Finally, in the third, the product of two propositions cannot be false unless one of them is false, for, if both were true, their product would be true (equal to 1).

58. **Law of Importation and Exportation.**—The fundamental equivalence $(a < b) = a' + b$ has many other interesting consequences. One of the most important of these is *the law of importation and exportation,* which is expressed by the following formula:

$$[a < (b < c)] = (ab < c).$$

"To say that if a is true b implies c, is to say that a and b imply c".

This equality involves two converse implications: If we infer the second member from the first, we *import* into the implication $(b < c)$ the hypothesis or condition a; if we infer the first member from the second, we, on the contrary, *export* from the implication $(ab < c)$ the hypothesis a.

Demonstration:

$$[a < (b < c)] = a' + (b < c) = a' + b' + c,$$
$$(ab < c) = (ab)' + c = a' + b' + c.$$

Cor. 1.—Obviously we have the equivalence

$$[a < (b < c)] = [b < (a < c)],$$

since both members are equal to $(ab < c)$, by the commutative law of multiplication.

Cor. 2.—We have also

$$[a < (a < b)] = (a < b),$$

for, by the law of importation and exportation,

$$[a < (a < b)] = (aa < b) = (a < b).$$

If we apply the law of importation to the two following formulas, of which the first results from the principle of identity and the second expresses the principle of contraposition,

$$(a < b) < (a < b), \qquad (a < b) < (b' < a'),$$

we obtain the two formulas

$$(a < b)\, a < b, \qquad (a < b)\, b' < a',$$

which are the two types of *hypothetical reasoning:* "If a implies b, and if a is true, b is true" (*modus ponens*); "If a implies b, and if b is false, a is false" (*modus tollens*).

Remark. These two formulas could be directly deduced by the principle of assertion, from the following

$$(a < b)\ (a = 1) < (b = 1),$$
$$(a < b)\ (b = 0) < (a = 0),$$

which are not dependent on the law of importation and which result from the principle of the syllogism.

From the same fundamental equivalence, we can deduce several paradoxical formulas:

1. $$a < (b < a), \qquad a' < (a < b).$$

"If a is true, a is implied by any proposition b; if a is false, a implies any proposition b". This agrees with the known properties of 0 and 1.

2. $$a < [(a < b) < b], \qquad a' < [(b < a) < b'].$$

"If a is true, then 'a implies b' implies b; if a is false, then 'b implies a' implies not-b."

These two formulas are other forms of hypothetical reasoning (*modus ponens* and *modus tollens*).

3. $$[(a < b) < a] = a,\text{[1]} \qquad [(b < a) < a'] = a',$$

"To say that, if a implies b, a is true, is the same as affirming a; to say that, if b implies a, a is false, is the same as denying a".

Demonstration:

$$[(a < b) < a] = (a' + b < a) = ab' + a = a,$$
$$[(b < a) < a'] = (b' + a < a') = a'b + a' = a'.$$

[1] This formula is BERTRAND RUSSELL's "principle of reduction". See *The Principles of Mathematics*, Vol. I, p. 17 (Cambridge, 1903).

In formulas (1) and (3), in which b is any term at all, we might introduce the sign $\prod$ with respect to b. In the following formula, it becomes necessary to make use of this sign.

4. $$\prod_x \left\{[a < (b < x)] < x\right\} = ab.$$

Demonstration:

$$\left\{[a < (b < x)] < x\right\} = \left\{[a' + (b < x)] < x\right\}$$
$$= [(a' + b' + x) < x] = abx' + x = ab + x.$$

We must now form the product $\prod_x (ab + x)$, where x can assume every value, including 0 and 1. Now, it is clear that the part common to all the terms of the form $(ab + x)$ can only be ab. For, (1) ab is contained in each of the sums $(ab + x)$ and therefore in the part common to all; (2) the part common to all the sums $(ab + x)$ must be contained in $(ab + 0)$, that is, in ab. Hence this common part is equal to ab[1], which proves the theorem.

59. Reduction of Inequalities to Equalities.—As we have said, the principle of assertion enables us to reduce inequalities to equalities by means of the following formulas:

$$(a \neq 0) = (a = 1), \qquad (a \neq 1) = (a = 0),$$
$$(a \neq b) = (a = b').$$

For,

$$(a \neq b) = (ab' + a'b \neq 0) = (ab' + ab' = 1) = (a = b').$$

Consequently, we have the paradoxical formula

$$(a \neq b) = (a = b').$$

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[1] This argument is general and from it we can deduce the formula

$$\prod_x (a + x) = a,$$

whence may be derived the correlative formula

$$\sum_x ax = a.$$