops differently, by merely numbering them

* Weber has given a determination of the constant A for a general linear transformation by means of such an integral, and thence, by means of multiple-Gaussian series. See *Crelle*, LXXIV. (1872), pp. 57 and 69.

differently—attaching the numbers 1, k to the period-loop-pairs which were formerly numbered k and 1. No further remark is therefore necessary in regard to this case.

The substitution B, which makes only the change given by

$$[\omega_{r,1}] = \omega'_{r,1}, \quad [\omega'_{r,1}] = -\omega_{r,1},$$

can be effected, as in § 320, by regarding the loop (b_1) as an $[a_1]$ loop, with retention of its positive direction; thus the direction of the (old) loop (a_1), which now becomes the $[b_1]$ loop, will be altered; the change is shewn by comparing the figure of § 18 (p. 21) with the annexed figure (13).

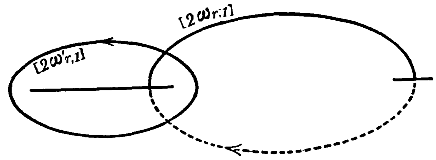

Fig. 13.

The change, due to the substitution C, which is given by

$$[\omega'_{r,1}] = \omega'_{r,1} - \omega_{r,1},$$

is to be effected by drawing the loop $[a_1]$ in such a way that a circuit of it (which gives rise to the value $[2\omega'_{r,1}]$ for the integral u_r) is equivalent to a circuit of the original loop (a_1) taken with a circuit of the loop (b_1) from the positive to the negative side of the original loop (a_1).

This may be effected by taking the loop $[a_1]$ as in the annexed figure (14) (cf. § 331).

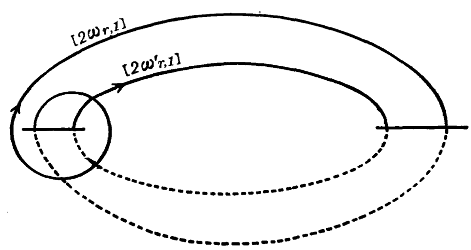

Fig. 14.

For the transformation D the only change introduced is that given by

$$[\omega'_{r,1}] = \omega'_{r,1} - \omega_{r,2}, \quad [\omega'_{r,2}] = \omega'_{r,2} - \omega_{r,1},$$

and this is effected by drawing the loops $[a_1]$, $[a_2]$, so that a circuit of

$[a_1]$ is equivalent to a circuit of the (original) loop (a_1) together with a circuit of (b_2), in a certain direction, and similarly for $[a_2]$. This may be done as in the annexed diagram (Fig. 15).

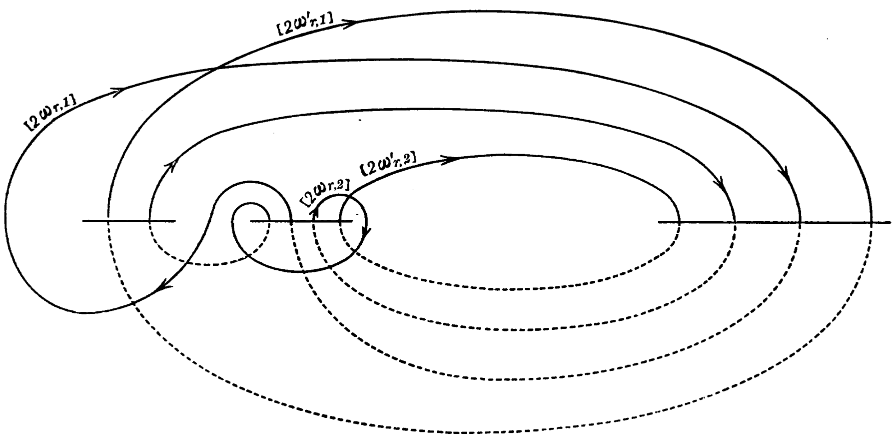

FIG. 15.

For instance the new loop $[a_2]$ in this diagram (Fig. 15) is a deformation of a loop which may be drawn as here (Fig. 16);

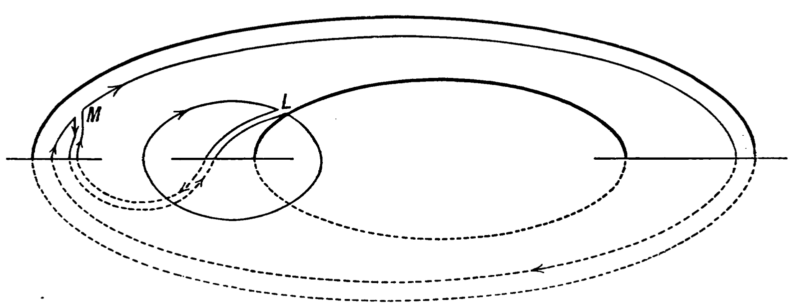

FIG. 16.

since the integrand of the Abelian integral u_r is single-valued on the Riemann surface, independently of the loops, the doubled portion from L to M is self-destructive; and a circuit of this new loop $[a_2]$ gives $\omega'_{r,2} - \omega_{r,1}$, as desired.

Hence the general transformation can be effected by a composition of the changes here given. It is immediately seen, for any of the linear transformations of § 326, that if the arguments there denoted by $U_1, \ldots, U_p$ be a set of normal integrals of the first kind for the original system of period loops, then $W_1, \ldots, W_p$ are a normal set for the new loops associated with the transformation.

337. Coming next to the question of how the theory of the vanishing of the Riemann theta function, which has been given in Chap. X., is modified

by the adoption of a different series of period loops, we prove first that when a change is made equivalent to the linear transformation

$$[\omega] = \omega\alpha + \omega'\alpha', \quad [\omega'] = \omega\beta + \omega'\beta',$$

the places $m_1, \dots, m_p$ of § 179, Chap. X., derived from any place m, upon which the theory of the vanishing of the theta function depends, become changed into places $m_1', \dots, m_p'$ which satisfy the p equations

$$u_i^{m_1', m_1} + \dots + u_i^{m_p', m_p} \equiv \tfrac{1}{2}[d(\alpha\bar{\beta})]_i + \tfrac{1}{2}\tau_{i,1}[d(\alpha'\bar{\beta}')]_1 + \dots + \tfrac{1}{2}\tau_{i,p}[d(\alpha'\bar{\beta}')]_p, \qquad (i = 1, \dots, p),$$

wherein $u_1, \dots, u_p$ denote the normal integrals of the first kind for the original system of period loops.

For let $w_1, \dots, w_p$ be the normal integrals of the first kind for the new period loops, and let $m_1', \dots, m_p'$ be the places derived from the place m, in connexion with the new system of period loops, just as $m_1, \dots, m_p$ were derived from the original system. In the equation of transformation

$$e^{\pi i \bar{a}'(a+\tau a')w^2}\,\Theta\left[u;\ \tau \left| \begin{matrix} \tfrac{1}{2}d(\alpha'\bar{\beta}') \\ \tfrac{1}{2}d(\alpha\bar{\beta}) \end{matrix}\right.\right] = A_1\Theta(w;\ \tau'),$$

put

$$w = w^{x,m} - w^{x_1, m_1'} - \dots - w^{x_p, m'_p},$$

so that the right-hand side of the equation vanishes when x is at any one of the places $m_1', \dots, m_p'$; then we also have

$$u = u^{x,m} - u^{x_1, m_1'} - \dots - u^{x_p, m_p'};$$

hence the function

$$\Theta\left[u^{x,m} - u^{x_1, m_1'} - \dots - u^{x_p, m_p'};\ \tau \left| \begin{matrix} \tfrac{1}{2}d(\alpha'\bar{\beta}') \\ \tfrac{1}{2}d(\alpha\bar{\beta}) \end{matrix}\right.\right]$$

vanishes when x is at any one of the places $x_1, \dots, x_p$; therefore, by a proposition previously given (Chap. X., § 184 (X.)), the places $m_1', \dots, m_p'$ satisfy the equivalence stated above.

It is easy to see that this equivalence may be stated in the form

$$w_i^{m_1', m_1} + \dots + w_i^{m'_p, m_p} \equiv \tfrac{1}{2}[d(\bar{\beta}\beta')]_i + \tfrac{1}{2}\tau'_{i,1}[d(\bar{a}\alpha')]_1 + \dots + \tfrac{1}{2}\tau'_{i,p}[d(\bar{a}\alpha')]_p, \qquad (i = 1, 2, \dots, p).$$

It may be noticed also that, of the elementary transformations associated with the matrices A_k, B, C, D, of § 333, only the transformation associated with the matrix C gives rise to a change in the places $m_1, \dots, m_p$; for each of the others the characteristic $[\tfrac{1}{2}d(\alpha\bar{\beta}), \tfrac{1}{2}d(\alpha'\bar{\beta}')]$ vanishes.

338. From the investigation of § 329 it follows, by interchanging the rows and columns of the matrix of transformation, that a linear trans-

formation can be taken for which the characteristic $[\frac{1}{2}d(\alpha\bar{\beta}), \frac{1}{2}d(\alpha'\bar{\beta}')]$ represents any specified even characteristic; thus all the $2^{p-1}(2^p+1)$ sets*, $m_1', \ldots, m_p'$, which arise by taking the characteristic $\frac{1}{2}\binom{\mu'}{\mu}$ in the equivalence

$$u^{m_1', m_1} + \ldots\ldots + u^{m'_p, m_p} \equiv \tfrac{1}{2}\Omega_{\mu, \mu'}$$

to be in turn all the even characteristics, can arise for the places $m_1', \ldots, m_p'$. In particular, if $\frac{1}{2}\Omega_{\mu, \mu'}$ be an even half-period for which $\Theta(\frac{1}{2}\Omega_{\mu, \mu'})$ vanishes, we may obtain for $m_1', \ldots, m_p'$ a set consisting of the place m and $p-1$ places $n_1', \ldots, n'_{p-1}$, in which $n_1', \ldots, n'_{p-1}$ are one set of a co-residual lot of sets of places in each of which a ϕ-polynomial vanishes to the second order (cf. Chap. X., § 185).

Ex. If in the hyperelliptic case, with $p=3$, the period loops be altered from those adopted in Chap. XI., in a manner equivalent to the linear transformation given in the Example of § 329, the function $\Theta(w; \tau')$, defined by means of the new loops, will vanish for $w=0$; and the places m_1', m_2', m_3', arising from the place a (§ 203, Chap. XI.), as $m_1, \ldots, m_p$ arise from m in § 179, Chap. X., will consist† of the place a itself and two arbitrary conjugate places, z and $\bar{z}$.

339. We have, on page 379 of the present volume, explained a method of attaching characteristics to root forms $\sqrt{X^{(1)}}$, $\sqrt{Y^{(3)}}$; we enquire now how these characteristics are modified when the period loops are changed. It will be sufficient to consider the case of $\sqrt{Y^{(3)}}$; the case of $\sqrt{X^{(1)}}$ arises (§ 244) by taking $\phi_0\sqrt{X^{(1)}}$ in place of $\sqrt{Y^{(3)}}$. Altering the notation of § 244, slightly, to make it uniform with that of this chapter, the results there obtained are as follows; the form $X^{(3)}$ is a polynomial of the third degree in the fundamental ϕ-polynomials, which vanishes to the second order in each of the places $A_1, \ldots, A_{2p-3}, m_1, \ldots, m_p$, where $A_1, \ldots, A_{2p-3}$ are, with the place m, the zeros of a ϕ-polynomial ϕ_0; the form $Y^{(3)}$ is a polynomial, also of the third degree in the fundamental ϕ-polynomials, which vanishes to the second order in each of the places $A_1, \ldots, A_{2p-3}, \mu_1, \ldots, \mu_p$; if

$$u_i^{\mu_1, m_1} + \ldots + u_i^{\mu_p, m_p} = \tfrac{1}{2}(q_i + q_i'\tau_{i,1} + \ldots + q_p'\tau_{i,p}), \quad (i = 1, 2, \ldots, p),$$

where $u_1, \ldots, u_p$ are the Riemann normal integrals of the first kind, the characteristic associated with the form $Y^{(3)}$ is that denoted by $\frac{1}{2}\binom{q'}{q}$; and‡ it may be defined by the fact that the function $\sqrt{Y^{(3)}}/\sqrt{X^{(3)}}$, which is single-valued on the dissected Riemann surface, takes the factors $(-1)^{q_i'}$, $(-1)^{q_i}$ respectively at the i-th period loops of the first and second kind.

Take now another set of period loops; let $m_1', \ldots, m_p'$ be the places

* Or lot of sets, when the equivalence has not an unique solution.

† Cf. the concluding remark of § 185.

‡ Integer characteristics being omitted.

which, for these loops, arise as $m_1, \ldots, m_p$ arise for the original set of period loops; let $Z^{(3)}$ be the form which, for the new loops, has the same character as has the form $X^{(3)}$ for the original loops, so that $Z^{(3)}$ vanishes to the second order in each of $A_1, \ldots, A_{2p-3}, m_1', \ldots, m_p'$; then from the equivalences (§ 337)

$$w_i^{m_1', m_1} + \ldots + w_i^{m'_p, m_p} \equiv \tfrac{1}{2}[d(\bar{\beta}\beta')]_i + \tfrac{1}{2}\tau'_{i,1}[d(\bar{\alpha}\alpha')]_1 + \ldots + \tfrac{1}{2}\tau'_{i,p}[d(\bar{\alpha}\alpha')]_p, \quad (i=1, \ldots, p),$$

where $w_1, \ldots, w_p$ are the normal integrals of the first kind, it follows, as in § 244, that the function $\sqrt{Z^{(3)}}/\sqrt{X^{(3)}}$ is single-valued on the Riemann surface dissected by the new system of period loops, and at the r-th new loops, respectively of the first and second kind, has the factors

$$e^{-\pi i[d(\bar{\alpha}\alpha')]_r}, \quad e^{\pi i[d(\bar{\beta}\beta')]_r}.$$

The equations of transformation,

$$[\omega] = \omega\alpha + \omega'\alpha', \quad [\omega'] = \omega\beta + \omega'\beta',$$

of which one particular equation is that given by

$$[\omega_{n,r}] = \omega_{n,1}\alpha_{1,r} + \ldots\ldots + \omega_{n,p}\alpha_{p,r} + \omega'_{n,1}\alpha'_{1,r} + \ldots + \omega'_{n,p}\alpha'_{p,r}, \quad (n, r = 1, \ldots, p),$$

express the fact (cf. § 322) that a negative circuit of the new loop $[b_r]$ is equivalent to $\alpha_{i,r}$ negative circuits of the original loop (b_i) and $\alpha'_{i,r}$ positive circuits of the original loop (a_i); thus a function which has the factors $e^{-\pi i q_i'}$, $e^{\pi i q_i}$ at the i-th original loops, will at the r-th new loop $[a_r]$ have the factor $e^{-\pi i l_r'}$, where l_r' is an integer which is given by

$$-l_r' \equiv \sum_{i=1}^{p}[-q_i'\alpha_{i,r} + q_i\alpha'_{i,r}], \qquad (\text{mod. } 2);$$

thus the factors of $\sqrt{Y^{(3)}}/\sqrt{X^{(3)}}$ at the new period loops are given by $e^{-\pi i l'}$, $e^{\pi i l}$, where l, l' are rows of integers such that

$$l' \equiv \bar{\alpha}q' - \bar{\alpha}'q, \quad -l \equiv \bar{\beta}q' - \bar{\beta}'q, \qquad (\text{mod. } 2).$$

Therefore the factors of $\sqrt{Y^{(3)}}/\sqrt{Z^{(3)}} = (\sqrt{Y^{(3)}}/\sqrt{X^{(3)}})/(\sqrt{Z^{(3)}}/\sqrt{X^{(3)}})$, at the new period loops, are given by $e^{-\pi i k'}$, $e^{\pi i k}$, where

$$k' \equiv \bar{\alpha}q' - \bar{\alpha}'q - d(\bar{\alpha}\alpha'), \quad -k \equiv \bar{\beta}q' - \bar{\beta}'q - d(\bar{\beta}\beta'), \qquad (\text{mod. } 2);$$

now the characteristic associated with $\sqrt{Y^{(3)}}$ corresponding to the original system of period loops may be defined by the factors of $\sqrt{Y^{(3)}}/\sqrt{X^{(3)}}$ at those loops; similarly the characteristic which belongs to $\sqrt{Y^{(3)}}$ for the new system of loops is defined by the factors of $\sqrt{Y^{(3)}}/\sqrt{Z^{(3)}}$, and is therefore $\frac{1}{2}\begin{pmatrix} k' \\ k \end{pmatrix}$; the equations just obtained prove therefore that *the characteristic associated with* $\sqrt{Y^{(3)}}$ *is transformed precisely as a theta characteristic.*

The same result may be obtained thus; the p equations of the form

$$u_i^{\mu_1, m_1} + \dots + u_i^{\mu_p, m_p} = \tfrac{1}{2}(q_i + q_1\tau_{i,1} + \dots + q_p\tau_{i,p}), \quad (i=1, \dots, p),$$

are immediately seen, by means of the equation $(a+\tau a')(\bar{\beta}' - \tau'\bar{a}') = 1$ to lead to p equations expressible by

$$w^{\mu_1, m_1} + \dots + w^{\mu_p, m_p} = \tfrac{1}{2}(\bar{\beta}'q - \bar{\beta}q') + \tfrac{1}{2}\tau'(\bar{a}q' - \bar{a}'q);$$

subtracting from these the equations

$$w_i^{m_1', m_1} + \dots + w_i^{m_p', m_p} \equiv \tfrac{1}{2}[d(\bar{\beta}\beta')]_i + \tfrac{1}{2}\tau'_{i,1}[d(\bar{a}a')]_1 + \dots + \tfrac{1}{2}\tau'_{i,p}[d(\bar{a}a')]_p, \quad (i=1, \dots, p),$$

we obtain equations from which (as in § 244) the characteristic of $\sqrt{Y^{(3)}}$, for the new loops, is immediately deducible.

Similar reasoning applies obviously to the characteristics of the forms $\sqrt{X^{(2\nu+1)}}$ considered on page 380 (§ 245). But the characteristic for a form $\sqrt{X^{(2\mu)}}$ (p. 381), which is obtained by consideration of the single-valued function $\sqrt{X^{(2\mu)}}/\Phi^{(\mu)}$—into which the form $\sqrt{X^{(3)}}$, depending on the places $m_1, \dots, m_p$, does not enter—is transformed in accordance with the equations

$$k' \equiv \bar{a}q' - \bar{a}'q, \quad -k \equiv \bar{\beta}q' - \bar{\beta}'q, \quad (\text{mod. } 2),$$

and may be described as a *period-characteristic*, as in § 328.

340. Having thus investigated the dependence of the characteristics assigned to radical forms upon the method of dissection of the Riemann surface, it is proper to explain, somewhat further, how these characteristics may be actually specified for a given radical form. The case of a form $\sqrt{X^{(2\mu)}}$ differs essentially from that of a form $\sqrt{X^{(2\nu+1)}}$. When the zeros of a form $\sqrt{X^{(2\mu)}}$ are known, and the Riemann surface is given with a specified system of period loops, the factors of a function $\sqrt{X^{(2\mu)}}/\Phi^{(\mu)}$ at these loops may be determined by following the value of the function over the surface, noticing the places at which the values of the function branch—which places are in general only the fixed branch places of the Riemann surface; the process is analogous to that whereby, in the case of elliptic functions, the values of $\sqrt{\wp(u+2\omega_1) - e_1}/\sqrt{\wp(u) - e_1}$, $\sqrt{\wp(u+2\omega_2) - e_1}/\sqrt{\wp(u) - e_1}$ may be determined, by following the values of $\sqrt{\wp(u) - e_1}$ over the parallelogram of periods. But it is a different problem to ascertain the factors of the function $\sqrt{Y^{(3)}}/\sqrt{X^{(3)}}$ at the period loops, because the form $\sqrt{X^{(3)}}$ depends upon the places $m_1, \dots, m_p$, and we have given no elementary method of determining these places; the geometrical interpretation of these places which is given in § 183 (Chap. X.), and the algebraic process resulting therefrom, does not distinguish them from other sets of places satisfying the same conditions; the distinction in fact, as follows from § 338, cannot be made algebraically unless the period loops are given by algebraical equations. Nevertheless we

may determine the characteristic of a form $Y^{(3)}$, and the places $m_1, \dots, m_p$, by the following considerations*:—It is easily proved, by an argument like that of § 245 (Chap. XIII.), that if there be a form $\sqrt{X^{(1)}}$ having the same characteristic as $\sqrt{Y^{(3)}}$, there exists an equation of the form $\sqrt{X^{(1)}}\sqrt{Y^{(3)}} = \Phi^{(2)}$; and conversely, if $q+1$ linearly independent polynomials, of the second degree in the p fundamental ϕ-polynomials, vanish in the zeros of $\sqrt{Y^{(3)}}$, and $\Psi^{(2)}$ denote the sum of these $q+1$ polynomials, each multiplied by an arbitrary constant, that we have an equation $\sqrt{Y^{(1)}}\sqrt{Y^{(3)}} = \Psi^{(2)}$, where $\sqrt{Y^{(1)}}$ is a linear aggregate of $q+1$ radical forms like $\sqrt{X^{(1)}}$, all having the same characteristic as $\sqrt{Y^{(3)}}$; in general, since a form $\Psi^{(2)}$ can contain at most $3(p-1)$ linearly independent terms (§ 111, Chap. VI.), and the number of zeros of $\sqrt{Y^{(3)}}$ is $3(p-1)$, we have $q+1=0$; in any case the value of $q+1$ is capable of an algebraic determination, being the number of forms $\Phi^{(2)}$ which vanish in assigned places. Now the number of linearly independent forms $\sqrt{X^{(1)}}$ with the same characteristic is even or odd according as the characteristic is even or odd (§§ 185, 186, Chap. X.); hence, without determining the characteristic of $\sqrt{Y^{(3)}}$ we can beforehand ascertain whether it is even or odd by finding whether $q+1$ is even or odd. Suppose now that $\mu_1, \dots, \mu_p$ and $\mu_1', \dots, \mu_p'$ are two sets of places such that

$$(m^3, A_1, \dots, A_{2p-3}) \equiv (\mu_1^2, \dots, \mu_p^2) \equiv (\mu_1'^2, \dots, \mu_p'^2),$$

m being an arbitrary place, and $m, A_1, \dots, A_{2p-3}$ being the zeros of any ϕ-polynomial ϕ_0; so that $\mu_1, \dots, \mu_p$ and $\mu_1', \dots, \mu_p'$ are two sets arbitrarily selected from 2^{2p} sets which can be determined geometrically as in § 183, Chap. X. (cf. § 244, Chap. XIII.); let $Y^{(3)}$ vanish to the second order in each of $\mu_1, \dots, \mu_p, A_1, \dots, A_{2p-3}$ and $Y_1^{(3)}$ vanish to the second order in each of $\mu_1', \dots, \mu_p', A_1, \dots, A_{2p-3}$; by following the values of the single-valued function $\sqrt{Y_1^{(3)}}/\sqrt{Y^{(3)}}$ on the Riemann surface, we can determine its factors at the period loops; at the r-th period loops of the first and second kind let these factors be $(-1)^{k_r'}$, $(-1)^{k_r}$ respectively; then if $\frac{1}{2}(q_1, \dots, q_p')$ and $\frac{1}{2}(Q_1, \dots, Q_p')$ be respectively the characteristics of $\sqrt{Y^{(3)}}$ and $\sqrt{Y_1^{(3)}}$, which we wish to determine, we have (§ 244)

$$k_r' \equiv Q_r' - q_r', \quad k_r \equiv Q_r - q_r, \quad (\text{mod. } 2).$$

Take now, in turn, for $\mu_1', \dots, \mu_p'$, all the possible 2^{2p} sets which, as in § 183, are geometrically determinable from the place m; and, for the same form $\sqrt{Y^{(3)}}$, determine the 2^{2p} characteristics of all the functions $\sqrt{Y_1^{(3)}}/\sqrt{Y^{(3)}}$ arising

* Noether, *Jahresbericht der Deutschen Mathematiker Vereinigung*, Bd. iii. (1894), p. 494, where the reference is to Fuchs, *Crelle*, LXXIII. (1871); cf. Prym, *Zur Theorie der Functionen in einer zweiblättrigen Fläche* (Zürich, 1866).

by the change of the forms $\sqrt{Y_1^{(3)}}$; then there exists one, and only one, characteristic, $\frac{1}{2}\begin{pmatrix} s' \\ s \end{pmatrix}$, satisfying the condition that the characteristic

$$\tfrac{1}{2}\begin{pmatrix} s' \\ s \end{pmatrix} + \tfrac{1}{2}\begin{pmatrix} k' \\ k \end{pmatrix}$$

is even when $\sqrt{Y_1^{(3)}}$ has an even characteristic and odd when $\sqrt{Y_1^{(3)}}$ has an odd characteristic; for, clearly, the characteristic $\frac{1}{2}\begin{pmatrix} q' \\ q \end{pmatrix}$ is a value for $\frac{1}{2}\begin{pmatrix} s' \\ s \end{pmatrix}$ which satisfies the condition, and if $\frac{1}{2}\begin{pmatrix} \sigma' \\ \sigma \end{pmatrix}$ were another possible value for $\frac{1}{2}\begin{pmatrix} s' \\ s \end{pmatrix}$ we should have

$$(k+\sigma)(k'+\sigma') \equiv (k+q)(k+q') \qquad (\text{mod. } 2),$$

or

$$k(\sigma'-q') + k'(\sigma - q) \equiv qq' - \sigma\sigma'$$

for all the 2^{2p} possible values of $\frac{1}{2}\begin{pmatrix} k' \\ k \end{pmatrix}$; and this is impossible (Chap. XVII., § 295).

Hence we have the following rule:—*Investigate the factors of* $\sqrt{Y_1^{(3)}}/\sqrt{Y^{(3)}}$ *for an arbitrary form* $\sqrt{Y^{(3)}}$ *and all* 2^{2p} *forms* $\sqrt{Y_1^{(3)}}$; *corresponding to each form* $\sqrt{Y_1^{(3)}}$ *determine, by the method explained in the earlier part of this Article, whether its characteristic is even or odd; then, denoting the factors of any function* $\sqrt{Y_1^{(3)}}/\sqrt{Y^{(3)}}$ *respectively at the first and second kinds of period loops by quantities of the form* $(-1)^{k'}, (-1)^{k}$, *determine the characteristic* $\frac{1}{2}\begin{pmatrix} q' \\ q \end{pmatrix}$, *satisfying the condition that the characteristic* $\frac{1}{2}\begin{pmatrix} q'+k' \\ q+k \end{pmatrix}$ *is, for every form* $\sqrt{Y_1^{(3)}}$, *even or odd according as the characteristic of that form,* $\sqrt{Y_1^{(3)}}$, *is even or odd; then* $\frac{1}{2}\begin{pmatrix} q' \\ q \end{pmatrix}$ *is the characteristic of the form* $\sqrt{Y^{(3)}}$; *this being determined the characteristic of every form* $\sqrt{Y_1^{(3)}}$ *is known; the particular form* $\sqrt{Y_1^{(3)}}$ *for which the characteristic, thus arising, is actually zero, is the form previously denoted by* $\sqrt{X^{(3)}}$—*namely the form vanishing in the places* $m_1, \dots, m_p$ *which are to be associated (as in* § 179, *Chap.* X.) *with the particular system of period loops of the Riemann surface which has been adopted.*

Thus the method determines the places $m_1, \dots, m_p$ and determines the characteristic of every form $\sqrt{Y^{(3)}}$; the characteristic of any other form $\sqrt{Y^{(2\nu+1)}}$ is then algebraically determinable by the theorems of § 245 (p. 380).

341. For the hyperelliptic case we have shewn, in Chap. XI., how to express the ratios of the 2^{2p} Riemann theta functions with half-integer characteristics by means of algebraic functions; the necessary modification

of these formulae when the period loops are taken otherwise than in Chap. XI., follows immediately from the results of this chapter. If the change in the period loops be that leading to the linear transformation which is associated with the Abelian matrix formed with the integer matrices α, β, α', β', we have (§ 324)

$$\vartheta\left[u;\ \tfrac{1}{2}\begin{pmatrix}q'\\ q\end{pmatrix}\right]=A\vartheta_1\left[w;\ \tfrac{1}{2}\begin{pmatrix}k'\\ k\end{pmatrix}\right],$$

where

$$k'=\bar{\alpha}q'-\bar{\alpha}'q-d(\bar{\alpha}\alpha'),\quad -k=\bar{\beta}q'-\bar{\beta}'q-d(\bar{\beta}\beta').$$

If now, considering as sufficient example the formula of § 208 (Chap. XI.), we have

$$u_r^{b,a}\equiv q_1\omega_{r,1}+\dots+q_p\omega_{r,p}+q_1'\omega'_{r,1}+\dots+q_p'\omega'_{r,p},$$

then we have

$$w_r^{b,a}\equiv l_1v_{r,1}+\dots+l_pv_{r,p}+l_1'v'_{r,1}+\dots+l_p'v_{r,p},$$

where

$$l'=\bar{\alpha}q'-\bar{\alpha}'q=k'+d(\bar{\alpha}\alpha'),\quad -l=\bar{\beta}q'-\bar{\beta}'q=-k+d(\bar{\beta}\beta');$$

therefore, if the characteristic $\frac{1}{2}(d(\bar{\beta}\beta'), d(\bar{\alpha}\alpha'))$ be denoted by μ, the function $\vartheta_1\left[w;\ \frac{1}{2}\begin{pmatrix}k'\\ k\end{pmatrix}\right]$ is a constant multiple of $\vartheta_1\left[w;\ \frac{1}{2}\begin{pmatrix}l'\\ l\end{pmatrix}+\mu\right]$; and we may denote the latter function by $\vartheta_1[w|w^{b,a}+\mu]$. Thus the formula of § 208 is equivalent to

$$\sqrt{(b-x_1)\dots(b-x_p)}=C\frac{\vartheta_1(w|w^{b,a}+\mu)}{\vartheta_1(w|\mu)},$$

where C is independent of the arguments $w_1, \dots, w_p$, and, as in § 206,

$$w_r=w_r^{x_1,a_1}+\dots+w_r^{x_p,a_p},\qquad (r=1, 2, \dots, p).$$

Similar remarks apply to the formula of §§ 209, 210. It follows from § 337 that the characteristic μ is that associated with the half-periods

$$w^{m_1',a_1}+\dots+w^{m_p',a_p},$$

where $m_1', \dots, m_p'$ are the places which, for the new system of period loops, play the part of the places $m_1, \dots, m_p$ of § 179, Chap. X. It has already (§ 337) been noticed that for the elementary linear substitutions A_k, B, D the characteristic μ is zero.

342. In case the roots $c_1, a_1, c_2, a_2, \dots, c$, in the equation associated with the hyperelliptic case

$$y^2=4(x-c_1)(x-a_1)(x-c_2)(x-a_2)\dots(x-c_p)(x-a_p)(x-c),$$

be *real* and in *ascending* order of magnitude, we may usefully modify the notation of § 200, Chap. XI. Denote these roots, in order, by $b_{2p}, b_{2p-1}, \dots, b_0$, so that b_{2i}, b_{2i-1} are respectively c_{p-i+1}, a_{p-i+1} and b_0 is c, and interchange the period loops (a_i), (b_i), with retention of the direction of (b_i), as in the figure annexed (Fig. 17).

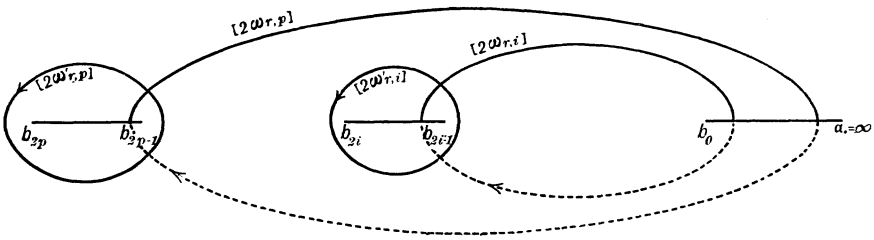

FIG. 17.

Then if $U_1^{x,a}, \ldots, U_p^{x,a}$ are linearly independent integrals of the first kind, such that $dU_r^{x,a}/dx = \psi_r/y$, where ψ_r is an integral polynomial in x, of degree $p-1$ at most, with only *real* coefficients, the half-periods

$$U_r^{b_{2i},\, b_{2i-1}} = [\omega_{r,i}], \quad U_r^{b_{2i-1},\, b_{2i-2}} = [\omega'_{r,i}] - [\omega'_{r,i-1}], \quad (i = 1, 2, \ldots, p;\ [\omega']_{r,0} = 0),$$

are respectively real and purely imaginary, so that $[\omega'_{r,i}]$ is also purely imaginary; if now $w_1^{x,a}, \ldots, w_p^{x,a}$ be the normal integrals, so that

$$U_r = [2\omega_{r,1}]\, w_1 + \ldots + [2\omega_{r,p}]\, w_p, \quad w_r = L_{r,1} U_1 + \ldots + L_{r,p} U_p,$$

then the second set of periods of $w_1^{x,a}, \ldots, w_p^{x,a}$, which are given by

$$\tau'_{r,i} = L_{r,1}[2\omega'_{1,i}] + \ldots + L_{r,p}[2\omega'_{p,i}], \qquad (r, s = 1, 2, \ldots, p),$$

are also purely imaginary*; forming with these the theta function $\Theta(w;\, \tau')$, the theta function of Chap. XI. is given (§ 335) by

$$e^{\pi i r w^2}\, \Theta\left(u;\ \tau \,\middle|\, {}^{Q'}_{Q}\right) = A e^{2\pi i QQ'}\, \Theta\left(w;\ \tau' \,\middle|\, {}^{-K}_{K'}\right),$$

where K, K' are obtainable from Q, Q' respectively by reversing the order of the p elements, and A is the constant $\sqrt{i/\Delta_1}\sqrt{i\Delta_1/\Delta_2}\sqrt{i\Delta_2/\Delta_3}\ldots$, in which $\Delta = \tau_{1,1}$, $\Delta_2 = \tau_{1,1}\tau_{2,2} - \tau^2_{1,2}$, etc. We find immediately that

$$U_r^{b_{2i-1},\, a} = -[\omega_{r,i}] - \ldots - [\omega_{r,p}] + [\omega'_{r,i}], \quad U_r^{b_{2i},a} = -[\omega_{r,i+1}] - \ldots - [\omega_{r,p}] + [\omega'_{r,i}],$$

$(i = 0, 1, \ldots, p)$, and may hence associate with b_{2i-1}, b_{2i} the respective odd and even characteristics

$$\{2i-1\} = \tfrac{1}{2}\begin{pmatrix} 0\ldots 0 & 1 & 0\ldots & 0 \\ 0\ldots 0 & -1 & -1\ldots & -1 \end{pmatrix}, \quad = \tfrac{1}{2}\begin{pmatrix} 0 \\ 0 \end{pmatrix}^{i-1}\begin{pmatrix} 1 \\ -1 \end{pmatrix}\begin{pmatrix} 0 \\ -1 \end{pmatrix}^{p-i},$$

$$\{2i\} = \tfrac{1}{2}\begin{pmatrix} 0\ldots 0 & 1 & 0\ldots & 0 \\ 0\ldots 0 & 0 & -1\ldots & -1 \end{pmatrix} = \tfrac{1}{2}\begin{pmatrix} 0 \\ 0 \end{pmatrix}^{i-1}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 0 \\ -1 \end{pmatrix}^{p-i},$$

* The quantities $\tau_{i,j}$ of Chap. XI. (of which the matrix is given in terms of the $\tau'_{i,j}$ of § 342 by $\tau\tau' = -1$) are also purely imaginary when $c_1, a_1, \ldots, c_p, a_p, c$ are real and in ascending order of magnitude.

and may denote the theta functions with these characteristics respectively by $\Theta_{2i-1}(w;\ \tau')$, $\Theta_{2i}(w;\ \tau')$; if $b_k, b_l, b_m, \dots$, be any of the places $b_{2p}, \dots, b_0$, not more than p in number, and if, with $0 \ngtr q_i < 2$, $0 \ngtr q_i' < 2$, we have

$$U_r^{b_k, a} + U_r^{b_l, a} + \dots \equiv -q_1[\omega_{r,1}] - \dots - q_p[\omega_{r,p}] + q_1'[\omega'_{r,1}] + \dots + q_p'[\omega'_{r,p}],$$

then the function whose characteristic is $\frac{1}{2}\begin{pmatrix} q' \\ -q \end{pmatrix}$ may be denoted by

$$\Theta_{k,l,m\dots}(w;\ \tau').$$

This function is equal to, or equal to the negative of, the function with characteristic $\frac{1}{2}\begin{pmatrix} q' \\ q \end{pmatrix}$, according as the characteristic is even or odd.

We have thus a number notation for the 2^{2p} half-integer characteristics*, equally whether the surface be hyperelliptic or not; this notation is understood to be that of Weierstrass (Königsberger, *Crelle*, LXIV. (1865), p. 20). For the numerical definition of the half-periods, which are given by the rule at the bottom of p. 297, precise conventions are necessary as to the allocation of the signs of the single valued functions $\sqrt{x-b_r}$ on the Riemann surface (cf. Chap. XXII.).

In the hyperelliptic case $p=2$, the characteristics of the theta functions given in the table of § 204 are supposed to consist of positive elements less than unity; when Q_1, Q_2, Q_1', Q_2' are each either 0 or $\frac{1}{2}$, the formula of the present article gives

$$e^{\pi i\tau w^2}\,\Theta\left[u\ ;\ \tau \,\middle|\, \begin{matrix} Q_1' & Q_2' \\ Q_1 & Q_2 \end{matrix}\right] = Ae^{-2\pi i QQ'}\,\Theta\left[w\ ;\ \tau' \,\middle|\, \begin{matrix} Q_2 & Q_1 \\ -Q_2' & -Q_1' \end{matrix}\right];$$

the number notations for the transformed characteristics are then immediately given by the table of § 204. The result is that the numbers

02, 24, 04, 1, 13, 3, 5, 23, 12, 2, 01, 0, 14, 4, 34, 03

are respectively replaced by

3, 1, 13, 24, 04, 02, 5, 0, 4, 2, 34, 23, 14, 12, 01, 03.

* For convenience in the comparison of results in the analytical theory of theta functions, it appears better to regard it as a notation for the characteristics rather than for the functions.