Chapter 5

Bohr-Jessen limit theorem

5.1 Log zeta function

To state the limit theorem, we begin with the definition of the log zeta function. We intend to make its definition given by Matsumoto $\big[$cf. [26, Chapter 6]$\big]$ clear.

Claim 5.1 *Put* $\Gamma^{\dagger 1} := \big\{t \in \mathbb{R} \setminus \{0\};\, {}^{\exists}\sigma \in (0,1)$ *s.t.* $\zeta(\sigma + \sqrt{-1}t) = 0\big\}$. *Then*

$$ {}^{\forall}t \in \mathbb{R},\ {}^{\exists}\delta > 0 \ \ s.t.\ \ \#\big(U_\delta(t) \cap \Gamma\big) \le 1, $$

where $U_\delta(t) = (t - \delta, t + \delta)$.

Proof. We prove it by a reduction to absurdity. Assume that

$$ {}^{\exists}t \in \mathbb{R} \ \text{ s.t. } \ \#\big(U_\delta(t) \cap \Gamma\big) \ge 2 \quad ({}^{\forall}\delta > 0). $$

Then, for ${}^{\forall}n \in \mathbb{N}$, ${}^{\exists}t_n \in \Gamma$ s.t. $0 < |t_n - t| < \frac{1}{n}$. By the definition of Γ, ${}^{\exists}\sigma_n \in (0,1)$ s.t. $\zeta(\sigma_n + \sqrt{-1}t_n) = 0$. Since $\{\sigma_n\}_{n=1}^{\infty}$ is bounded, ${}^{\exists}\{n'\}$: a subsequence, ${}^{\exists}\sigma \in [0,1]$ s.t. $\sigma_{n'} \to \sigma$.

In case $\sigma \ne 1$ or $\sigma = 1$ and $t \ne 0$,

$$ \begin{aligned} &\sigma_{n'} + \sqrt{-1}t_{n'} \to \sigma + \sqrt{-1}t \in \mathbb{C} \setminus \{1\}, \\ &\sigma_{n'} + \sqrt{-1}t_{n'} \ne \sigma + \sqrt{-1}t \ \ ({}^{\forall}n') \quad \big[\because\ t_{n'} \ne t\big], \\ &\{\sigma_{n'} + \sqrt{-1}t_{n'}\}_{n'} \subset \mathbb{C} \setminus \{1\} \quad \big[\because\ \sigma_{n'} \in (0,1)\big], \\ &\zeta(\sigma_{n'} + \sqrt{-1}t_{n'}) = 0 \ \ ({}^{\forall}n'). \end{aligned} $$

By the uniqueness theorem, this implies that $\zeta(\cdot) = 0$ on $\mathbb{C} \setminus \{1\}$.

In case $\sigma = 1$ and $t = 0$,

$$ \begin{aligned} &\sigma_{n'} + \sqrt{-1}t_{n'} \ne 1 \ \ ({}^{\forall}n') \quad \big[\because\ \sigma_{n'} \in (0,1)\big], \\ &\zeta(\sigma_{n'} + \sqrt{-1}t_{n'}) = 0 \ \ ({}^{\forall}n'), \\ &\sigma_{n'} + \sqrt{-1}t_{n'} \to 1. \end{aligned} $$

[†1] Here Γ is a subset of $\mathbb{R}$. It is entirely a different thing from the gamma function.

By Theorem 4.2, it is seen that

$$0 = \big(\sigma_{n'} + \sqrt{-1}t_{n'} - 1\big)\zeta(\sigma_{n'} + \sqrt{-1}t_{n'}) \to 1.$$

Since a contradiction occurs in either case, the assertion of the claim must hold. ■

Remark 5.1 For $-\infty < {}^\forall S < {}^\forall T < \infty$, $[S,T] \cap \Gamma$ is a finite set, and thus Γ is at most countable. In fact, for ${}^\forall t \in [S,T]$, ${}^\exists \delta_t > 0$ s.t. $\#\big(U_{\delta_t}(t) \cap \Gamma\big) \le 1$. Since $\{U_{\delta_t}(t)\}_{t\in[S,T]}$ is an open covering of $[S,T]$, the compactness of $[S,T]$ yields that

$${}^\exists t_1, \ldots, {}^\exists t_m \in [S,T] \ \text{ s.t. } \ [S,T] \subset \bigcup_{i=1}^m U_{\delta_{t_i}}(t_i).$$

Then

$$\#\big([S,T]\cap\Gamma\big) \le \#\bigcup_{i=1}^m \big(U_{\delta_{t_i}}(t_i)\cap\Gamma\big) \le \sum_{i=1}^m \#\big(U_{\delta_{t_i}}(t_i)\cap\Gamma\big) \le m,$$

which is just the first assertion. From an identity

$$\Gamma = \Gamma \cap \mathbb{R} = \Gamma \cap \sum_{n\in\mathbb{Z}} (n, n+1] = \sum_{n\in\mathbb{Z}} \Gamma \cap (n, n+1],$$

the second assertion is obvious.

Remark 5.2 $-\Gamma = \Gamma$. For, by Remark 4.1,

$$\begin{aligned}
t \in -\Gamma &\Leftrightarrow -t \in \Gamma \\
&\Leftrightarrow {}^\exists\sigma \in (0,1) \ \text{ s.t. } \ \zeta(\sigma - \sqrt{-1}t) = 0 \\
&\qquad\qquad\qquad\qquad\quad \| \\
&\qquad\qquad\qquad\quad \zeta\Big(\overline{\sigma + \sqrt{-1}t}\Big) = \overline{\zeta(\sigma + \sqrt{-1}t)} \\
&\Leftrightarrow {}^\exists\sigma \in (0,1) \ \text{ s.t. } \ \zeta(\sigma + \sqrt{-1}t) = 0 \\
&\Leftrightarrow t \in \Gamma.
\end{aligned}$$

Remark 5.3 For each $t \in \Gamma$, $\big\{\sigma \in (0,1); \zeta(\sigma + \sqrt{-1}t) = 0\big\}$ is symmetric relative to $\sigma = \frac{1}{2}$. In other words,

$$\zeta(\sigma + \sqrt{-1}t) = 0 \underset{\text{iff}}{\Longleftrightarrow} \zeta(1 - \sigma + \sqrt{-1}t) = 0.$$

For since, for $s = \sigma + \sqrt{-1}t$ $(0 < \sigma < 1,\ t \in \mathbb{R})$, $\Gamma(1-s)$[†2] $\ne 0$, $\sin\frac{\pi}{2}s \ne 0$ and $(2\pi)^{s-1} \ne 0$, it follows from Theorem 4.3(i) that

$$\zeta(s) = 0 \underset{\text{iff}}{\Longleftrightarrow} \zeta(1-s) = 0.$$

By this, together with Remark 4.1,

$$\begin{aligned}
\zeta(\sigma + \sqrt{-1}t) = 0 &\Leftrightarrow \zeta(1 - \sigma - \sqrt{-1}t) = 0 \\
&\qquad\qquad\qquad \| \\
&\qquad \zeta\Big(\overline{1 - \sigma + \sqrt{-1}t}\Big) = \overline{\zeta(1 - \sigma + \sqrt{-1}t)} \\
&\Leftrightarrow \zeta(1 - \sigma + \sqrt{-1}t) = 0.
\end{aligned}$$

[†2] This Γ is the gamma function.

Definition 5.1 For each $t \in \Gamma$, put

$$\sigma_t := \sup\{\sigma \in (0,1); \zeta(\sigma + \sqrt{-1}t) = 0\}.$$

Note that $\frac{1}{2} \le \sigma_t < 1$ by Theorem 4.4 and Remark 5.3. We now define

$$G := \mathbb{C} \setminus \Big(\bigcup_{t\in\Gamma} \{\sigma + \sqrt{-1}t; -\infty < \sigma \le \sigma_t\} \cup (-\infty, 1] \Big).$$

G is a domain, simply connected and $G \supset \{s \in \mathbb{C}; \operatorname{Re} s > 1\}$. Moreover $\zeta(\cdot)$ is holomorphic and has no zeros on G.

Claim 5.2 $^{\exists 1}l$*: holomorphic on* G *s.t.* $\begin{cases} l(2) = \log \dfrac{\pi^2}{6}, \\ \zeta(s) = e^{l(s)} \quad (s \in G). \end{cases}$

Proof. By (3.4) with $a = 2$, let $l(s) = \log \zeta(s)$. Then $e^{l(s)} = \zeta(s)$ $(s \in G)$ by (3.6), and $l(2) = \log \zeta(2) = \log \frac{\pi^2}{6}$ [cf. Claim A.10].

Next let l_0 be another holomorphic function on G, i.e., l_0 holomorphic on G s.t.

$$\begin{cases} l_0(2) = \log \dfrac{\pi^2}{6}, \\ \zeta(s) = e^{l_0(s)} \quad (s \in G). \end{cases}$$

Then $e^{l_0(s)} = \zeta(s) = e^{l(s)}$ $(s \in G)$. Differentiation in s gives that

$$\begin{aligned} e^{l_0(s)} l_0'(s) = e^{l(s)} l'(s) &\Leftrightarrow \zeta(s) l_0'(s) = \zeta(s) l'(s) \\ &\Leftrightarrow l_0'(s) = l'(s) \quad [\because \zeta(s) \neq 0 \text{ on } G] \\ &\Leftrightarrow {}^{\exists}a \in \mathbb{C} \text{ s.t. } l_0(s) = l(s) + a \ ({}^{\forall}s \in G). \end{aligned}$$

Since $l_0(2) = \log \frac{\pi^2}{6} = l(2)$, we have $a = 0$, and thus $l_0(s) = l(s)$ $({}^{\forall}s \in G)$. ■

Definition 5.2 We call $\log \zeta(\cdot)$ the *log zeta function*.

Remark 5.4 Under the Riemann hypothesis, $\sigma_t = \frac{1}{2}$ $({}^{\forall}t \in \Gamma)$, and so $G \supset \{s \in \mathbb{C}; \operatorname{Re} s > \frac{1}{2}\} \setminus (\frac{1}{2}, 1] =: G'$. Thus if we define the log zeta function as

$$\log \zeta(s) = \log \frac{\pi^2}{6} + \int_2^s \frac{\zeta'(z)}{\zeta(z)} dz, \quad s \in G',$$

then it becomes clear by not considering G. But, since the Riemann hypothesis is still open, we need some effort as above.

Claim 5.3 $\log \zeta(s) = \displaystyle\sum_{p:prime} -\log\Big(1 - \frac{1}{p^s}\Big)$, $\operatorname{Re} s > 1$.

Proof. Let $\sigma := \operatorname{Re} s > 1$ for simplicity. By (3.9),

$$\left|\log\left(1-\frac{1}{p^s}\right)\right| \le \frac{\frac{1}{p^\sigma}}{1-\frac{1}{p^\sigma}} \le \frac{\frac{1}{p^\sigma}}{1-\frac{1}{2^\sigma}}.$$

Since $\sum_{p:\text{prime}} \frac{1}{p^\sigma} < \infty$, $\sum_{p:\text{prime}} -\log\bigl(1-\frac{1}{p^s}\bigr)$ is uniformly convergent on every compact set of $\{s \in \mathbb{C}; \operatorname{Re} s > 1\}$, and thus it is holomorphic there.

By Claim 5.2, Claim 4.3 and (3.2),

$$\begin{aligned} e^{\log\zeta(s)} = \zeta(s) = \prod_p \frac{1}{1-\frac{1}{p^s}} = \prod_p e^{-\log(1-\frac{1}{p^s})} &= \lim_{n\to\infty} \prod_{i=1}^n e^{-\log(1-\frac{1}{p_i^s})} \\ &= \lim_{n\to\infty} e^{\sum_{i=1}^n -\log(1-\frac{1}{p_i^s})} \\ &= e^{\sum_{i=1}^\infty -\log(1-\frac{1}{p_i^s})} \\ &= e^{\sum_{p:\text{prime}} -\log(1-\frac{1}{p^s})}. \end{aligned}$$

In particular, when $s = 2$, $e^{\log\zeta(2)} = e^{\sum_{p:\text{prime}} -\log(1-\frac{1}{p^2})}$, and since $\log\zeta(2) = \log\frac{\pi^2}{6} \in \mathbb{R}$ and $\sum_{p:\text{prime}} -\log\bigl(1-\frac{1}{p^2}\bigr) \in \mathbb{R}$, it follows that $\log\zeta(2) = \sum_{p:\text{prime}} -\log(1-\frac{1}{p^2})$. This implies that $\log\zeta(s) = \sum_{p:\text{prime}} -\log(1-\frac{1}{p^s})$. ■

Claim 5.4 (i) *$s \in G$ implies $\overline{s} \in G$.*
(ii) $\overline{\log\zeta(s)} = \log\zeta(\overline{s})$ $(s \in G)$.

Proof. (i) Since, by Remark 5.2,

$$\begin{aligned} \sigma_t &= \sup\bigl\{\sigma \in (0,1); \zeta(\sigma+\sqrt{-1}t) = 0\bigr\} \\ &= \sup\bigl\{\sigma \in (0,1); \zeta(\sigma-\sqrt{-1}t) = 0\bigr\} = \sigma_{-t}, \end{aligned}$$

we have

$$\begin{aligned} G &= \mathbb{C} \setminus \left(\bigcup_{t\in\Gamma}((-\infty,\sigma_t]+\sqrt{-1}t) \cup (-\infty,1]\right) \\ &= \mathbb{C} \setminus \left(\left(\bigcup_{t\in\Gamma\cap[0,\infty)}((-\infty,\sigma_t]+\sqrt{-1}t)\right)\right. \\ &\qquad \left.\cup \left(\bigcup_{t\in\Gamma\cap(-\infty,0]}((-\infty,\sigma_t]+\sqrt{-1}t)\right) \cup (-\infty,1]\right) \\ &= \mathbb{C} \setminus \left(\left(\bigcup_{t\in\Gamma\cap[0,\infty)}((-\infty,\sigma_t]+\sqrt{-1}t)\right)\right. \\ &\qquad \left.\cup \left(\bigcup_{t\in\Gamma\cap[0,\infty)}((-\infty,\sigma_{-t}]-\sqrt{-1}t)\right) \cup (-\infty,1]\right) \\ &= \mathbb{C} \setminus \left(\left(\bigcup_{t\in\Gamma\cap[0,\infty)}((-\infty,\sigma_t]+\sqrt{-1}t)\right)\right. \end{aligned}$$

$$\cup \left(\bigcup_{t \in \Gamma \cap [0,\infty)} \left((-\infty, \sigma_t] - \sqrt{-1}t \right) \right) \cup (-\infty, 1] \Bigg),$$

from which the assertion (i) follows immediately.
(ii) Let $s = \sigma + \sqrt{-1}t \in G$. By (i), $\overline{s} = \sigma - \sqrt{-1}t \in G$. By (3.4),

$$\begin{aligned}
&\log \zeta(\sigma + \sqrt{-1}t) \\
&= \log \frac{\pi^2}{6} + \int_2^{\sigma+\sqrt{-1}t} \frac{\zeta'(z)}{\zeta(z)} dz \\
&= \log \frac{\pi^2}{6} + \int_2^{2+\sqrt{-1}t} \frac{\zeta'(z)}{\zeta(z)} dz + \int_{2+\sqrt{-1}t}^{\sigma+\sqrt{-1}t} \frac{\zeta'(z)}{\zeta(z)} dz \\
&= \log \frac{\pi^2}{6} + \int_0^1 \frac{\zeta'(2+\sqrt{-1}tv)}{\zeta(2+\sqrt{-1}tv)} (\sqrt{-1}t) dv + \int_2^\sigma \frac{\zeta'(u+\sqrt{-1}t)}{\zeta(u+\sqrt{-1}t)} du \\
&\quad \begin{bmatrix} \because \text{ change of variables: } z = 2 + \sqrt{-1}tv \text{ in the 2nd term;} \\ z = u + \sqrt{-1}t \text{ in the 3rd term} \end{bmatrix}.
\end{aligned}$$

Taking the conjugate, we have

$$\begin{aligned}
&\overline{\log \zeta(\sigma + \sqrt{-1}t)} \\
&= \log \frac{\pi^2}{6} + \int_0^1 \frac{\overline{\zeta'(2+\sqrt{-1}tv)}}{\overline{\zeta(2+\sqrt{-1}tv)}} (-\sqrt{-1}t) dv + \int_2^\sigma \frac{\overline{\zeta'(u+\sqrt{-1}t)}}{\overline{\zeta(u+\sqrt{-1}t)}} du \\
&= \log \frac{\pi^2}{6} + \int_0^1 \frac{\zeta'(2-\sqrt{-1}tv)}{\zeta(2-\sqrt{-1}tv)} (-\sqrt{-1}t) dv + \int_2^\sigma \frac{\zeta'(u-\sqrt{-1}t)}{\zeta(u-\sqrt{-1}t)} du \\
&\quad \begin{bmatrix} \because \text{ By Remark 4.1,} & \\ \overline{\zeta'(s)} = \overline{\lim_{h\to 0} \frac{\zeta(s+h)-\zeta(s)}{h}} & = \lim_{h\to 0} \frac{\overline{\zeta(s+h)}-\overline{\zeta(s)}}{\overline{h}} \\ & = \lim_{h\to 0} \frac{\zeta(\overline{s}+\overline{h})-\zeta(\overline{s})}{\overline{h}} \\ & = \zeta'(\overline{s}) \end{bmatrix} \\
&= \log \frac{\pi^2}{6} + \int_2^{2-\sqrt{-1}t} \frac{\zeta'(z)}{\zeta(z)} dz + \int_{2-\sqrt{-1}t}^{\sigma-\sqrt{-1}t} \frac{\zeta'(z)}{\zeta(z)} dz \\
&= \log \frac{\pi^2}{6} + \int_2^{\sigma-\sqrt{-1}t} \frac{\zeta'(z)}{\zeta(z)} dz \\
&= \log \zeta(\sigma - \sqrt{-1}t).
\end{aligned}$$

■

5.2 Presentation of the main theorem

We are now in a position to state the main theorem in this monograph.

Theorem 5.1 (Bohr-Jessen limit theorem) *Let* $\sigma_0 > \frac{1}{2}$. *As* $T \to \infty$, *the probability measure*

$$\frac{1}{2T} \mu\Big(t \in [-T, T]; \sigma_0 + \sqrt{-1}t \in G, \log \zeta(\sigma_0 + \sqrt{-1}t) \in \cdot \Big)$$

on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ *converges weakly to the distribution of*

$$\sum_{p:prime} -\log\Big(1-\frac{e(-\log p)}{p^{\sigma_0}}\Big).$$

Namely

$$\begin{aligned}&\frac{1}{2T}\mu\Big(t\in[-T,T];\sigma_0+\sqrt{-1}t\in G,\log\zeta(\sigma_0+\sqrt{-1}t)\in\cdot\Big)\\ &\to \mathbf{P}\Big(\sum_{p:prime}-\log\Big(1-\frac{e(-\log p)}{p^{\sigma_0}}\Big)\in\cdot\Big)\ \textit{weakly}\quad \big[\textit{cf.}\ \text{Definition A.1}\big].\end{aligned}$$

Here μ *is the* 1*-dimensional Lebesgue measure.*

Remark 5.5 (i) For ${}^{\forall}B\in\mathcal{B}(\mathbb{C})$,

$$\Big\{t\in[-T,T];\sigma_0+\sqrt{-1}t\in G,\log\zeta(\sigma_0+\sqrt{-1}t)\in B\Big\}\in\mathcal{B}(\mathbb{R}).$$

For, by the continuity of $G\ni z\mapsto\log\zeta(z)\in\mathbb{C}$, $(\log\zeta)^{-1}(B)\in\mathcal{B}(G)=\mathcal{B}(\mathbb{C})\cap G$, and by the continuity of $\mathbb{R}\ni t\mapsto\sigma_0+\sqrt{-1}t\in\mathbb{C}$, $\big\{t\in\mathbb{R};\sigma_0+\sqrt{-1}t\in(\log\zeta)^{-1}(B)\big\}\in\mathcal{B}(\mathbb{R})$. Thus the set above $=[-T,T]\cap\big\{t\in\mathbb{R};\sigma_0+\sqrt{-1}t\in(\log\zeta)^{-1}(B)\big\}\in\mathcal{B}(\mathbb{R})$.

(ii) $\frac{1}{2T}\mu\left(t\in[-T,T];\begin{matrix}\sigma_0+\sqrt{-1}t\in G,\\ \log\zeta(\sigma_0+\sqrt{-1}t)\in\cdot\end{matrix}\right)$ is a probability measure on $(\mathbb{C},\mathcal{B}(\mathbb{C}))$. For, first

$$\frac{1}{2T}\mu\left(t\in[-T,T];\begin{matrix}\sigma_0+\sqrt{-1}t\in G,\\ \log\zeta(\sigma_0+\sqrt{-1}t)\in\mathbb{C}\end{matrix}\right)=\frac{1}{2T}\mu\big(t\in[-T,T];\sigma_0+\sqrt{-1}t\in G\big).$$

Since, by the definition of G,

$$\begin{aligned}&\big\{t\in[-T,T];\sigma_0+\sqrt{-1}t\in G\big\}\\ &=\begin{cases}[-T,T] & \text{if } \sigma_0>1,\\ [-T,T]\setminus\{0\} & \text{if } \sigma_0=1,\\ \big\{t\in[-T,T]\cap\Gamma;\sigma_t<\sigma_0\big\}\cup\big([-T,T]\cap\Gamma^{\complement}\big) & \text{if } \frac12<\sigma_0<1,\end{cases}\end{aligned}$$

it follows from Remark 5.1 that

$$\frac{1}{2T}\mu\big(t\in[-T,T];\sigma_0+\sqrt{-1}t\in G\big)=\frac{1}{2T}\mu\big([-T,T]\big)=1.$$

(iii) The characteristic function of $\frac{1}{2T}\mu\left(t\in[-T,T];\begin{matrix}\sigma_0+\sqrt{-1}t\in G,\\ \log\zeta(\sigma_0+\sqrt{-1}t)\in\cdot\end{matrix}\right)$ is as follows $\big[$cf. Definition A.2(ii)$\big]$:

$$\begin{aligned}&\Big(\frac{1}{2T}\mu\Big(t\in[-T,T];\sigma_0+\sqrt{-1}t\in G,\log\zeta(\sigma_0+\sqrt{-1}t)\in\cdot\Big)\Big)^{\widehat{}}(w)\\ &=\frac{1}{2T}\int_{-T}^{T}\mathbf{1}_G\big(\sigma_0+\sqrt{-1}t\big)e^{\sqrt{-1}\langle\log\zeta(\sigma_0+\sqrt{-1}t),w\rangle}dt,\quad w\in\mathbb{C}.\end{aligned}$$

Here, for $z,w\in\mathbb{C}$, we let

$$\langle z,w\rangle:=(\operatorname{Re}z)\cdot(\operatorname{Re}w)+(\operatorname{Im}z)\cdot(\operatorname{Im}w). \tag{5.1}$$

Remark 5.6 Let U[†3] be a real random variable uniformly distributed on $[-1, 1]$. For $\sigma > \frac{1}{2}$ and $T > 0$, put a complex random variable $\widetilde{X}_T(\sigma)$ as

$$\widetilde{X}_T(\sigma) = \mathbf{1}_G(\sigma + \sqrt{-1}TU)\log\zeta(\sigma + \sqrt{-1}TU).$$

Then its distribution is just

$$\frac{1}{2T}\mu\Big(t \in [-T, T]; \sigma + \sqrt{-1}t \in G, \log\zeta(\sigma + \sqrt{-1}t) \in \cdot\Big).$$

Therefore Theorem 5.1 can be stated in the following form:

$$\widetilde{X}_T(\sigma) \textit{ converges in law to } \sum_p -\log\big(1 - \frac{e(-\log p)}{p^\sigma}\big) \textit{ as } T \to \infty.$$

This expression is familiar to our probabilists, though it is only a paraphrase. In the following, however, we go on with the previous one.

Remark 5.7 We make one more comment. If we only state Theorem 5.1 and / or the convergence in Remark 5.6, a complex random variable $\sum_p -\log\big(1 - \frac{e(-\log p)}{p^\sigma}\big)$ on $(\mathbb{R}^{\mathbb{B}}, \mathbf{P})$ is not necessary. On some probability space take a sequence $\{\Theta_p\}_{p:\text{prime}}$ of independent complex random variables such that each Θ_p is uniformly distributed on the unit circle of $\mathbb{C}$, i.e., $E\big[f(\Theta_p)\big] = \frac{1}{2\pi}\int_0^{2\pi} f(e^{\sqrt{-1}t})dt$ for $^\forall f : \mathbb{C} \to \mathbb{C}$ bounded Borel measurable $\big[$cf. Claim 3.1$\big]$. Then $\sum_p -\log\big(1 - \frac{\Theta_p}{p^\sigma}\big)$ is equivalent in law to $\sum_p -\log\big(1 - \frac{e(-\log p)}{p^\sigma}\big)$, and thus, for instance, the convergence in Remark 5.6 becomes

$$\widetilde{X}_T(\sigma) \textit{ converges in law to } \sum_p -\log\big(1 - \frac{\Theta_p}{p^\sigma}\big) \textit{ as } T \to \infty.$$

When we prove this, however, what $\sum_p -\log\big(1 - \frac{e(-\log p)}{p^\sigma}\big)$ is plays a great role for the proof. For this reason, we go ahead with this setting.

From Corollary 3.2, the following is obvious:

Corollary 5.1 *For* $-\infty < \alpha_1 \le \alpha_2 < \infty$, $-\infty < \beta_1 \le \beta_2 < \infty$, *let* $E_{\alpha_1,\alpha_2;\beta_1,\beta_2}$ *be a closed rectangle in* $\mathbb{C}$ *defined by* (3.14). *Then*

$$\begin{aligned}&\lim_{T\to\infty}\frac{1}{2T}\mu\Big(t \in [-T, T]; \sigma_0 + \sqrt{-1}t \in G, \log\zeta(\sigma_0 + \sqrt{-1}t) \in E_{\alpha_1,\alpha_2;\beta_1,\beta_2}\Big)\\ &= \mathbf{P}\bigg(\sum_p -\log\Big(1 - \frac{e(-\log p)}{p^{\sigma_0}}\Big) \in E_{\alpha_1,\alpha_2;\beta_1,\beta_2}\bigg).\end{aligned}$$

This is the original statement of the Bohr-Jessen limit theorem $\big[$cf. Theorem 2$\big]$.

Proof. For simplicity, put probability measures ν_T and ν on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ as

$$\nu_T(\cdot) := \frac{1}{2T}\mu\Big(t \in [-T, T]; \sigma_0 + \sqrt{-1}t \in G, \log\zeta(\sigma_0 + \sqrt{-1}t) \in \cdot\Big)$$

[†3]This U is a different thing from that in 2° in the proof of Claim 3.1.

$$\nu(\cdot) := \mathbf{P}\Big(\sum_p -\log\Big(1 - \frac{e(-\log p)}{p^{\sigma_0}}\Big) \in \cdot\Big).$$

Since, by Corollary 3.2, $E_{\alpha_1,\alpha_2;\beta_1,\beta_2}$ is a continuity set of ν, Theorem 5.1, together with Claim A.1, implies the assertion of the corollary. ■

From this corollary and Claim 3.4, the following is easily seen:

Corollary 5.2 *When* $\frac{1}{2} < \sigma_0 \leq 1$, *the set* $\{\log\zeta(\sigma_0+\sqrt{-1}t); t \in \mathbb{R}$ *with* $\sigma_0+\sqrt{-1}t \in G\}$ *is dense in* $\mathbb{C}$.

Proof. Let $\frac{1}{2} < \sigma_0 \leq 1$. Fix ${}^\forall z_0 \in \mathbb{C}$ and ${}^\forall \varepsilon > 0$. Put

$$Q_{z_0,\varepsilon} := \big\{z \in \mathbb{C}; |\operatorname{Re} z - \operatorname{Re} z_0| \leq \varepsilon,\ |\operatorname{Im} z - \operatorname{Im} z_0| \leq \varepsilon\big\}.$$

Clearly $Q_{z_0,\varepsilon} = E_{\operatorname{Re} z_0-\varepsilon,\operatorname{Re} z_0+\varepsilon;\operatorname{Im} z_0-\varepsilon,\operatorname{Im} z_0+\varepsilon}$. By Corollary 5.1,

$$\begin{aligned}
&\lim_{T\to\infty}\frac{1}{2T}\mu\Big(t \in [-T,T]; \sigma_0+\sqrt{-1}t \in G, \log\zeta(\sigma_0+\sqrt{-1}t) \in Q_{z_0,\varepsilon}\Big)\\
&= \mathbf{P}\Big(\sum_p -\log\Big(1-\frac{e(-\log p)}{p^{\sigma_0}}\Big) \in Q_{z_0,\varepsilon}\Big).
\end{aligned}$$

On the other hand, by Claim 3.4,

$$\begin{aligned}
&\mathbf{P}\Big(\sum_p -\log\Big(1-\frac{e(-\log p)}{p^{\sigma_0}}\Big) \in Q_{z_0,\varepsilon}\Big)\\
&= p_{\sigma_0}\Big(\big[\operatorname{Re} z_0-\varepsilon, \operatorname{Re} z_0+\varepsilon\big] \times \big[\operatorname{Im} z_0-\varepsilon, \operatorname{Im} z_0+\varepsilon\big]\Big) > 0.
\end{aligned}$$

Thus, for some $T > 0$,

$$\mu\Big(t \in [-T,T]; \sigma_0+\sqrt{-1}t \in G, \log\zeta(\sigma_0+\sqrt{-1}t) \in Q_{z_0,\varepsilon}\Big) > 0.$$

This tells us that ${}^\exists t_0 \in [-T,T]$ s.t. $\sigma_0+\sqrt{-1}t_0 \in G$, $\log\zeta(\sigma_0+\sqrt{-1}t_0) \in Q_{z_0,\varepsilon}$. Since $\varepsilon > 0$ is arbitrary, z_0 is an adherent point of the set $\{\log\zeta(\sigma_0+\sqrt{-1}t); t \in \mathbb{R}$ with $\sigma_0 + \sqrt{-1}t \in G\}$, so that we have the assertion of the corollary. ■

This corollary is just Theorem 1. Theorem 5.1 is generalized to the limit theorem for a class of more general zeta functions [cf. Laurinčikas [21, 22, 23], Matsumoto [24, 25]]. But, the generalization of Corollary 5.2 similar to that of Theorem 5.1 will be probably difficult, since the proof of this corollary is based on Claim 3.4, i.e., Lemma 3.2.

5.3 Proof of the main theorem

We divide the proof of Theorem 5.1 into two cases:

- the case where $\sigma_0 > 1$,
- the case where $\frac{1}{2} < \sigma_0 \leq 1$.

As will be seen below, the proof of the latter needs much efforts. Compared with this, that of the former is straightforward. In either case, to show is the following convergence of characteristic functions: For $^\forall w \in \mathbb{C}$,

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \mathbf{1}_G\big(\sigma_0+\sqrt{-1}t\big)e^{\sqrt{-1}\left\langle \log\zeta(\sigma_0+\sqrt{-1}t),w\right\rangle}dt$$
$$= E^{\mathbf{P}}\left[e^{\sqrt{-1}\left\langle \sum_{i=1}^{\infty} -\log(1-\frac{e(-\log p_i)}{p_i^{\sigma_0}}),w\right\rangle}\right]. \tag{5.2}$$

Proof of Theorem 5.1 *in the case where* $\sigma_0>1$. Fix $\sigma_0>1$. Note that $\{\sigma_0+\sqrt{-1}t; t\in\mathbb{R}\}\subset G$. By Claim 5.3,

$$\log\zeta(\sigma_0+\sqrt{-1}t) = \sum_p -\log\Big(1-\frac{1}{p^{\sigma_0+\sqrt{-1}t}}\Big)$$
$$= \sum_p -\log\Big(1-\frac{e^{-\sqrt{-1}t\log p}}{p^{\sigma_0}}\Big). \tag{5.3}$$

In the following, we divide the proof into three steps:

$\underline{1^\circ}$ For $^\forall N\in\mathbb{N}$,

$$\left|\log\zeta(\sigma_0+\sqrt{-1}t)-\sum_{i=1}^{N}-\log\Big(1-\frac{e^{-\sqrt{-1}t\log p_i}}{p_i^{\sigma_0}}\Big)\right| \le \frac{2^{\sigma_0}}{2^{\sigma_0}-1}\sum_{p>p_N}\frac{1}{p^{\sigma_0}}.$$

$\because$ By (5.3),

$$\begin{aligned}
&\left|\log\zeta(\sigma_0+\sqrt{-1}t)-\sum_{i=1}^{N}-\log\Big(1-\frac{e^{-\sqrt{-1}t\log p_i}}{p_i^{\sigma_0}}\Big)\right|\\
&=\left|\sum_{i=1}^{\infty}-\log\Big(1-\frac{e^{-\sqrt{-1}t\log p_i}}{p_i^{\sigma_0}}\Big)-\sum_{i=1}^{N}-\log\Big(1-\frac{e^{-\sqrt{-1}t\log p_i}}{p_i^{\sigma_0}}\Big)\right|\\
&=\left|\sum_{i=N+1}^{\infty}-\log\Big(1-\frac{e^{-\sqrt{-1}t\log p_i}}{p_i^{\sigma_0}}\Big)\right|\\
&\le\sum_{i=N+1}^{\infty}\left|\log\Big(1-\frac{e^{-\sqrt{-1}t\log p_i}}{p_i^{\sigma_0}}\Big)\right|\\
&\le\sum_{i=N+1}^{\infty}\frac{\frac{1}{p_i^{\sigma_0}}}{1-\frac{1}{p_i^{\sigma_0}}}\qquad\big[\text{cf. (3.9)}\big]\\
&\le\frac{1}{1-\frac{1}{2^{\sigma_0}}}\sum_{i=N+1}^{\infty}\frac{1}{p_i^{\sigma_0}}\\
&\qquad\Big[\because\ p_i\ge 2\Rightarrow p_i^{\sigma_0}\ge 2^{\sigma_0}\Rightarrow\tfrac{1}{p_i^{\sigma_0}}\le\tfrac{1}{2^{\sigma_0}}\Rightarrow\tfrac{1}{1-\frac{1}{p_i^{\sigma_0}}}\le\tfrac{1}{1-\frac{1}{2^{\sigma_0}}}\Big]\\
&=\frac{2^{\sigma_0}}{2^{\sigma_0}-1}\sum_{p>p_N}\frac{1}{p^{\sigma_0}}.
\end{aligned}$$

2° For $^\forall N \in \mathbb{N}$ and $^\forall w \in \mathbb{C}$,

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} e^{\sqrt{-1}\left\langle \sum_{i=1}^{N} -\log\left(1-\frac{e^{-\sqrt{-1}t\log p_i}}{p_i^{\sigma_0}}\right),w\right\rangle}dt$$
$$= E^{\mathbf{P}}\left[e^{\sqrt{-1}\left\langle \sum_{i=1}^{N} -\log\left(1-\frac{e(-\log p_i)}{p_i^{\sigma_0}}\right),w\right\rangle}\right].$$

∵ This is seen in the following way:

$$\frac{1}{2T}\int_{-T}^{T} e^{\sqrt{-1}\left\langle \sum_{i=1}^{N} -\log\left(1-\frac{e^{-\sqrt{-1}t\log p_i}}{p_i^{\sigma_0}}\right),w\right\rangle}dt$$
$$= \frac{1}{2T}\int_{-T}^{T} e^{\sqrt{-1}\left\langle \sum_{i=1}^{N} -\log\left(1-\frac{1}{p_i^{\sigma_0}}(\cos t(-\log p_i)+\sqrt{-1}\sin t(-\log p_i))\right),w\right\rangle}dt$$
$$= \int_{\mathbb{R}^{2N}} e^{\sqrt{-1}\left\langle \sum_{i=1}^{N} -\log\left(1-\frac{1}{p_i^{\sigma_0}}(x_i+\sqrt{-1}y_i)\right),w\right\rangle}$$
$$\times P_T^{(\cos(-\log p_1)\cdot,\sin(-\log p_1)\cdot,\ldots,\cos(-\log p_N)\cdot,\sin(-\log p_N)\cdot)}(dx_1dy_1\cdots dx_Ndy_N)$$
$$\to \int_{\mathbb{R}^{2n}} e^{\sqrt{-1}\left\langle \sum_{i=1}^{N} -\log\left(1-\frac{1}{p_i^{\sigma_0}}(x_i+\sqrt{-1}y_i)\right),w\right\rangle}$$
$$\times \mathbf{P}\circ\pi^{-1}_{(\cos(-\log p_1)\cdot,\sin(-\log p_1)\cdot,\ldots,\cos(-\log p_N)\cdot,\sin(-\log p_N)\cdot)}(dx_1dy_1\cdots dx_Ndy_N)$$

[∵ Lemma 2.1 and Theorem 2.1],

$$= \int_{\mathbb{R}^{\mathbb{B}}} e^{\sqrt{-1}\left\langle \sum_{i=1}^{N} -\log\left(1-\frac{1}{p_i^{\sigma_0}}(\pi_{\cos(-\log p_i)\cdot}+\sqrt{-1}\pi_{\sin(-\log p_i)\cdot})\right),w\right\rangle}d\mathbf{P}$$
$$= \int_{\mathbb{R}^{\mathbb{B}}} e^{\sqrt{-1}\left\langle \sum_{i=1}^{N} -\log\left(1-\frac{e(-\log p_i)}{p_i^{\sigma_0}}\right),w\right\rangle}d\mathbf{P}$$
$$= E^{\mathbf{P}}\left[e^{\sqrt{-1}\left\langle \sum_{i=1}^{N} -\log\left(1-\frac{e(-\log p_i)}{p_i^{\sigma_0}}\right),w\right\rangle}\right].$$

3° For $^\forall w \in \mathbb{C}$,

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} e^{\sqrt{-1}\langle\log\zeta(\sigma_0+\sqrt{-1}t),w\rangle}dt = E^{\mathbf{P}}\left[e^{\sqrt{-1}\left\langle \sum_{i=1}^{\infty} -\log\left(1-\frac{e(-\log p_i)}{p_i^{\sigma_0}}\right),w\right\rangle}\right].$$

Since $\sigma_0+\sqrt{-1}t \in G$, this is the convergence (5.2), and thus the assertion in the case where $\sigma_0 > 1$ is seen.

∵ By 1°,

$$\left|\frac{1}{2T}\int_{-T}^{T} e^{\sqrt{-1}\langle\log\zeta(\sigma_0+\sqrt{-1}t),w\rangle}dt - E^{\mathbf{P}}\left[e^{\sqrt{-1}\left\langle \sum_{i=1}^{\infty} -\log\left(1-\frac{e(-\log p_i)}{p_i^{\sigma_0}}\right),w\right\rangle}\right]\right|$$
$$= \left|\frac{1}{2T}\int_{-T}^{T} e^{\sqrt{-1}\langle\log\zeta(\sigma_0+\sqrt{-1}t),w\rangle}dt\right.$$
$$- \frac{1}{2T}\int_{-T}^{T} e^{\sqrt{-1}\left\langle \sum_{i=1}^{N} -\log\left(1-\frac{e^{-\sqrt{-1}t\log p_i}}{p_i^{\sigma_0}}\right),w\right\rangle}dt$$

$$
\begin{aligned}
&+ \frac{1}{2T}\int_{-T}^{T} e^{\sqrt{-1}\left\langle \sum_{i=1}^{N} -\log\left(1-\frac{e^{-\sqrt{-1}t\log p_i}}{p_i^{\sigma_0}}\right), w\right\rangle} dt \\
&- E^{\mathbf{P}}\left[e^{\sqrt{-1}\left\langle \sum_{i=1}^{N} -\log\left(1-\frac{e(-\log p_i)}{p_i^{\sigma_0}}\right), w\right\rangle}\right] \\
&+ E^{\mathbf{P}}\left[e^{\sqrt{-1}\left\langle \sum_{i=1}^{N} -\log\left(1-\frac{e(-\log p_i)}{p_i^{\sigma_0}}\right), w\right\rangle}\right] \\
&- E^{\mathbf{P}}\left[e^{\sqrt{-1}\left\langle \sum_{i=1}^{\infty} -\log\left(1-\frac{e(-\log p_i)}{p_i^{\sigma_0}}\right), w\right\rangle}\right]\Bigg|
\end{aligned}
$$

$$
\begin{aligned}
\le{}& \frac{1}{2T}\int_{-T}^{T} \left| e^{\sqrt{-1}\langle \log\zeta(\sigma_0+\sqrt{-1}t), w\rangle} - e^{\sqrt{-1}\left\langle \sum_{i=1}^{N} -\log\left(1-\frac{e^{-\sqrt{-1}t\log p_i}}{p_i^{\sigma_0}}\right), w\right\rangle}\right| dt \\
&+ \Bigg| \frac{1}{2T}\int_{-T}^{T} e^{\sqrt{-1}\left\langle \sum_{i=1}^{N} -\log\left(1-\frac{e^{-\sqrt{-1}t\log p_i}}{p_i^{\sigma_0}}\right), w\right\rangle} dt \\
&\quad - E^{\mathbf{P}}\left[e^{\sqrt{-1}\left\langle \sum_{i=1}^{N} -\log\left(1-\frac{e(-\log p_i)}{p_i^{\sigma_0}}\right), w\right\rangle}\right]\Bigg| \\
&+ E^{\mathbf{P}}\left[\left| e^{\sqrt{-1}\left\langle \sum_{i=1}^{N} -\log\left(1-\frac{e(-\log p_i)}{p_i^{\sigma_0}}\right), w\right\rangle} - e^{\sqrt{-1}\left\langle \sum_{i=1}^{\infty} -\log\left(1-\frac{e(-\log p_i)}{p_i^{\sigma_0}}\right), w\right\rangle}\right|\right]
\end{aligned}
$$

$$
\begin{aligned}
\le{}& \frac{1}{2T}\int_{-T}^{T} \left| \log\zeta(\sigma_0+\sqrt{-1}t) - \sum_{i=1}^{N} -\log\left(1-\frac{e^{-\sqrt{-1}t\log p_i}}{p_i^{\sigma_0}}\right)\right| |w| dt \\
&+ \Bigg| \frac{1}{2T}\int_{-T}^{T} e^{\sqrt{-1}\left\langle \sum_{i=1}^{N} -\log\left(1-\frac{e^{-\sqrt{-1}t\log p_i}}{p_i^{\sigma_0}}\right), w\right\rangle} dt \\
&\quad - E^{\mathbf{P}}\left[e^{\sqrt{-1}\left\langle \sum_{i=1}^{N} -\log\left(1-\frac{e(-\log p_i)}{p_i^{\sigma_0}}\right), w\right\rangle}\right]\Bigg| \\
&+ E^{\mathbf{P}}\left[\left| e^{\sqrt{-1}\left\langle \sum_{i=1}^{N} -\log\left(1-\frac{e(-\log p_i)}{p_i^{\sigma_0}}\right), w\right\rangle} - e^{\sqrt{-1}\left\langle \sum_{i=1}^{\infty} -\log\left(1-\frac{e(-\log p_i)}{p_i^{\sigma_0}}\right), w\right\rangle}\right|\right]
\end{aligned}
$$

$$
\left[\begin{aligned}
&\because \text{ For } z, z', w \in \mathbb{C}, \\
&|e^{\sqrt{-1}\langle z,w\rangle} - e^{\sqrt{-1}\langle z',w\rangle}| \le |\langle z, w\rangle - \langle z', w\rangle| \\
&\qquad\qquad\qquad\qquad\qquad = |\langle z - z', w\rangle| \\
&\qquad\qquad\qquad\qquad\qquad \le |z - z'|\,|w|
\end{aligned}\right]
$$

$$
\begin{aligned}
\le{}& \left(\frac{2^{\sigma_0}}{2^{\sigma_0}-1}\sum_{p>p_N}\frac{1}{p^{\sigma_0}}\right)|w| \\
&+ \Bigg| \frac{1}{2T}\int_{-T}^{T} e^{\sqrt{-1}\left\langle \sum_{i=1}^{N} -\log\left(1-\frac{e^{-\sqrt{-1}t\log p_i}}{p_i^{\sigma_0}}\right), w\right\rangle} dt \\
&\quad - E^{\mathbf{P}}\left[e^{\sqrt{-1}\left\langle \sum_{i=1}^{N} -\log\left(1-\frac{e(-\log p_i)}{p_i^{\sigma_0}}\right), w\right\rangle}\right]\Bigg|
\end{aligned}
$$

$$+ E^{\mathbf{P}}\Big[\Big|e^{\sqrt{-1}\left\langle \sum_{i=1}^{N} -\log\left(1-\frac{e(-\log p_i)}{p_i^{\sigma_0}}\right), w\right\rangle} - e^{\sqrt{-1}\left\langle \sum_{i=1}^{\infty} -\log\left(1-\frac{e(-\log p_i)}{p_i^{\sigma_0}}\right), w\right\rangle}\Big|\Big].$$

By 2°, letting $T \to \infty$ yields that

$$\begin{aligned}
&\overline{\lim_{T\to\infty}}\left|\frac{1}{2T}\int_{-T}^{T} e^{\sqrt{-1}\langle \log\zeta(\sigma_0+\sqrt{-1}t), w\rangle} dt - E^{\mathbf{P}}\Big[e^{\sqrt{-1}\left\langle \sum_{i=1}^{\infty} -\log\left(1-\frac{e(-\log p_i)}{p_i^{\sigma_0}}\right), w\right\rangle}\Big]\right| \\
&\le \Big(\frac{2^{\sigma_0}}{2^{\sigma_0}-1}\sum_{p>p_N}\frac{1}{p^{\sigma_0}}\Big)|w| \\
&\quad + E^{\mathbf{P}}\Big[\Big|e^{\sqrt{-1}\left\langle \sum_{i=1}^{N} -\log\left(1-\frac{e(-\log p_i)}{p_i^{\sigma_0}}\right), w\right\rangle} - e^{\sqrt{-1}\left\langle \sum_{i=1}^{\infty} -\log\left(1-\frac{e(-\log p_i)}{p_i^{\sigma_0}}\right), w\right\rangle}\Big|\Big].
\end{aligned}$$

We here let $N \to \infty$. Clearly the 1st term in R.H.S. $\to 0$. By Theorem 3.1, the 2nd term in R.H.S. $\to 0$. Thus we have the assertion of 3°. ■

Next is the proof in the case where $\frac{1}{2} < \sigma_0 \le 1$. For this, we need a definition and a claim.

Definition 5.3 For $N \in \mathbb{N}$ and $\operatorname{Re} s > 0$, put

$$\zeta_N(s) := \prod_{i=1}^{N}\frac{1}{1-\frac{1}{p_i^s}}. \tag{5.4}$$

Clearly $\zeta_N(\cdot)$ is holomorphic and has no zeros on $\{s \in \mathbb{C}; \operatorname{Re} s > 0\}$. By (3.4) with $a = 2$, $\log \zeta_N$ is defined. Then

$$\log\zeta_N(s) = \sum_{i=1}^{N} -\log\Big(1-\frac{1}{p_i^s}\Big), \quad s \in \mathbb{C} \text{ with } \operatorname{Re} s > 0 \tag{5.5}$$

holds. For, by (3.2) and (3.6),

$$\begin{aligned}
e^{\sum_{i=1}^{N} -\log(1-\frac{1}{p_i^s})} = \prod_{i=1}^{N} e^{-\log(1-\frac{1}{p_i^s})} &= \prod_{i=1}^{N}\frac{1}{1-\frac{1}{p_i^s}} \\
&= \zeta_N(s) = e^{\log\zeta_N(s)}, \quad {}^{\forall}s \in \mathbb{C} \text{ with } \operatorname{Re} s > 0.
\end{aligned}$$

Since $\sum_{i=1}^{N} -\log(1-\frac{1}{p_i^s}), \log\zeta_N(s) \in \mathbb{R}$ ($s \in (0,\infty)$), this, together with the uniqueness theorem, implies (5.5).

Also, by 1° and 2° in the proof of Claim 4.3, we note that for ${}^{\forall}\sigma > 1$,

$$\begin{aligned}
\Big|\frac{\zeta(s)}{\zeta_N(s)} - 1\Big| = \Big|\sum_{\substack{n\ge 2;\\ p_1\nmid n,\dots,p_N\nmid n}}\frac{1}{n^s}\Big| &\le \sum_{\substack{n\ge 2;\\ p_1\nmid n,\dots,p_N\nmid n}}\frac{1}{n^{\operatorname{Re} s}} \\
&\le \sum_{n>p_N}\frac{1}{n^{\sigma}} \to 0, \quad {}^{\forall}s \in \mathbb{C} \text{ with } \operatorname{Re} s \ge \sigma.
\end{aligned} \tag{5.6}$$

Claim 5.5 *Let* $\frac{1}{2} < \sigma_0 \le 1 < \sigma_1$ *and* $0 < \delta < \sigma_0 - \frac{1}{2}$. *Then*

$$
\begin{aligned}
&{}^\forall \eta > 0,\ {}^\exists N_0 = N_0(\sigma_0, \delta, \eta) \in \mathbb{N} \\
&\text{s.t.} \begin{cases} {}^\forall N \ge N_0,\ {}^\exists T_0 = T_0(N, \sigma_0, \sigma_1, \delta, \eta) \in [2, \infty) \\ \text{s.t.} \displaystyle\int_{\frac{1}{2}}^{T} \left| \frac{\zeta}{\zeta_N}(\sigma + \sqrt{-1}t) - 1 \right|^2 dt \le \eta T, \quad {}^\forall T \ge T_0, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \sigma_0 - \delta \le {}^\forall \sigma \le \sigma_1 + \delta. \end{cases}
\end{aligned}
$$

This claim will be proved at the end of the next chapter. Let us now recognize this and proceed to the proof in the case where $\frac{1}{2} < \sigma_0 \le 1$.

Proof of Theorem 5.1 *in the case where* $\frac{1}{2} < \sigma_0 \le 1$. Fix $\frac{1}{2} < \sigma_0 \le 1$. For $\sigma_1 = 2$, $0 < \delta < \sigma_0 - \frac{1}{2}$ and $0 < \eta < \delta^4$, take $N_0 = N_0(\sigma_0, \delta, \eta) \in \mathbb{N}$ as in Claim 5.5. Let $N \ge N_0$ be such that

$$\sum_{n > p_N} \frac{1}{n^{\frac{3}{2}}} \le \frac{1}{2}, \tag{5.7}$$

and take $T_0 = T_0(N, \sigma_0, \delta, \eta) \in [2, \infty)$ as in Claim 5.5. In the following, fix $T \ge T_0$. First

$$
\begin{aligned}
&\int_{\sigma_0 - \delta}^{2+\delta} d\sigma \int_{\frac{1}{2}}^{T} \left| \frac{\zeta}{\zeta_N}(\sigma + \sqrt{-1}t) - 1 \right|^2 dt \le (2 - \sigma_0 + 2\delta)\eta T. \\
&\qquad\qquad\qquad\qquad \| \\
&\int_{\frac{1}{2}}^{T} dt \int_{\sigma_0 - \delta}^{2+\delta} \left| \frac{\zeta}{\zeta_N}(\sigma + \sqrt{-1}t) - 1 \right|^2 d\sigma
\end{aligned}
\tag{5.8}
$$

Put $K_T \subset \{1, \dots, \lfloor T \rfloor - 1\}$ and $J_T \subset \left[\frac{1}{2} + \delta, T - \frac{1}{2} - \delta\right]$ as

$$
\begin{aligned}
K_T &:= \left\{ k \in \{1, \dots, \lfloor T \rfloor - 1\};\ \begin{array}{l} \displaystyle\int_{k-\frac{1}{2}}^{k+\frac{1}{2}} dt \int_{\sigma_0 - \delta}^{2+\delta} \left| \frac{\zeta}{\zeta_N}(\sigma + \sqrt{-1}t) - 1 \right|^2 d\sigma \\ \le (2 - \sigma_0 + 2\delta)\sqrt{\eta} \end{array} \right\}, \\
J_T &:= \sum_{k \in K_T} \left[k - \frac{1}{2} + \delta, k + \frac{1}{2} - \delta \right].
\end{aligned}
$$

Note that $\left\{[k - \frac{1}{2} + \delta, k + \frac{1}{2} - \delta]\right\}_{k \in K_T}$ is disjoint. This is seen from the following implications: For $k, l \in K_T$, $k < l$,

$$
\begin{aligned}
&l - \frac{1}{2} + \delta - \left(k + \frac{1}{2} - \delta\right) = l - k - 1 + 2\delta \ge 1 - 1 + 2\delta = 2\delta > 0 \\
&\Rightarrow k + \frac{1}{2} - \delta < l - \frac{1}{2} + \delta \\
&\Rightarrow \left[k - \frac{1}{2} + \delta, k + \frac{1}{2} - \delta\right] \cap \left[l - \frac{1}{2} + \delta, l + \frac{1}{2} - \delta\right] = \emptyset.
\end{aligned}
$$

We divide the proof into six steps:

<u>1°</u> $\mu(J_T) \ge \left(\lfloor T \rfloor - 1 - \sqrt{\eta}T\right)(1 - 2\delta)$.

$\because$ First, by the definition of J_T,

$$\mu(J_T) = \sum_{k \in K_T} \mu\Big(\Big[k - \frac{1}{2} + \delta, k + \frac{1}{2} - \delta\Big]\Big) = \big(\#K_T\big)(1 - 2\delta).$$

Next, by (5.8) and the definition of K_T,

$$\begin{aligned}
(2 - \sigma_0 + 2\delta)\eta T &\geq \int_{\frac{1}{2}}^{T} dt \int_{\sigma_0 - \delta}^{2+\delta} \left| \frac{\zeta}{\zeta_N}(\sigma + \sqrt{-1}t) - 1 \right|^2 d\sigma \\
&\geq \int_{\frac{1}{2}}^{\lfloor T \rfloor - \frac{1}{2}} dt \int_{\sigma_0 - \delta}^{2+\delta} \left| \frac{\zeta}{\zeta_N}(\sigma + \sqrt{-1}t) - 1 \right|^2 d\sigma \\
&\qquad \Big[\because \lfloor T \rfloor - \tfrac{1}{2} \leq T - \tfrac{1}{2} < T\Big] \\
&= \sum_{k=1}^{\lfloor T \rfloor - 1} \int_{k-\frac{1}{2}}^{k+\frac{1}{2}} dt \int_{\sigma_0 - \delta}^{2+\delta} \left| \frac{\zeta}{\zeta_N}(\sigma + \sqrt{-1}t) - 1 \right|^2 d\sigma \\
&\geq \sum_{k \in \{1,\ldots,\lfloor T \rfloor - 1\} \setminus K_T} \int_{k-\frac{1}{2}}^{k+\frac{1}{2}} dt \int_{\sigma_0 - \delta}^{2+\delta} \left| \frac{\zeta}{\zeta_N}(\sigma + \sqrt{-1}t) - 1 \right|^2 d\sigma \\
&\geq \sum_{k \in \{1,\ldots,\lfloor T \rfloor - 1\} \setminus K_T} (2 - \sigma_0 + 2\delta)\sqrt{\eta} \\
&= \#\big(\{1, \ldots, \lfloor T \rfloor - 1\} \setminus K_T\big)(2 - \sigma_0 + 2\delta)\sqrt{\eta} \\
&= \big(\lfloor T \rfloor - 1 - \#K_T\big)(2 - \sigma_0 + 2\delta)\sqrt{\eta},
\end{aligned}$$

which implies that $\#K_T \geq \lfloor T \rfloor - 1 - \sqrt{\eta}T$. Thus we have $\mu(J_T) \geq \big(\lfloor T \rfloor - 1 - \sqrt{\eta}T\big)(1 - 2\delta)$.

<u>2°</u> $\displaystyle \left| \frac{\zeta}{\zeta_N}(\sigma + \sqrt{-1}t) - 1 \right| < \Big(\frac{2}{\pi}\Big)^{\frac{1}{2}} \frac{\eta^{\frac{1}{4}}}{\delta}, \ (\sigma, t) \in [\sigma_0, 2] \times J_T.$

$\because$ Fix $(\sigma, t) \in [\sigma_0, 2] \times J_T$. By the definition of J_T,

$$^{\exists}k \in K_T \ \text{ s.t. } \ (\sigma, t) \in [\sigma_0, 2] \times \Big[k - \frac{1}{2} + \delta, k + \frac{1}{2} - \delta\Big].$$

Then, from the implications

$$\begin{aligned}
\big|s' - (\sigma + \sqrt{-1}t)\big| \leq \delta &\Rightarrow \big|\mathrm{Re}\, s' - \sigma\big| \leq \delta,\ \big|\mathrm{Im}\, s' - t\big| \leq \delta \\
&\Rightarrow \sigma - \delta \leq \mathrm{Re}\, s' \leq \sigma + \delta,\ t - \delta \leq \mathrm{Im}\, s' \leq t + \delta \\
&\Rightarrow \sigma_0 - \delta \leq \mathrm{Re}\, s' \leq 2 + \delta,\ k - \frac{1}{2} \leq \mathrm{Im}\, s' \leq k + \frac{1}{2},
\end{aligned}$$

it follows that

$$\begin{aligned}
&\overline{\delta\text{-neighborhood of } \sigma + \sqrt{-1}t} \\
&\quad \subset \Big\{s' = \sigma' + \sqrt{-1}t'; (\sigma', t') \in [\sigma_0 - \delta, 2 + \delta] \times \Big[k - \frac{1}{2}, k + \frac{1}{2}\Big]\Big\}.
\end{aligned} \tag{5.9}$$

By Cauchy's integral representation,

$$
\begin{aligned}
&\Big(\frac{\zeta}{\zeta_N}(\sigma+\sqrt{-1}t)-1\Big)^2\\
&=\frac{1}{2\pi\sqrt{-1}}\int_{|s'-(\sigma+\sqrt{-1}t)|=r}\frac{\big(\frac{\zeta}{\zeta_N}(s')-1\big)^2}{s'-(\sigma+\sqrt{-1}t)}ds'\\
&=\frac{1}{2\pi\sqrt{-1}}\int_0^{2\pi}\frac{\big(\frac{\zeta}{\zeta_N}(\sigma+\sqrt{-1}t+re^{\sqrt{-1}\theta})-1\big)^2}{re^{\sqrt{-1}\theta}}\sqrt{-1}re^{\sqrt{-1}\theta}d\theta\\
&\qquad\Big[\because \text{ change of variable: } s'=\sigma+\sqrt{-1}t+re^{\sqrt{-1}\theta}\Big]\\
&=\frac{1}{2\pi}\int_0^{2\pi}\Big(\frac{\zeta}{\zeta_N}(\sigma+\sqrt{-1}t+re^{\sqrt{-1}\theta})-1\Big)^2d\theta,\quad 0<{}^\forall r\le\delta.
\end{aligned}
$$

Taking the absolute value, we have

$$
\Big|\frac{\zeta}{\zeta_N}(\sigma+\sqrt{-1}t)-1\Big|^2\le\frac{1}{2\pi}\int_0^{2\pi}\Big|\frac{\zeta}{\zeta_N}(\sigma+\sqrt{-1}t+re^{\sqrt{-1}\theta})-1\Big|^2d\theta.
$$

Multiplying this by r, and then integrating it in $r\in[0,\delta]$, we have

$$
\begin{aligned}
&\Big|\frac{\zeta}{\zeta_N}(\sigma+\sqrt{-1}t)-1\Big|^2\frac{\delta^2}{2}\\
&=\int_0^\delta\Big|\frac{\zeta}{\zeta_N}(\sigma+\sqrt{-1}t)-1\Big|^2rdr\\
&\le\frac{1}{2\pi}\int_0^\delta rdr\int_0^{2\pi}\Big|\frac{\zeta}{\zeta_N}(\sigma+\sqrt{-1}t+re^{\sqrt{-1}\theta})-1\Big|^2d\theta\\
&=\frac{1}{2\pi}\int_0^{2\pi}\int_0^\delta\Big|\frac{\zeta}{\zeta_N}(\sigma+\sqrt{-1}t+re^{\sqrt{-1}\theta})-1\Big|^2rdrd\theta\\
&=\frac{1}{2\pi}\iint_{|\sigma'+\sqrt{-1}t'|\le\delta}\Big|\frac{\zeta}{\zeta_N}(\sigma+\sqrt{-1}t+\sigma'+\sqrt{-1}t')-1\Big|^2d\sigma'dt'\\
&=\frac{1}{2\pi}\iint_{|\sigma'+\sqrt{-1}t'-(\sigma+\sqrt{-1}t)|\le\delta}\Big|\frac{\zeta}{\zeta_N}(\sigma'+\sqrt{-1}t')-1\Big|^2d\sigma'dt'\\
&\le\frac{1}{2\pi}\int_{k-\frac12}^{k+\frac12}dt'\int_{\sigma_0-\delta}^{2+\delta}\Big|\frac{\zeta}{\zeta_N}(\sigma'+\sqrt{-1}t')-1\Big|^2d\sigma'\quad\Big[\because (5.9)\Big]\\
&\le\frac{1}{2\pi}(2-\sigma_0+2\delta)\sqrt{\eta}\quad\Big[\because \text{ since } k\in K_T\Big]\\
&<\frac{\eta^{\frac12}}{\pi}\quad\Big[\because 2-\sigma_0+2\delta<2-\sigma_0+2\sigma_0-1=\sigma_0+1\le 2\Big].
\end{aligned}
$$

Thus we obtain

$$
\Big|\frac{\zeta}{\zeta_N}(\sigma+\sqrt{-1}t)-1\Big|<\Big(\frac{2}{\pi}\Big)^{\frac12}\frac{\eta^{\frac14}}{\delta}.
$$

3° For $s = \sigma_0 + \sqrt{-1}t \in G$ and $t \in J_T$,

$$\left|\log\zeta(s) - \log\zeta_N(s)\right| \le \frac{\left(\frac{2}{\pi}\right)^{1/2}\frac{\eta^{1/4}}{\delta}}{1 - \left(\frac{2}{\pi}\right)^{1/2}\frac{\eta^{1/4}}{\delta}}.$$

∵ First, by (3.4),

$$\left.\begin{aligned}\log\zeta(s) &= \log\frac{\pi^2}{6} + \int_2^s \frac{\zeta'(z)}{\zeta(z)}dz,\\ \log\zeta_N(s) &= \log\zeta_N(2) + \int_2^s \frac{\zeta_N'(z)}{\zeta_N(z)}dz\end{aligned}\right\} \quad \text{on } G \cap \{s \in \mathbb{C}; \mathrm{Re}\, s > 0\}.$$

For simplicity, put a holomorphic function f on $G \cap \{s \in \mathbb{C}; \mathrm{Re}\, s > 0\}$ as

$$f(s) := \frac{\zeta(s)}{\zeta_N(s)} - 1.$$

Since $\zeta(s) = (f(s) + 1)\zeta_N(s)$,

$$\begin{aligned}\frac{\zeta'(s)}{\zeta(s)} - \frac{\zeta_N'(s)}{\zeta_N(s)} &= \frac{f'(s)\zeta_N(s) + (f(s)+1)\zeta_N'(s)}{(f(s)+1)\zeta_N(s)} - \frac{\zeta_N'(s)}{\zeta_N(s)}\\ &= \frac{f'(s)}{f(s)+1} + \frac{\zeta_N'(s)}{\zeta_N(s)} - \frac{\zeta_N'(s)}{\zeta_N(s)}\\ &= \frac{f'(s)}{f(s)+1},\end{aligned}$$

and thus

$$\begin{aligned}&\log\zeta(s) - \log\zeta_N(s)\\ &= \log\zeta(2) - \log\zeta_N(2) + \int_2^s \left(\frac{\zeta'(z)}{\zeta(z)} - \frac{\zeta_N'(z)}{\zeta_N(z)}\right)dz\\ &= \log\zeta(2) - \log\zeta_N(2) + \int_2^s \frac{f'(z)}{f(z)+1}dz \quad {}^\forall s \in G \cap \{s \in \mathbb{C}; \mathrm{Re}\, s > 0\}.\end{aligned} \tag{5.10}$$

Now, let $s = \sigma_0 + \sqrt{-1}t \in G$ and $t \in J_T$. By the definition of G,

$$0 < {}^\exists\varepsilon < \delta \text{ s.t. } \left\{\sigma + \sqrt{-1}\tau; \sigma_0 - \varepsilon < \sigma, |\tau - t| < \varepsilon\right\} \subset G \cap \{s \in \mathbb{C}; \mathrm{Re}\, s > 0\}.$$

Indeed, since, by Claim 5.1,

$$0 < {}^\exists\varepsilon_0 < \delta \text{ s.t. } U_{\varepsilon_0}(t) \cap \Gamma = \begin{cases}\{t\} & \text{if } t \in \Gamma,\\ \emptyset & \text{if } t \notin \Gamma,\end{cases}$$

we may take $\varepsilon = \begin{cases}(\sigma_0 - \sigma_t) \wedge t & \text{if } t \in \Gamma,\\ \varepsilon_0 & \text{if } t \notin \Gamma,\end{cases}$. From the continuity of f on $G \cap \{s \in \mathbb{C}; \mathrm{Re}\, s > 0\}$ and

$$\{\sigma + \sqrt{-1}t; \sigma \in [\sigma_0, 2]\}$$

$$\begin{aligned}&\subset \left\{\sigma+\sqrt{-1}\tau; \sigma_0-\varepsilon' \le \sigma \le 2+\varepsilon', |\tau-t| \le \varepsilon'\right\} \quad (0<\varepsilon'<\varepsilon)\\&\subset \left\{\sigma+\sqrt{-1}\tau; \sigma_0-\varepsilon < \sigma, |\tau-t| < \varepsilon\right\}\\&\subset G \cap \left\{s \in \mathbb{C}; \operatorname{Re} s > 0\right\},\end{aligned}$$

$$\begin{aligned}&\left\{\sigma+\sqrt{-1}\tau; \sigma_0-\varepsilon' \le \sigma \le 2+\varepsilon', |\tau-t| \le \varepsilon'\right\}\\&\searrow \left\{\sigma+\sqrt{-1}t; \sigma \in [\sigma_0, 2]\right\} \quad \text{as } \varepsilon' \searrow 0,\end{aligned}$$

it follows that

$$\begin{aligned}&\lim_{\varepsilon'\searrow 0} \max\left\{|f(\sigma+\sqrt{-1}\tau)|; \sigma_0-\varepsilon' \le \sigma \le 2+\varepsilon', |\tau-t| \le \varepsilon'\right\}\\&= \max\left\{|f(\sigma+\sqrt{-1}t)|; \sigma \in [\sigma_0, 2]\right\}\\&< \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{\eta^{\frac{1}{4}}}{\delta} \quad [\because 2^{\circ}].\end{aligned}$$

This implies that

$$0 < {}^{\exists}\varepsilon' < \varepsilon \ \text{ s.t. } \begin{cases} \left|f(\sigma+\sqrt{-1}\tau)\right| < \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{\eta^{\frac{1}{4}}}{\delta} < \left(\frac{2}{\pi}\right)^{\frac{1}{2}} < 1, \\ \text{for } {}^{\forall}(\sigma, \tau) \ \text{ with } \sigma_0-\varepsilon' < \sigma < 2+\varepsilon', |\tau-t| < \varepsilon'. \end{cases}$$

On the other hand, by (5.6) and (5.7),

$$\left|f(\sigma+\sqrt{-1}\tau)\right| \le \sum_{n>p_N} \frac{1}{n^{\frac{3}{2}}} \le \frac{1}{2}, \quad {}^{\forall}\sigma > \frac{3}{2}, {}^{\forall}\tau \in \mathbb{R}.$$

Therefore, if we let

$$\begin{aligned}G_0 &:= \left\{z; \sigma_0-\varepsilon' < \operatorname{Re} z < 2+\varepsilon', |\operatorname{Im} z - t| < \varepsilon'\right\} \cup \left\{z; \operatorname{Re} z > \frac{3}{2}\right\}\\&\subset G \cap \left\{z; \operatorname{Re} z > 0\right\},\end{aligned}$$

$|f(z)| < 1$ (${}^{\forall}z \in G_0$), so that $\log(f(z)+1) := \int_1^{f(z)+1} \frac{dw}{w}$ can be defined on G_0, and

$$\big(\log(f(z)+1)\big)' = \frac{f'(z)}{f(z)+1}$$

holds. Since $\sigma_0+\sqrt{-1}t, 2 \in G_0$ and G_0 is a simply connected domain of $\mathbb{C}$,

$$\begin{aligned}\int_2^{\sigma_0+\sqrt{-1}t} \frac{f'(z)}{f(z)+1} dz &= \Big[\log(f(z)+1)\Big]_2^{\sigma_0+\sqrt{-1}t}\\&= \log(f(\sigma_0+\sqrt{-1}t)+1) - \log(f(2)+1).\end{aligned}$$

Noting that $\log(f(2)+1) = \log \frac{\zeta(2)}{\zeta_N(2)} = \log\zeta(2) - \log\zeta_N(2)$, we have

$$\begin{aligned}&\left|\log\zeta(\sigma_0+\sqrt{-1}t) - \log\zeta_N(\sigma_0+\sqrt{-1}t)\right|\\&= \left|\log\zeta(2) - \log\zeta_N(2) + \int_2^{\sigma_0+\sqrt{-1}t} \frac{f'(z)}{f(z)+1} dz\right| \quad [\because (5.10)]\end{aligned}$$

$$
\begin{aligned}
&= \left|\log\zeta(2) - \log\zeta_N(2) + \log(f(\sigma_0+\sqrt{-1}t)+1) - \log(f(2)+1)\right| \\
&= \left|\log(f(\sigma_0+\sqrt{-1}t)+1)\right| \\
&\le \frac{|f(\sigma_0+\sqrt{-1}t)|}{1-|f(\sigma_0+\sqrt{-1}t)|} \quad [\because (3.9)] \\
&< \frac{\left(\frac{2}{\pi}\right)^{1/2}\frac{\eta^{1/4}}{\delta}}{1-\left(\frac{2}{\pi}\right)^{1/2}\frac{\eta^{1/4}}{\delta}} \quad [\because 2^{\circ}].
\end{aligned}
$$

$\underline{4^{\circ}}$ For $\forall w \in \mathbb{C}$,

$$
\begin{aligned}
&\left|\frac{1}{2T}\int_{-T}^{T} \mathbf{1}_G(\sigma_0+\sqrt{-1}t)e^{\sqrt{-1}\langle \log\zeta(\sigma_0+\sqrt{-1}t),w\rangle}dt \right.\\
&\left. -\frac{1}{2T}\int_{-T}^{T} e^{\sqrt{-1}\langle \log\zeta_N(\sigma_0+\sqrt{-1}t),w\rangle}dt\right| \\
&\le 2\frac{\left(\frac{2}{\pi}\right)^{1/2}\frac{\eta^{1/4}}{\delta}}{1-\left(\frac{2}{\pi}\right)^{1/2}\frac{\eta^{1/4}}{\delta}}|w| + 2\left(1-\frac{\lfloor T\rfloor - 1 - \sqrt{\eta}T}{T}(1-2\delta)\right).
\end{aligned}
$$

$\because$ Fix $w \in \mathbb{C}$. This estimate is seen in the following way:

$$
\begin{aligned}
&\left|\frac{1}{2T}\int_{-T}^{T} \mathbf{1}_G(\sigma_0+\sqrt{-1}t)e^{\sqrt{-1}\langle \log\zeta(\sigma_0+\sqrt{-1}t),w\rangle}dt \right.\\
&\left. -\frac{1}{2T}\int_{-T}^{T} e^{\sqrt{-1}\langle \log\zeta_N(\sigma_0+\sqrt{-1}t),w\rangle}dt\right| \\
&= \left|\frac{1}{2T}\int_{-T}^{T} \mathbf{1}_G(\sigma_0+\sqrt{-1}t)\Big(e^{\sqrt{-1}\langle \log\zeta(\sigma_0+\sqrt{-1}t),w\rangle}\right.\\
&\qquad\qquad \left. - e^{\sqrt{-1}\langle \log\zeta_N(\sigma_0+\sqrt{-1}t),w\rangle}\Big)dt\right|
\end{aligned}
$$

$$
\left[\begin{array}{l}
\because \text{ Since} \\
\qquad \{t \in \mathbb{R}; \sigma_0+\sqrt{-1}t \notin G\} \\
\qquad = \begin{cases} \{0\} & \text{if } \sigma_0 = 1, \\ \{t \in \Gamma; \sigma_0 \le \sigma_t\} \subset \Gamma & \text{if } \frac{1}{2} < \sigma_0 < 1 \end{cases} \\
\text{and } \Gamma \text{ is at most countable, } \mu\big(\{t \in \mathbb{R}; \sigma_0+\sqrt{-1}t \notin G\}\big) = 0
\end{array}\right]
$$

$$
\begin{aligned}
&\le \frac{1}{2T}\int_{-T}^{T} \mathbf{1}_G(\sigma_0+\sqrt{-1}t)\Big|e^{\sqrt{-1}\langle \log\zeta(\sigma_0+\sqrt{-1}t),w\rangle} \\
&\qquad\qquad - e^{\sqrt{-1}\langle \log\zeta_N(\sigma_0+\sqrt{-1}t),w\rangle}\Big|dt \\
&= \frac{1}{2T}\int_{-T}^{T} \mathbf{1}_{J_T}(t)\mathbf{1}_G(\sigma_0+\sqrt{-1}t)\Big|e^{\sqrt{-1}\langle \log\zeta(\sigma_0+\sqrt{-1}t),w\rangle} \\
&\qquad\qquad - e^{\sqrt{-1}\langle \log\zeta_N(\sigma_0+\sqrt{-1}t),w\rangle}\Big|dt \\
&\quad + \frac{1}{2T}\int_{-T}^{T} \mathbf{1}_{-J_T}(t)\mathbf{1}_G(\sigma_0+\sqrt{-1}t)\Big|e^{\sqrt{-1}\langle \log\zeta(\sigma_0+\sqrt{-1}t),w\rangle} \\
&\qquad\qquad - e^{\sqrt{-1}\langle \log\zeta_N(\sigma_0+\sqrt{-1}t),w\rangle}\Big|dt
\end{aligned}
$$

$$+\frac{1}{2T}\int_{[-T,T]\setminus(J_T\cup(-J_T))}\mathbf{1}_G(\sigma_0+\sqrt{-1}t)\big|e^{\sqrt{-1}\langle\log\zeta(\sigma_0+\sqrt{-1}t),w\rangle}-e^{\sqrt{-1}\langle\log\zeta_N(\sigma_0+\sqrt{-1}t),w\rangle}\big|dt$$

$$=\frac{1}{2T}\int_{-T}^{T}\mathbf{1}_{J_T}(t)\mathbf{1}_G(\sigma_0+\sqrt{-1}t)\big|e^{\sqrt{-1}\langle\log\zeta(\sigma_0+\sqrt{-1}t),w\rangle}-e^{\sqrt{-1}\langle\log\zeta_N(\sigma_0+\sqrt{-1}t),w\rangle}\big|dt$$

$$+\frac{1}{2T}\int_{-T}^{T}\mathbf{1}_{J_T}(\tau)\mathbf{1}_G(\sigma_0-\sqrt{-1}\tau)\big|e^{\sqrt{-1}\langle\log\zeta(\sigma_0-\sqrt{-1}\tau),w\rangle}-e^{\sqrt{-1}\langle\log\zeta_N(\sigma_0-\sqrt{-1}\tau),w\rangle}\big|d\tau$$

$$+\frac{1}{2T}\int_{[-T,T]\setminus(J_T\cup(-J_T))}\mathbf{1}_G(\sigma_0+\sqrt{-1}t)\big|e^{\sqrt{-1}\langle\log\zeta(\sigma_0+\sqrt{-1}t),w\rangle}-e^{\sqrt{-1}\langle\log\zeta_N(\sigma_0+\sqrt{-1}t),w\rangle}\big|dt$$

$\big[\because$ change of variable: $\tau=-t$ in the 2nd term$\big]$

$$=\frac{1}{2T}\int_{-T}^{T}\mathbf{1}_{J_T}(t)\mathbf{1}_G(\sigma_0+\sqrt{-1}t)\big|e^{\sqrt{-1}\langle\log\zeta(\sigma_0+\sqrt{-1}t),w\rangle}-e^{\sqrt{-1}\langle\log\zeta_N(\sigma_0+\sqrt{-1}t),w\rangle}\big|dt$$

$$+\frac{1}{2T}\int_{-T}^{T}\mathbf{1}_{J_T}(\tau)\mathbf{1}_G(\sigma_0+\sqrt{-1}\tau)\big|e^{\sqrt{-1}\langle\overline{\log\zeta(\sigma_0+\sqrt{-1}\tau)},w\rangle}-e^{\sqrt{-1}\langle\overline{\log\zeta_N(\sigma_0+\sqrt{-1}\tau)},w\rangle}\big|d\tau$$

$$+\frac{1}{2T}\int_{[-T,T]\setminus(J_T\cup(-J_T))}\mathbf{1}_G(\sigma_0+\sqrt{-1}t)\big|e^{\sqrt{-1}\langle\log\zeta(\sigma_0+\sqrt{-1}t),w\rangle}-e^{\sqrt{-1}\langle\log\zeta_N(\sigma_0+\sqrt{-1}t),w\rangle}\big|dt$$

$\Big[\because$ Since, by Definition 5.3, $\frac{\zeta_N'(s)}{\zeta_N(s)}=\sum_{i=1}^{N}\frac{\log\frac{1}{p_i}}{p_i^s-1}$, it follows that $\overline{\left(\frac{\zeta_N'(s)}{\zeta_N(s)}\right)}=\frac{\zeta_N'(\overline{s})}{\zeta_N(\overline{s})}$. In the same way as in the proof of Claim 5.4(ii), we have $\overline{\log\zeta_N(s)}=\log\zeta_N(\overline{s})\Big]$

$$\le\frac{1}{T}\int_{-T}^{T}\mathbf{1}_{J_T}(t)\mathbf{1}_G(\sigma_0+\sqrt{-1}t)\big|\log\zeta(\sigma_0+\sqrt{-1}t)-\log\zeta_N(\sigma_0+\sqrt{-1}t)\big|dt|w|$$

$$+\frac{1}{T}\mu\Big([-T,T]\setminus\big(J_T\cup(-J_T)\big)\Big)$$

$$\le 2\frac{\left(\frac{2}{\pi}\right)^{1/2}\frac{\eta^{1/4}}{\delta}}{1-\left(\frac{2}{\pi}\right)^{1/2}\frac{\eta^{1/4}}{\delta}}|w|+\frac{2}{T}\big(T-\mu(J_T)\big)\quad[\because 3^{\circ}]$$

$$\le 2\frac{\left(\frac{2}{\pi}\right)^{1/2}\frac{\eta^{1/4}}{\delta}}{1-\left(\frac{2}{\pi}\right)^{1/2}\frac{\eta^{1/4}}{\delta}}|w|+2\left(1-\frac{\lfloor T\rfloor-1-\sqrt{\eta}T}{T}(1-2\delta)\right)\quad[\because 1^{\circ}].$$

5° For $^\forall w \in \mathbb{C}$,

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} e^{\sqrt{-1}\langle \log \zeta_N(\sigma_0+\sqrt{-1}t), w\rangle} dt = E^{\mathbf{P}}\left[e^{\sqrt{-1}\left\langle \sum_{i=1}^{N} -\log\left(1-\frac{e(-\log p_i)}{p_i^{\sigma_0}}\right), w\right\rangle}\right].$$

$\because$ First, by (5.5),

$$\frac{1}{2T}\int_{-T}^{T} e^{\sqrt{-1}\langle \log \zeta_N(\sigma_0+\sqrt{-1}t), w\rangle} dt = \frac{1}{2T}\int_{-T}^{T} e^{\sqrt{-1}\left\langle \sum_{i=1}^{N} -\log\left(1-\frac{e^{-\sqrt{-1}t\log p_i}}{p_i^{\sigma_0}}\right), w\right\rangle} dt.$$

Next, in the same way as in 2° in the proof of Theorem 5.1 in the case where $\sigma_0 > 1$,

$$\begin{aligned}
&\frac{1}{2T}\int_{-T}^{T} e^{\sqrt{-1}\left\langle \sum_{i=1}^{N} -\log\left(1-\frac{e^{-\sqrt{-1}t\log p_i}}{p_i^{\sigma_0}}\right), w\right\rangle} dt \\
&\to \int_{\mathbb{R}^{\mathbb{B}}} e^{\sqrt{-1}\left\langle \sum_{i=1}^{N} -\log\left(1-\frac{1}{p_i^{\sigma_0}}(\pi_{\cos(-\log p_i)\cdot}+\sqrt{-1}\pi_{\sin(-\log p_i)\cdot})\right), w\right\rangle} d\mathbf{P} \\
&= E^{\mathbf{P}}\left[e^{\sqrt{-1}\left\langle \sum_{i=1}^{N} -\log\left(1-\frac{e(-\log p_i)}{p_i^{\sigma_0}}\right), w\right\rangle}\right] \quad \text{as } T\to\infty.
\end{aligned}$$

Combining these, we have the assertion of 5°.

6° For $^\forall w \in \mathbb{C}$,

$$\begin{aligned}
&\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \mathbf{1}_G(\sigma_0+\sqrt{-1}t) e^{\sqrt{-1}\langle \log \zeta(\sigma_0+\sqrt{-1}t), w\rangle} dt \\
&= E^{\mathbf{P}}\left[e^{\sqrt{-1}\left\langle \sum_{i=1}^{\infty} -\log\left(1-\frac{e(-\log p_i)}{p_i^{\sigma_0}}\right), w\right\rangle}\right],
\end{aligned}$$

from which, the assertion in the case where $\frac{1}{2} < \sigma_0 \leq 1$ is seen.

$\because$ By 4° and 5°,

$$\begin{aligned}
&\varlimsup_{T\to\infty} \Bigg| \frac{1}{2T}\int_{-T}^{T} \mathbf{1}_G(\sigma_0+\sqrt{-1}t) e^{\sqrt{-1}\langle \log \zeta(\sigma_0+\sqrt{-1}t), w\rangle} dt \\
&\qquad - E^{\mathbf{P}}\left[e^{\sqrt{-1}\left\langle \sum_{i=1}^{\infty} -\log\left(1-\frac{e(-\log p_i)}{p_i^{\sigma_0}}\right), w\right\rangle}\right]\Bigg| \\
&\leq \varlimsup_{T\to\infty} \Bigg| \frac{1}{2T}\int_{-T}^{T} \mathbf{1}_G(\sigma_0+\sqrt{-1}t) e^{\sqrt{-1}\langle \log \zeta(\sigma_0+\sqrt{-1}t), w\rangle} dt \\
&\qquad - \frac{1}{2T}\int_{-T}^{T} e^{\sqrt{-1}\langle \log \zeta_N(\sigma_0+\sqrt{-1}t), w\rangle} dt \Bigg| \\
&\quad + \lim_{T\to\infty} \Bigg| \frac{1}{2T}\int_{-T}^{T} e^{\sqrt{-1}\langle \log \zeta_N(\sigma_0+\sqrt{-1}t), w\rangle} dt \\
&\qquad - E^{\mathbf{P}}\left[e^{\sqrt{-1}\left\langle \sum_{i=1}^{\infty} -\log\left(1-\frac{e(-\log p_i)}{p_i^{\sigma_0}}\right), w\right\rangle}\right]\Bigg|
\end{aligned}$$

$$
\begin{aligned}
&\le \varlimsup_{T\to\infty}\left(2\frac{\left(\frac{2}{\pi}\right)^{1/2}\frac{\eta^{1/4}}{\delta}}{1-\left(\frac{2}{\pi}\right)^{1/2}\frac{\eta^{1/4}}{\delta}}|w| + 2\left(1-\left(\frac{\lfloor T\rfloor}{T}-\frac{1}{T}-\sqrt{\eta}\right)(1-2\delta)\right)\right)\\
&\quad+\left|E^{\mathbf{P}}\left[e^{\sqrt{-1}\left\langle\sum_{i=1}^{N}-\log\left(1-\frac{e(-\log p_i)}{p_i^{\sigma_0}}\right),w\right\rangle}\right]-E^{\mathbf{P}}\left[e^{\sqrt{-1}\left\langle\sum_{i=1}^{\infty}-\log\left(1-\frac{e(-\log p_i)}{p_i^{\sigma_0}}\right),w\right\rangle}\right]\right|\\
&= 2\frac{\left(\frac{2}{\pi}\right)^{1/2}\frac{\eta^{1/4}}{\delta}}{1-\left(\frac{2}{\pi}\right)^{1/2}\frac{\eta^{1/4}}{\delta}}|w| + 2\Big(1-\big(1-\sqrt{\eta}\big)(1-2\delta)\Big)\\
&\quad+\left|E^{\mathbf{P}}\left[e^{\sqrt{-1}\left\langle\sum_{i=1}^{N}-\log\left(1-\frac{e(-\log p_i)}{p_i^{\sigma_0}}\right),w\right\rangle}\right]-E^{\mathbf{P}}\left[e^{\sqrt{-1}\left\langle\sum_{i=1}^{\infty}-\log\left(1-\frac{e(-\log p_i)}{p_i^{\sigma_0}}\right),w\right\rangle}\right]\right|.
\end{aligned}
$$

By letting $N \to \infty$, the 3rd term $\to 0$; after that, by letting $\eta \searrow 0$, and then $\delta \searrow 0$, the 1st term + the 2nd term $\to 0$. Therefore we have the assertion of 6°. ■