From this remark, PORETSKY concluded that, in general, the solution of an equation is neither a consequence nor a cause of the equation. It is a cause of it in the particular case in which

$$ab = 0,$$

and it is a consequence of it in the particular case in which

$$(a'b' = 0) = (a + b = 1).$$

But if ab is not equal to 0, the equation is unsolvable and the formula of solution absurd, which fact explains the preceding paradox. If we have at the same time

$$ab = 0 \quad \text{and} \quad a + b = 1,$$

the solution is both consequence and cause at the same time, that is to say, it is equivalent to the equation. For when $a' = b$ the equation is determinate and has only the one solution

$$x = a' = b.$$

Thus, whenever an equation is solvable, its solution is one of its causes; and, in fact, the problem consists in finding a value of x which will verify it, *i. e.*, which is a cause of it.

To sum up, we have the following equivalence:

$$(ax + bx' = 0) = (ab = 0)\sum_u (x = a'u + bu')$$

which includes the following implications:

$$(ax + bx' = 0) < (ab = 0),$$

$$(ax + bx' = 0) < \sum_u (x = a'u + bu'),$$

$$(ab = 0)\sum_u (x = a'u + bu') < (ax + bx' = 0).$$

37. Elimination of Several Unknown Quantities.— We shall now consider an equation involving several unknown quantities and suppose it reduced to the normal form, *i. e.*, its first member developed with respect to the unknown quantities, and its second member zero. Let us first concern ourselves with the problem of elimination. We can eliminate the unknown quantities either one by one or all at once.

For instance, let

(1) $\varphi(x, y, z) = axyz + bxyz' + cxy'z + dxy'z' + fx'yz + gx'yz' + hx'y'z + kx'y'z' = 0$

be an equation involving three unknown quantities.

We can eliminate z by considering it as the only unknown quantity, and we obtain as resultant

$$(axy + cxy' + fx'y + hx'y')\ (bxy + dxy' + gx'y + kx'y') = 0$$

or

(2) $$abxy + cdxy' + fgx'y + hkx'y' = 0.$$

If equation (1) is possible, equation (2) is possible as well; that is, it is verified by some values of x and y. Accordingly we can eliminate y from the equation by considering it as the only unknown quantity, and we obtain as resultant

$$(abx + fgx')\ (cdx + hkx') = 0$$

or

(3) $$abcdx + fghkx' = 0.$$

If equation (1) is possible, equation (3) is also possible; that is, it is verified by some values of x. Hence we can eliminate x from it and obtain as the final resultant,

$$abcd \,.\, fghk = 0$$

which is a consequence of (1), independent of the unknown quantities. It is evident, by the principle of symmetry, that the same resultant would be obtained if we were to eliminate the unknown quantities in a different order. Moreover this result might have been foreseen, for since we have (§ 28)

$$abcdfghk < \varphi(x, y, z),$$

$\varphi(x, y, z)$ can vanish only if the product of its coefficients is zero:

$$[\varphi(x, y, z) = 0] < (abcdfghk = 0).$$

Hence we can eliminate all the unknown quantities at once by equating to 0 the product of the coefficients of the function developed with respect to all these unknown quantities.

We can also eliminate some only of the unknown quantities at one time. To do this, it is sufficient to develop the first member with respect to these unknown quantities and to equate the product of the coefficients of this development to o. This product will generally contain the other unknown quantities. Thus the resultant of the elimination of z alone, as we have seen, is

$$abxy + cdxy' + fgx'y + hkx'y' = 0$$

and the resultant of the elimination of y and z is

$$abcdx + fghkx' = 0.$$

These partial resultants can be obtained by means of the following practical rule: Form the constituents relating to the unknown quantities to be retained; give each of them, for a coefficient, the product of the coefficients of the constituents of the general development of which it is a factor, and equate the sum to o.

38. Theorem Concerning the Values of a Function:— *All the values which can be assumed by a function of any number of variables $f(x, y, z \ldots)$ are given by the formula*

$$abc \ldots k + u\,(a + b + c + \ldots + k),$$

in which u is absolutely indeterminate, and a, b. $c \ldots$, k are the coefficients of the development of f.

Demonstration.—It is sufficient to prove that in the equality

$$f(x, y, z \ldots) = abc \ldots k + u\,(a + b + c + \ldots + k)$$

u can assume all possible values, that is to say, that this equality, considered as an equation in terms of u, is indeterminate.

In the first place, for the sake of greater homogeneity, we may put the second member in the form

$$u'\,abc \ldots k + u(a + b + c + \ldots + k),$$

for

$$abc \ldots k = uabc \ldots k + u'\,abc \ldots k,$$

and

$$uabc \ldots k < u(a + b + c + \ldots + k).$$

Reducing the second member to o (assuming there are only three variables x, y, z)