PART II

THEORY OF INVARIANTS IN NON-SYMBOLIC NOTATION

15. Homogeneity of Invariants. We saw in § 11 that two binary quadratic forms f and f' have the invariants

$$d=ac-b^2, \qquad s=ac'+a'c-2bb'$$

of index 2. Note that s is of the first degree in the coefficients a, b, c of f and also of the first degree in the coefficients of f', and hence is homogeneous in the coefficients of each form separately. The latter is also true of d, but not of the invariant $s+2d$.

When an invariant of two or more forms is not homogeneous in the coefficients of each form separately, it is a sum of invariants each homogeneous in the coefficients of each form separately.

A proof may be made similar to that used in the following case. Grant merely that $s+2d$ is an invariant of index 2 of the binary quadratic forms f and f'. In the transformed forms (§ 11), the coefficients A, B, C of F are linear in a, b, c; the coefficients A', B', C' of F' are linear in a', b', c'. By hypothesis

$$AC'+A'C-2BB'+2(AC-B^2)=\Delta^2(s+2d).$$

The terms $2d\Delta^2$ of degree 2 in a, b, c on the right arise only from the part $2(AC-B^2)$ on the left. Hence d is itself an invariant of index 2; likewise s itself is an invariant.

However, *an invariant of a single form is always homogeneous.* For example, this is the case with the above discriminant d of f. We shall deduce this theorem from a more general one.

Let I be an invariant of r forms $f_1, \ldots, f_r$ of orders $p_1, \ldots, p_r$ in the same q variables $x_1, \ldots, x_q$. Let a particular term t of I be of degree d_1 in the coefficients of f_1, of degree d_2 in the coefficients of f_2, etc. Apply the special transformation

$$x_1=\alpha\xi_1, \qquad x_2=\alpha\xi_2, \ldots, \qquad x_q=\alpha\xi_q,$$

of determinant $\Delta=\alpha^q$. Then f_i is transformed into a form whose coefficients are the products of those of f_i by α^{p_i}. Hence in the function I of the transformed coefficients, the term corresponding to t equals the product of t by

$$(\alpha^{p_1})^{d_1} \ldots (\alpha^{p_r})^{d_r}=\alpha^{\Sigma d_i p_i}.$$

This factor therefore equals Δ^λ, if λ is the index of the invariant. Thus

$$\sum_{i=1}^{r} d_i p_i=\lambda q.$$

Hence $\Sigma d_i p_i$ is constant for all the terms of the invariant.

For the above two quadratic forms, $r=p_1=p_2=2$. For invariant d, we have $d_1=2$, $d_2=0$, $\Sigma d_i p_i=4=2\lambda$. For s, we have $d_1=d_2=1$, $\Sigma d_i p_i=4$. Again, the discriminant (§ 8) of the binary cubic form is of constant degree 4 and index $\lambda=6$; we have $\Sigma d_i p_i=4\cdot 3=2\lambda$.

If, as in the last example, we take $r=1$, we see that an invariant of index λ of a single q-ary form of order p is of constant degree d, where $dp=\lambda q$, and hence is homogeneous.

16. Weight of an Invariant I of a Binary Form f. Give to I and f the notations in § 7. Let

$$t=ca_0^{e_0}a_1^{e_1} \ldots a_p^{e_p}$$

be any term of I, and call

$$w=e_1+2e_2+3e_3+\ldots+pe_p$$

the *weight* of t. Thus w is the sum of the subscripts of the factors a_i each repeated as often as its exponent indicates. We shall prove that *the various terms of an invariant of a binary form are of constant weight*, and hence call the invariant *isobaric*. For example, $a_0x^2+2a_1xy+a_2y^2$ has the invariant $a_0a_2-a_1^2$, each of whose terms is of weight 2.

To prove the theorem, apply to f the transformation

$$x=\xi, \qquad y=\alpha\eta.$$

We obtain a form with the literal coefficients

$$A_0=a_0, \qquad A_1=a_1\alpha, \qquad A_2=a_2\alpha^2, \ . \ . \ . \ , \ A_p=a_p\alpha^p.$$

Hence if I is of index λ,

$$I(a_0, a_1\alpha, \ . \ . \ . \ , a_p\alpha^p)\equiv\alpha^\lambda I(a_0, a_1, \ . \ . \ . \ , a_p),$$

identically in α and the a's. The term of the left member which corresponds to the above term t of I is evidently

$$c_0a_0^{e_0} \ . \ . \ . \ a_p^{e_p}\alpha^w.$$

Hence $w=\lambda$. The weight of an invariant of degree d of a binary p-ic is thus its index and hence (§ 15) equals $\frac{1}{2}dp$.

17. Weight of an Invariant of any System of Forms. Let $f_1, \ . \ . \ . \ , f_n$ be forms in the same variables $x_1, \ . \ . \ . \ , x_q$. We define the weight of the coefficient of any term of f_i to be the exponent of x_q in that term, and the weight of a product of coefficients to be the sum of the weights of the factors. For $q=2$, this definition is in accord with that in § 16, where the coefficient a_k of $x_1^{p-k}x_2^k$ was taken to be of weight k. Again, in a ternary quadratic form, the coefficients of x_1^2, x_1x_2 and x_2^2 are of weight zero, those of x_1x_3 and x_2x_3 of weight unity, and that of x_3^2 of weight 2.

Under the transformation of determinant α,

$$x_1=\xi_1, \qquad . \ . \ . \ , \qquad x_{q-1}=\xi_{q-1}, \qquad x_q=\alpha\xi_q,$$

f_i becomes a form in which the coefficient c' corresponding to a coefficient c of weight k in f_i is $c\alpha^k$. If I is an invariant, $I(c')\equiv\alpha^\lambda I(c)$, identically in α. Hence every term of I is of weight λ.

Thus *any invariant of a single form is isobaric; any invariant of a system of two or more forms is isobaric on the whole, but not necessarily isobaric in the coefficients of each form separately.*

The index equals the weight and is therefore an integer $\geqq 0$.

EXERCISES

1. The invariant $a_0a'_2+a_2a'_0-2a_1a'_1$ of

$$a_0x^2+2a_1xy+a_2y^2, \qquad a'_0x^2+2a'_1xy+a'_2y^2$$

is of total weight 2, but is not of constant weight in a_0, a_1, a_2 alone.

2. Verify the theorem for the Jacobian of two binary linear forms.
3. Verify the theorem for the Hessian of a ternary quadratic form.
4. No binary form of odd order p has an invariant of odd degree d.

18. Products of Linear Transformations. The *product* TT' of

$$T: \qquad x=\alpha\xi+\beta\eta, \qquad y=\gamma\xi+\delta\eta, \qquad \Delta=\begin{vmatrix}\alpha & \beta\\ \gamma & \delta\end{vmatrix}\neq 0,$$

$$T': \qquad \xi=\alpha'X+\beta'Y, \quad \eta=\gamma'X+\delta'Y, \quad \Delta'=\begin{vmatrix}\alpha' & \beta'\\ \gamma' & \delta'\end{vmatrix}\neq 0,$$

is defined to be the transformation whose equations are obtained by eliminating ξ and η between the equations of the given transformations. Hence

$$TT': \quad \begin{cases} x=\alpha''X+\beta''Y, \quad y=\gamma''X+\delta''Y,\\ \alpha''=\alpha\alpha'+\beta\gamma', \beta''=\alpha\beta'+\beta\delta', \gamma''=\gamma\alpha'+\delta\gamma', \delta''=\gamma\beta'+\delta\delta'.\end{cases}$$

Its determinant is seen to equal $\Delta\Delta'$ and hence is not zero.

By solving the equations which define T, we get

$$\xi=\frac{\delta}{\Delta}x-\frac{\beta}{\Delta}y, \qquad \eta=\frac{-\gamma}{\Delta}x+\frac{\alpha}{\Delta}y.$$

These equations define the transformation T^{-1} *inverse* to T; each of the products TT^{-1} and $T^{-1}T$ is the *identity* transformation $x=X$, $y=Y$.

The product of transformation T_θ, defined in § 1, by $T_{\theta'}$ is seen to equal $T_{\theta+\theta'}$, in accord with the interpretation given there. The inverse of T_θ is

$$T_{-\theta}: \quad \xi=x\cos\theta+y\sin\theta, \quad \eta=-x\sin\theta+y\cos\theta.$$

Consider also any third linear transformation

$$T_1: \qquad X=\alpha_1U+\beta_1V, \qquad Y=\gamma_1U+\delta_1V.$$

To prove that the associative law

$$(TT')T_1=T(T'T_1)$$

holds, note that the first product is found by eliminating first ξ, η and then X, Y between the equations for T, T', T_1, while the second product is obtained by eliminating first X, Y and then ξ, η between the same equations. Thus the final eliminants must be the same in the two cases.

Hence we may write $TT'T_1$ for either product.

19. Generators of All Binary Linear Transformations. *Every binary linear homogeneous transformation is a product of the transformations*

$$\begin{array}{lll} T_n: & x=\xi+n\eta, & y=\eta; \\ S_k: & x=\xi, & y=k\eta \quad (k\neq 0); \\ V: & x=-\eta, & y=\xi. \end{array}$$

From these we obtain *

$$\begin{array}{lll} V^{-1}=V^3: & x=\eta, & y=-\xi; \\ V^{-1}T_{-n}V=T'_n: & x=x', & y=y'+nx'; \\ V^{-1}S_kV \;=S'_k: & x=kx', & y=y' \quad (k\neq 0). \end{array}$$

For $\delta\neq 0$, the transformation T in § 18 equals the product

$$S_\delta S'_{\Delta/\delta}T_{\beta\delta/\Delta}T'_{\gamma/\delta}.$$

For $\delta=0$, so that $\beta\gamma\neq 0$, T equals

$$S_\gamma S'_{-\beta}T_{-\alpha/\beta}V.$$

20. Annihilator of an Invariant of a Binary Form. The binary form in § 7 may be written as either of the sums

$$f=\sum_{i=0}^{p}\binom{p}{i}a_ix^{p-i}y^i=\sum_{i=0}^{p}\binom{p}{i}a_{p-i}x^iy^{p-i}.$$

Transformation V, of determinant unity, replaces the second sum by

$$\sum_{i=0}^{p}\binom{p}{i}a_{p-i}(-1)^i\xi^{p-i}\eta^i.$$

Comparing this with the first sum we see that an invariant of f must be unaltered when

(1) $\qquad a_i$ is replaced by $(-1)^ia_{p-i}\quad (i=0, 1, \ldots, p)$.

* The T's are of the nature of translations, and the S's stretchings.

By § 16, a function $I(a_0, \ldots, a_p)$ is invariant with respect to every transformation S_k if and only if it is isobaric.

Finally, the function must be invariant with respect to every T_n; under this transformation let

$$f=\sum_{i=0}^{p}\binom{p}{i} A_i\xi^{p-i}\eta^i.$$

Differentiating partially with respect to n, we get

$$0=\sum_{i=0}^{p}\binom{p}{i}\left\{\frac{\partial A_i}{\partial n}\xi^{p-i}\eta^i-A_i(p-i)\,\xi^{p-i-1}\,\eta^{i+1}\right\},$$

since $\eta=y$ is free of n, while $\xi=x-n\eta$. The total coefficient of $\xi^{p-j}\eta^j$ is

$$\binom{p}{j}\frac{\partial A_j}{\partial n}-\binom{p}{j-1}(p-j+1)A_{j-1}=0,$$

the second term being absent if $j=0$. But

$$\binom{p}{j}=\binom{p}{j-1}\frac{(p-j+1)}{j}.$$

Hence

$$\frac{\partial A_0}{\partial n}=0,\qquad \frac{\partial A_j}{\partial n}=jA_{j-1}\qquad (j=1,\ldots,p),$$

$$(2)\quad \frac{\partial I(A_0,\ldots,A_p)}{\partial n}=A_0\frac{\partial I}{\partial A_1}+2A_1\frac{\partial I}{\partial A_2}+3A_2\frac{\partial I}{\partial A_3}+\ldots+pA_{p-1}\frac{\partial I}{\partial A_p}.$$

Now $I(a_0, \ldots, a_p)$ is invariant with respect to every transformation T_n, of determinant unity, if and only if

$$I(A_0, \ldots, A_p)\equiv I(a_0, \ldots, a_p),$$

identically in n and the a's. This relation evidently implies

$$\frac{\partial I(A_0, \ldots, A_p)}{\partial n}\equiv 0.$$

Conversely, the latter implies that $I(A_0, \ldots, A_p)$ has the same value for all values of n and hence its value is that given by $n=0$, viz., $I(a_0, \ldots, a_p)$. Hence I has the desired property if and only if the right member of (2) is zero identically in n and the a's. But this is the case if and only if

$$\Omega I(a_0, \ldots, a_p)\equiv 0,$$

identically in the a's, where Ω is the differential operator

$$\Omega=a_0\frac{\partial}{\partial a_1}+2a_1\frac{\partial}{\partial a_2}+3a_2\frac{\partial}{\partial a_3}+\ldots+pa_{p-1}\frac{\partial}{\partial a_p}.$$

In other words, I must satisfy the partial differential equation $\Omega I=0$. In Sylvester's phraseology, I must be *annihilated* by the operator Ω.

From this section and the preceding we have the important

THEOREM. *A rational integral function I of the coefficients of the binary form f is an invariant of f if and only if I is isobaric, is unaltered by the replacement* (1), *and is annihilated by Ω.*

EXAMPLE

An invariant of degree d of the binary quartic (§ 6) is of weight $2d$ (end of § 16). For $d=1$, the only possible term is ka_2; since $0=\Omega(ka_2)=2ka_1$, we have $k=0$. For $d=2$, we have

$$I=ra_0a_4+sa_1a_3+ta_2^2,$$

$$\Omega I=(s+4r)a_0a_3+(4t+3s)a_1a_2\equiv 0,$$

$$s=-4r,\ t=3r,\ I=r(a_0a_4-4a_1a_3+3a_2^2).$$

EXERCISES

1. Every invariant of degree 3 of the binary quartic is the product of a constant by

$$J=a_0a_2a_4+2a_1a_2a_3-a_0a_3^2-a_1^2a_4-a_2^3.$$

2. The invariant of lowest degree of the binary cubic

$$a_0x^3+3a_1x^2y+3a_2xy^2+a_3y^3$$

is its discriminant $(a_0a_3-a_1a_2)^2-4(a_0a_2-a_1^2)(a_1a_3-a_2^2)$.

3. An invariant of two or more binary forms

$$a_0x^{p_1}+\ldots,\quad b_0x^{p_2}+\ldots,\quad c_0x^{p_3}+\ldots$$

is annihilated by the operator

$$\Sigma\Omega\equiv a_0\frac{\partial}{\partial a_1}+2a_1\frac{\partial}{\partial a_2}+\ldots+b_0\frac{\partial}{\partial b_1}+2b_1\frac{\partial}{\partial b_2}+\ldots+c_0\frac{\partial}{\partial c_1}+\ldots.$$

4. Every invariant of

$$a_0x^2+2a_1xy+a_2y^2,\quad b_0x^2+2b_1xy+b_2y^2$$

of the first degree in the a's and first degree in the b's is a multiple of $a_0b_2+a_2b_0-2a_1b_1$.

5. A binary quadratic and quartic have no such lineo-linear invariant.

6. Find the invariant of partial degrees 2, 1 of a binary linear and a quadratic form.

7. Find the invariant of partial degrees 1, 2 of a binary quadratic and a cubic form.

8. The first two properties in the theorem of § 20 imply that I is homogeneous. For, under replacement (1), any term $ca_0^{e_0}\ldots a_p^{e_p}$ of I, of weight $w=e_1+2e_2+\ldots+pe_p$, implies a term $\pm ca_0^{e_p}a_1^{e_{p-1}}\ldots a_p^{e_0}$ of weight $w=e_{p-1}+2e_{p-2}+\ldots+(p-1)e_1+pe_0$. Adding the two expressions for w, show that the degree $d=e_0+e_1+\ldots+e_p$ is the constant $2w/p$.

21. Homogeneity of Covariants. *A covariant which is not homogeneous in the variables is a sum of covariants each homogeneous in the variables.*

For, if $a, b, \ldots$ are the coefficients of the forms, and K is a covariant,

$$K(A, B, \ldots;\ \xi, \eta, \ldots)=\Delta^\lambda K(a, b, \ldots;\ x, y, \ldots).$$

When $x, y, \ldots$ are replaced by their linear expressions in $\xi, \eta, \ldots$, the terms of order ω in $x, y, \ldots$ on the right (and only such terms) give rise to terms of order ω in $\xi, \eta, \ldots$ on the left. Hence, if K_1 is the sum of all of the terms of order ω of K,

$$K_1(A, B, \ldots;\ \xi, \eta, \ldots)=\Delta^\lambda K_1(a, b, \ldots;\ x, y, \ldots),$$

and K_1 is a covariant. In this way, $K=K_1+K_2+\ldots$.

Henceforth, we shall restrict attention to covariants which are homogeneous in the variables, and hence of constant *order*.

A covariant K of constant order ω of a single form f is homogeneous in the coefficients, and hence of constant degree d.

For, let f have the coefficients $a, b, \ldots$ and order p, and apply the transformation $x=\alpha\xi$, $y=\alpha\eta, \ldots$. The coefficients of the resulting form are $A=\alpha^p a$, $B=\alpha^p b, \ldots$. Thus

$$K(\alpha^p a, \alpha^p b, \ldots;\ \alpha^{-1}x, \alpha^{-1}y, \ldots)\equiv(\alpha^q)^\lambda K(a, b, \ldots;\ x, y, \ldots),$$

identically in $\alpha, a, b, \ldots, x, y, \ldots$, since the left member equals $K(A, B, \ldots; \xi, \eta, \ldots)$. Now K is homogeneous in $x, y, \ldots$, of order ω; thus

$$\alpha^{-\omega}K(\alpha^p a, \alpha^p b, \ldots; x, y, \ldots) \equiv \alpha^{q\lambda}K(a, b, \ldots; x, y, \ldots).$$

Thus if K has a term of degree d in $a, b, \ldots$, then

$$\alpha^{-\omega}\cdot\alpha^{pd} = \alpha^{q\lambda}, \qquad pd - \omega = q\lambda,$$

so that d is the same for all terms of K.

If f is a form of order p in q variables and if K is a covariant of degree d, order ω and index λ, then $pd - \omega = q\lambda$.

22. Weight of a Covariant of a Binary Form. In

$$f = a_0x^p + pa_1x^{p-1}y + \ldots + \binom{p}{i}a_ix^{p-i}y^i + \ldots + a_py^p$$

the weight of a_k is k. We now attribute the weight 1 to x and the weight 0 to y, so that every term of f is of total weight p.

Apply to f the transformation $x = \xi$, $y = \alpha\eta$. The literal coefficients of the resulting form are

$$A_0 = a_0, \qquad A_1 = \alpha a_1, \quad \ldots, \quad A_p = \alpha^p a_p.$$

If K is a covariant of degree d, order ω, and index λ, then

$$K(A_0, \ldots, A_p; \xi, \eta) = \alpha^\lambda K(a_0, \ldots, a_p; x, y).$$

Any term on the left is of the form

$$cA_0^{e_0}A_1^{e_1}\ldots A_p^{e_p}\xi^r\eta^{\omega-r} \qquad (e_0+e_1+\ldots+e_p=d).$$

This equals

$$ca_0^{e_0}a_1^{e_1}\ldots a_p^{e_p}x^ry^{\omega-r}\alpha^{W-\omega} \qquad (W = r+e_1+2e_2+\ldots+pe_p).$$

This must equal a term of the right member, so that $W - \omega = \lambda$. But W is the total weight of that term. Hence every term of K is of the same total weight. *A covariant of index λ and order ω of a binary form is isobaric and its weight is $\omega + \lambda$.*

For a form f of order p in q variables, we attribute the weight 1 to $x_1, x_2, \ldots, x_{q-1}$ and the weight 0 to x_q; then (§ 17) every term of f is of total weight p. By a proof similar to the above, a covariant of index λ and order ω of f is isobaric and its weight is $\omega + \lambda$.

Consider a covariant K homogeneous and of total order ω in the variables $x_1, \ldots, x_q$ of two or more forms f_i. As in § 15, K need not be homogeneous in the coefficients of each form separately, but is a sum of covariants homogeneous in the coefficients of each. Let such a K be of degree d_i in the coefficients of f_i, of order p_i. As in § 21, $\Sigma p_i d_i - \omega = q\lambda$. The total weight of K is $\omega + \lambda$.

For example, if $p_1 = p_2 = q = 2$,

$$f_1 = a_0 x^2 + 2a_1 xy + a_2 y^2, \quad f_2 = b_0 x^2 + 2b_1 xy + b_2 y^2.$$

The Jacobian of f_1 and f_2 is $4K$, where

$$K = (a_0 b_1 - a_1 b_0)x^2 + (a_0 b_2 - a_2 b_0)xy + (a_1 b_2 - a_2 b_1)y^2.$$

Here

$$d_1 = d_2 = 1, \ \omega = 2, \ \lambda = 1, \text{ and } K \text{ is of weight } 3.$$

23. Annihilators of Covariants K of a Binary Form. Proceeding as in § 20, we have instead of (2)

$$\frac{\partial}{\partial n} K(A_0, \ldots, A_p;\ \xi, \eta) = \sum_{j=0}^{p} \frac{\partial K}{\partial A_j}\frac{\partial A_j}{\partial n} + \frac{\partial K}{\partial \xi}\frac{\partial \xi}{\partial n} + \frac{\partial K}{\partial \eta}\frac{\partial \eta}{\partial n}$$

$$= \sum_{j=1}^{p} jA_{j-1}\frac{\partial K}{\partial A_j} - \eta\frac{\partial K}{\partial \xi},$$

and obtain the following result: K is covariant with respect to every transformation $x = \xi + n\eta$, $y = \eta$, if and only if it is annihilated by *

$$(1) \qquad \Omega - y\frac{\partial}{\partial x} \qquad \left(\Omega = a_0\frac{\partial}{\partial a_1} + \ldots + pa_{p-1}\frac{\partial}{\partial a_p}\right).$$

The binary form is unaltered if we interchange x and y, a_i and a_{p-i} for $i = 0, 1, \ldots, p$. Hence K is covariant with respect to every transformation $x = \xi$, $y = \eta + n\xi$, if and only if it is annihilated by

$$(2) \quad O - x\frac{\partial}{\partial y} \qquad \left(O \equiv pa_1\frac{\partial}{\partial a_0} + (p-1)a_2\frac{\partial}{\partial a_1} + \ldots + a_p\frac{\partial}{\partial a_{p-1}}\right).$$

Denote a covariant of order ω of the binary p-ic by

$$K = Sx^\omega + S_1 x^{\omega-1}y + \ldots + S_\omega y^\omega.$$

* For another derivation, see the corollary in § 47.

By operating on K by (2), we must have

$$(OS-S_1)x^\omega+(OS_1-2S_2)x^{\omega-1}y+\ldots+(OS_{\omega-1}-\omega S_\omega)xy^{\omega-1}+OS_\omega y^\omega\equiv 0,$$

identically in x, y. Hence K becomes

$$(3)\quad K=Sx^\omega+OSx^{\omega-1}y+\tfrac{1}{2}O^2Sx^{\omega-2}y^2+\ldots+\frac{1}{\omega!}O^\omega Sy^\omega,$$

while, by $OS_\omega=0$,

$$(4)\qquad O^{\omega+1}S=0.$$

Hence a covariant is uniquely determined by its *leader* S. (Cf. § 25).

Similarly, K is annihilated by (1) if and only if

$$(5)\quad \Omega S=0,\quad \Omega S_1=\omega S,\quad \Omega S_2=(\omega-1)S_1,\quad \ldots,\quad \Omega S_\omega=S_{\omega-1}.$$

The function S of $a_0, \ldots, a_p$ must be homogeneous and isobaric (§§ 21, 22). If such a function S is annihilated by Ω, it is called a *seminvariant.* If we have S_ω, we may find $S_{\omega-1}$ by (5), then $S_{\omega-2}, \ldots$, and finally S_1. But if K is a covariant, we can derive S_ω from S. For, by § 20, the transformation $x=-\eta$, $y=\xi$ replaces f by a form in which $A_i=(-1)^i a_{p-i}$; by the covariance of K,

$$S(A)\xi^\omega+\ldots=S(A)y^\omega+\ldots\equiv S(a)x^\omega+\ldots+S_\omega(a)y^\omega,$$

so that $S_\omega(a)=S(A)$. Hence S_ω is derived from S by the replacement (1) in § 20.

When the seminvariant leader S is given, and hence also ω (see Ex. 1), the function (3) is actually a covariant of f; likewise the function whose coefficients are given by (5). Proof will be made in § 25. In the following exercises, indirect verification of the covariance is indicated.

EXERCISES

1. The weight of the leader S of a covariant of order ω of a binary form f is $W-\omega=\lambda$ and hence (§ 21) is $\frac{1}{2}(pd-\omega)$. Thus S and f determine ω.

2. The binary cubic has the seminvariant $S=a_0a_2-a_1^2$. A covariant with S as leader of is order $\omega=2$ and is

$$(a_0a_2-a_1^2)x^2+(a_0a_3-a_1a_2)xy+(a_1a_3-a_2^2)y^2.$$

Since this is the Hessian of the cubic, it is a covariant.

3. Find the covariant of the binary cubic f whose leader is $a_0^2a_3-3a_0a_1a_2+2a_1^3$, the only seminvariant of weight 3 and degree 3. It is the Jacobian of f and its Hessian.

4. A covariant of two or more binary forms is annihilated by

$$\Sigma\Omega-y\frac{\partial}{\partial x}, \qquad \Sigma O-x\frac{\partial}{\partial y}.$$

5. Find a seminvariant of weight 2 and partial degrees 1, 1 of a binary quadratic and cubic. Show that it is the leader of the covariant

$$(a_0b_2-2a_1b_1+a_2b_0)x+(a_0b_3-2a_1b_2+a_2b_1)y.$$

24. Alternants. Consider the annihilators

$$\Omega=\sum_{j=1}^{p} ja_{j-1}\frac{\partial}{\partial a_j}=\sum_{k=0}^{p-1}(k+1)a_k\frac{\partial}{\partial a_{k+1}},$$

$$O=\sum_{j=1}^{p}(p-j+1)a_j\frac{\partial}{\partial a_{j-1}}=\sum_{k=0}^{p-1}(p-k)a_{k+1}\frac{\partial}{\partial a_k}$$

of invariants of a binary form. We have

$$\Omega O=\sum_{j=1}^{p} ja_{j-1}\left\{(p-j+1)\frac{\partial}{\partial a_{j-1}}+\sum_{k=0}^{p-1}(p-k)a_{k+1}\frac{\partial^2}{\partial a_j\partial a_k}\right\},$$

$$O\Omega=\sum_{k=0}^{p-1}(p-k)a_{k+1}\left\{(k+1)\frac{\partial}{\partial a_{k+1}}+\sum_{j=1}^{p} ja_{j-1}\frac{\partial^2}{\partial a_k\partial a_j}\right\}.$$

The terms involving second derivatives are identical. Hence

$$\Omega O-O\Omega=\sum_{i=0}^{p-1}(i+1)(p-i)a_i\frac{\partial}{\partial a_i}-\sum_{i=1}^{p} i(p-i+1)a_i\frac{\partial}{\partial a_i}$$

$$=\sum_{i=0}^{p}(p-2i)a_i\frac{\partial}{\partial a_i},$$

since the first sum is the first sum in ΩO with j replaced by $i+1$, and the second is the first sum in $O\Omega$ with k replaced by $i-1$.

If S is a homogeneous function of $a_0, \ldots, a_p$ of total degree d and hence a sum of terms

$$ca_0^{e_0}a_1^{e_1}\ldots a_p^{e_p} \qquad (e_0+e_1+\ldots+e_p=d),$$

we readily verify Euler's theorem:

$$\sum_{i=0}^{p}\frac{\partial S}{\partial a_i}=dS.$$

If S is isobaric, it is a sum of terms

$$t=ca_0^{e_0}a_1^{e_1}\ .\ .\ .\ a_p^{e_p}\qquad (e_1+2e_2+.\ .\ .+pe_p=w)$$

where w is constant; then

$$\sum_{i=0}^{p} ia_i\frac{\partial t}{\partial a_i}=\sum_{i=0}^{p} ie_it=wt,\qquad \sum_{i=0}^{p} ia_i\frac{\partial S}{\partial a_i}=wS.$$

Hence *if S is both homogeneous (of degree d) and isobaric (of weight w) in $a_0, \ldots, a_p$, then*

(1) $$(\Omega O-O\Omega)S=\omega S,\qquad \omega=pd-2w.$$

A covariant with the leader S has the order ω. (Ex. 1, § 23.)

Since OS is of degree d and weight $w+1$, we have

$$\begin{aligned}(\Omega O^2-O^2\Omega)S&\equiv(\Omega O-O\Omega)OS+O(\Omega O-O\Omega)S\\ &=(\omega-2)OS+\omega OS=2(\omega-1)OS.\end{aligned}$$

Hence for $r=1$ and $r=2$, we have

(2) $$(\Omega O^r-O^r\Omega)S=r(\omega-r+1)O^{r-1}S.$$

To proceed by induction, note that (2) implies

$$\begin{aligned}(\Omega O^{r+1}-O^{r+1}\Omega)S&\equiv(\Omega O^r-O^r\Omega)OS+O^r(\Omega O-O\Omega)S\\ &=r(\omega-2-r+1)O^rS+\omega O^rS=(r+1)(\omega-r)O^rS,\end{aligned}$$

so that (2) holds also when r is replaced by $r+1$.

25. Seminvariants as Leaders of Binary Covariants.

LEMMA. *If S is a seminvariant, not identically zero, of degree d and weight w, of a binary p-ic, then $dp-2w\geqq 0$.*

Suppose on the contrary that S is a seminvariant for which $\omega<0$, where $\omega=dp-2w$. By the definition of a seminvariant, $\Omega S=0$. Hence, by (2), § 24,

(1) $$\Omega O^rS=r(\omega-r+1)O^{r-1}S\qquad (r=1,2,3,\ldots)$$

and no one of the coefficients on the right is zero. But

$$O^{dp-w+1}S\equiv 0,$$

being of degree d and weight $dp+1$; in fact, the largest weight of a function of $a_0, \ldots, a_p$ of degree d is dp, the weight of a_p^d. Then (1) for $r=dp-w+1$ gives $O^{dp-w}S=0$. Then (1) for $r=dp-w$ gives $O^{dp-w-1}S=0$, etc. Finally, we get $S=0$, contrary to hypothesis.

THEOREM. *There exists a covariant K of a binary p-ic whose leader is any given seminvariant S of the p-ic.*

The covariant K is in fact given by (3), § 23. By (1), for $r=\omega+1$,

$$\Omega O^{\omega+1}S=0.$$

Hence $O^{\omega+1}S$ is a seminvariant of degree d and weight

$$w' \equiv w+\omega+1=pd-w+1.$$

Then $dp-2w'=-(pd-2w)-2$ is negative. Hence (4), § 23, follows from the Lemma. Thus K is annihilated by the operator (2), § 23. Next, in

$$\left(\Omega-y\frac{\partial}{\partial x}\right)K,$$

the coefficient of $x^{\omega-r}y^r$ is

$$\frac{1}{r!}\Omega O^r S-\frac{1}{(r-1)!}(\omega-r+1)O^{r-1}S\equiv\frac{1}{r!}\{\Omega O^r S-r(\omega-r+1)O^{r-1}_{i}S\},$$

which is zero by (1). Hence K is covariant with respect to all of the transformations T_n and T'_n of § 19. Now

$$T_{-1}T'_1T_{-1}=V: \qquad x=-Y, \qquad y=X,$$

as shown by eliminating ξ, η, ξ_1, η_1 between

$$\begin{cases} x=\xi-\eta, \\ y=\quad\ \eta, \end{cases} \qquad \begin{cases} \xi=\qquad \xi_1, \\ \eta=\eta_1+\xi_1, \end{cases} \qquad \begin{cases} \xi_1=X-Y, \\ \eta_1=\qquad Y. \end{cases}$$

Since K is of constant weight, it is covariant with respect to every S_k (§ 16). Hence, by § 19, K is covariant with respect to all binary linear transformations.

26. Number of Linearly Independent Seminvariants.

LEMMA. *Given any homogeneous isobaric function S of $a_0, \ldots, a_p$ of degree d and weight w, where $\omega\equiv dp-2w>0$, we can find a homogeneous isobaric function S_1 of degree d and weight $w+1$ such that $\Omega S_1\equiv S$.*

In (2), § 24, replace S by $\Omega^{r-1}S$, whose degree is d and weight is $w-r+1$, so that its ω is $\omega+2r-2$. We get

$$\Omega O^r\Omega^{r-1}S-O^r\Omega^r S=r(\omega+r-1)O^{r-1}\Omega^{r-1}S.$$

Multiply this by

$$(-1)^{r-1}\frac{1}{r!\omega(\omega+1)\ .\ .\ .\ (\omega+r-1)}.$$

The new right member cancels the second term of the new left member after r is replaced by $r-1$ in the latter. Hence if we sum from $r=1$ to $r=w+1$, the terms not cancelling are those from the first terms of the left members, that from the right member for $r=1$, and that from the second term on the left for $r=w+1$. But the last is zero, since $\Omega^{w+1}S\equiv 0$, $\Omega^w S$ being of weight zero and hence a power of a_0. Hence we get $\Omega S_1\equiv S$, where

$$S_1\equiv \sum_{r=1}^{w+1}\frac{(-1)^{r-1}}{r!\omega(\omega+1)\ .\ .\ .\ (\omega+r-1)}O^r\Omega^{r-1}S.$$

THEOREM.* *The number of linearly independent seminvariants of degree d and weight w of the binary p-ic is zero if $pd-2w<0$, but is*

$$(w;\ d,\ p)-(w-1;\ d,\ p),$$

if $pd-2w\geqq 0$, where $(w;\ d,\ p)$ denotes the number of partitions of w into d integers chosen from $0, 1, \ldots, p$, with repetitions allowed.

If $p\geqq 4$, $(4;\ 2,\ p)=3$, since $4+0$, $3+1$, $2+2$ are the partitions of 4 into 2 integers. Also, $(3;\ 2,\ p)=2$, corresponding to $3+0$, $2+1$. Hence the theorem states that every seminvariant of degree 2 and weight 4 of the binary p-ic, $p\geqq 4$, is a numerical multiple of one such (see the Example in § 20).

The literal part of any term of a seminvariant S specified in the theorem is a product of d factors chosen from a_0, a_1, $\ldots$, a_p, with repetitions allowed, such that the sum of the subscripts of the d factors is w. Hence there are $(w;\ d,\ p)$ possible terms. Giving them arbitrary coefficients and operating on the sum of the resulting terms with Ω, we obtain a linear combination S' of the $(w-1;\ d,\ p)$ possible products

* Stated by Cayley; proved much later by Sylvester.

of degree d and weight $w-1$. By the Lemma there exists * an S for which ΩS is any assigned S'. Thus the coefficients of our $S' \equiv \Omega S$ are arbitrary and hence are linearly independent functions of the $(w; d, p)$ coefficients of S. Hence the condition $\Omega S \equiv 0$ imposes $(w-1; d, p)$ linearly independent linear relations between the coefficients of S and hence determines $(w-1; d, p)$ of the coefficients of S in terms of the remaining coefficients. Thus the difference gives the number of arbitrary constants in the general seminvariant S, and hence the number of linearly independent seminvariants S.

27. Hermite's Law of Reciprocity. Consider any partition

$$w = n_1 + n_2 + \ldots + n_\delta$$

of w into $\delta \leqq d$ positive integers such that $p \geqq n_1 \geqq n_2 \quad \ldots \quad \geqq n_\delta$. Write n_1 dots in a row; then in a second row write n_2 dots under the first n_2 dots of the first row; then in a third row write n_3 dots under the first n_3 dots of the second row, etc., until w dots have been written in δ rows.

Now count the dots by columns instead of by rows. The number m_1 of dots in the first (left-hand) column is δ; the number m_2 in the second column is $\leqq m_1$; etc. The number of columns is $n_1 \leqq p$. Hence we have a partition

$$w = m_1 + m_2 + \ldots + m_\pi$$

of w into $\pi \leqq p$ positive integers not exceeding d.

Hence to every one of the $(w; d, p)$ partitions of the first kind corresponds a unique one of the $(w; p, d)$ partitions of the second kind. The converse is true, since we may begin with an arrangement in columns and read off an arrangement by rows. The correspondence is thus one-to-one. Hence $(w; d, p) = (w; p, d)$.

By two applications of this result, we get

$$(w; d, p) - (w-1; d, p) = (w; p, d) - (w-1; p, d).$$

Hence, by the theorem of § 26, *the number of linearly independent*

* Provided $pd - 2(w-1) > 0$, which holds if $pd - 2w \geqq 0$. But if $pd - 2w < 0$, our theorem is true by the Lemma in § 25.

seminvariants of weight w and degree d of the binary p-ic equals the number of weight w and degree p of the binary d-ic.

Let $dp-2w\equiv\omega\geqq 0$. Then, by the theorem of § 25, each seminvariant in question uniquely determines a covariant of order ω.

The number of linearly independent covariants of degree d and order ω of the binary p-ic equals the number of linearly independent covariants of degree p and order ω of the binary d-ic.

The covariants are of course invariants if and only if $\omega=0$.

EXERCISES

1. Show by means of (1), § 24, that $w=\frac{1}{2}pd$ for an invariant.

2. Show that $(6;\ 6,\ 3)=7$, $(5;\ 6,\ 3)=5$. Find the two linearly independent seminvariants of weight 6 and degree 6 of the binary cubic.

3. There are only two linearly independent seminvariants of degree 4 and weight 4 of a binary quartic. Find them.

4. There is a single invariant or no invariant of degree 3 of the binary p-ic according as p is or is not a multiple of 4. (Cayley.)

Hint: Every invariant of the binary cubic is a product of a constant by a power of its discriminant, of order 4 (§ 30).

5. The binary p-ic has a single covariant or no covariant of order p and degree 2 according as p is or is not a multiple of 4. (Cayley.)

Hint: Every covariant of the binary quadratic f is of the type $c\ D^nf^m$, where c is a constant and D the discriminant of f (§ 29.) The degree $2n+m$ of the product equals its order $2m$ if $m=2n$. Thus f has a covariant of order and degree p if and only if $p=4n$, viz., $c\ D^nf^{2n}$.

6. No covariant of degree 2 has a leader of odd weight.

7. If S is of degree d_1 in the coefficients of a binary p_1-ic, of degree d_2 in the coefficients of a p_2-ic, . . . , and of total weight w, (2), § 24, holds with Ω and O replaced by $\Sigma\Omega$ and ΣO, and ω replaced by Σp_id_i-2w. For any such S, there exists an S_1 of partial degrees d_i and total weight $w+1$ for which $(\Sigma\Omega)S_1=S$. If S is a seminvariant, $\omega\geqq 0$. Generalize §§ 26, 27, using $(w;\ d_1,\ p_1;\ d_2,\ p_2;\ .\ .\ .)$ to denote the number of ways in which w can be expressed as a sum of d_1 or fewer positive integers $\leqq p_1$, of d_2 or fewer positive integers $\leqq p_2$, etc.

FUNDAMENTAL SYSTEM OF COVARIANTS OF A BINARY FORM, §§ 28–31

28. Certain Seminvariants. For $a_0 \neq 0$, we may set

$$f = a_0 x^p + p a_1 x^{p-1} y + \ldots + a_p y^p = a_0 (x - \alpha_1 y) \ldots (x - \alpha_p y).$$

Apply to f the transformation

$$T_n: \qquad x = \xi + n\eta, \qquad y = \eta.$$

Then each root α_i of $f = 0$ is diminished by n, since

$$x - \alpha_i y = \xi - (\alpha_i - n)\eta.$$

Hence the difference of any two roots is unaltered.

In particular, if $n = -a_1/a_0$, f is transformed into the reduced form

$$f' = a_0 \xi^p + \binom{p}{2} a'_2 \xi^{p-2} \eta^2 + \binom{p}{3} a'_3 \xi^{p-3} \eta^3 + \ldots,$$

where

$$a'_2 = a_2 - \frac{a_1^2}{a_0}, \qquad a'_3 = a_3 - 3\frac{a_1 a_2}{a_0} + 2\frac{a_1^3}{a_0^2}, \ \ldots,$$

and the roots of $f' = 0$ are $\alpha_i + a_1/a_0$ $(i = 1, \ldots, p)$. Since

$$\alpha_i + \frac{a_1}{a_0} = \alpha_i - \frac{\Sigma \alpha_1}{p} = \frac{(\alpha_i - \alpha_1) + \ldots + (\alpha_i - \alpha_p)}{p},$$

each root of $f' = 0$ is a linear function of the differences of the roots of $f = 0$ and hence is unaltered by every transformation T_n. The same is true of a'_2/a_0, a'_3/a_0, . . . , which equal numerical multiples of the elementary symmetric functions of the roots of $f' = 0$. Hence the polynomials

$$A_2 = a_0 a'_2 = a_0 a_2 - a_1^2,$$
$$A_3 = a_0^2 a'_3 = a_0^2 a_3 - 3a_0 a_1 a_2 + 2a_1^3,$$
$$A_4 = a_0^3 a'_4 = a_0^3 a_4 - 4a_0^2 a_1 a_3 + 6a_0 a_1^2 a_2 - 3a_1^4$$

are homogeneous and isobaric,* and are invariants of f with respect to all transformations T_n. By definition they are, therefore, seminvariants of f provided the subscript of each A in question does not exceed p.

* This is evident for A_2, A_3, A_4. Further A's will not be employed here. A general proof follows from § 34.

Since f' was derived from f by a linear transformation of determinant unity, any seminvariant S of f has the property

$$S(a_0, \ldots, a_p) = S(a_0, 0, a'_2, \ldots, a'_p) = S\left(a_0, 0, \frac{A_2}{a_0}, \ldots, \frac{A_p}{a_0^{p-1}}\right).$$

Hence any rational integral seminvariant is the quotient of a polynomial in $a_0, A_2, \ldots, A_p$ by a power of a_0. For $p \leqq 4$, we shall find which of these quotients equal rational integral functions of $a_0, \ldots, a_p$ and hence give rational integral seminvariants. The method is due to Cayley.

For $p=1$, S is evidently a numerical multiple of a power of a_0. Since a_0 is the leader of the covariant $f=a_0x+a_1y$ of f, we conclude that every covariant of a binary linear form f is a product of a power of f by a constant; in particular, there is no invariant.

29. Binary Quadratic Form. Since A_2 does not have the factor a_0, we conclude that every rational integral seminvariant is a polynomial in a_0 and A_2. Now A_2 is an invariant of f (§ 4), and a_0 is the leader of the covariant f of f. Hence *a fundamental system of rational integral covariants of the binary quadratic form f is given by f and its discriminant A_2.* We express in these words our result that any such covariant is a rational integral function of f and A_2.

30. Binary Cubic Form. We seek a polynomial $P(a_0, A_2, A_3)$ with the implicit, but not explicit, factor a_0. Write A'_i for the terms of A_i free of a_0:

(1) $$A'_2 = -a_1^2, \qquad A'_3 = 2a_1^3.$$

We desire that $P(0, A'_2, A'_3) \equiv 0$, identically in a_1. Now

$$4A'^3_2 + A'^2_3 \equiv 0,$$

(2) $$4A_2^3 + A_3^2 \equiv a_0^2 D,$$

where D is the discriminant of the cubic form,

$$D = a_0^2a_3^2 - 6a_0a_1a_2a_3 + 4a_0a_2^3 + 4a_1^3a_3 - 3a_1^2a_2^2.$$

By means of (2) we eliminate A_3^2 and higher powers of A_3 from $P(a_0, A_2, A_3)$ and conclude that any seminvariant is of the form π/a_0^k, where π is a polynomial in a_0, A_2, A_3, D, of degree 1 or 0 in A_3. If $k>0$, we may assume that not every term of π has the explicit factor a_0. In the latter case, π does not have the implicit factor a_0. For, if it did,

$$\pi'=\pi(0, A'_2, A'_3, D')\equiv 0, \qquad D'=4a_1^3a_3-3a_1^2a_2^2.$$

Since a_3 occurs in D', but not in A'_2 or A'_3, π' is free of D'. By (1), the first power of A'_3 is not cancelled by a power of A'_2. Hence π' is free of A'_3 and hence of A'_2.

A fundamental system of rational integral seminvariants of the binary cubic is given by a_0, A_2, A_3, D. *They are connected by the syzygy* (2).

A fundamental system of rational integral covariants of the binary cubic f *is given by* f, *its discriminant* D, *its Hessian* H, *and the Jacobian* J *of* f *and* H. *They are connected by the syzygy*

$$4H^3+J^2\equiv f^2D. \tag{3}$$

The last theorem follows from the first one and (2), since a_0, A_2, A_3 are the leaders of the covariants f, H, J.

31. Binary Quartic Form. We first seek polynomials $P(a_0, A_2, A_3, A_4)$ with the implicit, but not explicit, factor a_0. Thus

$$P'=P(0, A'_2, A'_3, A'_4)\equiv 0,\ A'_2=-a_1^2,\ A'_3=2a_1^3,\ A'_4=-3a_1^4.$$

The simplest P' is evidently $3A'^2_2+A'_4$. We get

$$A_4+3A_2^2=a_0^2I, \qquad I=a_0a_4-4a_1a_3+3a_2^2.$$

We drop A_4 and consider polynomials $\pi(a_0, A_2, A_3, I)$ with the implicit, but not explicit, factor, a_0. Such a polynomial is given by (2), § 30. For $a_0=0$, $D=-a_1^2I=A'_2I$. We have

$$A_2I-D=a_0J,$$

$$J=a_0a_2a_4-a_0a_3^2+2a_1a_2a_3-a_1^2a_4-a_2^3.$$

Eliminating D between this relation and (2), § 30, we get

$$a_0^3J-a_0^2A_2I+4A_2^3+A_3^2=0. \tag{1}$$

In view of their origin, I and J are seminvariants of the quartic f. Since they are unaltered by the replacement (1), § 20, they are invariants of f (*cf.* § 20, Example and Ex. 1). In view of (1), π equals a polynomial ϕ in a_0, A_2, A_3, I, J, of degree 0 or 1 in A_3. Suppose that ϕ does not have the explicit factor a_0. Then the equal function of $a_0, \ldots, a_4$ is not divisible by a_0. For, if it were,

$$\phi(0,\ -a_1^2,\ 2a_1^3,\ 3a_2^2-4a_1a_3,\ -a_1^2a_4+\ldots)\equiv 0.$$

In view of the term a_4, ϕ cannot involve J, and hence not I. Nor can ϕ be linear in A_3 in view of the odd power a_1^3. Hence ϕ is free of A_3 and hence of A_2.

A fundamental system of rational integral seminvariants of the binary quartic is given by a_0, A_2, A_3, I, J. *They are connected by the syzygy* (1).

A fundamental system of rational integral covariants of the binary quartic f is given by f, its invariants I and J, its Hessian H and the Jacobian G of f and H. They are connected by the syzygy

$$f^3J-f^2HI+4H^3+G^2\equiv 0. \tag{2}$$

The second theorem follows from the first one, since a_0, A_2, A_3 are the leaders of the covariants f, H, G.

It would be excessively laborious, if not futile, to apply the same method to the binary quintic, whose fundamental system is composed of 23 covariants,* most of which are very complex. The symbolic method is here superior both as to theory and as to compact notation (see Part III.).

Canonical Form of Binary Quartic. Solution of Quartic Equations

32. Theorem. *A binary quartic form f, whose discriminant is not zero, can be transformed linearly into the canonical form*

$$X^4+Y^4+6mX^2Y^2. \tag{1}$$

* Faà di Bruno, *Theorie der Binären Formen*, German tr. by Walter, 1881, pp. 199, 316–355. Salmon, *Modern Higher Algebra*, Fourth Edition, 1885, p. 227, p. 347.

The reason there is here a parameter m lies in the existence of two invariants I and J of weights (and hence indices) 4 and 6, and hence a rational *absolute* invariant I^3/J^2, i.e., one of index zero, and consequently having the same value for f and any form derived from f by linear transformation.

Since f vanishes for four values of x/y and hence is the product of four linear functions, it can be expressed (in three ways) as a product of two quadratic forms, say those in the right members of the next equations. To prove our theorem it suffices to show that there exist constant p, q, r, s (each $\neq 0$) and α, β ($\alpha \neq \beta$) such that

$$p(x+\alpha y)^2+q(x+\beta y)^2 \equiv ax^2+2bxy+cy^2,$$
$$r(x+\alpha y)^2+s(x+\beta y)^2 \equiv gx^2+2hxy+ky^2.$$

For, the product f of these becomes (1) by the transformation

$$X=\sqrt[4]{pr}\,(x+\alpha y), \qquad Y=\sqrt[4]{qs}\,(x+\beta y),$$

of determinant $\neq 0$. The conditions for the two identities are

$$p+q=a, \qquad p\alpha+q\beta=b, \qquad p\alpha^2+q\beta^2=c,$$
$$r+s=g, \qquad r\alpha+s\beta=h, \qquad r\alpha^2+s\beta^2=k.$$

The first three equations are consistent if

$$\begin{vmatrix} 1 & 1 & a \\ \alpha & \beta & b \\ \alpha^2 & \beta^2 & c \end{vmatrix} \div (\beta-\alpha) \equiv c-b(\alpha+\beta)+a\alpha\beta=0.$$

If $p=0$, or if $q=0$, the same equations give $b^2=ac$, so that the first quadratic factor of f and hence f would have a dcuble root. Similarly, the last three equations have solutions $r\neq 0$, $s\neq 0$, if

$$k-h(\alpha+\beta)+g\alpha\beta=0.$$

If the determinant $ah-bg$ is not zero, the last two relations determine $\alpha+\beta$ and $\alpha\beta$, and hence give α and β as the roots of *

$$(ah-bg)z^2-(ak-cg)z+bk-ch=0.$$

* Its left member is obtained by setting $x/y=-z$ in the Jacobian of the two quadratic factors of f.

If its roots were equal, the two relations would give

$$c-2b\alpha+a\alpha^2=0, \qquad k-2h\alpha+g\alpha^2=0,$$

and the two quadratic factors of f would vanish for $x/y=-\alpha$.

If $ah-bg=0$, but $ch-bk\neq 0$, we interchange x with y and proceed as before. If both determinants vanish, either $b\neq 0$ and the second quadratic factor is the product of the first by h/b, or else $b=0$ and hence $h=0$ and no transformation of f is needed.

33. Actual Determination of the Canonical Quartic. Let Δ denote the determinant of the coefficients of x, y in X, Y. Then f, its invariants I and J and Hessian H are related to the canonical form, its invariants and Hessian, as follows:

$$f=X^4+Y^4+6mX^2Y^2,$$
$$I=\Delta^4(1+3m^2), \qquad J=\Delta^6(m-m^3),$$
$$H=\Delta^2\{m(X^4+Y^4)+(1-3m^2)X^2Y^2\}.$$

Thus $\Delta^2 m$ may be found from the resolvent cubic equation

$$4(\Delta^2 m)^3-I(\Delta^2 m)+J=0.$$

Then Δ^4 may be found from I. We may select either square root as Δ^2 and hence find m. In fact, by replacing X by $X\sqrt{-1}$ in f, the signs of Δ^2 and m are changed. By eliminating X^4+Y^4, we get

$$\Delta^2 mf-H\equiv\Delta^2(9m^2-1)X^2Y^2.$$

If $9m^2=1$, f is the square of $X^2\pm Y^2$ and the discriminant of f would vanish. Hence we obtain XY by a root extraction. Thus X and Y are determined up to constant factors t and t^{-1}. We may find t by comparing the coefficients of x^4 and x^3y in f and the expansion of its canonical form, or by use of the Jacobian G of f and H:

$$G=\Delta^3(1-9m^2)XY(X^4-Y^4),$$

and combining the resulting X^4-Y^4 with the earlier X^4+Y^4. Or from f and XY we can find X^2+Y^2 and then $X\pm Y$.

To solve $f=0$, we have only to find the canonical form

SEMINVARIANTS, INVARIANTS, AND COVARIANTS OF A BINARY FORM f AS FUNCTIONS OF THE ROOTS OF $f=0$, §§ 34–37.

34. Seminvariants in Terms of the Roots. Give f the notation used in § 28, so that $\alpha_1, \ldots, \alpha_p$ are the roots of $f=0$. After removing possible factors a_0 from a given seminvariant of f, we obtain a seminvariant S not divisible by a_0. Let δ be the degree of the homogeneous function S of the a's. Thus S is the product of $a_0{}^\delta$ by a polynomial in $a_1/a_0, \ldots, a_p/a_0$ of degree δ. The latter equal numerical multiples of the elementary symmetric functions of $\alpha_1, \ldots, \alpha_p$, each of which is linear in every root. Hence our polynomial equals a symmetric polynomial σ in $\alpha_1, \ldots, \alpha_p$ of degree δ in every root.

Since S is of constant weight w and since a_i/a_0 equals a function of total degree i in the roots, σ is homogeneous in the roots and of total degree w in them.

Besides being homogeneous and isobaric in the a's, a seminvariant must be unaltered by every transformation T_n of § 28. Under that transformation, each root is diminished, by n (§ 28). Since

$$\alpha_i=\alpha_1+(\alpha_i-\alpha_1) \qquad (i=2, \ldots, p)$$

we can express σ as a polynomial $P(\alpha_1)$ whose coefficients are rational integral functions of the differences of the roots. If $P(\alpha_1)$ is of degree $\geqq 1$ in α_1, we have $P(\alpha_1)=P(\alpha_1-n)$, for all values of n. But an equation in n cannot have an infinitude of roots. Hence $P(\alpha_1)$ does not involve α_1, so that σ equals a polynomial in the differences of the roots.

Multiplying by the factors a_0 removed, we obtain the theorem:

Any seminvariant of degree d and weight w of the binary form $a_0x^p+\ldots$ equals the product of $a_0{}^d$ by a rational integral symmetric function σ of the roots, homogeneous (of total degree w) in the roots, of degree $\leqq d$ in any one root, and expressible as a polynomial in the differences of the roots.

Conversely, *any such product can be expressed as a polynomial in the* a's *and this polynomial is a seminvariant.*

Since the factor σ is symmetric in the roots, and is of degree $\leqq d$ in any one root, its product by $a_0{}^d$ equals a homogeneous polynomial in the a's whose degree is d. This polynomial is isobaric since σ is homogeneous, and is unaltered by every transformation T_n, since σ is expressible as a function of the differences of the roots.

The importance of these theorems is due mainly to the fact that they enable us to tell by inspection (without computation by annihilators) whether or not a given function of the roots and a_0 is a seminvariant. A like remark applies to the theorem in § 35 on invariants and that in § 36 on covariants.

EXAMPLE

The binary cubic has the seminvariant

$$a_0{}^2\sum_3(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)=a_0{}^2(\Sigma\alpha_1{}^2-\Sigma\alpha_1\alpha_2)$$

$$=a_0{}^2\{(\Sigma\alpha_1)^2-3\Sigma\alpha_1\alpha_2\}=a_0{}^2\left\{\left(\frac{-3a_1}{a_0}\right)^2-3\left(\frac{3a_2}{a_0}\right)\right\}=-9(a_0a_2-a_1{}^2).$$

35. Invariants in Terms of the Roots. A seminvariant of f is an invariant of f if and only if it is unaltered by the transformation $x=-\eta$, $y=\xi$ (§ 20). For the latter,

$$x-\alpha y=-\alpha\left(\xi+\frac{1}{\alpha}\eta\right),$$

so that α_r is replaced by $-1/\alpha_r$, and hence $\alpha_r-\alpha_s$ by

$$\frac{\alpha_r-\alpha_s}{\alpha_r\alpha_s}.$$

The coefficient of ξ^p in the transformed binary form is

$$A_0=(-1)^p\alpha_1\alpha_2\ \ldots\ \alpha_pa_0.$$

By § 34, any seminvariant of f is of the type

$$a_0{}^d\Sigma c_i(\text{product of } w \text{ factors like } \alpha_r-\alpha_s).$$

Hence this is an invariant if and only if it equals

$$(-1)^{pd}(\alpha_1\ \ldots\ \alpha_p)^da_0{}^d\Sigma c_i\left(\text{product of the } w \text{ corresponding } \frac{\alpha_r-\alpha_s}{\alpha_r\alpha_s}\right),$$

and hence if $\pm\alpha_1^d \ldots \alpha_p^d$ equals the product of the factors $\alpha_r\alpha_s$ in the denominators. This is the case if and only if each root occurs exactly d times in every term of the sum and if pd is even. By the total number of α's, $pd=2w$.

Any invariant of degree d and weight w of the binary form $a_0x^p+\ldots$ *equals the product of* a_0^d *by a sum of products of constants and certain differences of the roots, such that each root occurs exactly d times in every product; moreover, the sum equals a homogeneous symmetric function of the roots of total degree w. Conversely, the product of any such sum by* a_0^d *equals a rational integral invariant.*

EXERCISES

1. $a_0^2(\alpha_1-\alpha_2)^2$ is an invariant of the binary quadratic form. Any invariant is a numerical multiple of a power of this one.

2. $a_0^2\sum_3(\alpha_1-\alpha_2)^2(\alpha_3-\alpha_4)^2$ is an invariant of the binary quartic.

3. $a_0^2\sum_3(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)$ is not an invariant of the binary cubic.

4. If we multiply $a_0^{2(p-1)}$ by the product of the squares of the differences of the roots of the binary p-ic f, we obtain an invariant (discriminant of f). Also verify that $pd=2w$.

5. The sum of the coefficients of any seminvariant is zero.

Hint: Use $f=(x+y)^p$, whose roots are all equal.

6. Every invariant of the binary cubic is a power of its discriminant.

7. A function which satisfies the conditions in the theorem of § 35 except that of symmetry in the roots is called an *irrational invariant*. If $\alpha_1, \ldots, \alpha_4$ are the roots of a binary quartic f, and

$$u=(\alpha_1-\alpha_4)(\alpha_2-\alpha_3),\ v=(\alpha_2-\alpha_4)(\alpha_3-\alpha_1),\ w=(\alpha_1-\alpha_2)(\alpha_3-\alpha_4),$$

why are a_0u, a_0v, a_0w irrational invariants of f? They are the roots of $z^3-12Iz-\delta=0$, where δ^2 is the product of a_0^6 by the product of the squares of the differences of the roots and hence is the discriminant of f. Hints: $u+v+w=0$, and $s=uv+uw+vw$ is a symmetric function of $\alpha_1, \ldots, \alpha_4$ in which each α_i occurs twice in every product of differences, so that a_0^2s is an invariant of degree 2. By the Example in § 20, $a_0^2s=cI$, where c is a constant. To determine c, take $\alpha_1=1, \alpha_2=-1, \alpha_3=2, \alpha_4=-2$, so that $f=(x^2-y^2)(x^2-4y^2)$, $I=73/12$, $u=-9$, $v=1$, $w=8$, $s=-73$. Hence $c=-12$. As here, so always an irrational algebraic invariant is a root of an equation whose coefficients are rational invariants.

8. If α_1, α_2 are the roots of the binary quadratic form f, and α_3, α_4 the roots of f' in § 11, the simultaneous invariant

$$ac'+a'c-2bb'=aa'\{\alpha_3\alpha_4+\alpha_1\alpha_2-\tfrac{1}{2}(\alpha_1+\alpha_2)(\alpha_3+\alpha_4)\}=\tfrac{1}{2}a_0(u-v),$$

if the product ff' is identified with the quartic in Ex. 7. Hence a simultaneous invariant of the quadratic factors of a quartic is an irrational invariant of the quartic. Why *a priori* is the invariant three-valued?

9. The cross-ratios of the four roots of the quartic are $-v/u$, etc. These six are equal in sets of three if $I=0$. For, if $s=0$,

$$vw=u(-v-w)=u^2,\quad uw=v(-u-w)=v^2,\quad \frac{-v}{u}=\frac{-u}{w}=\frac{-w}{v}.$$

The remaining three are the reciprocals of these and are equal.

10. By Ex. 3, § 11, one of the cross-ratios is -1 if $ac'+\ldots=0$. Why does this agree with Ex. 8?

11. The product of the squares of the differences of the roots of the cubic equation in Ex. 7 is known * to be

$$-4(-12I)^3-27\delta^2=a_0^6(u-v)^2(u-w)^2(v-w)^2.$$

Also,* $\delta^2=256(I^3-27J^2)$. Hence the left member becomes $3^6\cdot 4^4J^2$. Thus

$$3^3\cdot 4^2J=\pm a_0^3(u-v)(u-w)(v-w).$$

Using J from § 31, and the special values in Ex. 7, show that the sign is plus. Verify that the cross-ratios equal $-1, -1, 2, 2, \frac{1}{2}, \frac{1}{2}$, if $J=0$.

36. Covariants in Terms of the Roots. Let $K(a_0, \ldots, a_p; x, y)$ be a covariant of constant degree d (in the coefficients) and constant order ω (in the variables) of the binary form $f=a_0x^p+\ldots$ Then

$$K=a_0{}^d y^\omega \kappa,$$

where κ is a polynomial in x/y and the roots $\alpha_1, \ldots, \alpha_p$ of $f=0$. Under the transformation T_n in § 28, let f become $A_0\xi^p+\ldots$, with the roots $\alpha'_1, \ldots, \alpha'_p$. Then

$$\frac{x}{y}-\alpha_i=\frac{\xi}{\eta}-\alpha'_i,\qquad \alpha_r-\alpha_s=\alpha'_r-\alpha'_s.$$

Making use of the identities

$$\frac{x}{y}=\left(\frac{x}{y}-\alpha_1\right)+\alpha_1,\qquad \alpha_i=(\alpha_i-\alpha_1)+\alpha_1,$$

* *Cf.* Dickson, *Elementary Theory of Equations*, p. 33, p. 42, Ex. 7.

we see that κ equals a polynomial $P(\alpha_1)$ whose coefficients are rational integral functions of the differences of $x/y, \alpha_1, \ldots, \alpha_p$ in pairs. Since

$$K(A_0, \ldots, A_p;\ \xi, \eta) = K(a_0, \ldots, a_p;\ x, y), \qquad A_0 = a_0, \quad \eta = y,$$

we have $$\kappa\left(\alpha'_1, \ldots, \alpha'_p, \frac{\xi}{\eta}\right) = \kappa\left(\alpha_1, \ldots, \alpha_p, \frac{x}{y}\right).$$

The left member equals $P(\alpha'_1)$ since

$$\alpha'_i = (\alpha_i - \alpha_1) + \alpha'_1, \qquad \frac{\xi}{\eta} = \left(\frac{x}{y} - \alpha_1\right) + \alpha'_1.$$

Hence

$$P(\alpha_1 - n) - P(\alpha_1) = 0$$

for every n. Hence α_1 does not occur in $P(\alpha_1)$, and κ is a polynomial in the differences of $x/y, \alpha_1, \ldots, \alpha_p$.

Let W be the weight of K and hence of the coefficient of y^ω. Then κ is of total degree W in the α's and of degree ω in x/y. Thus

$$\kappa = \Sigma c_i \left\{\text{product of } \omega \text{ differences like } \frac{x}{y} - \alpha_r\right\}$$

$$\cdot \{\text{product of } W-\omega \text{ differences like } \alpha_r - \alpha_s\}.$$

Hence

$$K = a_0{}^d \Sigma c_i \{\text{product of } \omega \text{ differences like } x - \alpha_r y\}$$

$$\cdot \{\text{product of } W-\omega \text{ differences like } \alpha_r - \alpha_s\}.$$

Next, for $x = -\eta$, $y = \xi$, f becomes $F = A_0 \xi^p + \ldots$ with a root $-1/\alpha_r$ corresponding to each root α_r of f. The function K for F is

$$A_0{}^d \Sigma c_i \left\{\text{product of } \omega \text{ differences like } \xi + \frac{1}{\alpha_r}\eta = \frac{(x - \alpha_r y)}{-\alpha_r}\right\}$$

$$\cdot \left\{\text{product of } W-\omega \text{ differences like } \frac{\alpha_r - \alpha_s}{\alpha_r \alpha_s}\right\}.$$

Using the value of A_0 in § 35, we see that the factor

$$(-1)^{pd} \alpha_1{}^d \ldots \alpha_p{}^d$$

must be cancelled by the $-\alpha_r$ and the $\alpha_r \alpha_s$ in the denominators.

Thus each term of the sum involves every root exactly d times. The signs agree since

$$dp=\omega+2(W-\omega),$$

as follows by counting the total number of α's.

Any covariant of degree d, order ω and weight W of

$$a_0(x-\alpha_1 y)\ \ldots\ (x-\alpha_p y)$$

equals the product of a_0^d by a sum of products of constants and ω differences like $x-\alpha_r y$ and $W-\omega$ differences like $\alpha_r-\alpha_s$, such that every root occurs in exactly d factors of each product; moreover, the sum equals a symmetric function of the roots. Conversely, the product of a_0^d by any such sum equals a rational integral covariant.

EXERCISES

1. $f=a_0x^3+3a_1x^2y+3a_2xy^2+a_3y^3$ has the covariant

$$K=a_0^2\underset{3}{\Sigma}(x-\alpha_1 y)^2(\alpha_2-\alpha_3)^2.$$

Show that the coefficient of x^2 in K equals $-18(a_0a_2-a_1^2)$. Why may we conclude that $K=-18H$, where H is the Hessian of f?

2. The same binary cubic has the covariant

$$a_0^2\underset{3}{\Sigma}(x-\alpha_1 y)(x-\alpha_2 y)(\alpha_2-\alpha_3)(\alpha_3-\alpha_1)=9H.$$

3. Every rational integral covariant of the binary quadratic f is a product of powers of f and its discriminant by a constant.

37. Covariant with a Given Leader S. If the seminvariant S has the factor a_0, and $S=a_0Q$, and if Q is the leader of a covariant K of f, then, since a_0 is the leader of f, S is the leader of the covariant fK. Hence it remains to consider only a seminvariant S not divisible by a_0. If S is of degree d and weight w,

$$S=a_0^d\Sigma c_i(\text{product of } w \text{ factors like } \alpha_r-\alpha_s),$$

where each product is of degree at most d in each root, and of degree exactly d in at least one root (§ 34). If each product is of degree d in every root, S is an invariant (§ 35) and hence is the required covariant. In the contrary case, let α_2, for example, enter to a degree less than d; we supply enough factors $x-\alpha_2 y$ to bring the degree in α_2 up to d. Then a_0^d

multiplied by the sum of the total products is a covariant with the leader S. For example,

$$a_0^2\sum_3(\alpha_2-\alpha_3)^2, \qquad a_0^2\sum_3(\alpha_2-\alpha_3)(\alpha_3-\alpha_1)$$

are the leaders of the covariants in Exs. 1, 2, § 36, of the binary cubic. The present result should be compared with the theorem in § 25.

We may now give a new proof of the lemma in § 25 that $dp-2w \geqq 0$ for any seminvariant S of degree d and weight w of the binary p-ic. Whether S has the factor a_0 or not, the first term of the resulting covariant K is Sx^ω, where $\omega=dp-2w$. For, in each product in the above S, the roots $\alpha_1, \ldots, \alpha_p$ occur $2w$ times in all. In K each root occurs d times. Hence we inserted $dp-2w$ factors $x-\alpha y$ in deriving K from S.

38. Differential Operators Producing Covariants. Let the transformation

$$T: \qquad x=\alpha\xi+\beta\eta, \qquad y=\gamma\xi+\delta\eta, \qquad \Delta=\alpha\delta-\beta\gamma\neq 0$$

replace $f(x, y)$ by $\phi(\xi, \eta)$. Then

$$\frac{\partial\phi}{\partial\xi}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial\xi}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial\xi}=\alpha\frac{\partial f}{\partial x}+\gamma\frac{\partial f}{\partial y},$$

$$\frac{\partial\phi}{\partial\eta}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial\eta}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial\eta}=\beta\frac{\partial f}{\partial x}+\delta\frac{\partial f}{\partial y}.$$

Solving, we get

$$\Delta\frac{\partial f}{\partial y}=\alpha\frac{\partial\phi}{\partial\eta}-\beta\frac{\partial\phi}{\partial\xi}, \qquad -\Delta\frac{\partial f}{\partial x}=\gamma\frac{\partial\phi}{\partial\eta}-\delta\frac{\partial\phi}{\partial\xi},$$

or $df=D\phi$, $d_1f=D_1\phi$, if we introduce the differential operators

$$d=\Delta\frac{\partial}{\partial y}, \qquad d_1=-\Delta\frac{\partial}{\partial x}, \qquad D=\alpha\frac{\partial}{\partial\eta}-\beta\frac{\partial}{\partial\xi}, \qquad D_1=\gamma\frac{\partial}{\partial\eta}-\delta\frac{\partial}{\partial\xi}.$$

As usual, write d^2d_1f for $d\{d(d_1f)\}$. Since the result of operating with d on df is the same as operating with D on the equal function $D\phi$ of ξ and η, we have $d^2f=D^2\phi$. Similarly,

$$\Sigma c_{rs}d^rd_1^sf=\Sigma c_{rs}D^rD_1^s\phi \qquad (r+s=\omega).$$

The right member is the result of operating on ϕ with the operator obtained by substituting D for $\partial/\partial\eta$ and D_1 for $-\partial/\partial\xi$ in

$$\Sigma c_{rs}\left(\frac{\partial}{\partial\eta}\right)^r\left(-\frac{\partial}{\partial\xi}\right)^s \qquad (r+s=\omega),$$

whose terms are partial derivatives of order ω. Hence, if the form

$$l(x, y)=\Sigma c_{rs}x^ry^s \qquad (r+s=\omega)$$

becomes $\lambda(\xi, \eta)$ under the transformation T, our right member is the result of operating on ϕ with $\lambda(\partial/\partial\eta, -\partial/\partial\xi)$. The left member is the result of operating on f with

$$l\left(\Delta\frac{\partial}{\partial y}, -\Delta\frac{\partial}{\partial x}\right)=\Delta^\omega l\left(\frac{\partial}{\partial y}, -\frac{\partial}{\partial x}\right).$$

Hence if T replaces the forms $f(x, y)$, $l(x, y)$ by $\phi(\xi, \eta)$, $\lambda(\xi, \eta)$, then

$$\left[\lambda\left(\frac{\partial}{\partial\eta}, -\frac{\partial}{\partial\xi}\right)\right]\phi(\xi, \eta)=\Delta^\omega\left[l\left(\frac{\partial}{\partial y}, -\frac{\partial}{\partial x}\right)\right]f(x, y)$$

is a consequence of the equations for T, if ω is the order of $l(x, y)$.

Let f and l be covariants of indices m and n of one or more binary forms f_i with the coefficients $c_1, c_2, \ldots$. Under T let the transformed forms have the coefficients $C_1, C_2, \ldots$. Then

$$f(C;\ \xi, \eta)=\Delta^m f(c;\ x, y), \qquad l(C;\ \xi, \eta)=\Delta^n l(c;\ x, y).$$

But $\phi(\xi, \eta)=f(c;\ x, y)$, by the earlier notation. Hence

$$\phi(\xi, \eta)=\Delta^{-m}f(C;\ \xi, \eta), \qquad \lambda(\xi, \eta)=\Delta^{-n}l(C;\ \xi, \eta).$$

Inserting these into the formula of the theorem, and multiplying by Δ^{m+n}, we get

$$\left[l\left(C;\ \frac{\partial}{\partial\eta}, -\frac{\partial}{\partial\xi}\right)\right]f(C;\ \xi, \eta)=\Delta^{\omega+m+n}\left[l\left(c;\ \frac{\partial}{\partial y}, -\frac{\partial}{\partial x}\right)\right]f(c; x,y).$$

The function in the right member is therefore a covariant of index $\omega+m+n$ of the f_i. We therefore have the theorem of Boole, one of the first known general theorems on covariants:

THEOREM. *If l and f are any covariants of a system of binary forms, we obtain a covariant (or invariant) of the system of forms by operating on f with the operator obtained from l by replacing x by $\partial/\partial y$ and y by $-\partial/\partial x$, i.e., $x^r y^s$ by $(-1)^s \partial^{r+s}/\partial y^r \partial x^s$.*

EXERCISES

1. Taking $l=f=ax^2+2bxy+cy^2$, obtain the invariant $4(ac-b^2)$ of f.

2. If $l=f$ is the binary quartic, the invariant is $2\cdot 4!\,I$ of § 31.

3. Using the binary quartic and its Hessian, obtain the invariant J.

4. Taking $l=a_0x^p+\dots, f=b_0x^p+\dots$, obtain their simultaneous invariant

$$\sum_{i=0}^{p}(-1)^i\binom{p}{i}a_ib_{p-i}.$$

If also $l\equiv f$, we have an invariant of f, which vanishes if p is odd. For $p=2$ and $p=4$, deduce the results in Exs. 1, 2.

5. A fundamental system of covariants of a quadratic and cubic

$$Q=Ax^2+2Bxy+Cy^2, \qquad f=ax^3+3bx^2y+3cxy^2+dy^3$$

is composed of 15 forms. We may take Q and its discriminant $AC-B^2$; f, its discriminant and Hessian h, given by (5) and (2) of § 8, the Jacobian J of f and H:

$$J=(a^2d-3abc+2b^3)x^3+3(abd+b^2c-2ac^2)x^2y$$
$$+3(2b^2d-acd-bc^2)xy^2+(3bcd-ad^2-2c^3)y^3;$$

the Jacobian of f and Q:

$$(Ab-Ba)x^3+(2Ac-Bb-Ca)x^2y+(Ad+Bc-2Cb)xy^2+(Bd-Cc)y^3;$$

the Jacobian of Q and h:

$$(As-Br)x^2+(At-Cr)xy+(Bt-Cs)y^2;$$

the result of operating on f with the operator obtained as in the theorem from $l=Q$:

$$L_1\equiv(aC+cA-2bB)x+(bC+dA-2cB)y;$$

the result of operating on Q with the operator obtained from L_1:

$$L_2=\{aBC-b(2B^2+AC)+3cAB-dA^2\}x$$
$$+\{aC^2-3bBC+c(AC+2B^2)-dAB\}y;$$

the result L_3, of operating on J with Q and the result L_4 of operating on Q with L_3 (so that L_3 and L_4 may be derived from L_1 and L_2 by replacing $a, \ldots, d$ by the corresponding coefficients of J); the intermediate invariant $At+Cr-2Bs$ of Q and h (§ 11); the resultant of Q and f:

$$a^2C^3-6abBC^2+6acC(2B^2-AC)+ad(6ABC-8B^3)+9b^2AC^2$$
$$-18bcABC+6bdA(2B^2-AC)+9c^2A^2C-6cdBA^2+d^2A^3;$$

the resultant of L_1 and L_4 (=resultant of L_2 and L_3), obtained at once as a determinant of order 2. Salmon, *Modern Higher Algebra*, § 198, gives geometrical interpretations. Hammond, *Amer. Jour. Math.*, vol. 8, obtains the syzygies between the 15 covariants.