CHAPTER XVI.

A DIRECT METHOD OF OBTAINING THE EQUATIONS CONNECTING ϑ-PRODUCTS.

290. THE result given as Ex. xi. of § 286, in the last chapter, is a particular case of certain equations which may be obtained by actually multiplying together the theta series and arranging the product in a different way. We give in this chapter three examples of this method, of which the last includes the most general case possible. The first two furnish an introduction to the method and are useful for comparison with the general theorem. The theorems of this chapter do not require the characteristics to be half-integers.

291. *Lemma.* If b be a symmetrical matrix of p^2 elements, $U, V, u, v, A, B, f, g, q, r, f', g', q', r', M, N, s', t', m, n$ be columns, each of p elements, subject to the equations

$$n + m = 2N + s', \quad q' + r' = f', \quad q + r = f, \quad U + V = 2u = A,$$
$$-n + m = 2M + t', \quad -q' + r' = g', \quad -q + r = g, \quad -U + V = 2v = B,$$

then

$$2U(n+q') + b(n+q')^2 + 2\pi i q(n+q') + 2V(m+r') + b(m+r')^2 + 2\pi i r(m+r')$$
$$= 2A\left(N + \frac{s'+f'}{2}\right) + 2b\left(N + \frac{s'+f'}{2}\right)^2 + 2\pi i f\left(N + \frac{s'+f'}{2}\right)$$
$$+ 2B\left(M + \frac{t'+g'}{2}\right) + 2b\left(M + \frac{t'+g'}{2}\right)^2 + 2\pi i g\left(M + \frac{t'+g'}{2}\right).$$

This the reader can easily verify.

Suppose now that the elements of s' and t' are each either 0 or 1, and that n and m take, independently, all possible positive and negative integer values. To any pair of values, the equations $n + m = 2N + s'$, $-n + m = 2M + t'$ give a corresponding pair of values for integers N and M, and a pair of values for s' and t'. Since $2m = 2N + 2M + s' + t'$, $s' + t'$ is even, and therefore, since each element of s' and t' is < 2, s' must be equal to t'. Hence by means of the 2^p possible values for s', the pairs (n, m) are divisible into 2^p sets, each characterised by a certain value of s'. Conversely to any assignable integer value for each of the pair (N, M) and any assigned value of s' (< 2) corresponds by the equations $n = N - M$, $m = N + M + s'$ a definite pair of integer columns n, m.

Hence, b being such a matrix that, for real x, bx^2 has its real part negative,

$$\Big[\sum_n e^{2U(n+q')+b(n+q')^2+2\pi i q(n+q')}\Big]\Big[\sum_m e^{2V(m+r')+b(m+r')^2+2\pi i r(m+r')}\Big]$$
$$= \sum_{s'}\left[\sum_N e^{2A\left(N+\frac{s'+f'}{2}\right)+2b\left(N+\frac{s'+f'}{2}\right)^2+2\pi i f\left(N+\frac{s'+f'}{2}\right)}\right]$$
$$\left[\sum_M e^{2B\left(M+\frac{s'+g'}{2}\right)+2b\left(M+\frac{s'+g'}{2}\right)^2+2\pi i g\left(M+\frac{s'+g'}{2}\right)}\right];$$

thus, if $\underline{\vartheta}(u;\ \lambda)$, or $\underline{\vartheta}\begin{pmatrix}u;\ \lambda' \\ \lambda\end{pmatrix}$, denote $\sum\limits_n e^{2u(n+\lambda')+b(n+\lambda')^2+2\pi i\lambda(n+\lambda')}$, $\underline{\underline{\vartheta}}(u,\ \lambda)$ or $\underline{\underline{\vartheta}}\begin{pmatrix}u;\ \lambda' \\ \lambda\end{pmatrix}$ denote $\sum\limits_n e^{4u(n+\lambda')+2b(n+\lambda')^2+2\pi i\lambda(n+\lambda')}$, we have

$$\underline{\vartheta}(u-v;\ q)\,\underline{\vartheta}(u+v;\ r) = \sum_{s'}\underline{\underline{\vartheta}}\begin{bmatrix}u;\ \frac{1}{2}(s'+q'+r') \\ q+r\end{bmatrix}\underline{\underline{\vartheta}}\begin{bmatrix}v;\ \frac{1}{2}(s'-q'+r') \\ -q+r\end{bmatrix},$$

where the equation on the right contains 2^p terms corresponding to all values of s', which is a column of p integers each either 0 or 1; all other quantities involved are quite unrestricted.

Therefore if a be a symmetrical matrix of p^2 elements and h any matrix of p^2 elements, we deduce, replacing u by hu, and v by hv, and multiplying both sides by $e^{au^2+av^2}$, the result

$$\vartheta(u-v;\ q)\,\vartheta(u+v;\ r) = \sum_{\epsilon'}\vartheta_1\begin{bmatrix}u;\ \frac{1}{2}(\epsilon'+q'+r') \\ q+r\end{bmatrix}\vartheta_1\begin{bmatrix}v;\ \frac{1}{2}(\epsilon'-q'+r') \\ -q+r\end{bmatrix},$$

where ϵ' denotes all possible 2^p columns of p elements, each either 0 or 1, and ϑ_1 differs from ϑ only by having $2a$, $2h$, $2b$ instead of a, h, b in the exponent; thus we may write, more fully,

$$\vartheta\left(u-v;\ \begin{matrix}q' \\ q\end{matrix}\,\middle|\,\begin{matrix}2\omega,\ 2\omega' \\ 2\eta,\ 2\eta'\end{matrix}\right)\vartheta\left(u+v;\ \begin{matrix}r' \\ r\end{matrix}\,\middle|\,\begin{matrix}2\omega,\ 2\omega' \\ 2\eta,\ 2\eta'\end{matrix}\right)$$
$$= \sum_{\epsilon'}\vartheta\left[u;\ \begin{matrix}\frac{1}{2}(\epsilon'+q'+r') \\ q+r\end{matrix}\,\middle|\,\begin{matrix}\omega,\ 2\omega' \\ 2\eta,\ 4\eta'\end{matrix}\right]\vartheta\left[v;\ \begin{matrix}\frac{1}{2}(\epsilon'-q'+r') \\ -q+r\end{matrix}\,\middle|\,\begin{matrix}\omega,\ 2\omega' \\ 2\eta,\ 4\eta'\end{matrix}\right].$$

Ex. i. When the characteristics q, r are equal half-integer characteristics, say

$$q = r = \tfrac{1}{2}\begin{pmatrix}a' \\ a\end{pmatrix},$$

the equation is

$$e^{\pi i a a'}\vartheta\left[u+v;\ \tfrac{1}{2}\begin{pmatrix}a' \\ a\end{pmatrix}\right]\vartheta\left[u-v;\ \tfrac{1}{2}\begin{pmatrix}a' \\ a\end{pmatrix}\right] = \sum_{\epsilon'} e^{\pi i a\epsilon'}\vartheta_1\begin{pmatrix}u;\ \frac{1}{2}(\epsilon'+a') \\ 0\end{pmatrix}\vartheta_1\begin{pmatrix}v;\ \frac{1}{2}\epsilon' \\ 0\end{pmatrix};$$

multiplying this equation by $e^{\pi i a n}$, when n denotes a definite row of integers, each either

0 or 1, and adding the equations obtained by ascribing to a all the 2^p possible sets of values in which each element of a is either 0 or 1, we obtain

$$2^p \vartheta_1\left(u\,;\begin{matrix}\tfrac{1}{2}(n+a')\\0\end{matrix}\right)\vartheta_1\left(v\,;\begin{matrix}\tfrac{1}{2}n\\0\end{matrix}\right)=\sum_a e^{\pi i a(n+a')}\,\vartheta\left[u+v\,;\tfrac{1}{2}\begin{pmatrix}a'\\a\end{pmatrix}\right]\vartheta\left[u-v\,;\tfrac{1}{2}\begin{pmatrix}a'\\a\end{pmatrix}\right];$$

for we have

$$\sum_a e^{\pi i a(\epsilon'+n)}=\prod_{i=1}^{p}[1+e^{\pi i(\epsilon'_i+n_i)}].$$

Ex. ii. Deduce from Ex. i. that when $p=1$, the ratio of the two functions

$$\vartheta\left[u+a\,;\tfrac{1}{2}\begin{pmatrix}0\\0\end{pmatrix}\right]\vartheta\left[u-a\,;\tfrac{1}{2}\begin{pmatrix}0\\0\end{pmatrix}\right]+i\vartheta\left[u+a\,;\tfrac{1}{2}\begin{pmatrix}1\\1\end{pmatrix}\right]\vartheta\left[u-a\,;\tfrac{1}{2}\begin{pmatrix}1\\1\end{pmatrix}\right],$$

$$\vartheta\left[u+b\,;\tfrac{1}{2}\begin{pmatrix}1\\0\end{pmatrix}\right]\vartheta\left[u-b\,;\tfrac{1}{2}\begin{pmatrix}1\\0\end{pmatrix}\right]+i\vartheta\left[u+b\,;\tfrac{1}{2}\begin{pmatrix}0\\1\end{pmatrix}\right]\vartheta\left[u-b\,;\tfrac{1}{2}\begin{pmatrix}0\\1\end{pmatrix}\right],$$

is independent of u.

Ex. iii. Prove that the 2^p functions $\vartheta_1\left(u\,;\begin{matrix}\tfrac{1}{2}(\epsilon'+a')\\0\end{matrix}\right)$, obtained by varying ϵ', are not connected by any linear equation with coefficients independent of u.

Ex. iv. Prove that if a, a' be integral,

$$\vartheta^2\left[u\,;\tfrac{1}{2}\begin{pmatrix}a'\\a\end{pmatrix}\right]=\sum_{\epsilon'} e^{\pi i a\epsilon'}\,\vartheta_1\left[0\,;\tfrac{1}{2}\begin{pmatrix}\epsilon'+a'\\0\end{pmatrix}\right]\vartheta_1\left[u\,;\tfrac{1}{2}\begin{pmatrix}\epsilon'\\0\end{pmatrix}\right].$$

From this set of equations we can obtain the linear relation connecting the squares of 2^p+1 (or less) assigned theta functions with half-integer coefficients.

Ex. v. Using the notation $|\lambda_{i,j}|$ for the matrix in which the j-th element of the i-th row is $\lambda_{i,j}$, prove that if $u_1, \dots, u_r, v_1, \dots, v_r$ be $2\,.\,2^p$ arguments, and $\tfrac{1}{2}\begin{pmatrix}a'\\a\end{pmatrix}$ any half-integer characteristic,

$$\left|\vartheta\left[u_i+v_j\,;\tfrac{1}{2}\begin{pmatrix}a'\\a\end{pmatrix}\right]\vartheta\left[u_i-v_j\,;\tfrac{1}{2}\begin{pmatrix}a'\\a\end{pmatrix}\right]\right|=\left|\vartheta_1\left[u_i\,;\begin{matrix}\tfrac{1}{2}\epsilon_j'a'\\a\end{matrix}\right]\right|\left|\vartheta_1\left[v_j\,;\begin{matrix}\tfrac{1}{2}\epsilon_i'\\0\end{matrix}\right]\right|,$$

and, denoting the determinant of the matrix on the left hand by $\{u_i, v_j\}$ and the determinant of the second matrix on the right hand by $\{v\}$, deduce that

$$\{v_i, v_j\}=e^{2^{p-1}\pi i A}\{v\}^2,\quad \{u_i, v_j\}=\sqrt{\{u_i, u_j\}\{v_i, v_j\}},$$

where A is the sum of the p elements of the row letter a. When the characteristic $\tfrac{1}{2}\begin{pmatrix}a'\\a\end{pmatrix}$ is odd, $\{u_i, u_j\}$ is a skew symmetrical determinant whose square root is* expressible rationally in terms of the constituents $\vartheta\left[u_i+u_j\,;\tfrac{1}{2}\begin{pmatrix}a'\\a\end{pmatrix}\right]\vartheta\left[u_i-u_j\,;\tfrac{1}{2}\begin{pmatrix}a'\\a\end{pmatrix}\right]$. For instance when $p=1$, we obtain, with a proper sign for the square root, the equation of three terms†.

Since any 2^p+1 functions of the form $\vartheta\left[u+v_\beta\,;\tfrac{1}{2}\begin{pmatrix}a'\\a\end{pmatrix}\right]\vartheta\left[u-v_\beta\,;\tfrac{1}{2}\begin{pmatrix}a'\\a\end{pmatrix}\right]$ are connected by a linear equation with coefficients independent of u, it follows that if $u_1, \dots, u_m$, $v_1, \dots, v_m$ be any $2m$ arguments, m being greater than 2^p, the determinant of m rows and columns, whose (i, j)th element is $\vartheta\left[u_i+v_j\,;\tfrac{1}{2}\begin{pmatrix}a'\\a\end{pmatrix}\right]\vartheta\left[u_i-v_j\,;\tfrac{1}{2}\begin{pmatrix}a'\\a\end{pmatrix}\right]$, vanishes identically. When $\tfrac{1}{2}\begin{pmatrix}a'\\a\end{pmatrix}$ is odd and m is even, for example equal to 2^p+2, this determinant is

* Scott, *Theory of determinants* (Cambridge, 1880), p. 71.

† Halphen, *Fonct. Ellip.* (Paris, 1886), t. I. p. 187.

a skew symmetrical determinant whose square root may be expressed rationally in terms of the functions $\vartheta\left[u_i+v_j\,;\ \tfrac{1}{2}\binom{a'}{a}\right]\vartheta\left[u_i-v_j\,;\ \tfrac{1}{2}\binom{a'}{a}\right]$. The result obtained may be written

$$\{u_i,\ v_j\}^{\frac{1}{2}}=0,$$

wherein* the determinant $\{u_i,\ v_j\}$ has m rows and columns, m being even and greater than 2^p. When m is odd the determinant $\{u_i,\ v_j\}$ itself vanishes.

A proof that for general values of the arguments the corresponding determinant $\{u_i,\ v_j\}$, of 2^p rows and columns, does not identically vanish is given by Frobenius, *Crelle*, XCVI. (1884), p. 102.

A more general formula for the product of two theta functions is given below Ex. ii. § 292.

292. We proceed now to another formula, for the product of four theta functions. Let J denote the substitution

$$\tfrac{1}{2}\begin{pmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{pmatrix},$$

and J_{rs} be the element of the matrix which is in the r-th row and the s-th column; then $\sum_{i=1}^{4} J_{ir}\,J_{is}=0$ or 1, according as $r\neq s$, or $r=s$ $(r,\ s=1,\ 2,\ 3,\ 4)$.

Let $u_1,\ u_2,\ u_3,\ u_4$ denote four columns, each of p quantities; written down together they will form a matrix of 4 columns and p rows. Let $U_1,\ U_2,\ U_3,\ U_4$ be four other such columns, such that the j-th row of the first matrix $(j=1,\ 2,\ \ldots,\ p)$ is associated with the j-th row of the second by the equation

$$((u_1)_j,\ (u_2)_j,\ (u_3)_j,\ (u_4)_j)=J\,((U_1)_j,\ (U_2)_j,\ (U_3)_j,\ (U_4)_j).$$

Let $v_1,\ v_2,\ v_3,\ v_4$ and $V_1,\ V_2,\ V_3,\ V_4$ be two other similarly associated sets, each of four columns of p elements. Then if h be any matrix whatever, of p rows and columns, we have

$$hu_1v_1+hu_2v_2+hu_3v_3+hu_4v_4=hU_1V_1+hU_2V_2+hU_3V_3+hU_4V_4\,;$$

this is quite easy to prove: an elementary direct verification is obtained by selecting on the left the term $h_{jk}(u_1)_k(v_1)_j+h_{jk}(u_2)_k(v_2)_j+h_{jk}(u_3)_k(v_3)_j+h_{jk}(u_4)_k(v_4)_j$

$$=h_{jk}\sum_{r=1}^{4}\,[J_{r1}(U_1)_k+J_{r2}(U_2)_k+J_{r3}(U_3)_k+J_{r4}(U_4)_k]\,[J_{r1}(V_1)_j+J_{r2}(V_2)_j$$
$$+J_{r3}(V_3)_j+J_{r4}(V_4)_j]$$
$$=h_{jk}\,\{(\textstyle\sum_r J^2_{r1})(U_1)_k(V_1)_j+(\sum_r J_{r1}J_{r2})\,[(U_1)_k(V_2)_j+(U_2)_k(V_1)_j]+\ldots\ldots\}$$
$$=h_{jk}\,\{(U_1)_k(V_1)_j+(U_2)_k(V_2)_j+(U_3)_k(V_3)_j+(U_4)_k(V_4)_j\},$$

and this is the corresponding element of $hU_1V_1+hU_2V_2+hU_3V_3+hU_4V_4$.

* The theorem was given by Weierstrass, *Sitzungsber. der Berlin. Ak.* 1882 (I.—XXVI., p. 506), with the suggestion that the theory of the theta functions may be *à priori* deducible therefrom, as is the case when $p=1$ (Halphen, *Fonct. Ellip.* (Paris (1886)), t. I. p. 188). See also Caspary, *Crelle*, XCVI. (1884), and *ibid.* XCVII. (1884), and Frobenius, *Crelle*, XCVI. (1884), pp. 101, 103.

Now we have

$$\vartheta(u_1, q_1)\,\vartheta(u_2, q_2)\,\vartheta(u_3, q_3)\,\vartheta(u_4, q_4)$$
$$= \sum_{n_1, n_2, n_3, n_4} e^{\Sigma a u_r^2 + 2\Sigma h u_r (n_r + q_r') + \Sigma b (n_r + q_r')^2 + 2\pi i \Sigma q_r (n_r + q_r')}.$$

In the exponent here there are four sets each of four columns of p quantities namely the sets

$$u_r,\ n_r,\ q_r,\ q'_r;$$

we suppose each of these transformed by the substitution J. Hence the exponent becomes

$$\sum_{N_1, N_2, N_3, N_4} e^{\Sigma a U_r^2 + 2\Sigma h U_r (N_r + Q_r') + \Sigma b (N_r + Q_r')^2 + 2\pi i \Sigma Q_r (N_r + Q'_r)}$$

wherein the summation extends to all values of N_{rj} given by

$$N_{rj} = \tfrac{1}{2}(n_{1j} + n_{2j} + n_{3j} + n_{4j} - 2n_{rj}),$$

for which all of n_{rj} are integers.

All the values N_{rj} will not be integral. But since $N_{rj} - N_{sj} = n_{sj} - n_{rj}$ the fractional parts of $N_{1j}, N_{2j}, N_{3j}, N_{4j}$ will be the same, $= \tfrac{1}{2}\epsilon_j'$, say, ($\epsilon_j' = 0$ or 1). Let m_{rj} be the integral part of N_{rj}. We arrange the terms of the right hand into 2^p classes according to the 2^p values of ϵ_j'. Then since

$$m_{rj} = \tfrac{1}{2}(n_{1j} + n_{2j} + n_{3j} + n_{4j} - 2n_{rj}) - \tfrac{1}{2}\epsilon_j',$$

every term of the left-hand product, arising from a certain set of values of the $4p$ integers n_{rj}, gives rise to a definite term of the transformed product on the right with a definite value for ϵ_j', while, since

$$n_{rj} = \tfrac{1}{2}(m_{1j} + m_{2j} + m_{3j} + m_{4j} - 2m_{rj}) + \tfrac{1}{2}\epsilon_j',$$

every assignable set of values of the $4p$ integers m_{rj} and value for ϵ_j' (which would correspond to a definite term of the transformed product) will arise, from a certain term on the right, *provided only the values assigned for m_{rj} be such that $\tfrac{1}{2}(m_{1j} + m_{2j} + m_{3j} + m_{4j} + \epsilon_j')$ is integral.*

Now we can specify an expression involving the quantities

$$\mu_j, = \tfrac{1}{2}(m_{1j} + m_{2j} + m_{3j} + m_{4j} + \epsilon_j'),$$

which is 1 or 0 according as $\mu = (\mu_1, \mu_2, \ldots, \mu_p)$ is a column of integers or not. In fact if $\epsilon = (\epsilon_1, \ldots, \epsilon_p)$ be a column of quantities each either 0 or 1—so that ϵ is capable of 2^p values—the expression

$$\frac{1}{2^p}\sum_{\epsilon} e^{2\pi i \epsilon\mu} = \frac{1}{2^p}(\Sigma e^{2\pi i \epsilon_1 \mu_1})\ldots(\Sigma e^{2\pi i \epsilon_p \mu_p}) = \frac{1}{2^p}(1 + e^{2\pi i \mu_1})(1 + e^{2\pi i \mu_2})\ldots(1 + e^{2\pi i \mu_p})$$

has this property; for when $\mu_1, \ldots, \mu_p$ are not integers they are half-integers.

Hence if the series $\frac{1}{2^p}\sum\limits_{\epsilon} e^{\pi i\epsilon(m_1+m_2+m_3+m_4+\epsilon')}$ be attached as factor to every term of the transformed product on the right we may suppose the summation to extend to *all* integral values of m_{rj}, *for every value of* ϵ'.

Then the transformed product is

$$\frac{1}{2^p}\sum_{m_1m_2m_3m_4\epsilon} e^{\Sigma a U_r^2+2\Sigma h U_r(m_r+\frac{1}{2}\epsilon'+Q'_r)+\Sigma b(m_r+\frac{1}{2}\epsilon'+Q'_r)^2+2\pi i\Sigma Q_r(m_r+\frac{1}{2}\epsilon'+Q'_r)+\pi i\epsilon(m_1+m_2+m_3+m_4+\epsilon')}$$

$$=\frac{1}{2^p}\sum\prod_r e^{aU_r^2+2hU_r(m_r+p'_r)+b(m_r+p'_r)^2+2\pi i p_r(m_r+p'_r)}\,.\,e^{-\pi i\epsilon(\Sigma p'_r-\epsilon')},$$

where

$$p_r=\tfrac{1}{2}\epsilon+Q_r,\quad p_r'=\tfrac{1}{2}\epsilon'+Q_r',$$

so that

$$\Sigma p_r'=2\epsilon'+\Sigma Q_r'=2\epsilon'+\Sigma q_r'.$$

Thus we have

$$\vartheta(u_1,q_1)\vartheta(u_2,q_2)\vartheta(u_3,q_3)\vartheta(u_4,q_4)$$
$$=\frac{1}{2^p}\sum_{(\epsilon,\epsilon')} e^{-\pi i\epsilon(\epsilon'+\Sigma q'_r)}\vartheta\left[U_1,Q_1+\tfrac{1}{2}\binom{\epsilon'}{\epsilon}\right]\vartheta\left[U_2,Q_2+\tfrac{1}{2}\binom{\epsilon'}{\epsilon}\right]\vartheta\left[U_3,Q_3+\tfrac{1}{2}\binom{\epsilon'}{\epsilon}\right]$$
$$\vartheta\left[U_4,Q_4+\tfrac{1}{2}\binom{\epsilon'}{\epsilon}\right].$$

This very general formula obviously includes the formula of Ex. xi., § 286, Chap. XV. It is clear moreover that a similar investigation can be made for the product of any number, k, of theta-functions, provided only we know of a matrix J, of k rows and columns, which will transform the exponent of the general term of the product into the exponent of the general term of the sum of other products.

It is for this more general case that the next Article is elaborated. It is not necessary for either case that the characteristics $q_1, q_2, \ldots$ should consist of half-integers.

Ex. i. If q be a half-integer characteristic, $=Q$, say, and we use the abbreviation

$$\phi(u,v,w,t\,;\,Q)=\vartheta(u\,;\,Q)\,\vartheta(v\,;\,Q)\,\vartheta(w\,;\,Q)\,\vartheta(t\,;\,Q),$$

we have

$$\phi(u+a,u-a,v+b,v-b,Q)=\frac{1}{2^p}\sum_{\epsilon,\epsilon'} e^{-\pi i\epsilon\epsilon'}\phi\left[u+b,u-b,v+a,v-a\,;\,Q+\tfrac{1}{2}\binom{\epsilon'}{\epsilon}\right],$$

where the summation on the right hand extends to all possible 2^{2p} half-integer characteristics $\frac{1}{2}\binom{\epsilon'}{\epsilon}$; putting $Q+\frac{1}{2}\binom{\epsilon'}{\epsilon}=R$, so that R also becomes all 2^{2p} half-integer characteristics, this is the same as

$$e^{\pi i|Q|}\phi(u+a,u-a,v+b,v-b\,;\,Q)=\frac{1}{2^p}\sum_R e^{\pi i|Q,R|+\pi i|R|}\phi(u+b,u-b,v+a,v-a\,;\,R),$$

where,

if $\quad Q=\frac{1}{2}\binom{a'}{a},\quad R=\frac{1}{2}\binom{\beta'}{\beta},\quad$ then $\;|Q|=aa',\;\;|R|=\beta\beta',\;\;|Q,R|=a\beta'-a'\beta.$

By adding, or subtracting, to this the formula derived from it by interchange of v and a, we obtain a formula in which only even or odd characteristics R occur on the right hand. Thus, for $p=1$, we derive the equation of three terms.

Ex. ii. If a, β, γ, δ be integers such that $a\gamma$ is positive and $\beta\delta$ is negative, $\rho=a\delta-\beta\gamma$, and r be the absolute value of ρ, prove that

$$\Theta\left(u;\ a\gamma\tau\left|\begin{matrix}0\\0\end{matrix}\right.\right)\Theta\left(v;\ -\beta\delta\tau\left|\begin{matrix}0\\0\end{matrix}\right.\right)=\sum_{\mu,\nu}\Theta\left(u\delta-v\gamma;\ \rho\gamma\delta\tau\left|\begin{matrix}\mu/\rho\\0\end{matrix}\right.\right)\Theta\left(-u\beta+va;\ -\rho a\beta\tau\left|\begin{matrix}\nu/\rho\\0\end{matrix}\right.\right)$$

$$=r^{-2p}\sum_{e,f,g,h}\Theta\left(u\delta-v\gamma;\ \rho\gamma\delta\tau\left|\begin{matrix}e/\rho\\-\gamma g+\delta h\end{matrix}\right.\right)\Theta\left(-u\beta+va;\ -\rho a\beta\tau\left|\begin{matrix}f/\rho\\ag-\beta h\end{matrix}\right.\right),$$

where $\Theta\left(u;\ \tau\left|\begin{matrix}\epsilon'\\\epsilon\end{matrix}\right.\right)$ denotes the theta function in which the exponent of the general term is

$$2\pi iu\,(n+\epsilon')+i\pi\tau\,(n+\epsilon')^2+2\pi i\epsilon\,(n+\epsilon'),$$

and μ, ν are row letters of p elements, all positive (or zero) and less than r, subject to the condition that $(\delta\mu-\beta\nu)/\rho$, $(a\nu-\gamma\mu)/\rho$ are integral, while e, f, g, h are row letters of p elements which are all positive (or zero) and less than r.

Ex. iii. Taking, in Ex. ii., a, β, γ, δ respectively equal to 1, 1, 1, $-k$, we find $\mu=\nu<k+1$, k being positive. Hence, taking $k=3$, prove the formula (Königsberger, *Crelle*, LXIV. (1865), p. 24), of which each side contains 2^p terms,

$$\sum_s\Theta\left(u;\ \tau\left|\begin{matrix}\tfrac12 s'\\\tfrac12 s\end{matrix}\right.\right)\Theta\left(u;\ 3\tau\left|\begin{matrix}\tfrac12 s'\\\tfrac12 s\end{matrix}\right.\right)=\sum_s e^{-\pi is's}\,\Theta\left(0;\ \tau\left|\begin{matrix}0\\\tfrac12 s\end{matrix}\right.\right)\Theta\left(2u;\ 3\tau\left|\begin{matrix}0\\\tfrac12 s\end{matrix}\right.\right),$$

s, s' being rows of p quantities each either 0 or 1.

293. We proceed now to obtain a formula* for the product of any number, k, of theta functions.

We shall be concerned with two matrices X, x, each of p rows and k columns; the original matrix, written with capital letters, is to be transformed into the new matrix by a substitution different for each of the p rows; for the j-th row this substitution is of the form

$$(X_{1,j},\ X_{2,j},\ \ldots,\ X_{r,j},\ \ldots,\ X_{k,j})=\frac{1}{r_j}\,\omega_j\,(x_{1,j},\ x_{2,j},\ \ldots,\ x_{r,j},\ \ldots,\ x_{k,j});$$

herein r_j is a positive integer; ω_j is a matrix of k rows and columns, consisting of integers; the determinant formed by the elements of this matrix is supposed other than zero, and denoted by μ_j; bearing in mind that throughout this Article the values of r are 1, 2, ..., k and the values of j are 1, 2, ..., p, we may write the substitution in the form

$$(X_{r,j})=\frac{1}{r_j}\,\omega_j\,(x_{r,j}).$$

The substitution formed with the first minors of the determinant of ω_j will be denoted by Ω_j; that formed from Ω_j by a transposition of its rows and columns will be denoted by $\overline{\Omega}_j$. Then the substitution inverse to $\frac{1}{r_j}\,\omega_j$ is $\frac{r_j}{\mu_j}\,\overline{\Omega}_j$; denoting the former substitution by λ_j, the latter is λ_j^{-1}.

* Prym und Krazer, *Neue Grundlagen...der allgemeinen thetafunctionen*, Leipzig, 1892.

If for any value of j a set of k integers, $P_{r,j}$, be known such that the k quantities

$$\lambda_j^{-1}(P_{r,j}) = \frac{r_j}{\mu_j}\bar{\Omega}_j(P_{r,j})$$

are integers, then it is clear that an infinite number of such sets can be derived; we have only to increase the integers $P_{r,j}$ by integral multiples of μ_j. But the number of such sets in which each of $P_{r,j}$ is positive (including zero) and less than the absolute value of μ_j is clearly finite, since each element has only a finite number of possible values. We shall denote this number by s_j and call it the number of *normal* solutions of the conditions

$$\frac{r_j}{\mu_j}\bar{\Omega}_j(P_{r,j}) = \text{integral};$$

it is the same as the number of sets of k integers, positive (or zero) and less than the absolute value of μ_j, which can be represented in the form $\lambda_j(p_{r,j})$, for integral values of the elements $p_{r,j}$.

The k theta functions to be multiplied together are at first taken to be those given by

$$\Theta_r = \Sigma e^{2V_rN_r + B_rN_r^2}, \qquad (r = 1, \ldots, k),$$

wherein B_r is such a symmetrical matrix that, for real values of the p quantities X, the real part of the quadratic form denoted (§ 174, Chap. X.) by B_rX^2 is negative. The p elements of the row-letters V_r, N_r are denoted by $V_{r,j}$, $N_{r,j}$ $(j = 1, \ldots, p)$. The substitutions λ_j are supposed to be such that the equations $(X_{r,j}) = \lambda_j(x_{r,j})$ transform the sum $\sum_{r=1}^{k} B_rX_r^2$ into a sum $\sum_{r=1}^{k} b_rx_r^2$, in which the matrices b_r are symmetrical and have the property that for real x_r the real part of $b_rx_r^2$ is negative.

Taking now quantities $m_{r,j}$, $v_{r,j}$ determined by

$$(m_{r,j}) = \lambda_j^{-1}(N_{r,j}) = \frac{r_j}{\mu_j}\bar{\Omega}_j(N_{r,j}), \quad (v_{r,j}) = \bar{\lambda}_j(V_{r,j}) = \frac{1}{r_j}\bar{\omega}_j(V_{r,j}),$$

the expressions $\sum_{r=1}^{k} B_rN_r^2$, $\sum_{r=1}^{k} N_rV_r$ are respectively transformed to $\sum_{r=1}^{k} b_rm_r^2$ and

$$\sum_{j=1}^{p} \lambda_j(m_{r,j})(V_{r,j}) = \sum_{j=1}^{p} \bar{\lambda}_j(V_{r,j})(m_{r,j}) = \sum_{r=1}^{k} v_rm_r;$$

hence the product $\prod_{r=1}^{k} \Theta_r$ is transformed into $\sum_{N_1, \ldots, N_k} e^{2\sum_r v_rm_r + \sum_r b_rm_r^2}$, where the quantities $m_{r,j}$ have every set of values such that the quantities $\lambda_j(m_{r,j})$ take all the integral values, $N_{r,j}$, of the original product.

As in the two cases previously considered in this chapter, we seek now to associate *integers* with the quantities $m_{r,j}$. Let $(P_{r,j})$ be any normal solution of the conditions

$$\frac{r_j}{\mu_j}\bar{\Omega}_j(P_{r,j}) = \text{integral}, \ = (p_{r,j}), \text{ say};$$

put, for every value of j,

$$(N_{r,j}) - (P_{r,j}) = \mu_j(M_{r,j}) + (E'_{r,j}), \qquad (r = 1, \ldots, k)$$

wherein $(M_{r,j})$ consists of integers, and $(E'_{r,j})$ consists of positive integers (including zero), of which each is less than the absolute value of μ_j. For an assigned set $(P_{r,j})$ this is possible in one way; then

$$(m_{r,j}) = \frac{r_j}{\mu_j}\bar{\Omega}_j(N_{r,j}) = (p_{r,j}) + r_j\bar{\Omega}_j(M_{r,j}) + \frac{r_j}{\mu_j}\bar{\Omega}_j(E'_{r,j})$$

$$= (n_{r,j}) + \frac{1}{\mu_j}(\epsilon'_{r,j}), \text{ say},$$

where

$$(n_{r,j}) = (p_{r,j}) + r_j\bar{\Omega}_j(M_{r,j}), \quad (\epsilon'_{r,j}) = r_j\bar{\Omega}_j(E'_{r,j});$$

by this means there is associated with $(N_{r,j})$, corresponding to an assigned set $(P_{r,j})$, a definite set of integers $(n_{r,j})$, and a definite set $(E'_{r,j})$. We do not thus obtain every possible set of integers for $(n_{r,j})$, for we have

$$\frac{1}{r_j}\omega_j(n_{r,j}) = \frac{1}{r_j}\omega_j(p_{r,j}) + \mu_j(M_{r,j}) = (P_{r,j}) + \mu_j(M_{r,j}),$$

so that the values of $n_{r,j}$ which arise are such that $\lambda_j(n_{r,j})$ are integers.

Conversely let $(n_{r,j})$ be any assigned integers such that $\lambda_j(n_{r,j})$ are integers; put

$$\lambda_j(n_{r,j}) = (P_{r,j}) + \mu_j(M_{r,j}),$$

wherein the quantities $M_{r,j}$ are integers, and the quantities $P_{r,j}$ are positive integers (or zero), which are all less than the absolute value of μ_j; this is possible in one way; then taking any set of assigned integers $(E'_{r,j})$, which are all positive (or zero) and less than the absolute value of μ_j, we can define a set of integers $N_{r,j}$ by the equations, wherein $\lambda_j^{-1}(P_{r,j}) = \text{integral}$,

$$(N_{r,j}) = (E'_{r,j}) + (P_{r,j}) + \mu_j(M_{r,j}) = (E'_{r,j}) + \lambda_j(n_{r,j}).$$

Thus, from any set of integers $(N_{r,j})$, arising with a term $e^{\sum_r(2V_rN_r+B_rN_r^2)}$ of the product $\prod_{r=1}^{k}\Theta_r$, we can, by association with a definite normal solution $(P_{r,j})$ of the conditions $\lambda_j^{-1}(P_{r,j}) = \text{integral}$, obtain a definite set $(E'_{r,j})$, and a definite set $(n_{r,j})$ such that $\lambda_j(n_{r,j})$ are integers. And conversely, from any set of integers $(n_{r,j})$ which are such that $\lambda_j(n_{r,j})$ are integral, we can, by association with a definite set $(E'_{r,j})$, obtain a definite normal solution $(P_{r,j})$ and a definite set $(N_{r,j})$.

It follows therefore that if the product $\prod_{r=1}^{k} \Theta_r$ be written down $s_1 \dots s_p$ times, a term $e^{\sum_r (2V_rN_r + B_rN_r^2)}$ being associated in turn with every one of the $s_1 \dots s_p$ normal solutions of the p conditions $\lambda_j^{-1}(P_j) =$ integral, then there will arise, once with every assigned set $(E'_{r,j})$, every possible set $(n_{r,j})$ for which $\lambda_j(n_{r,j})$ are integers.

We introduce now a factor which has the value 1 or 0 according as the integers $(n_{r,j})$ satisfy the conditions $\lambda_j(n_{r,j}) =$ integral, or not. Take k integers $(E_{r,j})$, which are positive (or zero), and less than r_j; put

$$(\epsilon_{r,j}) = \bar{\omega}_j(E_{r,j});$$

then

$$\sum_{j=1}^{p} \sum_{r=1}^{k} \frac{1}{r_j} \epsilon_{r,j} \left(n_{r,j} + \frac{\epsilon'_{r,j}}{\mu_j}\right) = \sum_j \sum_r \frac{1}{r_j} \epsilon_{r,j} m_{r,j} = \sum_j \bar{\lambda}_j (E_{r,j})(m_{r,j}) = \sum_j \lambda_j (m_{r,j})(E_{r,j})$$
$$= \sum_j (N_{r,j})(E_{r,j}) = N_r E_r,$$

and this is integral when N_r is integral, that is, for all the values $(n_{r,j})$ which actually occur; in fact the quantities $N_{r,j}$ defined by

$$(N_{r,j}) = \lambda_j(m_{r,j}) = \frac{1}{r_j}\omega_j\left(n_{r,j} + \frac{\epsilon'_{r,j}}{\mu_j}\right) = \frac{1}{r_j}\omega_j(n_{r,j}) + (E'_{r,j}) = \lambda_j(n_{r,j}) + (E'_{r,j})$$

are integral or not according as $\lambda_j(n_{r,j})$ are integers or not.

Hence, for a given set $n_{r,j}$, and a given set $E'_{r,j}$, the sum

$$\sum_E e^{2\pi i \sum_j \sum_r \frac{1}{r_j} \epsilon_{r,j} \left(n_{r,j} + \frac{\epsilon'_{r,j}}{\mu_j}\right)} = \sum_E e^{2\pi i \sum_r N_r E_r} = \prod_{r,j} \sum_{E_{r,j}} [e^{2\pi i N_{r,j}}]^{E_{r,j}},$$

wherein the summation extends to all positive (and zero) integer values of $(E_{r,j})$ less than r_j, is equal to $r_1^k \dots r_p^k$ when $(N_{r,j})$ are all integral, and otherwise contains a factor of the form

$$(e^{2\pi i r_j N_{r,j}} - 1)/(e^{2\pi i N_{r,j}} - 1),$$

which is zero because $r_j(N_{r,j})$ is certainly integral. Hence if we denote

$$\sum_j \sum_r \frac{1}{r_j} \epsilon_{r,j} \left(n_{r,j} + \frac{\epsilon'_{r,j}}{\mu_j}\right) \text{ by } \sum_r \frac{1}{R} \epsilon_r \left(n_r + \frac{\epsilon'_r}{\mu}\right),$$

R having the values $r_1, \dots, r_p$, then we can write

$$\frac{1}{(r_1 \dots r_p)^k} \sum_E e^{2\pi i \sum_r \frac{1}{R} \epsilon_r \left(n_r + \frac{\epsilon'_r}{\mu}\right)} = 1, \text{ or } 0,$$

according as $\lambda_j(n_{r,j})$ are all integers or not.

If then every term of the transformed series, in which, so far, only those values of $n_{r,j}$ arise for which $\lambda_j(n_{r,j})$ are integers, be multiplied by this factor, and the transformed series be completed by the introduction of terms of the same general form as those which naturally arise in this way, so that now all possible integer values of $(n_{r,j})$ are taken in, the value of the transformed series will be unaltered. In other words we have

$$\prod_r \Theta_r = \prod_r e^{2V_r N_r + B_r N_r^2} = \frac{1}{s_1 \ldots s_p (r_1 \ldots r_p)^k} \sum_{N_1, \ldots, N_k, E', E} \prod_r e^{2v_r m_r + b_r m_r^2 + 2\pi i \frac{1}{R} \epsilon_r \left(n_r + \frac{\epsilon'_r}{\mu}\right)},$$

$$= (s_1 \ldots s_p)^{-1} (r_1 \ldots r_p)^{-k} \sum_{n, E', E} \prod_r e^{2v_r \left(n_r + \frac{\epsilon'_r}{\mu}\right) + b_r \left(n_r + \frac{\epsilon'_r}{\mu}\right)^2 + 2\pi i \frac{1}{R} \epsilon_r \left(n_r + \frac{\epsilon'_r}{\mu}\right)},$$

wherein all possible integer values of $(n_{r,j})$ arise on the right; thus the right-hand side is equal to

$$(s_1 \ldots s_p)^{-1} (r_1 \ldots r_p)^{-k} \sum_{E', E} \prod_r \Theta_r \left(v_r;\ \begin{matrix} \epsilon_r'/\mu \\ \epsilon_r/R \end{matrix}\right);$$

and this is the desired form of the transformed product. For convenience we recapitulate the notations; E_r', E_r each denote a column of p integers, positive or zero, such that $E'_{r,j} < |\mu_j|$, $E_{r,j} < r_j$; $(\epsilon'_{r,j}) = r_j \overline{\Omega}_j (E'_{r,j})$; $(\epsilon_{r,j}) = \overline{\omega}_j (E_{r,j})$; s_j is the number of sets of integral solutions, positive or zero, each less than $|\mu_j|$, of the conditions $\mu_j^{-1} r_j \Omega_j (P_{r,j}) =$ integral; $(v_{r,j}) = r_j^{-1} \overline{\omega}_j (V_{r,j})$; the function Θ_r is a theta function in which the ordinary matrices a, b, h (§ 189) are respectively 0, b_r, 1; by linear transformation of the variables of the form $V_r = h_r W_r$, and, in case the matrices ω_j be suitable, multiplication by an exponential $e^{\sum_r A_r V_r^2}$, these particularities in the form of the theta functions may be removed.

The number of sets $(E_{r,j})$ is $(r_1 \ldots r_p)^k$; the number of sets $(E'_{r,j})$ is $|\mu_1^k \ldots \mu_p^k|$; the product of these numbers is the number of theta-products on the right-hand side of the equation.

Ex. i. We test this formula by applying it to the case already discussed where ω_j is an orthogonal substitution given by

$$\omega_j = \begin{pmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{pmatrix}, \quad = \omega \text{ say},$$

which is independent of j, $r_j = 2$, $b_r = b$, $k = 4$; then $\mu_j = -16$, $E_{r,j} < 2$, $E'_{r,j} < 16$, and

$$\frac{r_j}{\mu_j} \overline{\Omega}_j = \left[\frac{1}{r_j} \omega_j\right]^{-1} = \tfrac{1}{2}\omega, \quad \frac{1}{R}(\epsilon_{r,j}) = \tfrac{1}{2}\omega\, (E_{r,j}), \quad \frac{1}{\mu} \epsilon'_{r,j} = \tfrac{1}{2}\omega\, (E'_{r,j});$$

thence $\frac{1}{R}\epsilon_{1,j} - \frac{1}{R}\epsilon_{2,j} = E_{2,j} - E_{1,j} =$ integral, etc., so that the fractional part of $\frac{1}{R}\epsilon_{r,j}$ is independent of r: similarly the fractional part of $\frac{1}{\mu}(\epsilon'_{r,j})$ is independent of r and we may write $\frac{1}{\mu}(\epsilon'_{r,j}) = (\tfrac{1}{2}\epsilon'_j + L_{1,j},\ \tfrac{1}{2}\epsilon'_j + L_{2,j},\ \ldots,\ \tfrac{1}{2}\epsilon'_j + L_{4,j})$ wherein $2L_{r,j} + \epsilon'_j < 16$. By the formula

$\vartheta(v, q+N)=e^{2\pi i q' N}\vartheta(v, q)$, when N is integral, we know that $\Theta_r\left(v_r\,;\,\begin{smallmatrix}\epsilon'_r/\mu\\ \epsilon_r/R\end{smallmatrix}\right)$ is independent of the integral part of ϵ'_r/μ. Hence the $(16)^{4p}=2^{16p}$ terms on the right-hand side of the general formula, which, for a specified value of $\frac{1}{2}\omega(E_{r,j})$, correspond to all the values of $\frac{1}{2}\omega(E'_{r,j})$, reduce to 2^p terms, in which, since $(E'_{r,j})=\frac{1}{2}\omega(\frac{1}{2}\epsilon'_j+L_{1,j}, \ldots, \frac{1}{2}\epsilon'_j+L_{4,j})$, all values of $\epsilon'(<2)$ arise. Hence there is a factor 2^{15p} and instead of the summation in regard to E, E' we have a summation in regard to E, ϵ', the right hand being in fact

$$C\,.\,2^{15p}\sum_{E,\,\epsilon'}\prod_r\Theta\left(v_r,\ \begin{matrix}\frac{1}{2}\epsilon'\\ \frac{1}{2}\omega(E_{r,j})\end{matrix}\right)$$

and containing 2^{4p} terms.

Now put $$\tfrac{1}{2}(E_{1,j}+E_{2,j}+E_{3,j}+E_{4,j})=\tfrac{1}{2}\epsilon_j+M_j,$$

M_j being integral; then the factor of a general term of the expanded right-hand product which contains the quantities $\frac{1}{2}\omega(E_{r,j})$ is

$$\prod_r e^{2\pi i\frac{1}{2}k_r(n_r+\frac{1}{2}\epsilon')},$$

where

$$k_{r,j}=E_{1,j}+E_{2,j}+E_{3,j}+E_{4,j}-2E_{r,j}=\epsilon_j+2(M_j-E_{r,j}),$$

and

$$e^{\frac{1}{2}\pi i\epsilon'\sum_r k_r}=\prod_j e^{\frac{1}{2}\pi i\epsilon_j'(4\epsilon_j+8M_j-2\epsilon_j-4M_j)}=\prod_j e^{\pi i\epsilon_j\epsilon_j'}=e^{\pi i\epsilon\epsilon'},$$

while

$$\sum_j\sum_r\pi i k_{r,j}n_{r,j}\equiv\pi i\sum_j\sum_r\epsilon_j n_{r,j}\ (\text{mod. } 2),\ \equiv\pi i\epsilon\,.\sum_r n_r,$$

so that

$$\prod_r e^{2\pi i\frac{1}{2}k_r(n_r+\frac{1}{2}\epsilon')}=\left[\prod_r e^{2\pi i\frac{1}{2}\epsilon(n_r+\frac{1}{2}\epsilon')}\right]e^{-\pi i\epsilon\epsilon'};$$

therefore the right-hand product consists only of terms of the form $\left[\prod_r\Theta\left(v_r,\ \begin{smallmatrix}\frac{1}{2}\epsilon'\\ \frac{1}{2}\epsilon\end{smallmatrix}\right)\right]e^{-\pi i\epsilon\epsilon'}$. Hence the 2^{4p} terms arising, for a specified value of ϵ', for all the values of $E_{r,j}$, reduce to 2^p terms, and there is a further factor 2^{3p}—the right hand being

$$C\,.\,2^{18p}\sum_{\epsilon,\,\epsilon'}\left[\prod_r\Theta\left(v_r,\ \begin{matrix}\frac{1}{2}\epsilon'\\ \frac{1}{2}\epsilon\end{matrix}\right)\right]e^{-\pi i\epsilon\epsilon'},$$

where

$$C=(s_1\ldots s_p)^{-1}(r_1\ldots r_p)^{-k}=(s_1\ldots s_p)^{-1}2^{-4p}=s^{-p}2^{-4p}.$$

To determine the value of C we must know the number (s) of positive integral solutions, each less than 16, of the conditions $\frac{1}{2}\omega(x)=$integral, $=(y)$ say, namely of the conditions, $x_1+x_2+x_3+x_4=2(x_r+y_r)$. Now of these any positive values of x_1, x_2, x_3, x_4 (<16) are admissible for which $x_1+x_2+x_3+x_4$ is even. They must therefore either be all even, possible in 8^4 ways, or two even, possible in $6\,.\,8^2\,.\,8^2$ ways, or all odd, possible in 8^4 ways. Hence $s=8\,.\,8^4=2^{15}$. Hence $C=1/2^{15p}2^{4p}=1/2^{19p}$ and therefore $C\,.\,2^{18p}=\dfrac{1}{2^p}$.

Making now in the formula thus obtained, which is

$$\prod_r\Theta(V_r, 0)=\frac{1}{2^p}\sum_{\epsilon,\,\epsilon'}e^{-\pi i\epsilon\epsilon'}\prod_r\Theta\left[v_r,\ \tfrac{1}{2}\begin{pmatrix}\epsilon'\\ \epsilon\end{pmatrix}\right],$$

the substitution $V_r=hU_r$, we have $v_r=\frac{1}{2}(V_1+V_2+V_3+V_4-2V_r)=hu_r$, where $u_r=\frac{1}{2}(U_1+U_2+U_3+U_4-2U_r)$; and if we multiply the left hand by $e^{aU_1^2+aU_2^2+aU_3^2+aU_4^2}$, which is equal to $e^{au_1^2+au_2^2+au_3^2+au_4^2}$, we obtain

$$\prod_r\vartheta(U_r, 0)=\frac{1}{2^p}\sum e^{-\pi i\epsilon\epsilon'}\prod_r\vartheta\left[u_r,\ \tfrac{1}{2}\begin{pmatrix}\epsilon'\\ \epsilon\end{pmatrix}\right].$$

Therefore if Q_1, Q_2, Q_3, Q_4 denote any characteristics, and, as formerly, Ω_{Q_r} denote the period-part corresponding to Q_r, we have

$$\prod_r \vartheta(U_r, Q_r) = \prod_r e^{-\lambda(U_r, Q_r)} \vartheta(U_r + \Omega_{Q_r}, 0) = \prod_r e^{-\lambda(U_r, Q_r)} \prod_r \vartheta(U_r + \Omega_{Q_r}, 0),$$

of which the first factor is easily shewn to be $\prod_r e^{-\lambda(u_r, q_r)}$, if $(q_1, q_2, q_3, q_4) = \frac{1}{2}\omega(Q_1, Q_2, Q_3, Q_4)$; thus

$$\prod_r \vartheta(U_r, Q_r) = \frac{1}{2^p} \Sigma e^{-\pi i \epsilon\epsilon'} \prod_r e^{-\lambda(u_r, q_r)} \vartheta\left[u_r + \Omega_{q_r}, \tfrac{1}{2}\binom{\epsilon'}{\epsilon}\right]$$

$$= \frac{1}{2^p} \sum_{\epsilon, \epsilon'} e^{-\pi i \epsilon\epsilon'} \prod_r e^{-\lambda(u_r, q_r)} e^{\lambda(u_r, q_r) - 2\pi i q'_r(\frac{1}{2}\epsilon)} \vartheta\left[u_r, q_r + \tfrac{1}{2}\binom{\epsilon'}{\epsilon}\right]$$

$$= \frac{1}{2^p} \sum_{\epsilon, \epsilon'} e^{-\pi i \epsilon(\Sigma q'_r + \epsilon')} \vartheta\left[u_1, q_1 + \tfrac{1}{2}\binom{\epsilon'}{\epsilon}\right] \dots \vartheta\left[u_4, q_4 + \tfrac{1}{2}\binom{\epsilon'}{\epsilon}\right],$$

which is exactly the formula previously obtained (§ 292).

Ex. ii. More generally let $\lambda = \frac{1}{r_j}\omega_j$ be any matrix such that the linear equations $(X_r) = \lambda(x_r)$ give

$$X_1^2 + \dots\dots + X_k^2 = m(x_1^2 + \dots\dots + x_k^2),$$

wherein m is independent of $x_1, \dots, x_k$; then, since, by a property of all linear substitutions, the equations $(Y_r) = \lambda(y_r)$ lead to

$$Y_1 \frac{\partial}{\partial X_1} + \dots\dots + Y_k \frac{\partial}{\partial X_k} = y_1 \frac{\partial}{\partial x_1} + \dots\dots + y_k \frac{\partial}{\partial x_k},$$

we have also*

$$Y_1 X_1 + \dots\dots + Y_k X_k = m(y_1 x_1 + \dots\dots + y_k x_k).$$

Hence, if h be any matrix of p rows and columns and

$$(X_{r,j}) = \lambda(x_{r,j}), \qquad (j = 1, \dots, p),$$

we have

$$hX_1Y_1 + hX_2Y_2 + \dots + hX_kY_k = \sum_{i,j} h_{i,j} \sum_r X_{r,j} Y_{r,i} = m \sum_{i,j} h_{i,j} \sum_r x_{r,j} y_{r,i} = m(hx_1y_1 + \dots + hx_ky_k),$$

where X_1, x_1, etc. now denote rows of p quantities.

Thus any orthogonal substitution furnishes a case of our theorem. Taking a case where

$$m = 1,\ r_j = r,\ \omega_j = \omega,\ \mu = \pm r^k,\ E_{r,j} < r,\ E'_{r,j} < |\mu| < r^k,$$

we have

$$\frac{1}{R}(\epsilon_{r,j}) = \frac{1}{r}\omega(E_{r,j}),\ \frac{1}{\mu}(\epsilon'_{r,j}) = \frac{r\bar{\Omega}_j}{\mu}(E'_{r,j}) = \left[\frac{1}{r}\omega\right]^{-1}(E'_{r,j}) = \frac{1}{r}\bar{\omega}(E'_{r,j}),$$

so that the new characteristics will be r-th parts of integers.

Suppose now, in particular, that the substitution is

$$(X_1, \dots, X_r, \dots X_k) = \frac{1}{k}\begin{pmatrix} 2-k & 2 & \dots\dots & 2 \\ 2 & 2-k & \dots\dots & 2 \\ \dots & \dots & \dots & \dots \\ 2 & 2 & \dots\dots & 2-k \end{pmatrix}(x_1, \dots, x_r, \dots, x_k),$$

* Therefore $mxy = XY = \lambda x \,.\, \lambda y = \lambda\bar{\lambda}xy$, so that $\lambda\bar{\lambda} = m$; hence the determinant formed with the elements of λ has one of the values $\sqrt{m^k}$.

which gives

$$X_1^2+\ldots+X_k^2=\sum_r\left[\frac{2}{k}(x_1+\ldots+x_k)-x_r\right]^2=k\frac{4}{k^2}(x_1+\ldots+x_k)^2+\Sigma x_r^2-\frac{4}{k}\Sigma x_r(x_1+\ldots+x_k)=\Sigma x_r^2,$$

and

$$X_1+\ldots\ldots+X_k=x_1+\ldots\ldots+x_k,\quad X_1-X_2=x_2-x_1,\text{ etc.}$$

The previous example is a particular case, namely when $k=4$. In what follows we may suppose k odd so that $r_j=k$. When k is even r_j may be taken $=\frac{1}{2}k$. The work is arranged to apply to either case.

The fractional parts of $\frac{1}{\mu}(\epsilon'_{r,j})$ being independent of the suffix r—because

$$\frac{1}{\mu}\epsilon'_{1,j}-\frac{1}{\mu}\epsilon'_{2,j}=E'_{2,j}-E'_{1,j},\text{ etc.,}$$

—we may put $\frac{1}{\mu}(\epsilon'_{r,j})=\left(\frac{1}{r}\epsilon_j'+L_{1,j},\ \ldots,\ \frac{1}{r}\epsilon_j'+L_{k,j}\right)$, and may therefore write

$$\prod_r\Theta\left(v_r,\begin{matrix}\epsilon_r'/\mu\\ \epsilon_r/R\end{matrix}\right)\text{ in the form }\prod_r\Theta\left(v_r,\begin{matrix}\epsilon'/r\\ \epsilon_r/R\end{matrix}\right).$$

The equation

$$\left(\frac{1}{\mu}\epsilon_j'+L_{r,j}\right)=\frac{1}{r}\bar{\omega}(E'_{r,j})=\frac{1}{r}\omega(E'_{r,j})$$

shews that all values of $\frac{1}{r}\epsilon_j'\ (<1)$ do arise. Hence for a given value of $(E_{r,j})$ there are, instead of $|\mu|^{kp}=r^{k^2p}$ terms given by the general formula, only r^p, and the factor $r^{(k^2-1)p}$ divides out.

The values of $\frac{1}{R}(\epsilon_{r,j})$ given by the general formula are in number $|r|^{kp}$, corresponding to all the values of $(E_{r,j})$. As before the fractional part of $\frac{1}{R}(\epsilon_{r,j})$ is independent of r. Let

$$\frac{1}{k}(E_{1,j}+\ldots\ldots+E_{k,j})=\frac{\epsilon_j}{k}+M_j,$$

where $\frac{\epsilon_j}{k}<1$; then

$$\frac{1}{R}(\epsilon_{r,j})=\frac{1}{r}\omega(E_{r,j})=\left(\frac{2}{k}(E_{1,j}+\ldots\ldots+E_{k,j})-E_{r,j}\right)\equiv\left(\frac{2\epsilon_j}{k},\ \frac{2\epsilon_j}{k},\ldots\right),\ (\text{mod. }1).$$

The factor in the general term of the expanded product on the right hand which contains $\epsilon_{r,j}$ is

$$K=\prod_j\prod_r e^{2\pi i\frac{1}{r}\epsilon_{r,j}(n_{r,j}+\frac{1}{r}\epsilon_j')}.$$

Now

$$\sum_r\frac{1}{r}\epsilon_{r,j}=\sum_r(E_{r,j})=\epsilon_j+kM_j;$$

therefore, as r is k or a factor of k,

$$\prod_r e^{2\pi i\frac{1}{r}\epsilon_{r,j}\frac{1}{r}\epsilon_j'}=e^{2\pi i(\epsilon_j+kM_j)\frac{\epsilon_j'}{r}}=e^{2\pi i\frac{\epsilon_j\epsilon_j'}{r}}$$

and

$$\sum_r\frac{1}{r}\epsilon_{r,j}n_{r,j}=\sum_r\left[\frac{2}{k}(E_{1,j}+\ldots\ldots+E_{k,j})-E_{r,j}\right]n_{r,j}$$

$$=\sum_r\left[\frac{2\epsilon_j}{k}+2M_j-E_{r,j}\right]n_{r,j}\equiv\frac{2}{k}\sum_r\epsilon_jn_{r,j}\ (\text{mod. }1).$$

Hence the factor above is

$$K=\left[\prod_r e^{2\pi i\frac{2\epsilon}{k}\left(n_r+\frac{\epsilon'}{r}\right)}\right]e^{-4\pi i\frac{\epsilon\epsilon'}{r}}\cdot e^{2\pi i\frac{\epsilon\epsilon'}{r}},$$

and the general term of the right hand is

$$\left[\prod_r \Theta\left(v_r, \begin{matrix}\epsilon'/r\\ 2\epsilon/k\end{matrix}\right)\right]e^{-2\pi i\frac{\epsilon\epsilon'}{r}}.$$

Since $\frac{1}{R}(\epsilon_{r,j})=\left(\frac{2\epsilon_j}{k}+2M_j-E_{r,j}\right)$ we may suppose all values of $\epsilon_j<k$ to arise. Hence instead of r^{kp} we have k^p and a factor r^{kp}/k^p divides out.

To evaluate the factor $(r_1\ldots r_p)^{-1}(s_1\ldots s_p)^{-k}$, $=C$, say, we must enquire how many positive solutions exist of the conditions

$$\frac{2}{k}(x_1+\ldots\ldots+x_k)-x_r=\text{integral},$$

namely, how many solutions of the conditions

$$\frac{2}{k}(x_1+\ldots\ldots+x_k)=\text{integral},$$

exist, for which each of $x_1, \ldots, x_k<r^k$; let s be this number; then $C=s^{-p}r^{-kp}$, and

$$\prod_r \Theta(V_r, 0)=\frac{r^{(k^2-1)p}}{(sk)^p}\sum_{\epsilon,\epsilon'}\left[\prod_r \Theta\left(v_r, \begin{matrix}\epsilon'/r\\ 2\epsilon/k\end{matrix}\right)\right]e^{\frac{-2\pi i\epsilon\epsilon'}{r}},$$

where $\epsilon'<r$, $\epsilon<k$, the number of terms on the right being $(rk)^p$. For values of $\epsilon>\frac{k}{2}$ we may utilise the equation $\vartheta(v, q+N)=e^{2\pi iNq'}\vartheta(v, q)$. For example, when $k=r=3$ there are 3^{2p} terms, corresponding to characteristics $\left(\begin{matrix}\epsilon'/3\\ 2/3\end{matrix}\right)$. When $k=4$, $r=2$, the characteristics $\frac{2\epsilon}{k}=\frac{\epsilon}{2}$ will, effectively, repeat themselves. We can reduce the number of terms from 8^p or 2^{3p} to 2^{2p}. We shall thus get factors $\left(e^{2\pi iN\cdot\frac{\epsilon'}{2}}\right)^4=1$ and so the formula reduces to that already found.

Ex. iii. Apply the formula of the last example to the orthogonal case given by $\omega_j=\omega$,

$$(X, Y, Z, T, U, V)=\tfrac{1}{2}\omega(x, y, z, t, u, v),$$

$$\omega=\left(\begin{array}{rrrrrr} 1 & 1 & 0 & 0 & 1 & -1\\ 1 & 1 & 0 & 0 & -1 & 1\\ 1 & -1 & 1 & 1 & 0 & 0\\ -1 & 1 & 1 & 1 & 0 & 0\\ 0 & 0 & 1 & -1 & 1 & 1\\ 0 & 0 & -1 & 1 & 1 & 1\end{array}\right),\quad \omega^{-1}=\left(\begin{array}{rrrrrr} 1 & 1 & 1 & -1 & 0 & 0\\ 1 & 1 & -1 & 1 & 0 & 0\\ 0 & 0 & 1 & 1 & 1 & -1\\ 0 & 0 & 1 & 1 & -1 & 1\\ 1 & -1 & 0 & 0 & 1 & 1\\ -1 & 1 & 0 & 0 & 1 & 1\end{array}\right),$$

which lead to $\mu=64$ and

$$X^2+Y^2+Z^2+T^2+U^2+V^2=x^2+y^2+z^2+t^2+u^2+v^2$$
$$X+Y+Z+T+U+V=x+y+z+t+u+v$$
$$Z-T=x-y,\quad U-V=z-t,\quad X-Y=u-v,$$
$$X+Y=x+y,\quad Z+T=z+t,\quad U+V=u+v.$$