CHAPTER XV.

RELATIONS CONNECTING PRODUCTS OF THETA FUNCTIONS—INTRODUCTORY.

280. As preparatory to the general theory of multiply-periodic functions of several variables, and on account of the intrinsic interest of the subject, the study of the algebraic relations connecting the theta functions is of great importance. The multiplicity and the complexity of these relations render any adequate account of them a matter of difficulty; in this volume the plan adopted is as follows:—In the present chapter are given some preliminary general results frequently used in what follows, with some examples of their application. The following Chapter (XVI.) gives an account of a general method of obtaining theta relations by actual multiplication of the infinite series. In Chapter XVII. a remarkable theory of groups of half-integer characteristics, elaborated by Frobenius, is explained, with some of the theta relations that result; from these the reader will perceive that the theory is of great generality and capable of enormous development. References to the literature, which deals mostly with the case of half-integer characteristics, are given at the beginning of Chapter XVII.

281. Let $\phi(u_1, \dots, u_p)$ be a single-valued function of p independent variables $u_1, \dots, u_p$, such that, if $a_1, \dots, a_p$ be a set of *finite* values for $u_1, \dots, u_p$ respectively, the value of $\phi(u_1, \dots, u_p)$, for any set of finite values of $u_1, \dots, u_p$, is expressible by a converging series of ascending integral positive powers of $u_1 - a_1, u_2 - a_2, \dots, u_p - a_p$. Such a function is an integral analytical function. Suppose further that $\phi(u_1, \dots, u_p)$ has for each of its arguments, independently of the others, the period unity, so that if m be any integer, we have, for $\alpha = 1, 2, \dots, p$, the equation

$$\phi(u_1, \dots, u_\alpha + m, \dots, u_p) = \phi(u_1, \dots, u_p).$$

Then* the function $\phi(u_1, \dots, u_p)$ can be expressed by an infinite series of the form

$$\sum_{n_1=-\infty}^{\infty} \dots\dots \sum_{n_p=-\infty}^{\infty} A_{n_1, \dots, n_p} e^{2\pi i (u_1 n_1 + \dots + u_p n_p)},$$

* For the nomenclature and another proof of the theorem, see Weierstrass, *Abhandlungen aus der Functionenlehre* (Berlin, 1886), p. 159, etc.

wherein $n_1, \ldots, n_p$ are integers, each taking, independently of the others, all positive and negative values, and $A_{n_1, \ldots, n_p}$ is independent of $u_1, \ldots, u_p$.

Let the variables $u_1, \ldots, u_p$ be represented, in the ordinary way, each by the real points of an infinite plane. Put $x_1 = e^{2\pi i u_1}, \ldots, x_p = e^{2\pi i u_p}$; then to the finite part of the u_a-plane $(\alpha = 1, \ldots, p)$ corresponds the portion of an x_a-plane lying between a circle Γ_a of indefinitely great but finite radius R_a, whose centre is at $x_a = 0$, and a circle γ_a of indefinitely small but not zero radius r_a, whose centre is at $x_a = 0$. The annulus between these circles may be denoted by T_a. Let a_a be a value for x_a represented by a point in the annulus T_a; describe a circle (A_a) with centre at a_a, which does not cut the circle γ_a; then for values of x_a represented by points in the annulus T_a which are within the circle (A_a), u_a may be represented by a series of integral positive powers of $x_a - a_a$; and by the ordinary method of continuation, the values of u_a for all points within the annulus T_a may be successively represented by such series; the most general value of u_a, for any value of x_a, is of the form $x_a + m$, where m is an integer. Thus, in virtue of the definition, $\phi(u_1, \ldots, u_p)$ is a single-valued, and analytical, function of the variables $x_1, \ldots, x_p$, which is finite and continuous for values represented by points within the annuli $T_1, \ldots, T_p$ and upon the boundaries of these. So considered, denote it by $\psi(x_1, \ldots, x_p)$.

Take now the integral

$$\frac{1}{(2\pi i)^p} \iint \cdots \int \frac{\psi(t_1, \ldots, t_p)}{(t_1 - x_1) \ldots (t_p - x_p)} \, dt_1 \ldots dt_p,$$

wherein $x_1, \ldots, x_p$ are definite values such as are represented by points respectively within the annuli $T_1, \ldots, T_p$; let its value be formed in two ways;

(i) let the variable t_a be taken counter-clockwise round the circumference Γ_a and clockwise round the circumference γ_a $(\alpha = 1, \ldots, p)$; when t_a is upon the circumference Γ_a put

$$\frac{1}{t_a - x_a} = \frac{1}{t_a} + \frac{x_a}{t_a^2} + \frac{x_a^2}{t_a^3} + \ldots\ldots = \sum_{h_a=0}^{\infty} \frac{x_a^{h_a}}{t_a^{h_a+1}};$$

when t_a is upon the circumference γ_a put

$$\frac{1}{t_a - x_a} = -\left(\frac{1}{x_a} + \frac{t_a}{x_a^2} + \frac{t_a^2}{x_a^3} + \ldots\right) = -\sum_{k_a=-\infty}^{-1} \frac{x_a^{k_a}}{t_a^{k_a+1}};$$

then the integral is equal to

$$\frac{1}{(2\pi i)^p} \iint \cdots \int \psi(t_1, \ldots, t_p) \prod_{a=1}^{p} \left(dZ_a \sum_{h_a=0}^{\infty} \frac{x_a^{h_a}}{t_a^{h_a+1}} - dz_a \sum_{k_a=-\infty}^{-1} \frac{x_a^{k_a}}{t_a^{k_a+1}} \right),$$

where dZ_a represents an element dt_a taken counter-clockwise along the circumference Γ_a, and dz_a represents an element dt_a taken clockwise along the circumference γ_a; since the component series are uniformly and absolutely convergent, this is the same as

$$\frac{1}{(2\pi i)^p}\sum_{n_1=-\infty}^{\infty}\dots\sum_{n_p=-\infty}^{\infty}\iint\dots\int\psi(t_1,\dots,t_p)\frac{x_1^{n_1}\dots x_p^{n_p}}{t_1^{n_1+1}\dots t_p^{n_p+1}}dt_1\dots dt_p,$$

where for t_a the course of integration is a single complete circuit coincident with Γ_a when n_a is positive or zero, and a single complete circuit coincident with γ_a when n_a is negative, the directions in both cases being counter-clockwise; thus we obtain, as the value of the integral,

$$\sum_{n_1=-\infty}^{\infty}\dots\sum_{n_p=-\infty}^{\infty}A_{n_1,\dots,n_p}x_1^{n_1}\dots x_p^{n_p},$$

where

$$A_{n_1,\dots,n_p}=\frac{1}{(2\pi i)^p}\iint\dots\int\frac{\psi(t_1,\dots,t_p)}{t_1^{n_1+1}\dots t_p^{n_p+1}}dt_1\dots dt_p,$$

and the course of integration for t_a may be taken to be any circumference concentric with Γ_a and γ_a, not lying outside the region enclosed by them;

(ii) let the variable t_a be taken round a small circle, of radius ρ_a, whose centre is at the point representing x_a $(\alpha=1,\dots,n)$; putting

$$t_a=x_a+\rho_a e^{i\phi_a},$$

we obtain, as the value of the integral, $\psi(x_1,\dots,x_p)$.

The values of the integral obtained in these two ways are equal*; thus we have

$$\phi(u_1,\dots,u_p)=\sum_{n_1=-\infty}^{\infty}\dots\sum_{n_p=-\infty}^{\infty}A_{n_1,\dots,n_p}e^{2\pi i(n_1u_1+\dots+n_pu_p)},$$

where

$$A_{n_1,\dots,n_p}=\int_0^1\dots\int_0^1 e^{-2\pi i(n_1u_1+\dots+n_pu_p)}\phi(u_1,\dots,u_p)\,du_1\dots du_p.$$

By the nature of the proof this series is absolutely, and for all finite values of $u_1,\dots,u_p$, uniformly convergent. If $u_a=v_a+iw_a$ $(\alpha=1,\dots,p)$, and M be an upper limit to the value of the modulus of $\phi(u_1,\dots,u_p)$ for assigned finite upper limits of $w_1,\dots,w_p$, given suppose by $|w_a|\not>W_a$, we have

$$|A_{n_1,\dots,n_p}|\not>Me^{-2\pi(N_1W_1+\dots+N_pW_p)},$$

where $N_a=|n_a|$.

Ex. i. Prove that

$$\frac{\partial}{\partial w_a}\int_0^1\dots\dots\int_0^1 e^{-2\pi i(n_1v_1+\dots+n_pv_p)}\,e^{2\pi(n_1w_1+\dots+n_pw_p)}\,\phi(v_1+iw_1,\dots,v_p+iw_p)\,dv_1\dots dv_p=0.$$

Ex. ii. In the notation of § 174, Chap. X.,

$$e^{i\pi(\tau_{11}n_1^2+\dots+2\tau_{12}n_1n_2+\dots)}=\int_0^1\dots\int_0^1 e^{-2\pi i(n_1u_1+\dots+n_pu_p)}\,\Theta(u_1,\dots,u_p)\,du_1\dots du_p.$$

* Cf., for instance, Forsyth, *Theory of Functions*, p. 47. The reader may also find it of interest to compare Kronecker, *Vorlesungen über ...... Integrale* (Leipzig, 1894), p. 177, and Pringsheim, *Math. Annal.* XLVII. (1896), p. 121, ff.

282. Further it is useful to remark that the series obtained in § 281 is necessarily unique; in other words there can exist no relation of the form

$$\sum_{n_1=-\infty}^{\infty} \dots \sum_{n_p=-\infty}^{\infty} A_{n_1, \dots, n_p} x_1^{n_1} \dots x_p^{n_p} = 0,$$

valid for all values of $x_1, \dots, x_p$ which are given, in the notation of § 281, by $r_a < |x_a| < R_a$, unless each of $A_{n_1, \dots, n_p}$ be zero. For multiplying this equation by $x_1^{-n_1-1} \dots x_p^{-n_p-1} dx_1 \dots dx_p$, and integrating in regard to x_a round a circle, centre at $x_a = 0$, of radius lying between r_a and R_a, $(a = 1, \dots, p)$, we obtain

$$(2\pi i)^p A_{n_1, \dots, n_p} = 0.$$

An important corollary can be deduced. We have remarked (§ 175, Chap. X.) on the existence of 2^{2p} theta functions with half-integer characteristics; it is obvious now that these functions are not connected by any linear equation in which the coefficients are independent of the arguments. For an equation

$$\sum_{s=1}^{2^{2p}} C_{g_s, k_s} \sum_{n_1=-\infty}^{\infty} \dots \sum_{n_p=-\infty}^{\infty} e^{2hu(n+\frac{1}{2}k_s)+b(n+\frac{1}{2}k_s)^2+i\pi g_s(n+\frac{1}{2}k_s)} = 0,$$

where the notation is as in § 174, Chap. X., and k_s, g_s denote rows of p quantities each either 0 or 1, can be put into the form

$$\sum_{N_1=-\infty}^{\infty} \dots \sum_{N_p=-\infty}^{\infty} A_{N_1, \dots, N_p} e^{2\pi i(U_1 N_1 + \dots + U_p N_p)} = 0,$$

where $2\pi i U_1, \dots, 2\pi i U_p$ are the quantities denoted by hu, $A_{N_1, \dots, N_p}$ is given by

$$A_{N_1, \dots, N_p} = \sum_{g_s} C_{g_s, k_s} e^{b(n+\frac{1}{2}k_s)^2+i\pi g_s(n+\frac{1}{2}k_s)},$$

where the summation includes 2^p terms, and $N_1, \dots, N_p$ take the values arising, by the various values of n and k_s, for the quantities $2n + k_s$; it is clear that the aggregate of the values taken by $2n + k_s$ when n denotes a row of p unrestricted integers, and k_s a row of quantities each restricted to be either 0 or 1, is that of a row of unrestricted integers.

Hence by the result obtained above it follows that $A_{N_1, \dots, N_p} = 0$, for all values of n and k_s. Therefore, if λ denote a row of arbitrarily chosen quantities, each either 0 or 1, we have

$$e^{-b(n+\frac{1}{2}k_s)^2+i\pi\lambda(n+\frac{1}{2}k_s)} A_{N_1, \dots, N_p} = \sum_{g_s} C_{g_s, k_s} e^{i\pi(g_s+\lambda)(n+\frac{1}{2}k_s)} = 0;$$

adding the 2^p equations of this form in which the elements of n are each either 0 or 1, the value of k_s being the same for all, we have

$$\sum_{g_s} C_{g_s, k_s} e^{\frac{1}{2}i\pi k_s(g_s+\lambda)} [1 + e^{i\pi\mu_1}] \dots [1 + e^{i\pi\mu_p}],$$

where $\mu_1, \dots, \mu_p$ are the elements of the row letter μ given by $\mu = g_s + \lambda$; the product $(1 + e^{i\pi\mu_1}) \dots (1 + e^{i\pi\mu_p})$ is zero unless all of $\mu_1, \dots, \mu_p$ are even, that is, unless every element of g_s is equal to the corresponding element of λ. Hence we infer that $C_{\lambda, k_s} = 0$; and therefore, as λ is arbitrary, that all the 2^{2p} coefficients C_{g_s, k_s} are zero.

Similarly the r^{2p} possible theta functions whose characteristics are rth parts of unity are linearly independent.

283. Another* proof that the 2^{2p} theta functions with half-integer characteristics are linearly independent may conveniently be given here: we have (§ 190), if m and q be integral,

$$\vartheta(u + \Omega_m;\ \tfrac{1}{2}q) = e^{\lambda_m(u) + \pi i(mq' - m'q)}\, \vartheta(u;\ \tfrac{1}{2}q),$$

and therefore if k be integral and $Q' = q' + k'$, $Q = q + k$,

$$e^{-\lambda_m(u) + \pi i(mk' - m'k)}\, \vartheta(u + \Omega_m;\ \tfrac{1}{2}q) = e^{\pi i(mQ' - m'Q)}\, \vartheta(u;\ \tfrac{1}{2}q).$$

Therefore a relation

$$\sum_{s=1}^{2^{2p}} C_s \vartheta(u;\ \tfrac{1}{2}q_s) = 0$$

leads to

$$\sum_{s=1}^{2^{2p}} C_s e^{\pi i(mQ_s' - m'Q_s)}\, \vartheta(u;\ \tfrac{1}{2}q_s) = 0,$$

where $Q_s = q_s + k$, $Q_s' = q_s' + k'$; in this equation let (m, m') take in turn all the 2^{2p} possible values in which each element of m and m' is either 0 or 1; then as

$$\Sigma e^{\pi i(mQ_s' - m'Q_s)},\ = [1 + e^{\pi i(Q_s')_1}] \dots [1 + e^{\pi i(Q_s')_p}]\,[1 + e^{-\pi i(Q_s)_1}] \dots [1 + e^{-\pi i(Q_s)_p}]$$

is zero unless every one of the elements $(Q_s')_1, \dots, (Q_s)_p$ is an even integer, that is, unless $q_s = k$, $q_s' = k'$, we have

$$\sum_m \sum_{s=1}^{2^{2p}} C_s e^{\pi i(mQ_s' - m'Q_s)}\, \vartheta(u;\ \tfrac{1}{2}q_s) = 2^{2p} C_k \vartheta(u;\ \tfrac{1}{2}k) = 0;$$

thus, for any arbitrary characteristic (k, k'), $C_k = 0$. Thus all the coefficients in the assumed relation are zero.

284. We suppose now that we have four matrices ω, ω', η, η', each of p rows and columns, which satisfy the conditions, (i) that the determinant of ω is not zero, (ii) that the matrix $\omega^{-1}\omega'$ is symmetrical, (iii) that, for real values of $n_1, \dots, n_p$, the quadratic form $\omega^{-1}\omega' n^2$ has its imaginary part positive†, (iv) that the matrix $\eta\omega^{-1}$ is symmetrical, (v) that $\eta' = \eta\omega^{-1}\omega' - \frac{1}{2}\pi i\overline{\omega}^{-1}$; then the relations (B) of § 140, Chap. VII., are satisfied; we put $a = \frac{1}{2}\eta\omega^{-1}$, $h = \frac{1}{2}\pi i\omega^{-1}$, $b = \pi i\omega^{-1}\omega'$, so that (cf. Chap. X., § 190)

$$\eta = 2a\omega, \quad \eta' = 2a\omega' - \overline{h}, \quad h\omega = \tfrac{1}{2}\pi i, \quad h\omega' = \tfrac{1}{2}b;$$

* Frobenius, *Crelle*, LXXXIX. (1880), p. 200.

† Which requires that the imaginary part of the matrix $\omega^{-1}\omega'$ has not a vanishing determinant.

as in § 190 we use the abbreviation

$$\lambda_m(u) = H_m(u + \tfrac{1}{2}\Omega_m) - \pi i m m',$$

where

$$H_m = 2\eta m + 2\eta' m', \quad \Omega_m = 2\omega m + 2\omega' m'.$$

We have shewn (§ 190) that a theta function $\vartheta(u, q)$ satisfies the equation

$$\vartheta(u + \Omega_m, q) = e^{\lambda_m(u) + 2\pi i(mq' - m'q)} \vartheta(u, q),$$

m and m' each denoting a row of integers; it follows therefore that, when m, m' each denotes a row of integers, the product of r theta functions,

$$\Pi(u) = \vartheta(u, q^{(1)}) \vartheta(u, q^{(2)}) \dots\dots \vartheta(u, q^{(r)}),$$

satisfies the equation

$$\Pi(u + \Omega_m) = e^{r\lambda_m(u) + 2\pi i(mQ' - m'Q)} \Pi(u),$$

wherein Q_i, Q_i' are, for $i = 1, 2, \dots, p$, the sums of the corresponding components of the characteristics denoted by $q^{(1)}, \dots, q^{(r)}$.

Conversely*, Q, Q' denoting any assigned rows of p real rational quantities, we proceed to obtain the most general form of single-valued, integral, and analytical function, $\Pi(u)$, which, for all integral values of m and m', satisfies the equation just set down. We suppose r to be an integer, which we afterwards take positive. Under the assigned conditions for the matrices $\omega, \omega', \eta, \eta'$, such a function will be called *a theta function of order r, with the associated constants $2\omega, 2\omega', 2\eta, 2\eta'$, and the characteristic (Q, Q').*

Denoting the function $\vartheta(u; Q)$, of § 189, either by $\vartheta(u; 2\omega, 2\omega', 2\eta, 2\eta'; Q, Q')$ or $\vartheta(u; a, b, h; Q, Q')$, the function $\vartheta(u; 2\omega/r, 2\omega', 2\eta, 2r\eta'; Q, Q'/r)$ is a theta function of the first order with the associated constants $2\omega/r, 2\omega', 2\eta, 2r\eta'$, and $(Q, Q'/r)$ for characteristic; increasing u by $2\omega m + 2\omega' m'$, where m, m' are integral, the function is multiplied by a factor which characterises it also as a theta function of order r, with the associated constants $2\omega, 2\omega', 2\eta, 2\eta'$ and (Q, Q') for characteristic. We have, also,

$$\vartheta(u; ra, rb, rh) = \vartheta\left(u; \frac{2\omega}{r}, 2\omega', 2\eta, 2r\eta'\right) = \vartheta\left(ru; 2\omega, 2r\omega'; \frac{2\eta}{r}, 2\eta'\right) = \vartheta\left(ru; \frac{a}{r}, h, rb\right),$$

where the omitted characteristic is the same for each.

Let k_i be the least positive integer such that $k_i Q_i'$ is an integer, $= f_i$, say; denote the matrix of p rows and columns, of which every element is zero except those in the diagonal, which, in order, are $k_1, k_2, \dots, k_p$, by k; the inverse matrix k^{-1} is obtained from this by replacing $k_1, \dots$ respectively by

* Hermite, *Compt. Rend.* t. XL. (1855), and a letter from Brioschi to Hermite, *ibid.* t. XLVII. Schottky, *Abriss einer Theorie der Abel'schen Functionen von drei Variabeln* (Leipzig, 1880), p. 5. The investigation of § 284 is analogous to that of Clebsch and Gordan, *Abel. Funct.*, pp. 190, ff. The investigation of § 285 is analogous to that given by Schottky. Cf. Königsberger, *Crelle*, LXIV. (1865), p. 28.

$1/k_1, \ldots$; in place of the arguments u introduce arguments v determined by the p equations

$$h_{i,1} u_1 + \ldots\ldots + h_{i,p} u_p = k_i v_i, \qquad (i = 1, \ldots, p),$$

which we write $hu = kv$; then, by the equations $h\omega = \frac{1}{2}\pi i$, $h\omega' = \frac{1}{2}b$, it follows that the increments of the arguments v when the arguments u are increased by the quantities constituting the p rows of a period Ω_m, are given by the p rows of U_m defined by

$$kU_m = \pi i m + bm';$$

we shall denote the right-hand side of this equation by Υ_m; thus $U_m = k^{-1}\Upsilon_m = \pi i k^{-1} m + k^{-1} bm'$.

Now we have

$$a(u + \Omega_m)^2 - au^2 = 2au\Omega_m + a\Omega_m^2,$$

and, since* the matrix a is symmetrical, and $H_m = 2a\Omega_m - 2\bar{h}m'$, this is equal to

$$2a\Omega_m u + a\Omega_m^2 = 2a\Omega_m (u + \tfrac{1}{2}\Omega_m) = (H_m + 2\bar{h}m')(u + \tfrac{1}{2}\Omega_m)$$

and therefore equal to

$$\lambda_m(u) + \pi i m m' + 2hum' + h\Omega_m m'$$

or

$$\lambda_m(u) + \pi i m m' + 2kvm' + \Upsilon_m m';$$

thus, by the definition equation for the function $\Pi(u)$, we have

$$e^{-ra(u+\Omega_m)^2}\, \Pi(u + \Omega_m) = e^{-rau^2}\, \Pi(u) \,.\, e^{-r[\pi i m m' + 2(kv + \frac{1}{2}\Upsilon_m)m'] + 2\pi i (mQ' - m'Q)};$$

therefore, if $Q(v)$ denote $e^{-rau^2}\, \Pi(u)$,

$$Q(v + U_m) = Q(v)\, e^{-r[\pi i m m' + 2(kv + \frac{1}{2}\Upsilon_m)m'] + 2\pi i (mQ' - m'Q)};$$

now let $m' = 0$, and $m = ks$, where s denotes a row of integers $s_1, \ldots, s_p$; then $mQ' = ksQ' = k_1 s_1 Q_1' + \ldots\ldots + k_p s_p Q_p' = skQ'$, $= sf$, is also a row of integers; and $U_m = \pi i k^{-1} m + k^{-1} bm' = \pi i s$; thus we have

$$Q(v + \pi i s) = Q(v),$$

or, what is the same thing, the function $Q(v)$ is periodic for each of the arguments $v_1, \ldots, v_n$, separately, the period being πi; it follows then (§ 281) that the function is expressible as an infinite series of terms of the form $C_{n_1, n_2, \ldots, n_p} e^{2(n_1 v_1 + \ldots + n_p v_p)}$, where $n_1, \ldots, n_p$ are summation letters, each of which, independently of the others, takes all integral values from $-\infty$ to $+\infty$, and the coefficients $C_{n_1, \ldots, n_p}$ are independent of $v_1, \ldots, v_n$. This we denote by putting

$$Q(v) = e^{-rau^2}\, \Pi(u) = \Sigma C_n e^{2vn}.$$

To this relation, for the purpose of obtaining the values of the coefficients

* By a fundamental matrix equation, if μ be any matrix of p rows and columns, and u, v be row letters of p elements, $\mu\, u\, v = \bar{\mu}\, v\, u$.

C_n, we apply the equation, obtained above, which expresses the ratio to $Q(v)$ of $Q(v+U_m)$ or $Q(v+k^{-1}\Upsilon_m)$; thence we have

$$\sum_n C_n e^{2(v+k^{-1}\Upsilon_m)n} = [\sum_s C_s e^{2vs}]\, e^{-r[\pi i m m' + 2(kv+\frac{1}{2}\Upsilon_m)m'] + 2\pi i(mQ'-m'Q)};$$

in this equation, corresponding to a term of the left-hand side given by the summation letter n, consider the term of the right-hand side for which the summation letter s is such that

$$s_i = n_i + rk_i m_i', \qquad (i=1, 2, \ldots, p);$$

thus $s=n+rkm'$, and $2v_i s_i = 2v_i n_i + 2rk_i v_i m_i'$, or $2vs = 2vn + 2rkvm'$; hence we obtain

$$\sum_n C_n e^{2(v+k^{-1}\Upsilon_m)n} = [\sum_n C_{n+rkm'} e^{2vn}]\, e^{-r(\pi i m m' + \Upsilon_m m') + 2\pi i(mQ'-m'Q)};$$

therefore, equating coefficients of products of the same powers of the quantities $e^{2v_1}, \ldots, e^{2v_p}$, we have

$$C_{n+rkm'} = C_n \,.\, e^{2k^{-1}\Upsilon_m n + r(\pi i m m' + \Upsilon_m m') - 2\pi i(mQ'-m'Q)},$$

and this equation holds for all values of the integers denoted by n, m, m'.

By taking the particular case of this equation in which the integers m' are all zero we infer that the quantity

$$\frac{1}{\pi i}k^{-1}\Upsilon_m n - mQ', \ = \frac{1}{\pi i}k^{-1}(\pi i m)\, n - mQ', \ = \sum_{s=1}^{p} m_s\left(\frac{1}{k_s}n_s - Q_s'\right)$$

must be an integer for all integral values of the numbers m_s and n_s; therefore the only values of the integers n which occur are those for which the numbers $(n_s - k_s Q_s')/k_s$ are integers; thus, by the definition of k_s, we may put $n = f + kN$, N denoting a row of integers, and $f = kQ'$.

With this value we have

$$\begin{aligned} k^{-1}\Upsilon_m n - k^{-1}(\pi i m)\, n = k^{-1}(bm')\, n = \bar{k}^{-1} n\,(bm') = k^{-1}n\,.\,bm' \\ = (k^{-1}f+N)\,.\,bm' = (Q'+N)\,.\,bm' = bm'(Q'+N); \end{aligned}$$

hence, as $mQ' = k^{-1}mn - mN$, the equation connecting C_n and $C_{n+rkm'}$ becomes

$$\begin{aligned} C_{f+rkm'+kN} &= C_{f+kN} e^{2bm'(Q'+N) + [r(2\pi i m + bm') + 2\pi i Q]m'} \\ &= C_{f+kN} e^{rbm'^2 + 2bm'N + 2\pi i Q m' + 2bQ'm'}, \end{aligned}$$

$e^{2\pi i r m m'}$ being equal to unity because r is an integer, and $bm'Q' = \bar{b}Q'm' = bQ'm'$; therefore

$$e^{-\frac{1}{r}b(m'r+N)^2} C_{f+k(m'r+N)} = e^{-\frac{1}{r}bN^2} C_{f+kN}\,.\,e^{2\Upsilon_Q m'},$$

Υ_Q being $\pi i Q + bQ'$, or

$$e^{-\frac{1}{r}[b(N+rm')^2 + 2\Upsilon_Q(N+rm')]} C_{f+k(N+rm')} = e^{-\frac{1}{r}[bN^2 + 2\Upsilon_Q N]} C_{f+kN};$$

thus, if the right-hand side of this equation be denoted by D_N, we have, for every integral value of m', $D_{N+rm'} = D_N$; therefore every quantity D is equal to a quantity D for which the suffix is a row of positive integers (which may be zero) each less than the numerical value of the integer r. If then ρ be the numerical value of r, the series breaks up into a sum of ρ^p series; let D_μ be the coefficient, in one of these series, in which the integers μ are less than ρ; then the values of the integers N occurring in this series are given by $N = \mu + rM$, M being a row of integers, which, as appears from the work, may be any between $-\infty$ and ∞; and the general term of $Q(v)$ is

$$C_n e^{2vn} = C_n e^{2nv} = C_{f+kN} e^{2(f+kN)v} = D_N e^{\frac{1}{r}(bN^2 + 2\Upsilon_Q N) + 2k(Q'+N)v},$$

$$= D_{\mu+rM} e^{rb\left(M+\frac{\mu}{r}\right)^2 + 2\Upsilon_Q\left(M+\frac{\mu}{r}\right) + 2hu(rM+Q'+\mu)},$$

for $k.(Q'+N)v = \bar{k}v(Q'+N) = kv(Q'+N) = hu(Q'+N)$; thus the general term is

$$= D_\mu e^{2rhu\left(M+\frac{Q'+\mu}{r}\right) + rb\left(M+\frac{\mu}{r}\right)^2 + 2\Upsilon_Q\left(M+\frac{\mu}{r}\right)};$$

now, as $\Upsilon_Q = \pi i Q + bQ'$, and b is a symmetrical matrix, the quantity

$$rb\left(M+\frac{\mu}{r}\right)^2 + 2\Upsilon_Q\left(M+\frac{\mu}{r}\right)$$

is immediately seen to be equal to

$$rb\left(M+\frac{\mu+Q'}{r}\right)^2 + 2\pi i Q\left(M+\frac{\mu+Q'}{r}\right) - 2\pi i\frac{QQ'}{r} - \frac{1}{r}bQ'^2;$$

therefore the general term of $\Pi(u)$, or $e^{rau^2}Q(v)$, with the coefficient D_μ, is $e^{\psi+\chi}$, where

$$\psi = rau^2 + 2rhu\left(M+\frac{Q'+\mu}{r}\right) + rb\left(M+\frac{Q'+\mu}{r}\right)^2 + 2\pi i Q\left(M+\frac{Q'+\mu}{r}\right),$$

$$\chi = -2\pi i\frac{QQ'}{r} - \frac{1}{r}bQ'^2;$$

and this is the general term of the function

$$e^{-2\pi i\frac{QQ'}{r} - \frac{1}{r}bQ'^2}\,\underline{\vartheta}\left(u;\ Q,\ \frac{Q'+\mu}{r}\right),$$

where $\underline{\vartheta}$ denotes a theta function differing only from that before represented (§ 189, Chap. X.) by ϑ, in the change of the matrices a, b, h respectively into ra, rb, rh; the condition for the convergence of the series $\underline{\vartheta}$ requires that r be positive; thus $\rho = r$; recalling the formulae

$$h\Omega_P = \pi i P + bP', \quad \tfrac{1}{2}H_P = a\Omega_P - \bar{h}P',$$

we see, as already remarked on p. 448, that, instead of

$$\omega,\ \omega',\ \eta,\ \eta',$$

the quantities to be associated with the function $\mathfrak{S}$ are

$$\frac{\omega}{r},\ \omega',\ \eta,\ r\eta';$$

with this notation then we may write, as the necessary form of the function $\Pi(u)$,

$$\Pi(u) = \Sigma K_\mu \mathfrak{S}\left(u;\ Q,\ \frac{Q'+\mu}{r}\right),$$

wherein $K_\mu, = D_\mu e^{-2\pi i \frac{QQ'}{r} - \frac{1}{r} bQ'^2}$ is an unspecified constant coefficient, μ denotes a row of p integers each less than the positive integer r, and the summation extends to the r^p terms that arise by giving to μ all its possible values.

From this investigation an important corollary can be drawn; if a single-valued integral analytical function satisfying the definition equation of the function $\Pi(u)$ (p. 448), in which r is a positive integer and the quantities Q, Q' are rational real quantities, be called a theta function of the rth order with characteristic (Q, Q'), then* *any r^p+1 theta functions of the rth order, having the same associated quantities 2ω, $2\omega'$, 2η, $2\eta'$ and the same characteristic, or characteristics differing from one another by integers, are connected by a linear equation or by more than one linear equation, wherein the coefficients are independent of the arguments $u_1, \dots, u_p$; and therefore any of the functions can be expressed linearly by means of the other r^p functions, provided these latter are not themselves linearly connected.*

For the determining equation satisfied by $\Pi(u)$ is still satisfied if, in place of the characteristic (Q, Q'), we put $(Q+N,\ Q'+N')$, N and N' each denoting a row of p integers; and if

$$\mu + N' \equiv \nu \ (\text{mod. } r),\ \text{ say } \mu + N' = \nu + rL',$$

we have (§ 190, Chap. X.)

$$\mathfrak{S}\left(u;\ Q+N,\ \frac{Q'+N'+\mu}{r}\right) = \mathfrak{S}\left(u;\ Q+N,\ \frac{Q'+\nu}{r}+L'\right)$$
$$= e^{2\pi i N \frac{Q'+\nu}{r}} \mathfrak{S}\left(u;\ Q,\ \frac{Q'+\nu}{r}\right),$$

and therefore

$$\Sigma K_\mu \mathfrak{S}\left(u;\ Q+N,\ \frac{Q'+\mu+N'}{r}\right) = \Sigma H_\nu \mathfrak{S}\left(u;\ Q,\ \frac{Q'+\nu}{r}\right),$$

where $H_\nu = K_\mu e^{2\pi i N \frac{Q'+\nu}{r}}$; and the aggregate of the r^p values of $\dfrac{Q'+\nu}{r}$ is the same as that of the values of $\dfrac{Q'+\mu}{r}$.

Thus any r^p+1 theta functions of the rth order, with the same characteristic, or characteristics differing only by integers, and associated with the

* The theorem is attributed to Hermite: cf. *Compt. Rendus*, t. XL. (1855), p. 428.

same quantities 2ω, $2\omega'$, 2η, $2\eta'$, are all expressible as linear functions of the same r^p quantities $\underline{\underline{\vartheta}}\left(u\,;\ Q,\ \frac{Q'+\mu}{r}\right)$ with coefficients independent of $u_1, \ldots, u_p$. Hence the theorem follows as enunciated.

Ex. i. Prove that the r^p functions $\underline{\underline{\vartheta}}\left(u\,;\ Q,\ \frac{Q'+\nu}{r}\right)$ are linearly independent (§ 282).

Ex. ii. The function $\vartheta(u+a\,;\ Q)\ \vartheta(u-a\,;\ Q)$ is a theta function of order 2 with $(2Q,\ 2Q')$ as characteristic. Hence, if 2^p+1 values for the argument a be taken, the resulting functions are connected by a linear relation.

For example, when $p=1$, we have the equation

$$\sigma^2(a)\,\sigma(u-b)\,\sigma(u+b)-\sigma^2(b)\,\sigma(u-a)\,\sigma(u+a)=\sigma^2(u)\,.\,\sigma(a-b)\,\sigma(a+b).$$

Ex. iii. The function $\vartheta(ru,\ Q)$ is a theta function of order r^2 with $(rQ,\ rQ')$ as characteristic. Prove that if $\underline{\underline{\vartheta}}$ denote a theta function with the associated constants ω, $r^2\omega'$, $\frac{\eta}{r^2}$, η', in place of ω, ω', η, η' respectively, then we have the equations

$$\vartheta\left(u\,;\ \frac{Q}{r},\ Q'\right)=\sum_{\mu}\underline{\underline{\vartheta}}\left(ru\,;\ Q,\ \frac{Q'+\mu}{r}\right),\quad \underline{\underline{\vartheta}}\left(ru\,;\ Q,\ \frac{Q'+\mu}{r}\right)=\sum_{\nu}e^{-2\pi i\nu\frac{Q'+\mu}{r}}\,\vartheta\left(u\,;\ \frac{Q+\nu}{r},\ Q'\right),$$

where the summation letters μ, ν are row letters of p elements all less than r, and each summation contains r^p terms.

Ex. iv. The product of k theta functions, with different characteristics,

$$\vartheta(u+u^{(1)}\,;\ Q^{(1)})\ldots\ldots\vartheta(u+u^{(k)}\,;\ Q^{(k)})$$

is a theta function of order k for which the quantities

$$\left[\sum_{r=1}^{k}Q^{(r)}-2\bar{\eta}'\sum_{r=1}^{k}u^{(r)},\ \ \sum_{r=1}^{k}Q'^{(r)}+2\bar{\eta}\sum_{r=1}^{k}u^{(r)}\right],$$

enter as characteristic. Thus a simple case is when $u^{(1)}+\ldots+u^{(k)}=0$.

For $p=1$ a linear equation connects the five functions

$$\prod_{i=1}^{4}\sigma(u+u_i),\qquad \prod_{i=1}^{4}\sigma(u+u_i+\omega),\qquad \prod_{i=1}^{4}\sigma(u+u_i+\omega'),\qquad \prod_{i=1}^{4}\sigma(u+u_i+\omega+\omega'),$$

$$\sigma\left(2u+\frac{u_1+u_2+u_3+u_4}{4}\right).$$

Ex. v. Any $(p+2)$ theta functions of order r, for which the characteristic and the associated constants ω, ω', η, η' are the same, are connected by an equation of the form $P=0$, where P is an integral homogeneous polynomial in the theta functions. For the number of terms in such a polynomial, of degree N, is greater than $(Nr)^p$, when N is taken great enough. That such an equation does not generally hold for $(p+1)$ theta functions may be proved by the consideration of particular cases.

285. The following, though partly based on the investigation already given, affords an instructive view of the theorem of § 284.

Slightly modifying a notation previously used, we define a quantity, depending on the fundamental matrices ω, ω', η, η', by the equation

$$\begin{aligned}\lambda(u\,;\ P,\ P')&=H_P(u+\tfrac{1}{2}\Omega_P)-\pi iPP'\\&=(2\eta P+2\eta'P')(u+\omega P+\omega'P')-\pi iPP',\end{aligned}$$

where P, P' each denotes a row of p arbitrary quantities. The corresponding quantity arising when, in place of ω, ω', η, η' we take other matrices $\omega^{(1)}$, $\omega'^{(1)}$, $\eta^{(1)}$, $\eta'^{(1)}$ may be denoted by $\lambda^{(1)}(u;\ P, P')$. With this notation, and in case

$$\omega^{(1)},\ \omega'^{(1)},\ \eta^{(1)},\ \eta'^{(1)}$$

are respectively

$$\frac{\omega}{r}\ ,\ \omega'\ ,\ \eta\ ,\ r\eta',$$

where r is an arbitrary positive integer, we have the following identity

$$r\lambda\left[u+\frac{2\omega}{r}s;\ N,\ m'\right]$$
$$=\lambda^{(1)}[u+\Omega_m^{(1)};\ k,\ 0]+\lambda^{(1)}[u;\ m,\ m']-\lambda^{(1)}[u;\ s,\ 0]-2\pi i m'k,$$

where s, N, m, m', k each denotes a row of p arbitrary quantities subject to the relation

$$s+rN=m+k;$$

this the reader can easily verify; it is a corollary from the result of Ex. ii., § 190.

Let the abbreviation $R(u;f)$ be defined by the equation

$$R(u;f)=\sum_k e^{-2\eta k\left(u+\omega\frac{k}{r}\right)-2\pi i f\frac{k}{r}}\,\Pi\left(u+2\omega\frac{k}{r}\right),$$
$$=\sum_k e^{-\lambda^{(1)}[u;\,k,\,0]-2\pi i f\frac{k}{r}}\,\Pi\left(u+2\omega\frac{k}{r}\right),$$

wherein k denotes a row of p positive integers each less than r, and the summation extends to all the r^p values of k thus arising, f is a row of p arbitrary quantities, and $\Pi(u)$ denotes any theta function of order r.

Consider now the value of $R(u+\Omega_m^{(1)};f)$; by definition we have

$$\Pi\left[u+2\omega\frac{k}{r}+\Omega_m^{(1)}\right]=\Pi\left(u+2\omega\frac{k+m}{r}+2\omega'm'\right);$$

therefore, if $m+k\equiv s$ (mod. r), say $m+k=s+rN$, we have, by the definition equation (§ 284) satisfied by $\Pi(u)$,

$$\Pi\left[u+2\omega\frac{k}{r}+\Omega_m^{(1)}\right]=\Pi\left[u+2\omega^{(1)}s+2\omega N+2\omega'm'\right]$$
$$=\Pi\left(u+2\omega^{(1)}s\right)e^{r\lambda[u+2\omega^{(1)}s;\ N,\ m']+2\pi i(NQ'-m'Q)},$$

where (Q, Q') is the characteristic of $\Pi(u)$, and hence

$$R(u+\Omega_m^{(1)};f)=\sum_s e^{\psi}\,\Pi\left(u+2\omega^{(1)}s\right),$$

in which

$$\psi=-\lambda^{(1)}[u+\Omega_m^{(1)};\ k,\ 0]+r\lambda[u+2\omega^{(1)}s;\ N,\ m']-2\pi i f\frac{k}{r}+2\pi i(NQ'-m'Q);$$

by the identity quoted at the beginning of this Article, ψ can also be put into the form

$$\psi = \lambda^{(1)}[u;\ m, m'] - \lambda^{(1)}[u;\ s, 0] - 2\pi i m'k - 2\pi i f\frac{k}{r} + 2\pi i (NQ' - m'Q),$$
$$= \lambda^{(1)}[u;\ m, m'] - \lambda^{(1)}[u;\ s, 0] - 2\pi i m'k - 2\pi i m'Q + 2\pi i N(Q' - f) + 2\pi i f\frac{m-s}{r};$$

in the definition equation for $\Pi(u)$, the letters m, m' denote integers; and k has been taken to denote integers; if further f be chosen so that $Q' - f$ is a row of integers, we have, since, by definition, N denotes a row of integers,

$$R(u + \Omega_m^{(1)};\ f) = e^{\lambda^{(1)}[u;\ m, m'] + 2\pi i\left(m\frac{f}{r} - m'Q\right)} \sum_s e^{-\lambda^{(1)}(u;\ s,\ 0) - 2\pi i f\frac{s}{r}} \Pi(u + 2\omega^{(1)} s)$$
$$= e^{\lambda^{(1)}[u;\ m, m'] + 2\pi i\left(m\frac{f}{r} - m'Q\right)} R(u;\ f).$$

Hence $R(u;\ f)$ satisfies a determining equation of precisely the same form as that satisfied by $\Pi(u)$, the only change being in the substitution of $\frac{\omega}{r}$, ω', η, $r\eta'$ respectively for ω, ω', η, η'; so* considered $R(u;\ f)$ is a theta function of the first order with $\left(Q, \frac{f}{r}\right)$ as characteristic; putting, in accordance with the definition of f above, $f = Q' + \mu$, where μ is a row of p integers, we therefore have, by § 284,

$$R(u;\ Q' + \mu) = K_{Q'+\mu}\,\underline{\vartheta}\left(u;\ Q, \frac{Q' + \mu}{r}\right), = K_{Q'+\mu}\,\vartheta\left(ru;\ \frac{a}{r}, h, rb;\ \begin{matrix}\frac{Q'+\mu}{r}\\ Q\end{matrix}\right),$$

(p. 448) where $K_{Q+\mu}$ is a quantity independent of u, and $\underline{\vartheta}$ is the same theta function as that previously so denoted (§ 284), having, in place of the usual matrices a, b, h, respectively ra, rb, rh.

Remarking now that the series

$$\sum_\mu e^{-2\pi i\frac{\mu k}{r}},$$

wherein μ denotes a row of p integers (including zero), each less than r, and the summation extends to all the r^p terms thus arising, is equal to r^p when the p integers denoted by k are all zero, and is otherwise zero, we infer that the sum

$$\frac{1}{r^p}\sum_\mu R(u;\ Q' + \mu),$$

which, by the definition of $R(u, f)$, putting $f = Q' + \mu$, is equal to

$$\frac{1}{r^p}\sum_k \left[e^{-\lambda^{(1)}(u;\ k,\ 0) - 2\pi i Q'\frac{k}{r}}\, \Pi\left(u + 2\omega\frac{k}{r}\right) \sum_\mu e^{-2\pi i\mu\frac{k}{r}}\right],$$

* $R(u;\ f)$ may also be regarded as a theta function of order r, with the associated constants 2ω, $2\omega'$, 2η, $2\eta'$ and characteristic (Q, f).

is, in fact, equal to $\Pi(u)$. Hence as before we have the equation

$$\Pi(u) = \sum_\mu K_{Q+\mu} \mathfrak{S}\left(u;\ Q,\ \frac{Q'+\mu}{r}\right).$$

286. *Ex.* i. Suppose that m is an even half-integer characteristic, and that

$$a_1,\ a_2,\ \ldots\ldots,\ a_s$$

are $s, = 2^p$, half-integer characteristics such that the characteristic formed by adding the three characteristics m, a_i, a_j is always odd, when i is not equal to j. Thus when m is an integral, or zero, characteristic, the condition is that the characteristic formed by adding two different characteristics a_i, a_j may be odd. The characteristic whose elements are formed by the addition of the elements of two characteristics a, b may be denoted by $a+b$; when the elements of $a+b$ are reduced, by the subtraction of integers, to being less than unity and positive (or zero), the reduced characteristic may be denoted by ab.

For instance when $p=2$, if α, β, γ denote any three odd characteristics, so that* the characteristic $\alpha\beta\gamma$ is even, and if μ be any characteristic whatever, characteristics satisfying the required conditions are given by taking m, a_1, a_2, a_3, a_4 respectively equal to $\alpha\beta\gamma$, μ, $\mu\beta\gamma$, $\mu\gamma\alpha$, $\mu\alpha\beta$; in either case a characteristic ma_ia_j is one of the three α, β, γ and is therefore odd.

When $p=3$, corresponding to any even characteristic m, we can in 8 ways take seven other characteristics α, β, γ, κ, λ, μ, ν, such that the combinations α, β, γ, κ, λ, μ, ν, $m\alpha\beta$, $m\alpha\kappa$, $m\lambda\mu$ constitute all the 28 existent odd characteristics; this is proved in chapter XVII.; examples have already been given, on page 309. Hence characteristics satisfying the conditions here required are given by taking

$$m,\ a_1,\ a_2,\ a_3,\ \ldots,\ a_8$$

respectively equal to

$$m,\ m,\ \alpha,\ \beta,\ \ldots,\ \nu.$$

Now, by § 284, every 2^p+1 theta functions of the second order, with the same periods and the same characteristic, are connected by a linear equation. Hence, if p, q, r denote arbitrary half-integer characteristics, and v, w be arbitrary arguments, there exists an equation of the form

$$A\vartheta(u+w;\ q)\,\vartheta(u-w;\ r) = \sum_{\lambda=1}^{s} A_\lambda \vartheta[u+v;\ (q+r-p-a_\lambda)]\,\vartheta[u-v;\ (p+a_\lambda)],$$

wherein A, A_λ are independent of u; for each of the functions involved is of the second order, as a function of u, and of characteristic $q+r$.

We determine the coefficients A_λ by adding a half period to the argument u; for u put $u+\Omega_{m-a_j-p}$; then by the formula

$$\vartheta(u+\Omega_P,\ q) = e^{\lambda(u;\ P)-2\pi i P'q}\,\vartheta(u;\ P+q),$$

where

$$\lambda(u;\ P) = H_P(u+\tfrac{1}{2}\Omega_P) - \pi i PP',$$

noticing, what is easy to verify, that

$$\lambda(u+v;\ P)+\lambda(u-v;\ P)-\lambda(u+w;\ P)-\lambda(u-w;\ P)=0$$
$$= -\pi i P'[q+r-p-a_\lambda+p+a_\lambda-q-r],$$

* As the reader may verify from the table of § 204; a proof occurs in Chap. XVII.

we obtain

$$A\vartheta\,[u+w\,;\;(m-a_j-p+q)]\,\vartheta\,[u-w\,;\;(m-a_j-p+r)]$$
$$=\sum_{\lambda=1}^{s} A_\lambda \vartheta\,[u+v\,;\;(m-a_j-a_\lambda+q+r-2p)]\,\vartheta\,[u-v\,;\;(m-a_j+a_\lambda)].$$

But since $m-a_j+a_\lambda$ (which, save for integers, is the characteristic ma_ja_λ) is an odd characteristic when j is not the same as λ, we can hence infer, putting $u=v$, that

$$A_\lambda/A=\vartheta[v+w\,;(m-a_\lambda-p+q)]\vartheta[v-w\,;(m-a_\lambda-p+r)]/\vartheta[2v;(m-2a_\lambda+q+r-2p)]\vartheta[0;m].$$

Hence the form of the relation is entirely determined. The result can be put into various different shapes according to need. Denoting the characteristic $m+q+r$ momentarily by k, so that k consists of two rows, each of p half-integers, and similarly denoting the characteristic $a_\lambda+p$ momentarily by α_λ, and using the formula for integral M,

$$\vartheta\,(u\,;\;q+M)=e^{2\pi iMq'}\vartheta\,(u\,;\;q),$$

we have

$$\vartheta\,[2v\,;\;(m-2a_\lambda+q+r-2p)]=e^{-4\pi i\alpha_\lambda k'}\,\vartheta\,(2v\,;\;k)\,;$$

we shall denote the right-hand side of this equation by

$$e^{-4\pi i\,(a_\lambda+p)\,(m'+q'+r')}\,\vartheta\,[2v\,;\;(m+q+r)]\,;$$

hence the final equation can be put into the form

$$\vartheta\,[u+w\,;\;q]\,\vartheta\,[u-w\,;\;r]\,\vartheta\,[2v\,;\;(m+q+r)]\,\vartheta\,[0\,;\;m]$$
$$=\sum_{\lambda=1}^{2^p} e^{4\pi i\,(a_\lambda+p)\,(m'+q'+r')}\,\vartheta\,[u+v\,;\;(q+r-p-a_\lambda)]\,\vartheta\,[u-v\,;\;(p+a_\lambda)]$$
$$\vartheta\,[v+w\,;\;(m-a_\lambda-p+q)]\,\vartheta\,[v-w\,;\;(m-a_\lambda-p+r)].$$

It may be remarked that, with the notation of Chap. XI., if $b_1,\ldots,b_p$ be any finite branch places, and A_r denote the characteristic associated with the half-period $u^{b_r,a}$, and we take for the characteristics $a_1,\ldots,a_s$ the 2^p characteristics $A, AA_1\ldots A_k$, formed by adding an arbitrary half-integer characteristic A to the combinations of not more than p of the characteristics $A_1,\ldots,A_p$, and take for the characteristic m the characteristic associated with the half-period $u^{b_1,a_1}+\ldots+u^{b_p,a_p}$, then each of the hyperelliptic functions $\vartheta(0\,;\;ma_ia_j)$ vanishes (§ 206), though the characteristic ma_ia_j is not necessarily odd. Hence the formula here obtained holds for any hyperelliptic case when $m, a_1,\ldots,a_s$, have the specified values.

Ex. ii. When $p=2$, denoting three odd characteristics by α,β,γ, we can in Ex. i. take

$$p,\;q,\;r,\;m,\;a_1,\;a_2,\;a_3,\;a_4$$

respectively equal to

$$\alpha\beta\gamma,\;q,\;0,\;\alpha\beta\gamma,\;0,\;\beta\gamma,\;\gamma\alpha,\;\alpha\beta,$$

wherein 0 denotes the characteristic of which all the elements are zero, and $\beta\gamma$ denotes the reduced characteristic obtained* by adding the characteristics β and γ. Then the general formula of Ex. i. becomes, putting $v=0$ and retaining the notation m for the characteristic $\alpha\beta\gamma$,

$$\vartheta\,(u+w\,;\;q)\,\vartheta\,(u-w\,;\;0)\,\vartheta\,(0\,;\;q+m)\,\vartheta\,(0\,;\;m)$$
$$=\sum_{\lambda=1}^{4} e^{4\pi i(a_\lambda+m)(m'+q')}\,\vartheta\,(u\,;\;q-m-a_\lambda)\,\vartheta\,(u\,;\;m+a_\lambda)\,\vartheta\,(w\,;\;q-a_\lambda)\,\vartheta\,(w\,;\;a_\lambda).$$

* So that all the elements of $\beta\gamma$ are zero or positive and less than unity.

Ex. iii. As one application of the formula of Ex. ii. we put

$$q=\tfrac{1}{2}\begin{pmatrix}10\\10\end{pmatrix},\ a=\tfrac{1}{2}\begin{pmatrix}10\\10\end{pmatrix},\ \beta=\tfrac{1}{2}\begin{pmatrix}01\\11\end{pmatrix},\ \gamma=\tfrac{1}{2}\begin{pmatrix}01\\01\end{pmatrix},$$

and therefore

$$m=\tfrac{1}{2}\begin{pmatrix}10\\00\end{pmatrix},\ a_1=\tfrac{1}{2}\begin{pmatrix}00\\00\end{pmatrix},\ a_2=\tfrac{1}{2}\begin{pmatrix}00\\10\end{pmatrix},\ a_3=\tfrac{1}{2}\begin{pmatrix}11\\11\end{pmatrix},\ a_4=\tfrac{1}{2}\begin{pmatrix}11\\01\end{pmatrix};$$

hence we find, comparing the table of § 204, and using the formula

$$\vartheta(u;\ f+M)=e^{2\pi iMf'}\vartheta(u;\ f),$$

where $M, =\begin{pmatrix}M_1'M_2'\\M_1M_2\end{pmatrix}$, consists of integers, $f=\begin{pmatrix}f_1'f_2'\\f_1f_2\end{pmatrix}$, and $Mf'=M_1f_1'+M_2f_2'$, that*

$$\vartheta(u+w;\ q)=-\vartheta_{02}(u+w),\ \vartheta(u-w;\ 0)=\vartheta_5(u-w),\ \vartheta(0;\ q+m)=\vartheta_{12}(0),\ \vartheta(0;\ m)=\vartheta_{01}(0),$$

$$\begin{aligned}
&\vartheta(u;q-m-a_1)=\ \ \vartheta_{12}(u), \vartheta(u;m+a_1)=\ \ \vartheta_{01}(u), \vartheta(w;q-a_1)=-\vartheta_{02}(w), \vartheta(w;a_1)=\ \ \vartheta_5(w),\\
&\vartheta(u;q-m-a_2)=\ \ \vartheta_5(u), \vartheta(u;m+a_2)=-\vartheta_{02}(u), \vartheta(w;q-a_2)=\ \ \vartheta_{01}(w), \vartheta(w;a_2)=\ \ \vartheta_{12}(w),\\
&\vartheta(u;q-m-a_3)=\ \ \vartheta_{24}(u), \vartheta(u;m+a_3)=-\vartheta_{04}(u), \vartheta(w;q-a_3)=\ \ \vartheta_3(w), \vartheta(w;a_3)=\ \ \vartheta_{14}(w),\\
&\vartheta(u;q-m-a_4)=-\vartheta_{14}(u), \vartheta(u;m+a_4)=-\vartheta_3(u), \vartheta(w;q-a_4)=\ \ \vartheta_{04}(w), \vartheta(w;a_4)=-\vartheta_{24}(w),
\end{aligned}$$

all the factors of the form $e^{4\pi i(a_\lambda+m)(m'+q')}$ being equal to 1; by substitution of these results we therefore obtain

$$\vartheta_{02}(u+w)\,\vartheta_5(u-w)\,\vartheta_{12}(0)\,\vartheta_{01}(0)=\vartheta_{12}\vartheta_{01}\bar{\vartheta}_{02}\bar{\vartheta}_5+\vartheta_{02}\vartheta_5\bar{\vartheta}_{12}\bar{\vartheta}_{01}+\vartheta_{04}\vartheta_{24}\bar{\vartheta}_3\bar{\vartheta}_{14}+\vartheta_3\vartheta_{14}\bar{\vartheta}_{04}\bar{\vartheta}_{24},$$

where ϑ_{12} denotes $\vartheta_{12}(u)$, etc., and $\bar{\vartheta}_{02}$ denotes $\vartheta_{02}(w)$, etc.; this agrees with the formula of §§ 219, 220 (Chap. XI.).

Ex. iv. By putting in the formula of Ex. ii. respectively

$$a=\tfrac{1}{2}\begin{pmatrix}10\\10\end{pmatrix},\ \beta=\tfrac{1}{2}\begin{pmatrix}11\\01\end{pmatrix},\ \gamma=\tfrac{1}{2}\begin{pmatrix}01\\11\end{pmatrix},\ p=q=m=a\beta\gamma=0,$$

obtain the result

$$\vartheta_5(u+w)\,\vartheta_5(u-w)\,\vartheta_5^2=\vartheta_5^2(u)\,\vartheta_5^2(w)+\vartheta_{02}^2(u)\,\vartheta_{02}^2(w)+\vartheta_{24}^2(u)\,\vartheta_{24}^2(w)+\vartheta_{04}^2(u)\,\vartheta_{04}^2(w),$$

which is in agreement with the results of §§ 219, 220.

Dividing the result of Ex. iii. by that of Ex. iv. we obtain an addition formula for the theta quotient $\vartheta_{02}(u)/\vartheta_5(u)$, whereby $\vartheta_{02}(u+w)/\vartheta_5(u+w)$ is expressed by theta quotients with the arguments u and w.

Ex. v. The formula of Ex. ii. may be used in different ways to obtain an expression for the product $\vartheta(u+w;\ q)\ \vartheta(u-w;\ 0)$. It is sufficient that the characteristics m and $q+m$ be even and that the three odd characteristics a, β, γ have the sum m. Thus, starting with a given characteristic q, we express it, save for a characteristic of integers, as the sum of two even characteristics, m and $q+m$, which (unless q be zero) is possible in three ways†, and then express m as the sum of three odd characteristics, a, β, γ, which is possible in two ways‡; then§ we take $a_1=0$, $a_2=\beta\gamma$, $a_3=\gamma a$, $a_4=a\beta$. Taking $q=\tfrac{1}{2}\begin{pmatrix}10\\10\end{pmatrix}$, we have

* In Weierstrass's reduced characteristic symbol the upper row of elements is positive, and the lower row negative; cf. §§ 203, 204, and p. 337, foot-note.

† This is obvious from the table of § 204, or by using the two-letter notation; for instance the symbol $(a_1a_2)\equiv(a_1c)+(a_2c)\equiv(a_1c_1)+(a_2c_1)\equiv(a_1c_2)+(a_2c_2)$.

‡ For example, $(ac)\equiv(a_1a)+(a_2a)+(c_1c_2)\equiv(a_1a_2)+(c_1c)+(cc_2)$. See the final equation of § 201. The six odd characteristics form a set which is a particular case of sets considered in chapter XVII.

§ Moreover we may increase u and w by the same half-period. But the additions of the half-periods P, $P+\Omega_q$ lead to the same result; and, when q is one of a, β, γ, the same result is obtained by the addition of $P+\Omega_m$ and of $P+\Omega_m+\Omega_q$.

$$\tfrac{1}{2}\begin{pmatrix}10\\10\end{pmatrix}\equiv\tfrac{1}{2}\begin{pmatrix}10\\00\end{pmatrix}+\tfrac{1}{2}\begin{pmatrix}00\\10\end{pmatrix}\equiv\tfrac{1}{2}\begin{pmatrix}01\\10\end{pmatrix}+\tfrac{1}{2}\begin{pmatrix}11\\00\end{pmatrix}\equiv\tfrac{1}{2}\begin{pmatrix}00\\11\end{pmatrix}+\tfrac{1}{2}\begin{pmatrix}10\\01\end{pmatrix};$$

putting $m=\tfrac{1}{2}\begin{pmatrix}10\\00\end{pmatrix}$, we may take

$$a=\tfrac{1}{2}\begin{pmatrix}11\\01\end{pmatrix},\quad \beta=\tfrac{1}{2}\begin{pmatrix}11\\10\end{pmatrix},\quad \gamma=\tfrac{1}{2}\begin{pmatrix}10\\11\end{pmatrix}.$$

Hence obtain the result

$$\vartheta_{02}(u+w)\,\vartheta_5(u-w)\,\vartheta_{12}(0)\,\vartheta_{01}(0)=\vartheta_{12}\vartheta_{01}\bar\vartheta_{02}\bar\vartheta_5+\vartheta_{04}\vartheta_{24}\bar\vartheta_{14}\bar\vartheta_3+\vartheta_4\vartheta_{13}\bar\vartheta_{23}\bar\vartheta_{03}+\vartheta_{34}\vartheta_1\bar\vartheta_2\bar\vartheta_0,$$

where, on the right hand, ϑ_{12} denotes $\vartheta_{12}(u)$, etc., and $\bar\vartheta_{02}$ denotes $\vartheta_{02}(w)$, etc. Comparing this result with the result of Ex. iii., namely

$$\vartheta_{02}(u+w)\,\vartheta_5(u-w)\,\vartheta_{12}(0)\,\vartheta_{01}(0)=\vartheta_{12}\vartheta_{01}\bar\vartheta_{02}\bar\vartheta_5+\vartheta_{02}\vartheta_5\bar\vartheta_{12}\bar\vartheta_{01}+\vartheta_{04}\vartheta_{24}\bar\vartheta_3\bar\vartheta_{14}+\vartheta_3\vartheta_{14}\bar\vartheta_{04}\bar\vartheta_{24},$$

we deduce the remarkable identity

$$\vartheta_4(u)\,\vartheta_{13}(u)\,\vartheta_{23}(w)\,\vartheta_{03}(w)+\vartheta_1(u)\,\vartheta_{34}(u)\,\vartheta_0(w)\,\vartheta_2(w)$$
$$=\vartheta_{02}(u)\,\vartheta_5(u)\,\vartheta_{12}(w)\,\vartheta_{01}(w)+\vartheta_3(u)\,\vartheta_{14}(u)\,\vartheta_{04}(w)\,\vartheta_{24}(w),$$

wherein u, w are arbitrary arguments; this is one of a set of formulae obtained by Caspary, to which future reference will be made.

Ex. vi. By taking in Ex. v. the characteristics q, m to be respectively

$$\tfrac{1}{2}\begin{pmatrix}10\\10\end{pmatrix},\quad \tfrac{1}{2}\begin{pmatrix}10\\01\end{pmatrix},$$

and resolving m into the sum $a+\beta+\gamma$ in the two ways

$$\tfrac{1}{2}\begin{pmatrix}10\\10\end{pmatrix}+\tfrac{1}{2}\begin{pmatrix}11\\01\end{pmatrix}+\tfrac{1}{2}\begin{pmatrix}11\\10\end{pmatrix},\quad \tfrac{1}{2}\begin{pmatrix}01\\11\end{pmatrix}+\tfrac{1}{2}\begin{pmatrix}01\\01\end{pmatrix}+\tfrac{1}{2}\begin{pmatrix}10\\11\end{pmatrix},$$

respectively, obtain the formulae

$$\vartheta_{02}(u+w)\,\vartheta_5(u-w)\,\vartheta_0(0)\,\vartheta_2(0)=\vartheta_5\vartheta_{02}\bar\vartheta_2\bar\vartheta_0+\vartheta_0\vartheta_2\bar\vartheta_{02}\bar\vartheta_5\;-\vartheta_4\vartheta_{13}\bar\vartheta_{24}\bar\vartheta_{04}-\vartheta_{04}\vartheta_{24}\bar\vartheta_{13}\bar\vartheta_4,$$

$$\vartheta_{02}(u+w)\,\vartheta_5(u-w)\,\vartheta_0(0)\,\vartheta_2(0)=\vartheta_0\vartheta_2\bar\vartheta_{02}\bar\vartheta_5-\vartheta_{24}\vartheta_{04}\bar\vartheta_4\bar\vartheta_{13}-\vartheta_{14}\vartheta_3\bar\vartheta_{03}\bar\vartheta_{23}+\vartheta_{34}\vartheta_1\bar\vartheta_{01}\bar\vartheta_{12},$$

and the identity

$$\vartheta_{34}\vartheta_1\bar\vartheta_{01}\bar\vartheta_{12}+\vartheta_4\vartheta_{13}\bar\vartheta_{24}\bar\vartheta_{04}=\vartheta_5\vartheta_{02}\bar\vartheta_0\bar\vartheta_2+\vartheta_{14}\vartheta_3\bar\vartheta_{03}\bar\vartheta_{23}.$$

Putting in this equation $w=0$, we obtain a formula quoted without proof on page 340.

Ex. vii. Obtain the two formulae for $\vartheta_{02}(u+w)\;\vartheta_5(u-w)$ which arise, similarly to those in Exs. v. vi., by taking for m the characteristic $\tfrac{1}{2}\begin{pmatrix}01\\10\end{pmatrix}$, the characteristic q being unaltered.

Ex. viii. Obtain the formulae, for $p=2$,

$$\vartheta_{23}(u+w)\,\vartheta_{23}(u-w)\,\vartheta_5^2(0)=\vartheta_5^2\bar\vartheta_{23}^2+\vartheta_1^2\bar\vartheta_{04}^2-\vartheta_3^2\bar\vartheta_2^2-\vartheta_{13}^2\bar\vartheta_{12}^2,$$

$$\vartheta_{23}(u+w)\,\vartheta_5(u-w)\,\vartheta_5(0)\,\vartheta_{23}(0)=\vartheta_5\vartheta_{23}\bar\vartheta_5\bar\vartheta_{23}+\vartheta_1\vartheta_{04}\bar\vartheta_1\bar\vartheta_{04}-\vartheta_3\vartheta_2\bar\vartheta_3\bar\vartheta_2-\vartheta_{13}\vartheta_{12}\bar\vartheta_{13}\bar\vartheta_{12},$$

where the notation is as in Ex. v.

For tables of such formulae the reader may consult Königsberger, *Crelle*, LXIV. (1865), p. 28, and *ibid.*, LXV. (1866), p. 340. Extensive tables are given by Rosenhain, *Mém. par divers Savants*, (Paris, 1851), t. XI., p. 443; Cayley, *Phil. Trans.* (London, 1881), Vol. 171, pp. 948, 964; Forsyth, *Phil. Trans.* (London, 1883), Vol. 173, p. 834.

Ex. ix. We proceed now to apply the formula of Ex. i. to the case $p=3$; taking the argument $v=0$, the characteristics p, r both zero, and the characteristics $m, a_1, a_2, \ldots\ldots, a_8$ to be respectively $m, m, \alpha, \beta, \ldots\ldots, \nu$, where $\alpha, \beta, \gamma, \kappa, \lambda, \mu, \nu$ are seven characteristics such that the combinations $\alpha, \beta, \gamma, \kappa, \lambda, \mu, \nu, m\alpha\beta, m\alpha\kappa, m\lambda\mu$ are all odd characteristics, m being an even characteristic, and removing the negative signs in the characteristics by such steps* as

$$\begin{aligned}\vartheta(-w;\ m-a_\lambda-p)&=\vartheta(w;\ a_\lambda+p-m)=\vartheta(w;\ a_\lambda+p+m-2m)\\&=e^{-4\pi i m(a'_\lambda+p'+m')}\,\vartheta(w;\ a_\lambda+p+m)\\&=e^{-4\pi i m(p'+a'_\lambda)}\,\vartheta(w;\ p+a_\lambda+m),\end{aligned}$$

the formula becomes†

$$\begin{aligned}&\vartheta(u+w;\ q)\,\vartheta(u-w;\ 0)\,\vartheta(0;\ q+m)\,\vartheta(0;\ m)\\&\qquad=\sum_{\lambda=1}^{8} e^{-4\pi i(ma'_\lambda+q'a_\lambda)}\,\vartheta(u;\ q+a_\lambda)\,\vartheta(u;\ a_\lambda)\,\vartheta(w;\ q+m+a_\lambda)\,\vartheta(w;\ m+a_\lambda).\end{aligned}$$

In order that the left-hand side of this equation may not vanish, the characteristic $q+m$ must be even; now it can be shewn that every characteristic (q), except the zero characteristic, can be resolved into the sum of two even characteristics (m and $q+m$) in ten ways, and that, to every even characteristic (m) there are 8 ways of forming such a set as $\alpha, \beta, \gamma, \kappa, \lambda, \mu, \nu$ (cf. p. 309, Chap. XI.). Hence, for any characteristic q there are various ways of forming such an expression of $\vartheta(u+w;\ q)\,\vartheta(u-w;\ 0)$ in terms of theta functions of u and w; moreover by the addition of the same half-period to u and w, the form of the right-hand side is altered, while the left-hand side remains effectively unaltered. In all cases in which q is even we may obtain a formula by taking $m=0$.

Ex. x. Taking, in Ex. ix., the characteristics q, m both zero, prove in the notation of § 205, when $\alpha, \beta, \ldots\ldots, \nu$ are the characteristics there associated with the suffixes $1, 2, \ldots\ldots, 7$, that

$$\vartheta(u+w)\,\vartheta(u-w)\,\vartheta^2=\sum_{i=0}^{7}\vartheta_i^2(u)\,\vartheta_i^2(w).$$

Prove also, taking $m=0$, $q=\frac{1}{2}\begin{pmatrix}111\\000\end{pmatrix}$, that $\vartheta_{456}(u+w)\,\vartheta(u-w)\,\vartheta_{456}\,\vartheta$ is equal to

$$\begin{aligned}&\vartheta(u)\,\vartheta(w)\,\vartheta_{456}(u)\,\vartheta_{456}(w)+\vartheta_4(u)\,\vartheta_4(w)\,\vartheta_{56}(u)\,\vartheta_{56}(w)+\vartheta_5(u)\,\vartheta_5(w)\,\vartheta_{64}(u)\,\vartheta_{64}(w)\\&\qquad\qquad+\vartheta_6(u)\,\vartheta_6(w)\,\vartheta_{45}(u)\,\vartheta_{45}(w)\\&-\vartheta_7(u)\,\vartheta_7(w)\,\vartheta_{123}(u)\,\vartheta_{123}(w)-\vartheta_1(u)\,\vartheta_1(w)\,\vartheta_{237}(u)\,\vartheta_{237}(w)-\vartheta_2(u)\,\vartheta_2(w)\,\vartheta_{317}(u)\,\vartheta_{317}(w)\\&\qquad\qquad-\vartheta_3(u)\,\vartheta_3(w)\,\vartheta_{127}(u)\,\vartheta_{127}(w),\end{aligned}$$

where ϑ, ϑ_{456} denote respectively $\vartheta(0)$, $\vartheta_{456}(0)$.

Hence we immediately obtain an expression for $\vartheta_{456}(u+w)/\vartheta(u+w)$ in terms of theta quotients $\vartheta_i(u)/\vartheta(u)$, $\vartheta_i(w)/\vartheta(w)$.

Ex. xi. The formula of Ex. i. can by change of notation be put into a more symmetrical form which has theoretical significance. As before let m be any half-integer even characteristic, and let $a_1, \ldots\ldots, a_s$ be $s, =2^p$, half-integer characteristics such that every

* Wherein the notation is that the characteristic p is written $\begin{pmatrix}p_1' & p_2' & p_3'\\ p_1 & p_2 & p_3\end{pmatrix}$ and p' denotes the row (p_1', p_2', p_3'); and similarly for the characteristics m, a_λ.

† This formula is given by Weber, *Theorie der Abel'schen Functionen vom Geschlecht* 3 (Berlin, 1876), p. 38.

combination ma_ia_j, in which i is not equal to j, is an odd characteristic; let f, g, h be arbitrary half-integer characteristics; let J denote the matrix of substitution given by

$$J=\tfrac{1}{2}\begin{pmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{pmatrix},$$

and from the arbitrary arguments u, v, w determine other arguments U, V, W, T by the reciprocal linear equations

$$(U_i,\ V_i,\ W_i,\ T_i)=J(u_i,\ v_i,\ w_i,\ 0), \qquad (i=1,\ 2,\ \ldots\ldots,\ p),$$

or, as we may write them,

$$(U,\ V,\ W,\ T)=J(u,\ v,\ w,\ 0);$$

further determine the new characteristics F, G, H, K by means of equations of the form

$$(F,\ G,\ H,\ K)=J(f,\ g,\ h,\ m),$$

noticing that there are $2p$ such sets of four equations, one for every set of corresponding elements of the characteristics;

then deduce from the equation of Ex. i. that

$$\vartheta(0;\ m)\,\vartheta(u;\ f)\,\vartheta(v;\ g)\,\vartheta(w;\ h)$$
$$=\sum_{\lambda=1}^{2^p} e^{4\pi i a_\lambda a'_\lambda+2\pi i a_\lambda(f'+g'+h'+m')}\vartheta(T;\ K+a_\lambda)\,\vartheta(U;\ F+a_\lambda)\,\vartheta(V;\ G+a_\lambda)\,\vartheta(W;\ H+a_\lambda)$$
$$=\sum_{\lambda=1}^{2^p} e^{4\pi i a_\lambda m'}\vartheta(U;\ F-a_\lambda)\,\vartheta(V;\ G-a_\lambda)\,\vartheta(W;\ H-a_\lambda)\,\vartheta(T;\ K+a_\lambda).$$

Putting $m=0$, we derive the formula

$$\vartheta(0;\ 0)\,\vartheta(v+w;\ g+h)\,\vartheta(w+u;\ h+f)\,\vartheta(u+v;\ f+g)$$
$$=\sum_{\lambda=1}^{2^p}\vartheta(u+v+w;\ f+g+h+a_\lambda)\,\vartheta(u;\ f-a_\lambda)\,\vartheta(v;\ g-a_\lambda)\,\vartheta(w;\ h-a_\lambda),$$

wherein u, v, w are any arguments and f, g, h are any half-integer characteristics.

Ex. xii. Deduce from Ex. i. that when $p=2$ there are twenty sets of four theta functions, three of them odd and one even, such that the square of any theta function can be expressed linearly by the squares of these four.

287. The number, r^p, of terms in the expansion of $\Pi(u)$ may be expected to reduce in particular cases by the vanishing of some coefficients on the right-hand side. We proceed to shew* that this is the case, for instance, when $\Pi(u)$ is either an odd function, or an even function of the arguments u. We prove first that a necessary condition for this is that the characteristic $(Q,\ Q')$ consist of half-integers.

For, if $\Pi(-u)=\epsilon\Pi(u)$, where ϵ is $+1$ or -1, the equation

$$\Pi(u+\Omega_m)=e^{r\lambda_m(u)+2\pi i(mQ'-m'Q)}\,\Pi(u)$$

gives

$$\epsilon\Pi(-u-\Omega_m)=e^{r\lambda_m(u)+2\pi i(mQ'-m'Q)}\,\epsilon\Pi(-u),$$

* Schottky, *Abriss einer Theorie der Abel'schen Functionen von drei Variabeln* (Leipzig, 1880).

while, the left-hand side of this equation is, by the same fundamental equation, equal to

$$\epsilon e^{r\lambda_{-m}(-u)-2\pi i(mQ'-m'Q)}\,\Pi(-u);$$

hence, for all values of the integers m, m', the expression

$$r\,[\lambda_m(u)-\lambda_{-m}(-u)]+4\pi i\,(mQ'-m'Q)$$

must be an integral multiple of $2\pi i$; since, however,

$$\lambda_m(u)=H_m(u+\tfrac{1}{2}\Omega_m)-\pi imm'=\lambda_{-m}(-u),$$

this requires that $2(mQ'-m'Q)$ be an integer; thus $2Q$, $2Q'$ are necessarily integers.

Suppose now that Q, Q' are half-integers; denote them by q, q'; and suppose that $\Pi(u)=\epsilon\Pi(-u)$, where ϵ is $+1$ or -1. Then from the equation

$$\Pi(u)=\sum_\mu K_\mu \underline{\vartheta}\left(u;\ q,\ \frac{q'+\mu}{r}\right),$$

since, for any characteristic, $\vartheta(u,q)=\vartheta(-u,-q)$, we obtain

$$\Pi(u)=\epsilon\Pi(-u)=\epsilon\sum_\mu K_\mu\underline{\vartheta}\left(-u;\ q,\ \frac{\mu+q'}{r}\right)=\epsilon\sum_\mu K_\mu\underline{\vartheta}\left(u;\ -q,\ -\frac{\mu+q'}{r}\right)$$

$$=\epsilon\sum_\mu K_\mu\underline{\vartheta}\left[u;\ q-2q,\ \frac{\nu+q'}{r}-\frac{\mu+\nu+2q'}{r}\right],$$

where ν is a row of positive integers, each less than r, so chosen that

$$\nu\equiv-(\mu+2q'),\ (\text{mod. } r);$$

thus the aggregate of the values of ν is the same as the aggregate of the values of μ; therefore, by the formula (§ 190), $\vartheta(u;\ q+M,\ q'+M')=e^{2\pi iMq'}\vartheta(u;\ q,\ q')$, wherein M, M' are integers, we have

$$\sum K_\nu\underline{\vartheta}\left(u;\ q,\ \frac{\nu+q'}{r}\right)=\Pi(u)=\epsilon\sum_\mu K_\mu e^{-4\pi iq\frac{\nu+q'}{r}}\underline{\vartheta}\left(u;\ q,\ \frac{\nu+q'}{r}\right);$$

comparing these two forms for $\Pi(u)$ we see that in the formula

$$\Pi(u)=\sum_\mu K_\mu\underline{\vartheta}\left(u;\ q,\ \frac{\mu+q'}{r}\right)$$

the values of μ that arise may be divided into two sets; (i) those for which $2\mu+2q'\equiv 0$ (mod. r); for such terms the value of ν defined by the previously written congruence is equal to μ, and the transformation effected with the help of the congruence only reproduces the term to which it is applied; thus, *for all such values of* μ *which occur*, $e^{-4\pi iq\frac{\mu+q'}{r}}$ is equal to ϵ; (ii) those terms for which $2\mu + 2q' \not\equiv 0$ (mod. r); for such terms $K_\nu = \epsilon K_\mu e^{-4\pi i q \frac{\nu+q'}{r}}$. Hence on the whole $\Pi(u)$ can be put into the form

$$\sum_\mu K_\mu \vartheta\left(u;\ q,\ \frac{\mu+q'}{r}\right) + \sum_\mu K_\mu \left\{\vartheta\left(u;\ q,\ \frac{\mu+q'}{r}\right) + \epsilon e^{-4\pi i q \frac{\nu+q'}{r}} \vartheta\left(u;\ q,\ \frac{\nu+q'}{r}\right)\right\},$$

where the first summation extends to those values of μ for which $2\mu + 2q' \equiv 0$ (mod. r), and the second summation extends to half those values of μ for which $2\mu + 2q' \not\equiv 0$ (mod. r). The single term

$$\phi(u, \mu) = \vartheta\left(u;\ q,\ \frac{\mu+q'}{r}\right) + \epsilon e^{-4\pi i q \frac{\nu+q'}{r}} \vartheta\left(u;\ q,\ \frac{\nu+q'}{r}\right),$$

which can also be written in the form

$$\vartheta\left(u;\ q,\ \frac{\mu+q'}{r}\right) + \epsilon e^{4\pi i q \frac{\mu+q'}{r}} \vartheta\left(u;\ q,\ -\frac{\mu+q'}{r}\right),$$

is even or odd according as $\Pi(u)$ is even or odd; and this is also true for the term $\vartheta\left(u;\ q,\ \frac{\mu+q'}{r}\right)$ arising when $2\mu + 2q' \equiv 0$ (mod. r).

Hence if x be the number of values of μ, incongruent for modulus r, which satisfy the congruence $2\mu + 2q' \equiv 0$ (mod. r), and y be the number of these solutions for which also the condition $e^{-4\pi i q \frac{\mu+q'}{r}} = \epsilon$ is satisfied, the number of undetermined coefficients in $\Pi(u)$ is reduced to, at most,

$$y + \tfrac{1}{2}(r^p - x).$$

288. We proceed now to find x and y; we notice that y vanishes when x vanishes, for the terms whose number is y are chosen from among possible terms whose number is x. The result is that *when r is even and the characteristic (q, q') is integer or zero, and $\Pi(-u) = \epsilon\Pi(u)$, the number of terms in $\Pi(u)$ is $\tfrac{1}{2} r^p + 2^{p-1}\epsilon$; while, when r is odd, or when r is even and the half-integer characteristic (q, q') does not consist wholly of integers, or zeros, the number of terms in $\Pi(u)$ is $\tfrac{1}{2} r^p + \tfrac{1}{4}[1 - (-)^r]\,\epsilon e^{4\pi i q q'}$.*

Suppose r is even; then the congruence $2\mu + 2q' \equiv 0$ (mod. r) is satisfied by taking $\mu = M\frac{r}{2} - q'$, and in no other way, M denoting a row of p arbitrary integers. Thus unless q' consists of integers, x is zero, and therefore, as remarked above, y is zero, and the number of terms in $\Pi(u)$ is $\tfrac{1}{2} r^p$. While, when q' is integral, the incongruent values for μ (modulus r) are obtained by taking the incongruent values for M for modulus 2, in number 2^p; in that case $x = 2^p$; the condition $e^{-4\pi i q \frac{\mu+q'}{r}} = \epsilon$ is the same as $e^{-2\pi i q M} = \epsilon$; when q is integral, this is satisfied by all the 2^p values of M, or by no values of M, according as ϵ is $+1$ or is -1; in both cases $y = 2^{p-1}(1+\epsilon)$; when q is not integral, $p-1$ of the elements of M can be taken arbitrarily and the condition $e^{-2\pi i q M}=\epsilon$ determines the other element, so that $y=2^{p-1}$. Thus, when r is even, we have

(1) when q, q' are both rows of integers (including zero), $x=2^p$, $y=2^{p-1}(1+\epsilon)$, and the number of terms in $\Pi(u)$ is

$$2^{p-1}(1+\epsilon)+\tfrac{1}{2}(r^p-2^p)=\tfrac{1}{2}r^p+2^{p-1}\epsilon,$$

as stated, there being $\frac{1}{2}r^p+2^{p-1}$ terms when $\Pi(u)$ is an even function, and $\frac{1}{2}r^p-2^{p-1}$ terms when $\Pi(u)$ is an odd function;

(2) when q' is integral, and q is not integral, $x=2^p$, $y=2^{p-1}$, and therefore the number of terms in $\Pi(u)$ is

$$2^{p-1}+\tfrac{1}{2}(r^p-2^p)=r^p,$$

in accordance with the result stated;

(3) when q' is not integral, both x and y are zero, and the number of terms is $\frac{1}{2}r^p$, also agreeing with the given formula.

Suppose now that r is odd, then the equation

$$2\mu+2q'=rM, \text{ or } \mu=\frac{rM-2q'}{2}, = \text{integer}+\frac{M-2q'}{2},$$

wherein M is a row of integers, requires M to have the form $2q'+2N$, where N is a row of integers, and therefore

$$\frac{\mu}{r}=\frac{rM-2q'}{2r}, =N+q'\left(1-\frac{1}{r}\right);$$

this equation, since μ consists of positive integers all less than r, determines the value of N uniquely; hence $x=1$. The condition

$$e^{-4\pi i q\frac{\mu+q'}{r}}=\epsilon, \text{ or } e^{-4\pi i q(q'+N)}=\epsilon, \text{ or } e^{-4\pi i q q'}=\epsilon$$

determines $y=1$ or $y=0$ according as $\epsilon e^{4\pi i q q'}=+1$ or $=-1$; hence the number of terms in $\Pi(u)$ is

$$1+\tfrac{1}{2}(r^p-1), \text{ or } \tfrac{1}{2}(r^p-1),$$

according as $\epsilon e^{4\pi i q q'}=+1$ or -1; this agrees with the given result when r is odd, the number of terms being always one of the numbers $\frac{1}{2}(r^p\pm 1)$.

289. It follows from the investigation just given that if we take products of theta functions, forming odd or even theta functions of order r, with the same half-integer characteristic (q, q'), and associated with the same constants 2ω, $2\omega'$, 2η, $2\eta'$, then when r is even, the number of these which are linearly independent is, at most, $\frac{1}{2}r^p+2^{p-1}\epsilon$ when the characteristic is integral or zero, and is otherwise $\frac{1}{2}r^p$; while, when r is odd, the number which are linearly independent is, at most, $\frac{1}{2}(r^p+\epsilon e^{4\pi i q q'})$, ϵ being ± 1 according as the products are even or odd functions.

Ex. i. In case $p=2$ there are six odd characteristics, and the sum of any three of them is even*, as the reader can easily verify by the table of page 303. Let $a, \beta, \gamma, \delta, \epsilon, \zeta$ denote the odd characteristics, in any order, and let $a\beta\gamma$ denote the characteristic formed by adding the characteristics a, β, γ. Then the product

$$\Pi(u)=\vartheta(u, a)\,\vartheta(u, \beta)\,\vartheta(u, \gamma)\,\vartheta(u, a\beta\gamma)$$

is an odd theta function of the fourth order with integral characteristic. Hence this product can be written in the form

$$\Pi(u)=\Sigma A_\mu \underline{\vartheta}\left(u;\, 0, \frac{\mu}{4}\right),$$

where μ has the 4^2 values arising by giving to each of the two elements of μ, independently of the other, the values 0, 1, 2, 3. Changing the sign of u we have

$$\Pi(u)=-\Sigma A_\mu \underline{\vartheta}\left(-u;\, 0, \frac{\mu}{4}\right),\quad =-\Sigma A_\mu \underline{\vartheta}\left(u;\, 0, -\frac{\mu}{4}\right),\quad =-\Sigma A_\mu \underline{\vartheta}\left(u;\, 0, \frac{\nu}{4}-\frac{\mu+\nu}{4}\right),$$

where ν is chosen so that

$$\mu+\nu\equiv 0 \pmod{4}.$$

This congruence gives 16 values of ν corresponding to the 16 values of μ; of these there are 4 values for which $\mu=\nu$ and $2\mu\equiv 0 \pmod{4}$; these are the values

$$\mu=(0,0),\ (0,2),\ (2,0),\ (2,2),$$

greater values for the elements of μ being excluded by the condition that these elements must be less than 4. We have by the formula (§ 190) $\vartheta(u;\, q+M)=e^{2\pi i M q'}\vartheta(u)$,

$$\Pi(u)=-\Sigma A_\mu \underline{\vartheta}\left(u;\, 0, \frac{\nu}{4}\right);$$

comparing this with the original formula for $\Pi(u)$, we see that

$$A_\nu=-A_\mu,$$

so that the terms in the original formula for $\Pi(u)$ for which $\nu=\mu$ are absent, and the remaining twelve terms may be arranged as six terms in the form

$$\Pi(u)=\Sigma A_\mu\left[\underline{\vartheta}\left(u;\, 0, \frac{\mu}{4}\right)-\underline{\vartheta}\left(u;\, 0, -\frac{\mu}{4}\right)\right]=\Sigma A_\mu\left[\underline{\vartheta}\left(u;\, 0, \frac{\mu}{4}\right)-\underline{\vartheta}\left(-u;\, 0, \frac{\mu}{4}\right)\right],$$

where the summation extends to the following values of μ,

$$\mu=(0,1),\ (1,0),\ (1,1),\ (1,2),\ (1,3),\ (2,3);$$

these values may be interchanged respectively with

$$\mu=(0,3),\ (3,0),\ (3,3),\ (3,2),\ (3,1),\ (2,1),$$

if a proper corresponding change be made in the coefficients A_μ.

The number 6 is that obtained from the formula $\frac{1}{2}r^p+2^{p-1}\epsilon$, by putting $r=4$, $\epsilon=-1$, $p=2$.

Ex. ii. In case $p=2$, denoting the odd characteristics by $a, \beta, \gamma, \delta, \epsilon, \zeta$, and the sum of two of them, say a and β, by $a\beta$, and so on, each of the four products

$$\vartheta(u, a)\,\vartheta(u, a\epsilon\zeta),\quad \vartheta(u, \beta)\,\vartheta(u, \beta\epsilon\zeta),\quad \vartheta(u, \gamma)\,\vartheta(u, \gamma\epsilon\zeta),\quad \vartheta(u, \delta)\,\vartheta(u, \delta\epsilon\zeta),$$

or, in Weierstrass's notation, if $a, \beta, \gamma, \delta, \epsilon, \zeta$ be taken in the order in which they occur in the table of page 303, each of the products

$$\vartheta_{02}(u)\,\vartheta_{34}(u),\quad \vartheta_{24}(u)\,\vartheta_{03}(u),\quad \vartheta_{04}(u)\,\vartheta_{23}(u),\quad \vartheta_1(u)\,\vartheta_5(u),$$

* This is a particular case of a result obtained in chapter XVII.

is an odd theta function of order 2, and of characteristic differing only by integers from the characteristic denoted by $\epsilon\zeta$, or, in the arrangement here taken, $\frac{1}{2}\begin{pmatrix}10\\11\end{pmatrix}$; thus any three of these products are connected by a linear equation whose coefficients do not depend upon u.

Similarly each of the products

$$\vartheta(u, \alpha\delta\epsilon)\,\vartheta(u, \alpha\delta\zeta),\quad \vartheta(u, \beta\delta\epsilon)\,\vartheta(u, \beta\delta\zeta),\quad \vartheta(u, \gamma\delta\epsilon)\,\vartheta(u, \gamma\delta\zeta),\quad \vartheta(u, \epsilon)\,\vartheta(u, \zeta),$$

or, in Weierstrass's notation, if $\alpha, \beta, \gamma, \delta, \epsilon, \zeta$ be taken in the order in which they occur in the table of p. 303, each of the products

$$\vartheta_{14}(u)\,\vartheta_4(u),\quad \vartheta_{01}(u)\,\vartheta_0(u),\quad \vartheta_{12}(u)\,\vartheta_2(u),\quad \vartheta_{13}(u)\,\vartheta_3(u),$$

is an even theta function of order 2, and of characteristic differing only by integers from the characteristic denoted by $\epsilon\zeta$, or, in the arrangement here taken, $\frac{1}{2}\begin{pmatrix}10\\11\end{pmatrix}$; thus any three of these products are connected by a linear equation whose coefficients do not depend upon u.

Ex. iii. For $p=2$ the number of linearly independent even theta functions of the fourth order and of integral characteristic is $\frac{1}{2}4^2+2=10$. If q, r be any half-integer characteristics, it follows that any eleven functions of the form $\vartheta^2(u, q)\,\vartheta^2(u, r)$ are connected by a linear equation. Taking now, with Weierstrass's notation, the four functions*

$$t=\vartheta_5(u),\quad x=\vartheta_{34}(u),\quad y=\vartheta_{12}(u),\quad z=\vartheta_0(u),$$

it follows that there exists an identical equation

$$A_0t^4+A_1x^4+A_2y^4+A_3z^4+2Ctxyz+F_1y^2z^2+F_2x^2t^2+G_1z^2x^2+G_2y^2t^2+H_1x^2y^2+H_2z^2t^2=0,$$

in which the eleven coefficients $A_0, \ldots\ldots, H_2$ are independent of u.

The characteristics of the theta functions $\vartheta_5(u)$, $\vartheta_{34}(u)$, $\vartheta_{12}(u)$, $\vartheta_0(u)$ may be taken, respectively, to be (cf. § 220, Chap. XI.)

$$\begin{pmatrix}0, 0\\0, 0\end{pmatrix};\ \begin{pmatrix}0, 0\\0, \frac{1}{2}\end{pmatrix}=\begin{pmatrix}p_1', p_2'\\p_1, p_2\end{pmatrix}, \text{ say};\ \begin{pmatrix}0, 0\\\frac{1}{2}, 0\end{pmatrix}=\begin{pmatrix}q_1', q_2'\\q_1, q_2\end{pmatrix}, \text{ say};\ \begin{pmatrix}0, 0\\\frac{1}{2}, \frac{1}{2}\end{pmatrix}=\begin{pmatrix}r_1', r_2'\\r_1, r_2\end{pmatrix}, \text{ say};$$

hence, by the formulae (§ 190)

$$\vartheta(u+\Omega_P;\ q)=e^{\lambda_P(u)-2\pi iP'q}\,\vartheta(u;\ q+P),\quad \vartheta(u;\ q+M)=e^{2\pi iMq'}\,\vartheta(u;\ q),$$

wherein M denotes a row of integers, we obtain

$$\vartheta_5(u+\Omega_p)=e^{\lambda_p(u)}\,\vartheta_{34}(u),\quad \vartheta_{34}(u+\Omega_p)=e^{\lambda_p(u)}\,\vartheta_5(u),\quad \vartheta_{12}(u+\Omega_p)=e^{\lambda_p(u)}\,\vartheta_0(u),$$
$$\vartheta_0(u+\Omega_p)=e^{\lambda_p(u)}\,\vartheta_{12}(u);$$

hence the substitution of $u+\Omega_p$ for u in the identity replaces t, x, y, z respectively by x, t, z, y. Comparing the new form with the original form we infer that

$$A_0=A_1,\quad A_2=A_3,\quad G_1=G_2,\quad H_1=H_2.$$

Similarly the substitution of $u+\Omega_q$ for u replaces t, x, y, z respectively by y, z, t, x; making this change, and then comparing the old form with the derived form, we infer that

$$A_0=A_2,\quad A_1=A_3,\quad F_1=F_2,\quad H_1=H_2.$$

* Which are all even and such that the square of every other theta function is a linear function of the squares of these functions. It can be proved that these functions are not connected by any quadratic relation.

Thus the identity is of the form

$$t^4+x^4+y^4+z^4+2Ctxyz+F(y^2z^2+x^2t^2)+G(z^2x^2+y^2t^2)+H(x^2y^2+z^2t^2)=0.$$

Taking now the three characteristics

$$\begin{pmatrix} f_1', f_2' \\ f_1, f_2 \end{pmatrix}=\begin{pmatrix} 0, \frac{1}{2} \\ 0, 0 \end{pmatrix}, \quad \begin{pmatrix} g_1', g_2' \\ g_1, g_2 \end{pmatrix}=\begin{pmatrix} \frac{1}{2}, 0 \\ 0, 0 \end{pmatrix}, \quad \begin{pmatrix} h_1', h_2' \\ h_1, h_2 \end{pmatrix}=\begin{pmatrix} \frac{1}{2}, \frac{1}{2} \\ 0, 0 \end{pmatrix},$$

and adding to the argument u, in turn, the half-periods Ω_f, Ω_g, Ω_h and then putting $u=0$, we obtain the three equations

$$\vartheta_4^4+\vartheta_{03}^4+G\vartheta_4^2\vartheta_{03}^2=0, \quad \vartheta_{01}^4+\vartheta_2^4+F\vartheta_{01}^2\vartheta_2^2=0, \quad \vartheta_{23}^4+\vartheta_{14}^4+H\vartheta_{23}^2\vartheta_{14}^2=0,$$

where ϑ_4^4 denotes $\vartheta_4^4(0)$, etc., and the notation is Weierstrass's, as in § 220. By these equations the constants F, G, H are determined in terms of zero values of the theta functions. The value of C can then be determined by putting $u=0$ in the identity itself.

Thus we may regard the equation as known; it coincides with that considered in Exx. i. and iv. § 221, Chap. XI., and represents a quartic surface with sixteen nodes. With the assumption of certain relations connecting the zero values of the theta functions, proved by formulae occurring later (Chap. XVII. § 317, Ex. iv.), we can express the coefficients in the equation in terms of the four constants $\vartheta_5(0)$, $\vartheta_{34}(0)$, $\vartheta_{12}(0)$, $\vartheta_0(0)$. We have in fact, if these constants be respectively denoted by d, a, b, c

$$\begin{aligned}
&\vartheta_{01}^4+\vartheta_2^4=d^4+a^4-b^4-c^4, \quad &&\vartheta_4^4+\vartheta_{03}^4=d^4-a^4+b^4-c^4, \quad &&\vartheta_{23}^4+\vartheta_{14}^4=d^4-a^4-b^4+c^4,\\
&\vartheta_{01}^2\vartheta_2^2\ \ =d^2a^2-b^2c^2 \quad , &&\vartheta_4^2\vartheta_{03}^2\ \ =d^2b^2-c^2a^2 \quad , &&\vartheta_{23}^2\vartheta_{14}^2\ \ =d^2c^2-a^2b^2;
\end{aligned}$$

hence the identity under consideration can be put into the form

$$t^4+x^4+y^4+z^4-\frac{d^4+a^4-b^4-c^4}{d^2a^2-b^2c^2}(t^2x^2+y^2z^2)-\frac{d^4+b^4-c^4-a^4}{d^2b^2-c^2a^2}(t^2y^2+z^2x^2)$$

$$-\frac{d^4+c^4-a^4-b^4}{d^2c^2-a^2b^2}(t^2z^2+x^2y^2)+2\frac{dabc\prod\limits^{\epsilon_1,\,\epsilon_2}[d^2+\epsilon_1a^2+\epsilon_2b^2+\epsilon_1\epsilon_2c^2]}{(d^2a^2-b^2c^2)(d^2b^2-c^2a^2)(d^2c^2-a^2b^2)}txyz=0,$$

where the $\prod\limits^{\epsilon_1,\,\epsilon_2}$ denotes the product of the four factors obtained by giving to each of ϵ_1, ϵ_2 both the values $+1$ and -1. The quartic surface represented by this equation can be immediately proved to have a node at each of the sixteen points which are obtainable from the four,

$$(d, a, b, c), \ (d, a, -b, -c), \ (d, -a, b, -c), \ (d, -a, -b, c),$$

by writing respectively, in place of d, a, b, c,

$$\text{(i) } (d, a, b, c), \ \text{(ii) } (a, d, c, b), \ \text{(iii) } (b, c, d, a), \ \text{(iv) } (c, b, a, d).$$

Ex. iv. We have in Ex. iii. obtained a relation connecting the functions

$$\vartheta_5(u), \ \vartheta_{34}(u), \ \vartheta_{12}(u), \ \vartheta_0(u);$$

in Ex. iv. § 221 we have obtained the corresponding relation connecting the functions

$$\vartheta_5(u), \ \vartheta_{01}(u), \ \vartheta_4(u), \ \vartheta_{23}(u);$$

and in Ex. i. § 221 we have explained how to obtain the corresponding relation connecting the functions

$$\vartheta_5(u), \ \vartheta_{23}(u), \ \vartheta_{04}(u), \ \vartheta_1(u).$$

There are* in fact sixty sets of four functions among which such a relation holds; and these sixty sets break up into fifteen lots each consisting of four sets of four functions, such that in every lot all the sixteen theta functions occur, and such that in every lot one of the sets of four consists wholly of even functions while each of the three other sets consists of two odd functions and two even functions. This can be seen as follows: using the letter notation for the sixteen functions, as in § 204, and the derived letter notation for the fifteen ratios of which the denominator is $\vartheta(u)$, as at the top of page 338, it is immediately obvious, as on page 338, that any four ratios of the form

$$1,\ q_{k,\,l},\ q_{k_1,\,l_1},\ q_{k_2},$$

in which the letters $k,\ l,\ k_1,\ l_1,\ k_2$ *constitute in some order the letters* $a_1,\ a_2,\ c,\ c_1,\ c_2$, are connected by a relation of the form in question. Now such a set of four ratios can be formed in fifteen ways; there are firstly six such sets in which all the ratios are even functions of u, obtainable from the set

$$1,\ q_c,\ q_{a_1,\,c_1},\ q_{a_2,\,c_2}$$

by permuting the three letters $c,\ c_1,\ c_2$ among themselves in all possible ways; and nextly nine such sets in which two of the ratios are odd functions, obtainable from the set

$$1,\ q_c,\ q_{a_1,\,a_2},\ q_{c_1,\,c_2}$$

by taking instead of the pair a_1a_2 each of the three pairs† $a_1a_2,\ aa_1,\ aa_2$, and instead of the pair c_1c_2 each of the three pairs $c_1c_2,\ cc_1,\ cc_2$. Since (§ 204) the letter notation for an odd function consists always of two a-s or two c-s, and for an even function consists of one a and one c, the number of odd and even functions will remain unaltered. Further from each of these fifteen sets we can obtain three other sets of four ratios by the addition of half-periods to the argument u, in such a way that all the sixteen theta functions enter into each lot of sets. The fifteen lots obtained may all be represented by the scheme

$$\begin{array}{cccc} 1, & \alpha, & \beta, & \alpha\beta \\ \alpha_1, & \alpha\alpha_1, & \beta\alpha_1, & \alpha\beta\alpha_1 \\ \beta_1, & \alpha\beta_1, & \beta\beta_1, & \alpha\beta\beta_1 \\ \alpha_1\beta_1, & \alpha\alpha_1\beta_1, & \beta\alpha_1\beta_1, & \alpha\beta\alpha_1\beta_1, \end{array}$$

where $1,\ \alpha,\ \beta,\ \alpha\beta$ denote the characteristics of one of the fifteen sets of four theta functions just described, such as $\vartheta(u),\ \vartheta_c(u),\ \vartheta_{a_1c_1}(u),\ \vartheta_{a_2c_2}(u)$, or $\vartheta(u),\ \vartheta_c(u),\ \vartheta_{a_1a_2}(u),\ \vartheta_{c_1c_2}(u)$, $\alpha\beta$ denoting the characteristic formed by the addition of the characteristics $\alpha,\ \beta$; and $\alpha_1,\ \beta_1$ denote any other two characteristics other than $\alpha,\ \beta$, or $\alpha\beta$, and such that $\alpha\beta$ is not the same characteristic as $\alpha_1\beta_1$. This scheme must contain all the sixteen theta functions; for any repetition (such as $\alpha=\beta\alpha_1\beta_1$, for example) would be inconsistent with the hypothesis as to the choice of $\alpha_1,\ \beta_1$ (would be equivalent to $\alpha\beta=\alpha_1\beta_1$). It is easily seen, by writing down a representative of the six schemes in which the first row consists wholly of even functions, and a representative of the nine schemes in which the first row contains two odd functions, that in every scheme there are three rows in which two odd functions occur‡.

Ex. v. There are cases in which the number of linearly connected theta functions, as given by the general theorem, is subject to further reduction. For instance, suppose we

* Borchardt, *Crelle*, LXXXIII. (1877), p. 237. Each of the sixty sets of four functions may be called a Göpel tetrad.

† The letter a, when it occurs in a suffix, is omitted.

‡ A table of the sixty sets of four theta functions is given by Krause, *Hyperelliptische Functionen* (Leipzig, 1886), p. 27.

have $m=2^{p-1}$ odd half-integer characteristics $A_1, \ldots, A_m$, and another half-integer characteristic P, not (integral or) zero, such that the characteristics* $A_1P, \ldots, A_mP$, obtained by adding P to each of $A_1, \ldots, A_m$, are also odd†; suppose further that A is an even half-integer characteristic, and that AP is also an even characteristic, and that the theta functions $\vartheta(u\,;\,A)$, $\vartheta(u\,;\,AP)$ do not vanish for zero values of the argument. Then, by § 288 the $2^{p-1}+1$ following theta functions of order 2,

$$\vartheta(u\,;\,A)\,\vartheta(u\,;\,AP),\ \vartheta(u\,;\,A_1)\,\vartheta(u\,;\,A_1P),\ \ldots,\ \vartheta(u\,;\,A_m)\,\vartheta(u\,;\,A_mP),$$

which are all even functions with a characteristic differing only by integers from the characteristic P, are connected by a linear equation with coefficients independent of u. But in fact, if we put $u=0$, all these functions vanish except the first. Hence we infer that the coefficient of the first function is zero, and that in fact the other 2^{p-1} functions are themselves connected by a linear equation.

Ex. vi. In illustration of the case considered in Ex. v. we take the following :—When $p=3$, it is possible‡, if P be any characteristic whatever, to determine six odd characteristics $A_1, \ldots, A_6$, whose sum is zero, such that the characteristics $A_1P, \ldots, A_6P$ are also odd, and such that all the combinations of three of these, denoted by $A_iA_jA_k$, $A_iA_jA_kP$, are even. By the previous example there exists an equation

$$\lambda\vartheta(u\,;\,A_4)\,\vartheta(u\,;\,A_4P)$$
$$=\lambda_1\vartheta(u\,;\,A_1)\,\vartheta(u\,;\,A_1P)+\lambda_2\vartheta(u\,;\,A_2)\,\vartheta(u\,;\,A_2P)+\lambda_3\vartheta(u\,;\,A_3)\,\vartheta(u\,;\,A_3P),$$

wherein $\lambda, \lambda_1, \lambda_2, \lambda_3$ are independent of u. Adding to u any half-period Ω_Q, this equation becomes

$$\lambda\vartheta(u\,;\,A_4Q)\,\vartheta(u\,;\,A_4PQ)$$
$$=\lambda_1\epsilon_1\vartheta(u\,;\,A_1Q)\,\vartheta(u\,;\,A_1PQ)+\lambda_2\epsilon_2\vartheta(u\,;\,A_2Q)\,\vartheta(u\,;\,A_2PQ)+\lambda_3\epsilon_3\vartheta(u\,;\,A_3Q)\,\vartheta(u\,;\,A_3PQ),$$

where $\epsilon_i\,(i=1,\ 2,\ 3)$ is a certain square root of unity depending on the characteristics A_4, A_i, P, Q, whose value is determined in the following example. Taking in particular for Ω_Q the half-period associated with the characteristic A_2A_3, so that the characteristics A_2PQ, A_3PQ become respectively the odd characteristics A_3P, A_2P, and putting $u=0$, we infer

$$\lambda\vartheta(0\,;\,A_4A_2A_3)\,\vartheta(0\,;\,A_4A_2A_3P)=\lambda_1\epsilon_1'\vartheta(0\,;\,A_1A_2A_3)\,\vartheta(0\,;\,A_1A_2A_3P),$$

where ϵ_1' is the particular value of ϵ_1 when Q is A_2A_3. This equation determines the ratio of λ_1 to λ; similarly the ratios $\lambda_2:\lambda$ and $\lambda_3:\lambda$ are determinable.

Ex. vii. If $\tfrac{1}{2}r$, $\tfrac{1}{2}q$ be half-integer characteristics whose elements are either 0 or $\tfrac{1}{2}$, and $\tfrac{1}{2}k=\tfrac{1}{2}rq$ be their reduced sum, with elements either 0 or $\tfrac{1}{2}$, prove§ that

$$k_a=r_a+q_a-2r_aq_a,\quad k_a'=r_a'+q_a'-2r_a'q_a',\qquad (a=1,\ 2,\ \ldots,\ p),$$

and thence, by the formulae (§ 190)

$$\vartheta(u+\Omega_P\,;\,q)=e^{\lambda(u\,;\,P)-2\pi iP'q}\,\vartheta(u\,;\,P+q),\quad \vartheta(u\,;\,q+M)=e^{2\pi iMq'}\,\vartheta(u\,;\,q),$$

* A characteristic formed by adding two characteristics A, P is denoted by $A+P$. Its *reduced* value, in which each of its elements is 0 or $\tfrac{1}{2}$, is denoted by AP.

† It is proved in chapter XVII. that, when $p>2$, the characteristic P may be arbitrarily taken, and the characteristics $A_1, \ldots, A_m$ thence determined in a finite number of ways.

‡ This is proved in chapter XVII.

§ Schottky, *Crelle*, CII. (1888), pp. 308, 318.

where M is integral, prove that

$$\vartheta(u+\tfrac{1}{2}\Omega_r;\ \tfrac{1}{2}q)=e^{\lambda(u;\ \frac{1}{2}r)+\pi i\sum_{\alpha=1}^{p}(r_\alpha q_\alpha q_\alpha'+q_\alpha r_\alpha r_\alpha')-\frac{1}{2}\pi i\sum_{\alpha=1}^{p}r_\alpha' q_\alpha}\,\vartheta(u;\ \tfrac{1}{2}rq).$$

If $\frac{1}{2}r$, $\frac{1}{2}a$, $\frac{1}{2}q$ be any reduced characteristics, infer that

$$\frac{\vartheta(u+\frac{1}{2}\Omega_r;\ \frac{1}{2}a)\,\vartheta(u+\frac{1}{2}\Omega_r;\ \frac{1}{2}aq)}{\vartheta(u+\frac{1}{2}\Omega_r;\ \frac{1}{2}q)}=e^{\lambda(u;\ \frac{1}{2}r)}\,\epsilon\,\frac{\vartheta(u;\ \frac{1}{2}ar)\,\vartheta(u;\ \frac{1}{2}aqr)}{\vartheta(u;\ \frac{1}{2}qr)},$$

where

$$\epsilon=e^{i\pi\sum_{\alpha=1}^{p}[r_\alpha q_\alpha a_\alpha'+(r_\alpha q_\alpha'+r_\alpha' q_\alpha+r_\alpha')\,a_\alpha]}.$$

Ex. viii. If A_1, A_2, A_3, A_4 denote four odd characteristics, for $p=2$, and B denote an even characteristic, the $\frac{1}{2}2^p+2^{p-1}+1=5$ theta functions, of order 2 and zero (or integral) characteristic, $\vartheta^2(u;\ B)$, $\vartheta^2(u;\ A_1)$, ..., $\vartheta^2(u;\ A_4)$ are, by § 288, connected by a linear equation. As in Ex. v. we hence infer an equation of the form

$$\lambda\vartheta^2(u;\ A_4)=\lambda_1\vartheta^2(u;\ A_1)+\lambda_2\vartheta^2(u;\ A_2)+\lambda_3\vartheta^2(u;\ A_3);$$

adding to u the half-period associated with the characteristic A_2A_3, and putting $u=0$, we deduce by Ex. vii. that

$$\lambda e^{i\pi k_1' a_4}\,\vartheta^2(0;\ A_4A_2A_3)=\lambda_1 e^{i\pi k_1' a_1}\,\vartheta^2(0;\ A_1A_2A_3),$$

where $A_2A_3=\frac{1}{2}k_1$, $A_1=\frac{1}{2}a_1$, $A_4=\frac{1}{2}a_4$. Hence we obtain an equation which we may write in the form

$$\vartheta^2(0;\ A_1A_2A_3)\,\vartheta^2(u;\ A_4)=\begin{pmatrix}A_2A_3\\A_1A_4\end{pmatrix}\vartheta^2(0;\ A_4A_2A_3)\,\vartheta^2(u;\ A_1)$$
$$+\begin{pmatrix}A_3A_1\\A_2A_4\end{pmatrix}\vartheta^2(0;\ A_4A_3A_1)\,\vartheta^2(u;\ A_2)+\begin{pmatrix}A_1A_2\\A_3A_4\end{pmatrix}\vartheta^2(0;\ A_4A_1A_2)\,\vartheta^2(u;\ A_3),$$

where $\begin{pmatrix}A_2A_3\\A_1A_4\end{pmatrix}$ denotes a certain square root of unity. Such a relation holds between every four of the odd theta functions.

If A_1, ..., A_6 be the odd characteristics, and Q be any other characteristic, the six characteristics A_1Q, ..., A_6Q are said to form a Rosenhain hexad. It follows that the squares of every four theta functions of the same hexad are connected by a linear relation.