Introduction

We would describe more explicitly what was stated in Preface.

The Riemann zeta function $\zeta(\cdot)$ is defined on the half-plane $\operatorname{Re} s > 1$ by the Dirichlet series $\sum_{n=1}^{\infty} \frac{1}{n^s}$ where $n^s = e^{s \log n}$. This function is holomorphic there and is also expressed by the Euler product $\prod_{p:\text{prime}} \frac{1}{1-\frac{1}{p^s}}$. Moreover $\zeta(\cdot)$ is analytically continuable to a meromorphic function on the whole complex plane $\mathbb{C}$ that is holomorphic except $s = 1$ and has a simple pole at $s = 1$ with residue 1. Let us denote this meromorphic function by the same $\zeta(\cdot)$.

By the Euler product above, $\zeta(s) \neq 0$ for $\operatorname{Re} s > 1$. It is shown that $\zeta(s) \neq 0$ on the line $\operatorname{Re} s = 1$, and thus there are no zeros of $\zeta(\cdot)$ in the closed half-plane $\operatorname{Re} s \geq 1$. From the functional equation of $\zeta(\cdot)$, it is seen that its zeros in the closed half-plane $\operatorname{Re} s \leq 0$ are negative even integers $-2, -4, -6, \ldots$, which are called the *trivial zeros*. The Riemann hypothesis is concerned with *non-trivial zeros*, namely the zeros in the strip $0 < \operatorname{Re} s < 1$, and states that all non-trivial zeros are on the line $\operatorname{Re} s = \frac{1}{2}$. Since $\zeta(s) \neq 0$ for $\operatorname{Re} s > \frac{1}{2}$ under this hypothesis, the log zeta function $\log \zeta$ can be defined as a holomorphic function on the domain G', the half-plane $\operatorname{Re} s > \frac{1}{2}$ excluding the line segment $(\frac{1}{2}, 1]$ with the derivative $\frac{\zeta'(s)}{\zeta(s)}$ and the value $\log \frac{\pi^2}{6}$ at $s = 2$. In other words, $\log \zeta$ is a primitive function of $\frac{\zeta'}{\zeta}$ on G' with the value $\log \frac{\pi^2}{6}$ at $s = 2$. Unfortunately, the Riemann hypothesis is not yet proved to hold valid at the moment, so that a modification of this definition is necessary. In fact, in place of G', we have only to take a simply connected domain G of $\mathbb{C}$ such that G contains the half-plane $\operatorname{Re} s > 1$ but no zeros of $\zeta(\cdot)$.

When $\sigma > 1$, it is easily seen from the Dirichlet series expression of $\zeta(\cdot)$ that $|\zeta(\sigma + \sqrt{-1}t)| \leq \zeta(\sigma)$ for all $t \in \mathbb{R}$. Thus $\zeta(\sigma + \sqrt{-1}t)$ takes the value in the closed disc with center at origin and radius $\zeta(\sigma)$. Then how is the case when $\sigma \leq 1$? Since, by the functional equation of $\zeta(\cdot)$, the value of $\zeta(1 - s)$ is computed from that of $\zeta(s)$ immediately, we may restrict ourselves to the case when $\frac{1}{2} \leq \sigma \leq 1$. In the early 1910s, H. Bohr obtained many results about the behavior of $\zeta(\cdot)$ on the strip $\frac{1}{2} < \operatorname{Re} s \leq 1$. Among them, the following is an answer of the question above $\big[$cf. Bohr [2]$\big]$:

Theorem 1 *For $\frac{1}{2} < \sigma \leq 1$, the set $\left\{\log \zeta(\sigma + \sqrt{-1}t); t \in \mathbb{R} \text{ with } \sigma + \sqrt{-1}t \in G\right\}$ is dense in $\mathbb{C}$.*

In fact, since $\zeta(\sigma + \sqrt{-1}t) = e^{\log \zeta(\sigma + \sqrt{-1}t)}$, it is seen from the theorem that the set $\left\{\zeta(\sigma + \sqrt{-1}t); t \in \mathbb{R}\right\}$ is dense in $\mathbb{C}$ $\big[$cf. Bohr-Courant [4]$\big]$. According as $\sigma > 1$ or $\sigma \leq 1$, the behavior of $\zeta(\sigma + \sqrt{-1}\cdot)$, i.e., $\zeta(\cdot)$ on the line $\operatorname{Re} s = \sigma$ changes drastically.

Bohr much studied about the value-distribution of $\zeta(\cdot)$, which motivated him to develop the theory of almost periodic functions. It was in the 1930s that he, together with

B. Jessen, arrived at the following result $\big[$cf. Bohr-Jessen [5]$\big]$:

Theorem 2 *Let R be any closed rectangle in $\mathbb{C}$ with edges parallel to the axes. Then, for each $\sigma > \frac{1}{2}$,*

$$\lim_{T\to\infty} \frac{1}{2T}\mu\left(t \in [-T,T]; \sigma + \sqrt{-1}t \in G, \log\zeta(\sigma+\sqrt{-1}t) \in R\right)$$

exists. Here μ is the 1-dimensional Lebesgue measure.

This is the original version of the Bohr-Jessen limit theorem. The current version is stated in terms of convergence of probability measures:

Theorem 3 *For each $\sigma > \frac{1}{2}$, the probability measure*

$$\frac{1}{2T}\mu\left(t \in [-T,T]; \sigma + \sqrt{-1}t \in G, \log\zeta(\sigma+\sqrt{-1}t) \in \cdot\right)$$

on $\big(\mathbb{C}, \mathcal{B}(\mathbb{C})\big)$ is weakly convergent as $T \to \infty$. Here $\mathcal{B}(\mathbb{C})$ is the Borel σ-algebra of $\mathbb{C}$.

This proof can be found in Jessen-Wintner [16], Borchsenius-Jessen [6], Laurinčikas [21, 22, 23], Matsumoto [24, 25] and so on. It should be noted that Laurinčikas and Matsumoto studied the limit theorem as above for a class of more general zeta functions such as Dirichlet L-functions and Dedekind zeta functions.

Let us here restrict ourselves to the case of the Riemann zeta function.

Viewing their proofs, we have a feeling that the Bohr-Jessen limit theorem is treated in the framework of probability theory. In this monograph, we aim at refining their works. Specifically, in Theorem 3, we make the limit visible. In fact, we present a complex random variable whose distribution is just the limit distribution of the Bohr-Jessen limit theorem and from the form of which the required convergence is roughly seen. We wish that our present approach to the proof of Theorem 3 is accepted as natural and appropriate.

Several facts about $\zeta(\cdot)$ mentioned at the beginning are all well-known, but their proofs will be not familiar to many readers except some specialists. So we give detailed proofs for those facts and related matters. Some readers being mathematicians may feel tedious and impatient for our proofs which are to be given step by step. However we wish readers being graduate students to be satisfied with them.

Let us explain the organization of this monograph.

In Chapter 1, following Yosida [35], we view the almost periodic function due to Bohr. The important thing is the fact that every almost periodic function always has the mean value. In Chapter 2, based on these mean values and applying Kolmogorov's extension theorem, we introduce a probability $\mathbf{P}$ on $\mathbb{R}^{\mathbb{B}} = \big\{(x_f)_{f\in\mathbb{B}}; x_f \in \mathbb{R}\ (f \in \mathbb{B})\big\}$ where $\mathbb{B} :=$ the set of all real-valued almost periodic functions.

In Chapter 3, we present the complex random variable mentioned above. When $\sigma > 1$, expressing $\zeta(\sigma + \sqrt{-1}t)$ by the Euler product and then taking the logarithm yield that

$$\log\zeta(\sigma+\sqrt{-1}t) = \sum_{p:\text{prime}} -\log\left(1 - \frac{\cos(-\log p)t + \sqrt{-1}\sin(-\log p)t}{p^{\sigma}}\right). \quad (1)$$

R.H.S. (= the right-hand side) of (1) is absolutely, uniformly convergent in t, so that this identity is valid. On the contrary, when $\frac{1}{2} < \sigma \le 1$, this infinite series is not convergent. But, if, for each prime p, we consider a counterpart of $-\log\bigl(1-\frac{\cos(-\log p)t+\sqrt{-1}\sin(-\log p)t}{p^\sigma}\bigr)$ to be a complex random variable on the probability space $(\mathbb{R}^{\mathbb{B}}, \mathbf{P})$, then their summation over p is convergent $\mathbf{P}$-a.e. whenever $\sigma > \frac{1}{2}$. This sum is the desired complex random variable. In the rest of Chapter 3, the distribution of this one is investigated.

In Chapter 4, we prove several facts about $\zeta(\cdot)$ mentioned above in detail. In Chapter 5, we define $\log\zeta$, i.e., a simply connected domain G, and present Claim 5.5, which states that (1) holds in a certain sense, namely

$$\sum_{i=1}^{N} -\log\left(1 - \frac{\cos(-\log p_i)t + \sqrt{-1}\sin(-\log p_i)t}{p_i^\sigma}\right) \tag{2}$$

is convergent to $\log\zeta(\sigma+\sqrt{-1}t)$ as $N \to \infty$ in a certain sense. Taking this claim for granted and following Matsumoto [26, Chapter 6], we prove the Bohr-Jessen limit theorem, i.e., Theorem 3. Roughly speaking, the convergence stated in Theorem 3 is obtained by combining the convergence of (2) to $\log\zeta(\sigma+\sqrt{-1}t)$ in a certain sense with the almost sure convergence of its counterpart. As corollaries of Theorem 3, Theorems 1 and 2 are derived.

In Chapter 6, we give a proof of Claim 5.5, which needs considerable efforts. Following Matsumoto [26, Chapter 9], we present a general theorem — Carlson's mean value theorem to show it. To this end, we study the following two matters:

- Square mean value estimate of $\zeta(\cdot)$, in other words, asymptotics of $\int_1^T |\zeta(\sigma+\sqrt{-1}t)|^2 dt$ as $T \to \infty$,
- Exponential decay of $\Gamma^{(l)}(\sigma+\sqrt{-1}t)$ as $|t| \to \infty$, where $\Gamma^{(l)}$ is the lth derivative of the gamma function.

Using these results, we prove Carlson's mean value theorem, and then apply this theorem to complete the proof of Claim 5.5.

For those readers who are not familiar with some such facts in analytic number theory as in Chapters 4 and 6, we give detailed proofs. So it will be not necessary to consult other books or papers for matters related to this field. However, we expect the reader to be familiar with basic facts in measure theory and complex function theory.

In Appendix, we present various facts which are all necessary in this monograph. Several facts from probability theory are stated without their proofs, for which we wish to consult literatures cited. Some properties of the gamma function, and the second mean value theorem for integrals are stated with their proofs.

In June 2009 the author wrote a private note to collect our thoughts, the English version of which is just this monograph. The note [33] is a revised version of this note, where some further properties of the limit distribution were studied.