Chapter 5

Noneffectively hyperbolic Cauchy problem II

5.1 C^∞ well-posedness

We continue to assume that $\Sigma = \{(x,\xi) \mid p(x,\xi) = 0, dp(x,\xi) = 0\}$ is a C^∞ manifold and (4.1.1) is verified. In this chapter we study the case

$$\text{Ker} F_p^2(\rho) \cap \text{Im} F_p^2(\rho) \neq \{0\}. \tag{5.1.1}$$

As we have seen in Theorem 3.5.1 the following two assertions are equivalent

(i) $H_S^3\, p(\rho) = 0$, $\rho \in \Sigma$,

(ii) p admits an elementary decomposition at every $\rho \in \Sigma$

where S is any smooth function verifying (3.4.1) and (3.4.2). As we shall prove in Chapter 7, the condition (ii) is still equivalent to

(5.1.2) there is no null bicharacteristic of p having a limit point in Σ.

In this chapter we discuss the C^∞ well-posedness of the Cauchy problem assuming (5.1.2) (equivalently assuming (i) in Theorem 3.5.1) under the strict Ivrii-Petkov-Hörmander condition.

Theorem 5.1.1 *Assume* (4.1.1), (5.1.1), (5.1.2) *and the subprincipal symbol P_{sub} verifies the strict Ivrii-Petkov-Hörmander condition on Σ. Then the Cauchy problem for P is C^∞ well posed.*

Let fix any $\rho \in \Sigma$. Thanks to Proposition 3.5.1 near ρ we have an elementary decomposition of $p = -\xi_0^2 + \sum_{j=1}^r \phi_j^2$ such that

$$p = -(\xi_0 + \lambda)(\xi_0 - \lambda) + Q$$

where $\lambda = \phi_1 + O(\sum_{j=1}^r \phi_j^2)$. The main difference from the case that we have studied in the previous chapter is that we have no control of ϕ_1^2 by Q, that is the best we can expect is the inequality

$$CQ \geq \sum_{j=2}^{r} \phi_j^2 + \phi_1^4 |\xi'|^{-2}.$$

Another serious difficulty is that it seems to be hard to get a local (not microlocal) elementary decomposition. To overcome this difficulty we follow [31], [24] in the next section.

5.2 Parametrix with finite propagation speed of wave front sets

Recall that we are working with operators of the form

$$P(x, D) = -D_0^2 + A_1(x, D')D_0 + A_2(x, D') \tag{5.2.1}$$

where $A_j(x, \xi') \in S(\langle \xi' \rangle^j, g_0)$. Let $I = (-\tau, \tau)$ be an open interval containing the origin and we denote by $C^k(I, H^p)$ the set of all k-times continuously differentiable functions from I to $H^p = H^p(\mathbb{R}^n)$ and denote by $C^k(I, H^p)^+$ the set of all $f \in C^k(I, H^p)$ vanishing in $x_0 < 0$. We put $H^\infty = \cap_k H^k$ and $H^{-\infty} = \cup_k H^k$.

Definition 5.2.1 *Let T be a linear operator from $C^0(I, H^{-\infty})^+$ to $C^1(I, H^\infty)^+$. We say that $T \in \mathcal{R}$ if there is a positive constant $\delta(T)$ such that*

$$\|D_0^k T f(t, \cdot)\|_{(q)}^2 \leq c_{pq} \int^t \|f(\tau, \cdot)\|_{(p)}^2 d\tau, \quad \forall t \leq \delta(T)$$

for $k = 0, 1$ and for any $p, q \in \mathbb{R}$ and $f \in C^0(I, H^p)^+$.

Definition 5.2.2 ([31]) *Let $(0, \hat{x}', \hat{\xi}') = (0, \rho')$. We say that G is a parametrix of P at $(0, \rho')$ with finite propagation speed of wave front sets with loss of β derivatives if G satisfies the following conditions*

(i) *for any $h = h(x', D') \in S(1, g_0)$ supported near ρ' we have $PGh - h \in \mathcal{R}$,*

(ii) *we have*

$$\|D_0^j G f(t, \cdot)\|_{(p)}^2 \leq c_p \int^t \|f(\tau, \cdot)\|_{(p+j+\beta)}^2 d\tau, \quad j = 0, 1$$

for any $p \in \mathbb{R}$ and for any $f \in C^0(I, H^{p+1+\beta})^+$,

(iii) *for any $h_1(x', D') \in S(1, g_0)$ which is supported near ρ' and for any $h_2(x', D') \in S(1, g_0)$ with $\operatorname{supp} h_2 \subset\subset \mathbb{R}^{2n} \setminus (\operatorname{supp} h_1)$, one has*

$$D_0^j h_2 G h_1 \in \mathcal{R}, \quad j = 0, 1.$$

Let $\tilde{P}$ be another operator of the form (5.2.1) then we say

$$P \equiv \tilde{P} \quad \text{near} \quad (0, \rho')$$

if one can write

$$P - \tilde{P} = \sum_{j=0}^{2} B_j(x, D') D_0^{2-j}$$

with $B_j \in S(\langle \xi' \rangle^j, g_0)$ which are in $S^{-\infty} = \cap_k S(\langle \xi' \rangle^k, g_0)$ near ρ' uniformly in x_0 when $|x_0|$ is small.

In what follows, to simplify notations, we abbreviate a parametrix with finite propagation speed of wave front sets as just "parametrix". The next lemma is clear from the definition.

Lemma 5.2.1 *Let $\tilde{P} \equiv P$ near $(0, \rho')$ and let $\tilde{G}$ be a parametrix of $\tilde{P}$ at $(0, \rho')$ with loss of β derivatives. Then $\tilde{G}$ is a parametrix of P at $(0, \rho')$ with loss of β derivatives.*

Let $T(x, D') \in S(1, g_0)$ be elliptic near $(0, \rho')$ uniformly in x_0 with small $|x_0|$. Then

Proposition 5.2.1 *Let P, $\tilde{P}$ be operators of the form (5.2.1). Assume that $PT \equiv T\tilde{P}$ near $(0, \rho')$. If $\tilde{P}$ has a parametrix at $(0, \rho')$ with loss of β derivatives then so does P.*

Let χ be a local homogeneous canonical transformation from a neighborhood of $(\hat{y}_0, \hat{y}', \hat{\eta}_0, \hat{\eta}')$ to a neighborhood of $(\hat{x}_0, \hat{x}', \hat{\xi}_0, \hat{\xi}')$ such that $y_0 = x_0$. Since χ preserves y_0 coordinate, the generating function of this canonical transformation has the form

$$x_0 \eta_0 + H(x, \eta').$$

We work with a Fourier integral operator F associated with χ which is represented as

$$Fu(x) = \int e^{-iy'\eta' + iH(x,\eta')} a(x, \eta') u(x_0, y') dy' d\eta'$$

(in a convenient y' coordinates) and elliptic near $(\hat{x}, \hat{\xi}, \hat{y}, \hat{\eta})$, where x_0 is regarded as a parameter. We assume that F is bounded from $H^k(\mathbb{R}^n_{y'})$ to $H^k(\mathbb{R}^n_{x'})$ for any $k \in \mathbb{R}$ uniformly in x_0 with small $|x_0|$ (see [10], [17], Theorem 25.3.11 in [19]).

Proposition 5.2.2 *Let χ, F be as above and $P(x, D)$, $\tilde{P}(y, D)$ be operators of the form (5.2.1). Assume that*

$$PF \equiv F\tilde{P} \quad \textit{near} \quad (0, \hat{y}', \hat{\eta}').$$

If $\tilde{P}$ has a parametrix at $(0, \hat{y}', \hat{\eta}')$ with loss of β derivatives then so does P at $(0, \hat{x}', \hat{\xi}')$ with loss of β derivatives.

Proposition 5.2.3 ([31]) *Let P be an operator of the form* (5.2.1). *Assume that P has a parametrix at $(0,0,\xi')$ with loss of $\beta(\xi')$ derivatives for every ξ' with $|\xi'|=1$. Then the Cauchy problem for P is locally solvable near $(0,0)$ in C^∞. More precisely there is an open neighborhood $J\times\omega$ of $(0,0)$ such that for every $f\in C^0(I,H^{p+\nu})^+$ $(p+\nu\geq 0)$ there exists $u\in\cap_{j=0}^1 C^j(J,H^{p-j})^+$ satisfying*

$$Pu=f\quad in\quad J\times\omega$$

where $\nu=\sup_{|\xi'|=1}\beta(\xi')$.

In the following sections, assuming that P satisfies the strict Ivrii-Petkov-Hörmander condition on Σ, we prove the existence of parametrix of P at every $(0,0,\xi')$ with $|\xi'|=1$, hence we can conclude the C^∞ well-posedness.

5.3 Preliminaries

Let fix $\rho\in\Sigma$ and we work near ρ. Thanks to Proposition 3.5.1 p admits an elementary decomposition verifying the conditions stated there. We extend these ϕ_j (given in Proposition 3.5.1) outside a neighborhood of ρ so that they belong to $S(\langle\xi'\rangle,g_0)$ and zero outside another neighborhood of ρ. Using such extended ϕ_j we define λ by the same formula in Proposition 3.5.1

$$\lambda=\phi_1+L(\phi')\phi_1+\gamma\phi_1^3\langle\xi'\rangle^{-2}$$

where the coefficients of L are extended outside a neighborhood of ρ. Choosing a neighborhood enough small we may assume that

$$\lambda=w\phi_1 \tag{5.3.1}$$

where $c_1\leq w(x,\xi')\leq c_2$, $w\in S(1,g_0)$ with some $c_i>0$. Let us write

$$p=-(\xi_0+\lambda)(\xi_0-\lambda)+Q.$$

Recall

$$Q=\sum_{j=2}^r\phi_j^2+a(\phi)\phi_1^4\langle\xi'\rangle^{-2}+b(\phi')L(\phi')\phi_1^2\geq c(|\phi'|^2+\phi_1^4\langle\xi'\rangle^{-2})$$

with some $c>0$ where $\phi'=(\phi_2,...,\phi_r)$. Take $0\leq\chi_i(x',\xi')\leq 1$, homogeneous of degree 0 in ξ' $(|\xi'|\geq 1)$, which are 1 in conic neighborhoods of ρ', $\rho=(0,\rho')$ and supported in another small conic neighborhoods of ρ' such that $\chi_2=1$ on the support of χ_1. We can assume that Proposition 3.5.1 holds in a neighborhood of the support of χ_2. We now define $f(x,\xi')$ solving

$$\{\xi_0-\lambda,f\}=0,\quad f(0,x',\xi')=(1-\chi_1(x',\xi'))\langle\xi'\rangle. \tag{5.3.2}$$

Note that $f(x,\xi')=\langle\xi'\rangle$ outside some neighborhood of ρ' because $\lambda=0$ and $\chi_1=0$ outside some neighborhood of ρ'.

Lemma 5.3.1 *Let $f(x,\xi')$ be as above. Taking $M>0$ large and $\tau>0$ small we have a decomposition*

$$p=-(\xi_0+\lambda)(\xi_0-\lambda)+\hat{Q}$$

in $|x_0|<\tau$ with $\hat{Q}=Q+M^2f(x,\xi')^2$ such that

$$|\{\xi_0-\lambda,\hat{Q}\}|\le C\hat{Q},\quad |\{\xi_0+\lambda,\xi_0-\lambda\}|\le C\big(\sqrt{\hat{Q}}+|\lambda|\big).$$

Proof: By a compactness argument there are $c>0$ and $\tau>0$ such that we have

$$f(x,\xi')\ge c|\xi'|$$

outside the support of χ_2 if $|x_0|\le\tau$. Let us consider

$$|\{\xi_0-\lambda,\hat{Q}\}|$$

which is bounded by CQ on the support of χ_2 by Proposition 3.5.1 and by CM^2f^2 outside the support of χ_2, thus bounded by $C\hat{Q}$. Noting that $\{\xi_0+\lambda,\xi_0-\lambda\}=2\{\lambda,\xi_0-\lambda\}$ and $\{\phi_j,\xi_0-\lambda\}$ is a linear combination of ϕ_j, $j=1,...,r$ and $\lambda=\phi_1+L(\phi')\phi_1+\gamma\phi_1^3\langle\xi'\rangle^{-2}$ on the support of χ_2 repeating the same arguments we conclude that

$$|\{\xi_0+\lambda,\xi_0-\lambda\}|\le C(\sqrt{\hat{Q}}+|\lambda|)$$

which is the second assertion. □

Let f_1 be defined as (5.3.2) with $\tilde{\chi}_1$ of which support is smaller than that of χ_1 and consider

$$\tilde{P}=p^w+P_1+M_1f_1(x,\xi')+P_0,\quad p=-(\xi_0+\lambda)(\xi_0-\lambda)+\hat{Q}$$

which coincides with the original P near ρ. In what follows to simplify notations we denote this operator by P, $\hat{Q}$ by Q and $P_1+M_1f_1$ by P_1 again:

$$\tilde{P}\ \text{by}\ P,\quad \hat{Q}\ \text{by}\ Q,\quad P_1+M_1f_1\ \text{by}\ P_1.$$

We sometimes denote

$$\phi_{r+1}(x,\xi')=Mf(x,\xi').$$

Here we make a general remark. Let $a(x,\xi')\in S(\langle\xi'\rangle,g_0)$ be an extended symbol of some symbol which vanishes near ρ on Σ. Then repeating the same arguments as in the proof of Lemma 5.3.1 one can write a as

$$a(x,\xi')=\sum_{j=1}^{r+1}c_j\phi_j(x,\xi')$$

with some $c_j\in S(1,g_0)$.

5.4 Microlocal energy estimates

We study $P = (p + P_{sub})^w + R$ with $R \in S(1, g_0)$ where p is the symbol defined in the previous section. Recall that P coincides with the original P near ρ. We assume that the original P satisfies the strict Ivrii-Petkov-Hörmander condition. In this section we follow the arguments in [24] (also see [6]). We start with

Proposition 5.4.1 *There exists $a \in S(1, g_0)$ such that we can write*

$$P = -\tilde{M}\tilde{\Lambda} + Q + \hat{P}_1 + B\tilde{\Lambda} + \hat{P}_0$$

where $\tilde{\Lambda} = (\xi_0 - \lambda - a)^w$, $\tilde{M} = (\xi_0 + \lambda + a)^w$ and B, $\hat{P}_0 \in S(1, g_0)$ moreover we have

$$\mathsf{Im}\,\hat{P}_1 = \sum_{j=2}^{r+1} c_j\phi_j, \quad c_j \in S(1, g_0),$$

$$\mathrm{Tr}^+ Q_\rho + \mathsf{Re}\,\hat{P}_1(\rho) \geq c\langle\xi'\rangle, \quad \rho \in \Sigma, \quad \hat{P}_1 \in S(\langle\xi'\rangle, g_0)$$

with some $c > 0$.

Proof: As before let us write $P_{sub} = P_s + b(\xi_0 - \lambda)$. Then since λ vanishes on Σ we have

$$P_{sub}\big|_\Sigma = P_s\big|_{\{\phi_1=0,\dots,\phi_r=0\}}.$$

Since the strict Ivrii-Petkov-Hörmander condition is verified then we conclude that

$$\mathsf{Im}\,P_s = 0$$

on Σ near ρ. We note that

$$\begin{aligned} p^w &= -(\xi_0 + \lambda)^w(\xi_0 - \lambda)^w + Q^w - \frac{i}{2}\{\xi_0 + \lambda, \xi_0 - \lambda\} + R \\ &= -M\Lambda + Q^w - \frac{i}{2}\{\xi_0 + \lambda, \xi_0 - \lambda\} + R, \quad R \in S(1, g_0) \end{aligned}$$

with $\Lambda = (\xi_0 - \lambda)^w$, $M = (\xi_0 + \lambda)^w$. Since $\{\xi_0 + \lambda, \xi_0 - \lambda\}$ and $\mathsf{Im}\,P_s$ are linear combinations of ϕ_j, $j = 1, ..., r$ near ρ then, as we remarked as before, we can write

$$\mathsf{Im}\,\hat{P}_1 = \mathsf{Im}\,P_s - \frac{1}{2}\{\xi_0 + \lambda, \xi_0 - \lambda\} = \sum_{j=1}^{r+1} c_j\phi_j \tag{5.4.1}$$

with some real $c_j \in S(1, g_0)$. Recalling

$$w\phi_1 = \frac{1}{2}\big((\xi_0 + \lambda) - (\xi_0 - \lambda)\big)$$

one can write

$$-M\Lambda + (ic_1\phi_1)^w = -(\xi_0 + \lambda + iw^{-1}c_1/2)^w(\xi_0 - \lambda - iw^{-1}c_1/2)^w + r$$

with some $r \in S(1, g_0)$. Since it is clear $B\Lambda = B(\xi_0 - \lambda - iw^{-1}c_1/2)^w + r'$, $r' \in S(1, g_0)$ we get the assertion on $\mathsf{Im}\, \hat{P}_1$.

Lemma 4.5.1 and the strict Ivrii-Petkov-Hörmander condition shows that

$$\mathrm{Tr}^+ Q_\rho + \mathsf{Re}\, P_s(\rho) > 0$$

on Σ near the reference point, say in V. Outside V we have $f_1(x, \xi') \geq c\langle \xi' \rangle$ with some $c > 0$ and hence the second assertion. □

From Proposition 5.4.1 we can write

$$P = -\tilde{M}\tilde{\Lambda} + B\tilde{\Lambda} + \tilde{Q}$$

where

$$\begin{cases} \tilde{M} = \xi_0 + \lambda + a = \xi_0 - \tilde{m}, \\ \tilde{\Lambda} = \xi_0 - \lambda - a = \xi_0 - \tilde{\lambda}, \\ \tilde{Q} = Q + \hat{P}_1 + \hat{P}_0. \end{cases}$$

Recall that Proposition 4.3.2 gives

$$\begin{aligned} 2\mathsf{Im}(P_\theta u, \tilde{\Lambda}_\theta u) \geq & \frac{d}{dx_0}(\|\tilde{\Lambda}_\theta u\|^2 + ((\mathsf{Re}\, \tilde{Q})u, u) + \theta^2\|u\|^2) \\ & +\theta\|\tilde{\Lambda}_\theta u\|^2 + 2\theta\mathsf{Re}(\tilde{Q}u, u) + 2((\mathsf{Im}\, B)\tilde{\Lambda}_\theta u, \tilde{\Lambda}_\theta u) \\ & +2((\mathsf{Im}\, \tilde{m})\tilde{\Lambda}_\theta u, \tilde{\Lambda}_\theta u) + 2\mathsf{Re}(\tilde{\Lambda}_\theta u, (\mathsf{Im}\, \tilde{Q})u) \\ & +\mathsf{Im}([D_0 - \mathsf{Re}\, \tilde{\lambda}, \mathsf{Re}\, \tilde{Q}]u, u) + 2\mathsf{Re}((\mathsf{Re}\, \tilde{Q})u, (\mathsf{Im}\, \tilde{\lambda})u) \\ & +\theta^3\|u\|^2 + 2\theta^2((\mathsf{Im}\, \tilde{\lambda})u, u). \end{aligned} \tag{5.4.2}$$

Since $\mathsf{Im}\, \tilde{m}$, $\mathsf{Im}\, \tilde{\lambda} \in S(1, g_0)$ then it is clear that

$$|((\mathsf{Im}\, \tilde{m})\tilde{\Lambda}_\theta u, \tilde{\Lambda}_\theta u)| \leq C\|\tilde{\Lambda}_\theta u\|^2, \quad |((\mathsf{Im}\, \tilde{\lambda})u, u)| \leq C\|u\|^2. \tag{5.4.3}$$

It is also clear

$$((\mathsf{Im}\, B)\tilde{\Lambda}_\theta u, \tilde{\Lambda}_\theta u) \geq -C\|\tilde{\Lambda}_\theta u\|^2 \tag{5.4.4}$$

with some $C > 0$ because $\mathsf{Im}\, B \in S(1, g_0)$. To simplify notations let us denote

$$\Phi = (\Phi_2, ..., \Phi_r, \Phi_{r+1}, \Phi_{r+2}) = (\phi_2, ..., \phi_r, f, \phi_1^2\langle \xi' \rangle^{-1})$$

where we recall $\Phi_j \in S(\langle \xi' \rangle, g_0)$.

Lemma 5.4.1 *There exist $C_i > 0$ such that we have*

$$\sum_{j=2}^{r+2} \|\Phi_j u\|^2 \leq C_1(Qu, u) + C_2\|u\|^2.$$

Proof: Take $C_1 > 0$ so that $C_1 Q - \sum_{j=2}^{r+2} \Phi_j^2 \geq 0$. Then from the Fefferman-Phong inequality it follows that

$$C_1(Qu, u) \geq \big((\sum_{j=2}^{r+2} \Phi_j^2)^w u, u\big) - C_2\|u\|^2.$$

Noting that

$$\sum_{j=2}^{r+2} \Phi_j^2 = \sum_{j=2}^{r+2} \Phi_j \# \Phi_j + R, \quad R \in S(1, g_0)$$

the proof is immediate. $\square$

We now study

$$\mathsf{Re}\,\tilde{Q} = Q + \mathsf{Re}\,\hat{P}_1 + \mathsf{Re}\,\hat{P}_0, \quad \mathsf{Re}\,\hat{P}_1 \in S(\langle\xi'\rangle, g_0).$$

From Proposition 5.4.1 taking sufficiently small $\epsilon_0 > 0$ we have

$$(1-\epsilon_0)\mathrm{Tr}^+ Q_\rho + \mathsf{Re}\,\hat{P}_1(\rho) \geq c\langle\xi'\rangle, \quad \rho \in \Sigma$$

with some $c > 0$ and then from the Melin's inequality [35] it follows that

$$\mathsf{Re}((Q + \mathsf{Re}\,\hat{P}_1)u, u) \geq \epsilon_0 \mathsf{Re}(Qu, u) + c'\|u\|^2_{(1/2)} - C\|u\|^2 \tag{5.4.5}$$

with some $c' > 0$. Thus we conclude

$$\mathsf{Re}(\tilde{Q}u, u) \geq \epsilon_0(Qu, u) + c\|u\|^2_{(1/2)} - C\|u\|^2 \tag{5.4.6}$$

with some $c > 0$.

We now examine the term $\mathsf{Re}((\mathsf{Re}\,\tilde{Q})u, (\mathsf{Im}\,\tilde{\lambda})u)$. Since $\mathsf{Im}\,\tilde{\lambda} \in S(1, g_0)$ we have $\mathsf{Re}(\mathsf{Im}\,\tilde{\lambda} \# Q) = \mathsf{Im}\tilde{\lambda}\, Q + R$ with $R \in S(1, g_0)$ and hence

$$\mathsf{Re}(Qu, (\mathsf{Im}\,\tilde{\lambda})u) \leq (\mathsf{Im}\,\tilde{\lambda}\, Qu, u) + C'\|u\|^2.$$

Take $C > 0$ so that $C - \mathsf{Im}\,\tilde{\lambda} \geq 0$ then $C(Qu, u) - (\mathsf{Im}\,\tilde{\lambda}\, Qu, u) \geq -C_1\|u\|^2$ by the Fefferman-Phong inequality because $0 \leq (C - \mathsf{Im}\,\tilde{\lambda})Q \in S(\langle\xi'\rangle^2, g_0)$. Thus we have

$$C(Qu, u) \geq \mathsf{Re}(Qu, (\mathsf{Im}\,\tilde{\lambda})u) - C_2\|u\|^2.$$

Noting $|((\mathsf{Re}\,\hat{P}_1)u, (\mathsf{Im}\,\tilde{\lambda})u)| \leq C\|u\|^2_{(1/2)}$ for $\mathsf{Re}\,\hat{P}_1 \in S(\langle\xi'\rangle, g_0)$ it follows from (5.4.6) that

$$C_3 \mathsf{Re}(\tilde{Q}u, u) + 2\mathsf{Re}((\mathsf{Re}\,\tilde{Q})u, (\mathsf{Im}\,\tilde{\lambda})u) \geq -C\|u\|^2 \tag{5.4.7}$$

with some $C_3 > 0$.

Recall that

$$\mathsf{Im}\,\tilde{Q} = \mathsf{Im}\,\hat{P}_1 + \mathsf{Im}\,\hat{P}_0$$

and note

$$\mathsf{Im}\,\hat{P}_1 = \sum_{j=2}^{r+1} c_j \# \Phi_j + r, \quad c_j,\ r \in S(1, g_0)$$

by (5.4.1). Thus it is easy to see

$$\begin{aligned}|(\tilde{\Lambda}_\theta u, (\mathsf{Im}\,\hat{P}_1)u)| &\le C\|\tilde{\Lambda}_\theta u\|^2 + C\sum_{j=2}^{r+1}\|\Phi_j u\|^2 + C\|u\|^2\\ &\le C\|\tilde{\Lambda}_\theta u\|^2 + C'(Qu,u) + C'\|u\|^2\end{aligned}$$

by Lemma 5.4.1. Thus we get

$$|(\tilde{\Lambda}_\theta u, (\mathsf{Im}\,\tilde{Q})u)| \le C\|\tilde{\Lambda}_\theta u\|^2 + C(Qu,u) + C\|u\|^2. \tag{5.4.8}$$

We consider $\mathsf{Im}([D_0 - \mathsf{Re}\,\tilde{\lambda}, \mathsf{Re}\,\tilde{Q}]u,u)$. Recall that

$$\xi_0 - \mathsf{Re}\,\tilde{\lambda} = \xi_0 - \lambda + R, \quad R \in S(1, g_0).$$

Since

$$[D_0 - \lambda, Q] - \frac{1}{i}\{\xi_0 - \lambda, Q\}^w \in S(1, g_0)$$

and $|\{\xi_0 - \lambda, Q\}| \le CQ$ by Lemma 5.3.1 it follows from the Fefferman-Phong inequality that

$$|([D_0 - \lambda, Q]u,u)| \le C(Qu,u) + C\|u\|^2.$$

Since $[D_0 - \lambda, \mathsf{Re}\,\hat{P}_1 + \mathsf{Re}\,\hat{P}_0] \in S(\langle \xi' \rangle, g_0)$ we get

$$|([D_0 - \lambda, (\mathsf{Re}\,\tilde{Q})]u,u)| \le C(Qu,u) + C\|u\|^2_{(1/2)}.$$

Summarizing we get

$$\mathsf{Im}([D_0 - \mathsf{Re}\,\tilde{\lambda}, \mathsf{Re}\,\tilde{Q}]u,u) \le C(Qu,u) + C\|u\|^2_{(1/2)}. \tag{5.4.9}$$

Taking

$$\|\Lambda_\theta u\|^2 \le C\|\tilde{\Lambda}_\theta u\|^2 + C\|u\|^2$$

into account from (5.4.6), (5.4.7), (5.4.4), (5.4.8) and (5.4.9) we have

Proposition 5.4.2 *For $\theta \ge \theta_0$ we have*

$$\begin{aligned}c\big(\|\Lambda_\theta u(t)\|^2 + \|u(t)\|^2_{(1/2)} + \theta^2\|u(t)\|^2\big)&\\ +c\theta\int_\tau^t \big(\|\Lambda_\theta u(x_0,\cdot)\|^2 + \mathsf{Re}(Qu,u)&\\ +\|u(x_0,\cdot)\|^2_{(1/2)} + \theta^2\|u(x_0,\cdot)\|^2\big)dx_0&\\ +c\int_\tau^t \|\Lambda_\theta u(x_0,\cdot)\|^2 dx_0 \le C\int_\tau^t \|P_\theta u(x_0,\cdot)\|^2 dx_0&\end{aligned}$$

with some $c > 0$, $C > 0$ for any $u \in C^2([T_2, T_1]; C_0^\infty(\mathbb{R}^n))$ vanishing in $x_0 \le \tau$.

We now derive estimates for higher order derivatives of u.

Lemma 5.4.2 *We can write*

$$\langle D'\rangle^s P = (-\tilde{M}\tilde{\Lambda} + \tilde{B}\tilde{\Lambda} + Q + \tilde{P}_1 + \tilde{P}_0)\langle D'\rangle^s$$

where $\tilde{\Lambda} = (\xi_0 - \lambda - \tilde{a})^w$, $\tilde{M} = (\xi_0 + \lambda + \tilde{a})^w$ *with a pure imaginary* $\tilde{a} \in S(1, g_0)$ *and* $\tilde{B}$, $\tilde{P}_0 \in S(1, g_0)$. *Moreover* $\tilde{P}_1$ *verifies the same conditions as in Proposition 5.4.1.*

Proof: Recall that we have

$$P = -\Lambda^2 + B\Lambda + \tilde{Q}$$

where

$$\begin{cases} \Lambda = \xi_0 - \lambda - R, \\ B = -2\lambda + R, \\ \tilde{Q} = Q + \hat{P}_1 + R \end{cases}$$

with $R \in S(1, g_0)$. Noting

$$[\Lambda, \langle D'\rangle^s] \in S(\langle \xi'\rangle^s, g_0), \ \ [\Lambda, [\Lambda, \langle D'\rangle^s]] \in S(\langle \xi'\rangle^s, g_0)$$

it is easy to check that

$$[\Lambda^2, \langle D'\rangle^s] = R_1\Lambda\langle D'\rangle^s + R_2\langle D'\rangle^s$$

with some $R_i \in S(1, g_0)$.

We turn to consider $[B\Lambda, \langle D'\rangle^s]$. Let us write $[B\Lambda, \langle D'\rangle^s] = B[\Lambda, \langle D'\rangle^s] + [B, \langle D'\rangle^s]\Lambda$ and note

$$B[\Lambda, \langle D'\rangle^s]\langle D'\rangle^{-s} = (T_1\lambda + T_2)^w\langle D'\rangle^s$$

where $T_i \in S(1, g_0)$ and $T_1 = -2i\{\lambda, \langle \xi'\rangle^s\}\langle \xi'\rangle^{-s}$ is pure imaginary. Note that one can write

$$T_1\lambda = i\sum_{j=1}^{r+1} a_j\phi_j$$

with $a_j \in S(1, g_0)$. It is clear that we can write

$$[B, \langle D'\rangle^s]\Lambda = R_1\Lambda\langle D'\rangle^s + R_2\langle D'\rangle^s$$

with $R_i \in S(1, g_0)$. We finally check the term $[\tilde{Q}, \langle D'\rangle^s]$. Since

$$[\tilde{Q}, \langle D'\rangle^s]\langle D'\rangle^{-s} - [Q, \langle D'\rangle^s]\langle D'\rangle^{-s} \in S(1, g_0)$$

it suffices to consider $[Q, \langle D'\rangle^s]\langle D'\rangle^{-s}$. Note that

$$[Q, \langle D'\rangle^s]\langle D'\rangle^{-s} - \frac{1}{i}\{Q, \langle \xi'\rangle^s\}\langle \xi'\rangle^{-s} \in S(1, g_0)$$

and it is clear that we can write

$$\{Q, \langle \xi'\rangle^s\}\langle \xi'\rangle^{-s} = \sum_{j=1}^{r+1} c_j\phi_j$$

with real $c_j \in S(1, g_0)$ and hence

$$[Q, \langle D' \rangle^s] = -\big(i(\sum_{j=1}^{r+1} c_j \phi_j)^w + r\big)\langle D' \rangle^s$$

with some $r \in S(1, g_0)$. Repeating the same arguments as in the proof of Proposition 5.4.1 we move $i(a_1 + c_1)\phi_1$ to Λ to get the desired assertion. $\square$

Repeating the same arguments as deriving Proposition 5.4.2 for

$$\mathsf{Im}\,(\langle D' \rangle^s Pu, \tilde{\Lambda}\langle D' \rangle^s u)$$

we obtain energy estimates of $\langle D' \rangle^s u$. To formulate thus obtained estimate let us set

$$N_s(u) = \|\Lambda u\|^2_{(s)} + \mathsf{Re}(Qu, u)_{(s)} + \|u\|^2_{(s+1/2)}$$

where $(u, v)_{(s)} = (\langle D' \rangle^s u, \langle D' \rangle^s v)$ and $\Lambda = D_0 - \lambda^w$ again. Here we remark that

$$\langle \xi' \rangle^s \# Q \# \langle \xi' \rangle^{-s} - Q - \frac{1}{i}\{\langle \xi' \rangle^s, Q\}\langle \xi' \rangle^{-s} \in S(1, g_0)$$

so that

$$|\mathsf{Re}(\langle D' \rangle^s Qu, \langle D' \rangle^s u) - (Q\langle D' \rangle^s u, \langle D' \rangle^s u)| \le C\|u\|^2_{(s)}.$$

We also note that $\tilde{\Lambda}\langle D' \rangle^s = \langle D' \rangle^s \Lambda + r\langle D' \rangle^s$ with $r \in S(1, g)$ so that

$$\|\Lambda u\|^2_{(s)} \le C\|\tilde{\Lambda}\langle D' \rangle^s u\|^2 + C\|u\|^2_{(s)}.$$

Since $e^{\theta x_0} P_\theta e^{-\theta x_0} = P$, $e^{\theta x_0}\Lambda_\theta e^{-\theta x_0} = \Lambda$, choosing and fixing θ enough large we have

Proposition 5.4.3 *We have*

$$N_s(u(t)) + \int_\tau^t N_s(u(x_0))dx_0 \le C(s, T_i)\int_\tau^t \mathsf{Im}(\langle D' \rangle^s Pu, \tilde{\Lambda}\langle D' \rangle^s u)dx_0$$

for any $s \in \mathbb{R}$ and any $u \in C^2([T_2, T_1]; H^\infty(\mathbb{R}^n))$ vanishing in $x_0 \le \tau$.

Corollary 5.4.1 *We have*

$$N_s(u(t)) + \int_\tau^t N_s(u(x_0))dx_0 \le C(s, T_i)\int_\tau^t \|Pu\|^2_{(s)}dx_0$$

for any $s \in \mathbb{R}$ and any $u \in C^2([T_2, T_1]; H^\infty(\mathbb{R}^n))$ vanishing in $x_0 \le \tau$.

Let us put $P_-(x, D) = P(-x_0, x', -D_0, D')$ then it is clear that P_- verifies the same conditions as P. Note that $P_-^*(x, D)$ satisfies the strict Ivrii-Petkov-Hörmander condition by (4.4.6). Repeating the same arguments as proving Proposition 5.4.2 and Corollary 5.4.1 we conclude that Corollary 5.4.1 holds for P_-^*. Since

$$P^*(x, D) = P_-^*(-x_0, x', -D_0, D')$$

we get

Proposition 5.4.4 *We have*

$$N_s(u(t)) + \int_t^\tau N_s(u(x_0))dx_0 \le C(s, T_i)\int_t^\tau \|P^*u\|_{(s)}^2 dx_0$$

for any $s \in \mathbb{R}$ *and any* $u \in C^2([T_2, T_1]; H^\infty(\mathbb{R}^n))$ *vanishing in* $x_0 \ge \tau$.

5.5 Finite propagation speed of WF

Thanks to Proposition 5.4.4 repeating the same arguments on functional analysis in Section 4.4 we conclude that for any given $f \in C^0([T_2, T_1]; H^\infty(\mathbb{R}^n))$ vanishing in $x_0 \le 0$ there is a unique $u \in C^2([T_2, T_1]; H^\infty(\mathbb{R}^n))$ vanishing in $x_0 \le 0$ such that $Pu = f$. Let us denote

$$u = Gf$$

then it is clear that G verifies (i) and (ii) in Definition 5.2.2 with $\beta = -1/2$. Therefore in order to show that G is a parametrix of P with finite propagation speed of WF it remains to prove (iii). To prove that G verifies (iii) we introduce symbols of spatial type following [24].

Definition 5.5.1 *Let* $f(x, \xi) \in S(1, g_0)$. *We say that* f *is of spatial type if* f *satisfies*

$$\begin{aligned}\{\xi_0 - \lambda, f\} \ge \delta > 0, \quad \{\xi_0 + \lambda, f\}\{\xi_0 - \lambda, f\} \ge \delta > 0,\\ \{f, Q\}^2 \le 4c\big(\{\xi_0 - \lambda, f\}^2 + 2\{\lambda, f\}\{\xi_0 - \lambda, f\}\big)Q\\ = 4c\{\xi_0 + \lambda, f\}\{\xi_0 - \lambda, f\}Q\end{aligned}$$

with some $\delta > 0$ *and* $0 < c < 1$ *for* $|x_0| \le \tau$ *with small* $\tau > 0$.

Let $\chi(x') \in C_0^\infty(\mathbb{R}^n)$ be equal to 1 near $x' = 0$ and vanish in $|x'| \ge 1$. Set

$$d_\epsilon(x', \xi'; \bar\rho') = \{\chi(x' - y')|x' - y'|^2 + |\xi'\langle\xi'\rangle^{-1} - \eta'\langle\eta'\rangle^{-1}|^2 + \epsilon^2\}^{1/2}$$

with $\bar\rho' = (y', \eta')$. Set

$$f(x', \xi'; \bar\rho') = x_0 - \tau + \nu d_\epsilon(x', \xi'; \bar\rho')$$

for small $\nu > 0$, $\epsilon > 0$. Then it is easy to examine that f is a symbol of spatial type for $0 < \nu \le \nu_0$ if ν_0 is small. Indeed since $0 \le Q \in S(\langle\xi'\rangle^2, g_0)$ it follows that

$$\{Q, \nu d_\epsilon\}^2 \le C\nu^2 Q \tag{5.5.1}$$

with $C > 0$ independent of $\epsilon > 0$. On the other hand since it is clear that $\{\xi_0 + \lambda, f\}\{\xi_0 - \lambda, f\} = 1 + O(\nu)$ then we get the assertion taking ν_0 small. Note that ν_0 is independent of $\bar\rho'$ and $\epsilon > 0$.

Recall that one can write

$$P = -\Lambda^2 + B\Lambda + \tilde{Q}$$

where $\Lambda = \xi_0 - \lambda$, $B = -2\lambda + R$ with $R \in S(1, g_0)$ and

$$\tilde{Q} = Q + \hat{P}_1 + \hat{P}_0, \quad \hat{P}_1 \in S(\langle \xi' \rangle, g_0).$$

Let $f(x, \xi')$ be of spatial type. We define Φ by

$$\Phi(x, \xi') = \begin{cases} \exp\left(1/f(x, \xi')\right) & \text{if } f < 0 \\ 0 \quad \text{otherwise} \end{cases}$$

and also set

$$\Phi_1 = f^{-1}\{\Lambda, f\}^{1/2}\Phi.$$

Note that Φ, $\Phi_1 \in S(1, g_0)$ and

$$\Phi - (f\{\Lambda, f\}^{-1/2})\#\Phi_1 \in S(\langle \xi' \rangle^{-1}, g_0). \tag{5.5.2}$$

Consider

$$\mathsf{Im}(P\Phi u, \Lambda\Phi u)_{(s)} = \mathsf{Im}([P, \Phi]u, \Lambda\Phi u)_{(s)} + \mathsf{Im}(\Phi P u, \Lambda \Phi u)_{(s)}. \tag{5.5.3}$$

To estimate the term $\mathsf{Im}([P, \Phi]u, \Lambda\Phi u)_{(s)}$ we follow the arguments in [24].

Definition 5.5.2 *Let $T(u)$, $S(u)$ be two real functionals of u. Then we say $T(u) \sim S(u)$ and $T(u) \preceq S(u)$ if*

$$\begin{aligned} |T(u) - S(u)| &\le C(N_s(\Phi u) + N_{s-1/4}(u)), \\ T(u) &\le C(S(u) + N_s(\Phi u) + N_{s-1/4}(u)) \end{aligned}$$

respectively with some $C > 0$.

We first consider

$$-([\Lambda^2, \Phi]u, \Lambda\Phi u)_{(s)} = -(\Lambda[\Lambda, \Phi]u, \Lambda\Phi u)_{(s)} - ([\Lambda, \Phi]\Lambda u, \Lambda\Phi u)_{(s)}.$$

Note

$$(\Lambda[\Lambda, \Phi]u, \Phi\Lambda u)_{(s)} = -i\frac{d}{dx_0}([\Lambda, \Phi]u, \Phi\Lambda u)_{(s)} + ([\Lambda, \Phi]u, \Lambda\Phi\Lambda u)_{(s)}$$

for λ is real. Since it is clear that $([\Lambda, \Phi]u, [\Lambda, \Phi]\Lambda u)_{(s)} \sim 0$ we have

$$-\mathsf{Im}(\Lambda[\Lambda, \Phi]u, \Phi\Lambda u)_{(s)} \sim \frac{d}{dx_0}\mathsf{Re}([\Lambda, \Phi]u, \Phi\Lambda u)_{(s)} - \mathsf{Im}([\Lambda, \Phi]u, \Phi\Lambda^2 u)_{(s)}.$$

We next examine that

$$-\mathsf{Im}([\Lambda, \Phi]\Lambda u, \Lambda\Phi u)_{(s)} \sim -\|\Lambda\Phi_1 u\|^2_{(s)}.$$

Indeed since $\{\Lambda,\Phi\}-i\{\Lambda,f\}f^{-2}\Phi\in S(\langle\xi'\rangle^{-1},g_0)$ and hence

$$-\mathsf{Im}([\Lambda,\Phi]\Lambda u,\Lambda\Phi u)_{(s)}\sim-\mathsf{Re}((\{\Lambda,f\}f^{-2}\Phi)^w\Lambda u,\Lambda\Phi u)_{(s)}.$$

Since $\Phi=(\{\Lambda,f\}^{-1/2}f)\#\Phi_1+T$, $T\in S(\langle\xi'\rangle^{-1},g_0)$ which follows from (5.5.2) and

$$(\{\Lambda,f\}^{-1/2}f)\#\langle\xi'\rangle^{2s}\#(\{\Lambda,f\}f^{-2}\Phi)=\langle\xi'\rangle^{2s}\#\Phi_1+S(\langle\xi'\rangle^{2s-1},g_0)$$

one conclude easily the assertion. Therefore we have

$$\begin{aligned}-\mathsf{Im}([\Lambda^2,\Phi]u,\Lambda\Phi u)_{(s)}&\sim\frac{d}{dx_0}\mathsf{Re}([\Lambda,\Phi]u,\Phi\Lambda u)_{(s)}\\&-\mathsf{Im}([\Lambda,\Phi]u,\Phi\Lambda^2u)_{(s)}-\|\Lambda\Phi_1u\|_{(s)}^2.\end{aligned}\tag{5.5.4}$$

We turn to consider

$$([B\Lambda,\Phi]u,\Lambda\Phi u)_{(s)}=(B[\Lambda,\Phi]u,\Lambda\Phi u)_{(s)}+([B,\Phi]\Lambda u,\Lambda\Phi u)_{(s)}.$$

Write

$$\begin{aligned}(B[\Lambda,\Phi]u,\Lambda\Phi u)_{(s)}=2i((\mathsf{Im}B_s)\langle D'\rangle^s[\Lambda,\Phi]u,\langle D'\rangle^s\Lambda\Phi u)\\+(B_s^*\langle D'\rangle^s[\Lambda,\Phi]u,\langle D'\rangle^s\Lambda\Phi u)\\=2i((\mathsf{Im}B_s)\langle D'\rangle^s[\Lambda,\Phi]u,\langle D'\rangle^s\Lambda\Phi u)+([\Lambda,\Phi]u,B\Lambda\Phi u)_{(s)}\end{aligned}$$

with $B_s=\langle D'\rangle^sB\langle D'\rangle^{-s}$ and note $\mathsf{Im}B_s=\mathsf{Im}B+r$, $\mathsf{Im}B\in S(1,g_0)$, $r\in S(\langle\xi'\rangle^{-1},g_0)$. Then we see

$$\begin{aligned}&|(\mathsf{Im}B_s)\langle D'\rangle^s[\Lambda,\Phi]u,\langle D'\rangle^s\Lambda\Phi u)|\\&\quad\le C\|\Lambda\Phi u\|_{(s)}^2+C\|u\|_{(s)}^2\sim0.\end{aligned}$$

Thus we have

$$\mathsf{Im}(B[\Lambda,\Phi]u,\Lambda\Phi u)_{(s)}\sim\mathsf{Im}([\Lambda,\Phi]u,\Phi B\Lambda u)_{(s)}.$$

On the other hand recalling $B=-2\lambda+R$ with $R\in S(1,g_0)$ we see

$$[B,\Phi]=i\{2\lambda-R,\Phi\}^w+T,\quad T\in S(\langle\xi'\rangle^{-2},g_0)$$

and hence $\mathsf{Im}([B,\Phi]\Lambda u,\Lambda\Phi u)_{(s)}\sim\mathsf{Re}((\{2\lambda-R,\Phi\})^w\Lambda u,\Lambda\Phi u)_{(s)}$. Since $\{2\lambda-R,\Phi\}=-\{2\lambda-R,f\}f^{-2}\Phi$ and $\{R,f\}\in S(\langle\xi'\rangle^{-1},g_0)$ then repeating the same arguments as before we get

$$\mathsf{Im}([B,\Phi]\Lambda u,\Lambda\Phi u)_{(s)}\preceq-2\Big((\{\Lambda,f\}^{-1}\{\lambda,f\})^w\Lambda\Phi_1u,\Lambda\Phi_1u\Big)_{(s)}$$

and hence

$$\begin{aligned}\mathsf{Im}([B\Lambda,\Phi]u,\Lambda\Phi u)_{(s)}&\preceq\mathsf{Im}([\Lambda,\Phi]u,\Phi B\Lambda u)_{(s)}\\&-2\mathsf{Re}\Big((\{\Lambda,f\}^{-1}\{\lambda,f\})^w\Lambda\Phi_1u,\Lambda\Phi_1u\Big)_{(s)}.\end{aligned}\tag{5.5.5}$$

We finally consider $([\tilde{Q}, \Phi]u, \Lambda\Phi u)_{(s)}$. Noting that $\hat{P}_1 \in S(\langle\xi'\rangle, g_0)$ and hence

$$\left|([\hat{P}_1, \Phi]u, \Lambda\Phi u)_{(s)}\right| \le C\|u\|^2_{(s)} + C\|\Lambda\Phi u\|^2_{(s)} \sim 0.$$

Since $[Q, \Phi] = (-i\{Q, \Phi\})^w + R$ with $R \in S(\langle\xi'\rangle^{-1}, g_0)$ it follows from the same arguments that

$$\mathsf{Im}([Q + \hat{P}_1, \Phi]u, \Lambda\Phi u)_{(s)} \sim \mathsf{Re}\big((\{Q, f\}\{\Lambda, f\}^{-1})^w\Phi_1 u, \Lambda\Phi_1 u\big)_{(s)}.$$

Thus we obtain

$$\mathsf{Im}([\tilde{Q}, \Phi]u, \Lambda\Phi u)_{(s)} \preceq \mathsf{Re}\big((\{Q, f\}\{\Lambda, f\}^{-1})^w\Phi_1 u, \Lambda\Phi_1 u\big)_{(s)}. \tag{5.5.6}$$

Note that the sum of the second and the first term on the right-hand side of (5.5.4) and (5.5.5) yields

$$\mathsf{Im}([\Lambda, \Phi]u, \Phi(-\Lambda^2 + B\Lambda)u)_{(s)}.$$

Taking into account $-\Lambda^2 + B\Lambda = P - \tilde{Q}$ let us study

$$-\mathsf{Im}([\Lambda, \Phi]u, \Phi\tilde{Q}u)_{(s)}.$$

Write $\tilde{Q} = Q + \mathsf{Re}\,\hat{P}_1 + i\mathsf{Im}\,\hat{P}_1$ because $\hat{P}_0$ is irrelevant. Note that

$$\mathsf{Re}([\Lambda, \Phi]u, \Phi\,\mathsf{Im}\,\hat{P}_1 u)_{(s)} \sim -\mathsf{Im}\,(\{\Lambda, \Phi\}^w u, \Phi\,\mathsf{Im}\,\hat{P}_1 u)_{(s)} \sim 0.$$

Hence one has

$$\begin{aligned} -\mathsf{Im}([\Lambda, \Phi]u, \Phi\tilde{Q}u)_{(s)} &\sim -\mathsf{Im}([\Lambda, \Phi]u, \Phi(Q + \mathsf{Re}\,\hat{P}_1)u)_{(s)} \\ &= -\mathsf{Im}(\Phi\langle D'\rangle^{2s}[\Lambda, \Phi]u, (Q + \mathsf{Re}\,\hat{P}_1)u). \end{aligned}$$

Here we note that $\Phi\langle D'\rangle^{2s}[\Lambda, \Phi] = (i\Phi_1\langle\xi'\rangle^{2s}\Phi_1)^w + T_1 + T_2$ where $T_1 \in S(\langle\xi'\rangle^{2s-1}, g_0)$ is real and $T_2 \in S(\langle\xi'\rangle^{2s-2}, g_0)$. Since

$$-\mathsf{Im}((T_1 + T_2)u, (Q + \mathsf{Re}\,\hat{P}_1)u) \sim -\mathsf{Im}\big(T_1 u, Qu\big) \sim 0$$

it follows that

$$-\mathsf{Im}([\Lambda, \Phi]u, \Phi\tilde{Q}u)_{(s)} \sim -\mathsf{Re}(\Phi_1 u, \Phi_1(Q + \mathsf{Re}\,\hat{P}_1)u)_{(s)}.$$

Note

$$\begin{aligned} (\Phi_1 u, \Phi_1(Q + \mathsf{Re}\,\hat{P}_1)u)_{(s)} &= (\Phi_1 u, (Q + \mathsf{Re}\,\hat{P}_1)\Phi_1 u)_{(s)} \\ &\quad + (\Phi_1 u, [\Phi_1, Q + \mathsf{Re}\,\hat{P}_1]u)_{(s)} \\ &\sim (\Phi_1 u, (Q + \mathsf{Re}\,\hat{P}_1)\Phi_1 u)_{(s)} + (\Phi_1 u, [\Phi_1, Q]u)_{(s)} \end{aligned}$$

where we have $\mathsf{Re}(\Phi_1 u, [\Phi_1, Q]u)_{(s)} \sim 0$ since

$$[\Phi_1, Q] + (i\{\Phi_1, Q\})^w \in S(\langle\xi'\rangle^{-1}, g_0).$$

Thus we have

$$
\begin{aligned}
\mathsf{Im}([\Lambda,\Phi]u,\Phi(-\Lambda^2+B\Lambda)u)_{(s)} &= \mathsf{Im}([\Lambda,\Phi]u,\Phi Pu)_{(s)}\\
-\mathsf{Im}([\Lambda,\Phi]u,\Phi\tilde{Q}u)_{(s)} &\preceq \mathsf{Im}([\Lambda,\Phi]u,\Phi Pu)_{(s)}\\
&-\mathsf{Re}((\Phi_1 u,(Q+\mathsf{Re}\,\hat{P}_1)\Phi_1 u)_{(s)}.
\end{aligned}
\tag{5.5.7}
$$

From (5.5.4), (5.5.5), (5.5.6) and (5.5.7) we conclude that

$$
\begin{aligned}
\mathsf{Im}([P,\Phi]u,\Lambda\Phi u)_{(s)} \preceq \frac{d}{dx_0}&\mathsf{Re}([\Lambda,\Phi]u,\Phi\Lambda u)_{(s)}\\
-\|\Lambda\Phi_1 u\|_{(s)}^2 - &\mathsf{Re}((Q+\mathsf{Re}\,\hat{P}_1)\Phi_1 u,\Phi_1 u)_{(s)}\\
-2\mathsf{Re}&\big((\{\Lambda,f\}^{-1}\{\lambda,f\})^w\Lambda\Phi_1 u,\Lambda\Phi_1 u\big)_{(s)}\\
+\mathsf{Re}&\big((\{\Lambda,f\}^{-1}\{Q,f\})^w\Phi_1 u,\Lambda\Phi_1 u\big)_{(s)}\\
&+\mathsf{Im}([\Lambda,\Phi]u,\Phi Pu)_{(s)}.
\end{aligned}
$$

We remark that setting

$$
a=(1+2\{\Lambda,f\}^{-1}\{\lambda,f\})^{1/2},\quad b=a^{-1}\{\Lambda,f\}^{-1}\{Q,f\}
$$

we see that

$$
\begin{aligned}
&\|\Lambda\Phi_1 u\|_{(s)}^2+2\mathsf{Re}((\{\Lambda,f\}^{-1}\{\lambda,f\})^w\Lambda\Phi_1 u,\Lambda\Phi_1 u)_{(s)}\\
&\qquad\sim \|a^w\Lambda\Phi_1 u\|_{(s)}^2,\\
&\|a\Lambda\Phi_1 u\|_{(s)}^2+\mathsf{Re}((Q+\mathsf{Re}\,\hat{P}_1)\Phi_1 u,\Phi_1 u)_{(s)}\\
&\qquad-\mathsf{Re}((\{\Lambda,f\}^{-1}\{Q,f\})^w\Phi_1 u,\Lambda\Phi_1 u)_{(s)}\\
&\sim \|(a^w\Lambda-\frac{b^w}{2})\Phi_1 u\|_{(s)}^2+\mathsf{Re}((Q+\mathsf{Re}\,\hat{P}_1-\frac{1}{4}(b^2)^w)\Phi_1 u,\Phi_1 u)_{(s)}
\end{aligned}
$$

because

$$
\begin{aligned}
a\#a-a^2\in S(\langle\xi'\rangle^{-1},g_0),\quad &b\#b-b^2\in S(1,g_0),\\
&a\#b-ab\in S(1,g_0).
\end{aligned}
$$

From the assumption we have

$$
\begin{aligned}
&\hat{Q}=Q-\frac{1}{4}b^2=\frac{1}{4}\{\Lambda,f\}^{-2}a^{-2}\\
&\times\Big(4Q\big(\{\Lambda,f\}^2+2\{\Lambda,f\}\{\lambda,f\}\big)-\{Q,f\}^2\Big)\geq 0.
\end{aligned}
$$

but we note that the positive trace $\mathrm{Tr}^+\hat{Q}_\rho$ can be smaller than $\mathrm{Tr}^+ Q_\rho$ in general.

To avoid this inconvenience we choose f carefully. We first recall that

$$\mathrm{rank}\,(\{\phi_i,\phi_j\})_{0\le i,j\le r}=\mathrm{rank}\,(\{\phi_i,\phi_j\})_{1\le i,j\le r}=2k$$

is constant on Σ by assumption. Let $\rho\in\Sigma$ and take a new homogeneous symplectic coordinates system (X,Ξ) around ρ such that $\Xi_0=\xi_0-\phi_1$ and $X_0=x_0$ (see Appendix). Since $\{\Xi_0,\phi_j\}=0$ on Σ, $j=1,...,r$ then Σ is cylindrical in the X_0 direction and defined near ρ by $\Xi_0=0$, $\phi_j(0,X',\Xi')=0$, $j=1,...,r$. From Theorem 21.2.4 in [19] there are homogeneous symplectic coordinates y', η' such that $\Sigma'=\{\phi_j(0,X',\Xi')=0, j=1,...,r\}$ is defined by

$$y_1=\cdots=y_k=\eta_1=\cdots=\eta_k=0,\;\;\eta_{k+1}=\cdots=\eta_{k+\ell}=0$$

where $r=2k+\ell$. Let $\{y_{k+1},...,y_n,\eta_{k+\ell+1},...,\eta_n\}$ be given by $\psi_1(x',\xi')$,..., $\psi_s(x',\xi')$, $s=2n-(2k+\ell)$ in the original coordinates. We denote by the same $\psi_j(x',\xi')$ their extended symbols and define

$$d_{Q,\epsilon}(x,\xi';\bar\rho')=\big\{Q(x,\xi')\langle\xi'\rangle^{-2}+\sum_{j=1}^{s}(\tilde\psi_j(x',\xi')-\tilde\psi_j(\bar\rho'))^2+\epsilon^2\big\}^{1/2}$$

with $\tilde\psi_j=\psi_j\langle\xi'\rangle^{-1}$. Here we note that

$$\mathrm{Tr}^+\,Q_\rho=\mathrm{Tr}^+\,\big(Q-\frac{1}{4}\{Q,d_{Q,\epsilon}\}^2\big)_\rho \tag{5.5.8}$$

on Σ which is examined without difficulties because in the coordinates y', η' above we see that $\{Q,d_{Q,\epsilon}\}^2_\rho$ is a quadratic form in $(\eta_{k+1},...,\eta_{k+\ell})$ which is symplectically independent from $\{y_1,...,y_k,\eta_1,...,\eta_k\}$. It is easy to see that

$$C^{-1}d_0(x',\xi';\bar\rho')\le d_{Q,0}(x,\xi';\bar\rho')\le Cd_0(x',\xi';\bar\rho')$$

with some $C>0$ for (x',ξ') near $\bar\rho'$ and x_0 close to 0. Here we define Φ using f_Q

$$f_Q(x,\xi';\bar\rho')=x_0-\tau+\nu d_{Q,\epsilon}(x,\xi';\bar\rho'). \tag{5.5.9}$$

From (5.5.8) it follows that there is $\nu_0>0$ such that for $0<\nu\le\nu_0$

$$\mathrm{Tr}^+\,\hat Q_\rho+\mathsf{Re}\,\hat P_1(\rho)\ge c\langle\xi'\rangle \tag{5.5.10}$$

with some $c>0$. Then the Melin's inequality gives

$$\mathsf{Re}((Q+\mathsf{Re}\,\hat P_1-\frac{1}{4}(b^2)^w)\Phi_1,\Phi_1u)_{(s)}\ge c'\|\Phi_1u\|^2_{(s+1/2)}-C\|u\|^2_{(s)}$$

with some $c'>0$. We summarize what we have proved in

Lemma 5.5.1 *Let Φ be defined by f_Q. Then there exists $\nu_0>0$ such that for any $0<\nu\le\nu_0$ we have*

$$\begin{aligned}\mathsf{Im}([P,\Phi]u,\Lambda\Phi u)_{(s)}\preceq\;&\frac{d}{dx_0}\mathsf{Re}([\Lambda,\Phi]u,\Phi\Lambda u)_{(s)}\\&+\mathsf{Im}([\Lambda,\Phi]u,\Phi Pu)_{(s)}.\end{aligned}$$

We turn to $\mathsf{Im}(P\Phi u, \Lambda\Phi u)_{(s)}$. Let $\tilde{\Lambda} = \Lambda + a$ with $a \in S(1, g_0)$ where a is pure imaginary. Since a is pure imaginary, repeating similar arguments as above we see

$$\mathsf{Im}(\langle D'\rangle^s [P, \Phi]u, a\langle D'\rangle^s \Phi u) \sim 0$$

and hence

$$\begin{aligned}\mathsf{Im}(\langle D'\rangle^s P\Phi u, a\langle D'\rangle^s \Phi u) &\sim \mathsf{Im}(\langle D'\rangle^s \Phi P u, a\langle D'\rangle^s \Phi u)\\ &\geq -C\|\Phi P u\|_{(s)}^2 - C\|\Phi u\|_{(s)}^2\end{aligned}$$

so that

$$\mathsf{Im}(\langle D'\rangle^s P\Phi u, \tilde{\Lambda}\langle D'\rangle^s \Phi u) \succeq \mathsf{Im}(\langle D'\rangle^s P\Phi u, \Lambda\langle D'\rangle^s \Phi u) - C\|\Phi P u\|_{(s)}^2.$$

Noting $[\Lambda, \langle D'\rangle^s] + (i\{\Lambda, \langle \xi'\rangle^s\})^w \in S(\langle \xi'\rangle^{s-2}, g_0)$ the same reasoning shows that

$$\mathsf{Im}(\langle D'\rangle^s [P, \Phi]u, [\Lambda, \langle D'\rangle^s]\Phi u) \sim 0$$

and then we conclude that

$$\mathsf{Im}(P\Phi u, \Lambda\Phi u)_{(s)} \succeq \mathsf{Im}(\langle D'\rangle^s P\Phi u, \tilde{\Lambda}\langle D'\rangle^s \Phi u) - C\|\Phi P u\|_{(s)}^2.$$

From (5.5.3) and Lemma 5.5.1 it follows that

$$\begin{aligned}c\|\Phi_1 u\|_{(s+1/2)}^2 + c\|\Lambda\Phi_1 u\|_{(s)}^2 + \mathsf{Im}(\langle D'\rangle^s P\Phi u, \tilde{\Lambda}\langle D'\rangle^s \Phi u)&\\ \preceq \frac{d}{dx_0}\mathsf{Re}([\Lambda, \Phi]u, \Phi\Lambda u)_{(s)} + C\|\Phi P u\|_{(s)}^2&.\end{aligned}$$

Integrating in x_0 and applying Proposition 5.4.3 we get

Proposition 5.5.1 *Let Φ be as in Lemma 5.5.1. Then we have*

$$\begin{aligned}N_s(\Phi u(t)) + \int_\tau^t N_s(\Phi u)dx_0&\\ \leq C(s, T_i)\Big(N_{s-1/4}(u(t)) + \int_\tau^t \big(\|\Phi P u\|_{(s)}^2 + N_{s-1/4}(u)\big)dx_0\Big)&\end{aligned}$$

for any $s \in \mathbb{R}$ and any $u \in C^2([T_2, T_1]; H^\infty(\mathbb{R}^n))$ vanishing in $x_0 \leq \tau$.

REMARK: It is clear that Proposition 5.5.1 holds for any Φ defined by spatial type f satisfying (5.5.10).

Let Γ_i $(i = 0, 1, 2)$ be open conic sets in $\mathbb{R}^{2n} \setminus \{0\}$ with relatively compact basis such that $\Gamma_0 \subset\subset \Gamma_1 \subset\subset \Gamma_2$. Here $\Gamma_i \subset\subset \Gamma_{i+1}$ means that the base of Γ_i is relatively compact in that of Γ_{i+1}. Let us take $h_i(x', \xi') \in S(1, g_0)$ with $\mathrm{supp}\, h_1 \subset \Gamma_0$ and $\mathrm{supp}\, h_2 \subset \Gamma_2 \setminus \Gamma_1$. We consider the solution $u \in C^1(I; H^\infty)$ to $Pu = h_1 f$ with $f \in C^0(I; H^\infty)$ where $u = f = 0$ in $x_0 < \tau$, with $\tau \in I$. Arguing exactly as in [31] (Lemma 5.2.1 and Proposition 5.2.3) we have

Proposition 5.5.2 *Notations being as above. Then there is $\delta = \delta(\Gamma_i) > 0$ such that*

$$\|D_0^j h_2 u(t)\|_{(p)}^2 \leq C_{pq} \int^t \|f(x_0)\|_{(q)}^2 dx_0$$

for $j = 0, 1$ and $\tau \leq t \leq \tau + \delta$ and any p, $q \in \mathbb{R}$. In particular, there is a parametrix of the Cauchy problem for P with finite propagation speed of WF.

REMARK: Repeating the same arguments as in [31] one can estimate the wave front set applying Proposition 5.5.1. If we have more spatial type symbols verifying (5.5.10) then the estimate of wave front set becomes more precise. See [45].

Proof of Theorem 5.1.1: Thanks to Proposition 6.4.5 then P has a parametrix with finite propagation speed of WF at every $(0, 0, \xi')$ with $|\xi'| = 1$. Then the C^∞ well-posedness of the Cauchy problem follows from Proposition 5.2.3 immediately. □

Repeating similar arguments (with necessary modifications) proving Theorem 5.1.1 we can prove

Theorem 5.5.1 *Assume* (4.1.1), (5.1.1), (5.1.2) *and* $\mathrm{Tr}^+ F_p = 0$ *on* Σ. *Then in order that the Cauchy problem for P is C^∞ well posed it is necessary and sufficient that P satisfies the Levi condition on Σ.*

Note that Σ is neither involutive nor symplectic in this case. To prove energy estimates in Proposition 5.4.3 under the assumption $\mathrm{Tr}^+ F_p = 0$ we use the following

Lemma 5.5.2 *Let $a \in S(1, g_0)$. Then we have*

$$|(a\phi_1 u, u)| \leq C(\|\Phi_2 u\|^2 + \|\Phi_{r+1} u\|^2 + \|\Phi_{r+2} u\|^2) + C'\|u\|^2$$

with some C, $C' > 0$.

Lemma 5.5.3 *We have*

$$\|\langle D'\rangle^{1/3} u\|^2 \leq C(\|\Phi_2 u\|^2 + \|\Phi_{r+1}\|^2 + \|\Phi_{r+2} u\|^2 + \|u\|^2)$$

with some $C > 0$.