Chapter 3

Noneffectively hyperbolic characteristics

3.1 Elementary decomposition

In what follows we assume that the doubly characteristic set

$$\Sigma = \{(x,\xi) \mid p(x,\xi) = dp(x,\xi) = 0\}$$

of p is a smooth conic manifold. In this section we study p of the form

$$p = -\xi_0^2 + a_1(x,\xi')\xi_0 + a_2(x,\xi')$$

which is hyperbolic with respect to ξ_0.

Definition 3.1.1 *We say that $p(x,\xi)$ admits an elementary decomposition if there exist real valued symbols $\lambda(x,\xi')$, $\mu(x,\xi')$, $Q(x,\xi')$ defined near $x = 0$, depending smoothly on x_0, homogeneous of degree 1, 1, 2 respectively and $Q(x,\xi') \geq 0$ such that*

$$p(x,\xi) = -\Lambda(x,\xi)M(x,\xi) + Q(x,\xi'),$$
$$\Lambda(x,\xi) = \xi_0 - \lambda(x,\xi'),\ M(x,\xi) = \xi_0 - \mu(x,\xi'),$$
$$|\{\Lambda(x,\xi), Q(x,\xi')\}| \leq CQ(x,\xi'), \tag{3.1.1}$$
$$|\{\Lambda(x,\xi), M(x,\xi)\}| \leq C(\sqrt{Q(x,\xi')} + |\Lambda(x,\xi') - M(x,\xi')|) \tag{3.1.2}$$

with some constant C. If we can find such symbols defined in a conic neighborhood of ρ then we say that $p(x,\xi)$ admits an elementary decomposition at ρ.

Lemma 3.1.1 ([26]) *Assume that p admits an elementary decomposition. Then there is no null bicharacteristic which has a limit point in Σ.*

Proof: Note that $\Sigma = \{(x,\xi) \mid \Lambda(x,\xi) = M(x,\xi) = Q(x,\xi') = 0\}$ because $\partial_{\xi_0} p = -(\Lambda(x,\xi) + M(x,\xi)) = 0$ and $p(x,\xi) = 0$ implies $\Lambda(x,\xi)^2 + Q(x,\xi') = 0$. Let $\gamma(s)$ be a null bicharacteristic of p which lies outside Σ. Since $p(\gamma(s)) = 0$ we may assume that $dx_0(s)/ds = -\Lambda(\gamma(s)) - M(\gamma(s)) < 0$ so that we can take x_0 as a parameter:

$$\frac{d}{dx_0}\Lambda(\gamma(x_0)) = \frac{d}{ds}\Lambda(\gamma(s))\frac{ds}{dx_0} = \{p,\Lambda\}(\gamma(s))\frac{ds}{dx_0}.$$

Since $M\Lambda = Q \geq 0$ we have $\Lambda(\gamma(s)) \geq 0$ and $M(\gamma(s)) \geq 0$. Noting $p = -M\Lambda + Q$ we see on $\gamma(s)$

$$\begin{aligned} |\{p,\Lambda\}| &\leq C(Q + \Lambda\sqrt{Q} + \Lambda|\Lambda - M|) \\ &= C\Lambda(M + \sqrt{\Lambda M} + |\Lambda - M|). \end{aligned}$$

Since

$$\frac{M + \sqrt{\Lambda M} + |\Lambda - M|}{\Lambda + M} \leq 3$$

one has

$$\left|\frac{d}{dx_0}\Lambda(\gamma(x_0))\right| \leq C\Lambda(\gamma(x_0)). \tag{3.1.3}$$

Suppose that $\gamma(x_0) \notin \Sigma$ for $x_0 \neq 0$ and $\lim_{x_0 \to 0}\gamma(x_0) \in \Sigma$ so that $\Lambda(\gamma(0)) = 0$. From (3.1.3) it follows that $\Lambda(\gamma(x_0)) = 0$ and hence $Q(\gamma(x_0)) = 0$ for $p(\gamma(x_0)) = 0$. Since Q is non-negative it follows that $\{Q, M\}(\gamma(x_0)) = 0$. This proves

$$\left|\frac{d}{dx_0}M(\gamma(x_0))\right| \leq CM(\gamma(x_0))$$

and hence $M(\gamma(x_0)) = 0$ so that $\gamma(x_0) \in \Sigma$ which is a contradiction. □

3.2 Case $\operatorname{Im} F_p^2 \cap \operatorname{Ker} F_p^2 = \{0\}$

Here we work with

$$p(x,\xi) = -\xi_0^2 + q(x,\xi'), \quad q(x,\xi') \geq 0.$$

We assume that the doubly characteristic set

$$\Sigma = \{(x,\xi) \mid p(x,\xi) = dp(x,\xi) = 0\}$$

is a smooth manifold near $\bar{\rho}$ such that

$$\dim T_\rho\Sigma = \dim \operatorname{Ker} F_p(\rho), \quad \rho \in \Sigma \tag{3.2.1}$$

that is, the codimension of Σ is equal to the rank of the Hessian of p at every point on Σ and

$$\operatorname{rank}\big(\sigma|_\Sigma\big) = \text{constant} \quad \text{on } \Sigma \tag{3.2.2}$$

where $\sigma = \sum d\xi_j \wedge dx_j$ and finally we assume that p is noneffectively hyperbolic at every $\rho \in \Sigma$ and

$$\mathrm{Ker}\, F_p^2(\rho) \cap \mathrm{Im}\, F_p^2(\rho) = \{0\}, \quad \forall \rho \in \Sigma. \tag{3.2.3}$$

From the hypothesis (3.2.1), near every $\bar{\rho} \in \Sigma$, one can write

$$p(x,\xi) = -\xi_0^2 + \sum_{j=1}^{r} \phi_j(x,\xi')^2 \tag{3.2.4}$$

where $d\phi_j$ are linearly independent at $\bar{\rho}$ and Σ is given by

$$\Sigma = \{(x,\xi) \mid \phi_j(x,\xi) = 0, j = 0, ..., r\}$$

near $\bar{\rho}$ where we have set $\phi_0(x,\xi) = \xi_0$. Let $Q(u,v)$ be the polar form of $p_{\bar{\rho}}$. Since

$$\frac{1}{2}Q(u,v) = -d\phi_0(u)d\phi_0(v) + \sum_{j=1}^{r} d\phi_j(u)d\phi_j(v)$$

where $d\phi_j(u) = d\phi_j(\bar{\rho}; u)$ then it follows that

$$\frac{1}{2}Q(u,v) = -\sigma(u, H_{\phi_0})\sigma(v, H_{\phi_0}) + \sum_{j=1}^{r} \sigma(u, H_{\phi_j})\sigma(v, H_{\phi_j})$$

$$= \sigma\Big(u, -\sigma(v, H_{\phi_0})H_{\phi_0} + \sum_{j=1}^{r} \sigma(v, H_{\phi_j})H_{\phi_j}\Big) = \sigma(u, F_p(\bar{\rho})v).$$

Thus we have

$$F_p(\bar{\rho})v = -\sigma(v, H_{\xi_0})H_{\xi_0} + \sum_{j=1}^{r} \sigma(v, H_{\phi_j}(\bar{\rho}))H_{\phi_j}(\bar{\rho}). \tag{3.2.5}$$

In particular we see

$$\mathrm{Im}\, F_p(\bar{\rho}) = \langle H_{\phi_0}(\bar{\rho}), H_{\phi_1}(\bar{\rho}), ..., H_{\phi_r}(\bar{\rho}) \rangle. \tag{3.2.6}$$

It is also clear that

$$\begin{aligned} \mathrm{Ker}\, F_p(\bar{\rho}) &= \{v \in \mathbb{R}^{2(n+1)} \mid \sigma(v, H_{\phi_j}) = 0, j = 0, 1, ..., r\} \\ &= \langle H_{\phi_0}, H_{\phi_1}, ..., H_{\phi_r} \rangle^{\sigma} = (\mathrm{Im}\, F_p(\bar{\rho}))^{\sigma} = T_{\bar{\rho}}\Sigma. \end{aligned} \tag{3.2.7}$$

Here we remark

Lemma 3.2.1 *The condition* (3.2.2) *is equivalent to*

$$\mathrm{rank}(\{\phi_i, \phi_j\})(\rho) = \mathrm{const}, \quad \rho \in \Sigma.$$

Proof: Note that

$$(T_\rho\Sigma)^\sigma = \langle H_{\phi_0}(\rho), ..., H_{\phi_r}(\rho)\rangle$$

and $\sigma(H_{\phi_i}(\rho), H_{\phi_j}(\rho)) = \{\phi_i, \phi_j\}(\rho)$. From this it is enough to show that (3.2.2) is equivalent to

$$\mathrm{rank}\left(\sigma|_{(T_\rho\Sigma)^\sigma}\right) = \mathrm{const.}$$

Let us consider the map

$$L : T_\rho\Sigma \ni v \mapsto \sum_{j=1}^{s} \sigma(v, f_j(\rho))f_j(\rho) \in T_\rho\Sigma$$

where $T_\rho\Sigma = \langle f_1(\rho), ..., f_s(\rho)\rangle$. The assumption (3.2.2) implies that the rank of the matrix $(\sigma(f_i(\rho), f_j(\rho)))$ is constant and hence

$$\dim \mathrm{Ker}\, L = \dim\left(T_\rho\Sigma \cap (T_\rho\Sigma)^\sigma\right) = \mathrm{const.}$$

This proves the desired assertion because the kernel of

$$\tilde{L} : (T_\rho\Sigma)^\sigma \ni v \mapsto \sum_{j=0}^{r} \sigma(v, H_{\phi_j}(\rho))H_{\phi_j}(\rho) \in (T_\rho\Sigma)^\sigma$$

is just $\mathrm{Ker}\, L$. □

Assume (3.2.3) then from Corollary 2.3.1 the quadratic form p_ρ takes the form, in a suitable symplectic coordinates

$$p_\rho = -\xi_0^2 + \sum_{j=1}^{k} \mu_j^2(x_j^2 + \xi_j^2) + \sum_{j=k+1}^{k+\ell} \xi_j^2 \tag{3.2.8}$$

where we have

Lemma 3.2.2 *The number k in* (3.2.8) *is independent of $\rho \in \Sigma$.*

Proof: With $\{\psi_j\} = \{\xi_0, x_j, \xi_j, 1 \le j \le k, \xi_j, k+1 \le j \le k+\ell\}$ it follows from Lemma 3.2.1 that the rank of $(\{\psi_i, \psi_j\})$ is constant. This shows that k is independent of $\rho \in \Sigma$. □

Lemma 3.2.3 *There exist a conic neighborhood V of $\bar{\rho}$ and a smooth vector $h(\rho)$ defined in $V \cap \Sigma$ such that*

$$h(\rho) \in \mathrm{Ker}\, F_p^2(\rho), \;\; p_\rho(h(\rho)) < 0, \;\; \sigma(H_{x_0}, F_p(\rho)h(\rho)) = -1 \tag{3.2.9}$$

on $\rho \in V \cap \Sigma$.

Proof: Let p_ρ take the form (3.2.8). Then from (3.2.5)

$$F_p^2(\rho)v = \sum_{j=1}^{k} \mu_j^2\left(\sigma(v, H_{\xi_j})H_{x_j} - \sigma(v, H_{x_j})H_{\xi_j}\right)$$

so that

$$\mathrm{Ker}\, F_p^2(\rho) = \{v \mid \sigma(v, H_{\xi_j}) = 0, \sigma(v, H_{x_j}) = 0, j = 1, ..., k\}$$

and hence $\dim \mathrm{Ker}\, F_p^2(\rho) = 2n + 2 - 2k$ which is independent of $\rho \in \Sigma$ by Lemma 3.2.2. Let $p_{\bar\rho}$ take the form (3.2.8). Since we have that $F_p^2(\bar\rho)H_{x_0} = 0$, $p_{\bar\rho}(H_{x_0}) = -1$ and $\sigma(H_{x_0}, F_p(\bar\rho)H_{x_0}) = -1$ then there is a conic neighborhood V of $\bar\rho$ such that one can choose smooth $h(\rho)$ defined in $V \cap \Sigma$ such that

$$h(\rho) \in \mathrm{Ker}\, F_p^2(\rho), \;\; p_\rho(h(\rho)) < 0, \;\; \sigma(H_{x_0}, F_p(\rho)h(\rho)) = -1 \tag{3.2.10}$$

for $\rho \in V \cap \Sigma$. We can assume that $h(\rho)$ is homogeneous of degree 0 in ξ, for if not we can just restrict to the sphere $|\xi| = 1$ and extend the restriction so that it becomes homogeneous of degree 0. □

Lemma 3.2.4 *Assume that $h(\rho)$ satisfies* (3.2.10). *Then we have*

$$\sigma(v, F_p(\rho)h(\rho)) = 0 \Longrightarrow p_\rho(v) > 0.$$

Proof: Let us fix $\rho \in V \cap \Sigma$. We can assume that p_ρ has the form (3.2.8). Set $w = F_p(\rho)h(\rho)$ and hence $w \in \mathrm{Ker}\, F_p(\rho)$. From (3.2.10) one can put $h(\rho) = (y_0, ..., y_n, -1, \eta_1, ..., \eta_n)$ where $y_1 = \cdots = y_k = 0$, $\eta_1 = \cdots = \eta_k = 0$ then we see

$$1 > \sum_{j=k+1}^{k+\ell} \eta_j^2, \quad w = H_{\xi_0} - \sum_{j=k+1}^{k+\ell} \eta_j H_{\xi_j}$$

because $p_\rho(h(\rho)) < 0$ and $\sigma(H_{x_0}, w) = -1$. Let $v = (x_0, ..., x_n, \xi_0, ..., \xi_n)$ and $\sigma(v, w) = 0$ hence $\xi_0 - \sum_{j=k+1}^{k+\ell} \eta_j \xi_j = 0$ so that we conclude

$$\xi_0^2 < \sum_{j=k+1}^{k+\ell} \xi_j^2$$

and hence the assertion otherwise we would have

$$\xi_0^2 = (\sum_{j=k+1}^{k+\ell} \eta_j \xi_j)^2 \le (\sum_{j=k+1}^{k+\ell} \eta_j^2)(\sum_{j=k+1}^{k+\ell} \xi_j^2) \le \delta \xi_0^2$$

with some $\delta < 1$ which is a contradiction. □

Proposition 3.2.1 *Assume that* $\mathrm{Ker}\, F_p^2 \cap \mathrm{Im}\, F_p^2 = \{0\}$ *on* Σ. *Then* p *admits an elementary decomposition*

$$p = -M\Lambda + Q$$

such that $|M - \Lambda| \le C\sqrt{Q}$ *with some* $C > 0$.

Proof: We first work in a neighborhood V of any $\bar{\rho} \in \Sigma$. Let $h(\rho)$ be in Lemma 3.2.3 and put $w(\rho) = F_p(\rho)h(\rho)$. Since $\operatorname{Im} F_p(\rho) = \langle H_{\xi_0}, H_{\phi_1}, ..., H_{\phi_r}\rangle$ then one can write

$$w(\rho) = \gamma_0 H_{\xi_0} - \sum_{j=1}^{r} \gamma_j H_{\phi_j}$$

where $\gamma_j(\rho)$ are smooth in $V \cap \Sigma$. From $\sigma(H_{x_0}, w(\rho)) = -1$ we have $\gamma_0 = 1$. As remarked above we can assume that γ_j are homogeneous of degree 0 in ξ. Let us put

$$\lambda = \sum_{j=1}^{r} \gamma_j(x, \xi')\phi_j(x, \xi')$$

so that $w(\rho) = H_{\xi_0 - \lambda}$ on $V \cap \Sigma$. Let us write

$$p = -(\xi_0 + \lambda)(\xi_0 - \lambda) + \hat{q}, \quad \hat{q} = \sum_{j=1}^{r} \phi_j^2 - (\sum_{j=1}^{r} \gamma_j \phi_j)^2 = q - \lambda^2.$$

We now check that

$$\sum_{j=1}^{r} \gamma_j^2 < 1.$$

From Lemma 3.2.4 it follows that

$$\sigma(v, H_{\xi_0 - \lambda}) = 0 \Longrightarrow \sigma(v, F_p(\rho)v) = \sigma(v, F_{\hat{q}}(\rho)v) > 0.$$

This implies that

$$\sigma(v, H_{\xi_0 - \lambda}) = 0 \Longrightarrow \sum_{j=1}^{r} \sigma(v, H_{\phi_j})^2 - (\sum_{j=1}^{r} \gamma_j \sigma(v, H_{\phi_j}))^2 > 0.$$

Note that the map

$$\langle H_{\xi_0 - \lambda}\rangle^{\sigma} / T_\rho \Sigma \ni v \mapsto (\sigma(v, H_{\phi_j}))_{j=1,\dots,r} \in \mathbb{R}^r$$

is surjective. Indeed if $\sigma(v, H_{\xi_0 - \lambda}) = 0$, $\sigma(v, H_{\phi_j}) = 0$ for $j = 1, ..., r$ then it follows that

$$v \in \langle H_{\xi_0 - \lambda}, H_{\phi_1}, ..., H_{\phi_r}\rangle^{\sigma} = \langle H_{\xi_0}, H_{\phi_1}, ..., H_{\phi_r}\rangle^{\sigma} = \operatorname{Ker} F_p(\rho) = T_\rho \Sigma.$$

From this it follows that $\langle \gamma, t\rangle^2 < |t|^2$ for any $t \in \mathbb{R}^r$ and hence we conclude

$$|\gamma(\rho)| = (\sum_{j=1}^{r} \gamma_j(\rho)^2)^{1/2} < 1.$$

We extend $\gamma_j(\rho)$ $(\rho \in V \cap \Sigma)$ to V such a way that $|\gamma| < 1$ in V. This proves that

$$\hat{q} \geq c \sum_{j=1}^{r} \phi_j(x, \xi')^2$$

with some $c > 0$ and hence we have

$$|\lambda|^2 \le \delta q$$

with some $\delta < 1$. Recall that $H_{\xi_0 - \lambda} \in \mathrm{Ker}\, F_p$ in $V \cap \Sigma$ and this shows that $\{\xi_0 - \lambda, \phi_j\} = 0$ in $V \cap \Sigma$ and hence

$$\{\xi_0 - \lambda, \lambda\} = 0 \quad \text{in} \quad V \cap \Sigma.$$

Thus we can find a family of conic open sets $\{V_i\}$ and smooth $\{\lambda_i\}$ defined on V_i, homogeneous of degree 0 such that one can write in V_i

$$\begin{aligned} p &= -\xi_0^2 + q = -\xi_0^2 + \sum_{\alpha=1}^{r(i)} \phi_{i\alpha}^2 \\ &= -(\xi_0 + \lambda_i)(\xi_0 - \lambda_i) + q_i, \;\; q_i = q - \lambda_i^2, \\ |\lambda_i| &\le \sqrt{\delta}\sqrt{q} \quad \text{in} \quad V_i, \\ \{\xi_0 - \lambda_i, \phi_{i\alpha}\} &= 0 \quad \text{on} \quad V_i \cap \Sigma, \;\; \alpha = 1, ..., r(i). \end{aligned}$$

Take a partition of unity $\{\chi_i\}$ subordinate to $\{V_i\}$ such that $0 \le \chi_i \le 1$, $\chi_i \in C_0^\infty(V_i)$, homogeneous of degree 0 and $\sum \chi_i = 1$. Define

$$\begin{aligned} &\lambda = \sum \chi_i \lambda_i, \\ &p = -(\xi_0 + \lambda)(\xi_0 - \lambda) + Q, \quad Q = q - \lambda^2. \end{aligned}$$

Here we note that

$$\begin{aligned} |\lambda| \le \sum \chi_i |\lambda_i| \le \sqrt{\delta}\sqrt{q} \sum \chi_i = \sqrt{\delta}\sqrt{q}, \\ Q = q - \lambda^2 \ge q - \delta^2 q = (1 - \delta^2) q \ge 0. \end{aligned}$$

We now show that this gives an elementary decomposition. Note that

$$\begin{aligned} \{\xi_0 - \lambda, Q\} &= \sum \chi_i \{\xi_0 - \lambda_i, Q\} + \sum (\xi_0 - \lambda_i)\{\chi_i, Q\} \\ &= \sum \chi_i \{\xi_0 - \lambda_i, Q\} - \sum \lambda_i \{\chi_i, Q\} \end{aligned}$$

because $\sum \{\chi_i, Q\} = 0$. Recall that $\{\xi_0 - \lambda_i, \phi_{i\alpha}\} = 0$ on $V_i \cap \Sigma$ and hence $\{\xi_0 - \lambda_i, \phi_{i\alpha}\}$ is a linear combination of $\{\phi_{i\alpha}\}$ there. Since $c_1 \sum \phi_{i\alpha}^2 \ge Q \ge c_2 \sum \phi_{i\alpha}^2$ in V_i with some $c_i > 0$ and hence $Q = \sum Q_{\alpha\beta} \phi_{i\alpha} \phi_{i\beta}$ then on the support of χ_i we have

$$|\{\xi_0 - \lambda_i, Q\}| \le C \sum_\alpha \phi_{i\alpha}^2 \le C' q_i \le C' q \le C'' Q.$$

On the other hand we have $|\{\chi_i, Q\}| \le C\sqrt{Q}$ because $Q \ge 0$ and

$$|\lambda_i| \le \delta \sqrt{q} \le C\sqrt{Q}$$

then we get

$$|\{\xi_0-\lambda, Q\}|\leq CQ. \tag{3.2.11}$$

We now study $|\{\xi_0-\lambda,\xi_0+\lambda\}|=2|\{\xi_0-\lambda,\lambda\}|$. Note that

$$\{\xi_0-\lambda,\lambda\}=\sum\chi_i\{\xi_0-\lambda,\lambda_i\}+\sum\lambda_i\{\xi_0-\lambda,\chi_i\}$$

and

$$\chi_i\{\xi_0-\lambda,\lambda_i\}=\chi_i\sum\chi_k\{\xi_0-\lambda_k,\lambda_i\}-\chi_i\sum\lambda_k\{\chi_k,\lambda_i\}.$$

Since we have $\{\xi_0-\lambda_k,\lambda_i\}=0$ on $V_k\cap V_i\cap\Sigma$ the same arguments as above give

$$|\{\xi_0-\lambda_k,\lambda_i\}|\leq C\sqrt{q_i}\leq C\sqrt{q}\leq C'\sqrt{Q}.$$

We check other terms

$$|\lambda_k\{\chi_k,\lambda_i\}|\leq C\sqrt{q_k}\leq C\sqrt{q}\leq C'\sqrt{Q}\quad\text{on}\quad V_k,$$
$$|\lambda_i\{\xi_0-\lambda,\chi_i\}|\leq C\sqrt{q_i}\leq C\sqrt{q}\leq C'\sqrt{Q}\quad\text{on}\quad V_i.$$

Hence we have

$$|\{\xi_0-\lambda,\lambda\}|\leq C\sqrt{Q}$$

which shows $|\{\xi_0-\lambda,\xi_0+\lambda\}|\leq C\sqrt{Q}$. This together with (3.2.11) proves the assertion. □

3.3 Case $\mathrm{Im}\,F_p^2\cap\mathrm{Ker}\,F_p^2\neq\{0\}$

We next discuss the same problem studied in Section 4.2 for the case $\mathrm{Im}\,F_p^2\cap\mathrm{Ker}F_p^2\neq\{0\}$. In particular we give a necessary and sufficient condition in order that p admits an elementary decomposition for general case in terms of some vector field defined near the doubly characteristic manifold.

Recall that we are working with

$$p(x,\xi)=-\xi_0^2+q(x,\xi'),\quad q(x,\xi')\geq 0$$

where $p(x,\xi)$ is noneffectively hyperbolic and verifies the conditions (3.2.1), (3.2.2) and

$$\mathrm{Ker}\,F_p^2(\rho)\cap\mathrm{Im}\,F_p^2(\rho)\neq\{0\},\ \ \forall\rho\in\Sigma. \tag{3.3.1}$$

This means that the Hamilton map $F_p(\rho)$ has a Jordan block of size four at every $\rho\in\Sigma$. Recall that from the hypothesis (3.2.1) one can write

$$p(x,\xi)=-\xi_0^2+\sum_{j=1}^{r}\phi_j(x,\xi')^2$$

near every $\rho \in \Sigma$ where $d\phi_j$ are linearly independent at ρ and Σ is given by

$$\Sigma = \{\phi_j(x,\xi) = 0, j = 0, ..., r\}$$

where $\phi_0(x,\xi) = \xi_0$ as before. Assume (3.3.1) then by Theorem 2.3.1 the quadratic form $Q = p_\rho$ takes the form, in a suitable symplectic coordinates

$$Q = (-\xi_0^2 + 2\xi_0\xi_1 + x_1^2)/\sqrt{2} + \sum_{j=2}^{k} \mu_j(x_j^2 + \xi_j^2) + \sum_{j=k+1}^{k+\ell} \xi_j^2. \tag{3.3.2}$$

Lemma 3.3.1 *The number k in* (3.3.2) *is independent of $\rho \in \Sigma$.*

Proof: With $\{\psi_j\} = \{\xi_0, \xi_1, x_1, x_j, \xi_j, 2 \le j \le k, \xi_j, k+1 \le j \le k+\ell\}$ it follows from Lemma 3.2.1 that the rank of $(\{\psi_i, \psi_j\})$ is constant. This shows that k is independent of $\rho \in \Sigma$. □

Examining the standard canonical model (3.3.2) it is easy to see that

$$\dim \operatorname{Im} F_p^2(\rho) = 2 + 2(k-1), \quad \dim \operatorname{Im} F_p^3(\rho) = 1 + 2(k-1)$$

which are independent of ρ as we observed above. Since

$$\dim \big(\operatorname{Ker} F_p(\rho) \cap \operatorname{Im} F_p^3(\rho)\big) = 1, \quad \dim \big(\operatorname{Ker} F_p^2(\rho) \cap \operatorname{Im} F_p^2(\rho)\big) = 2 \tag{3.3.3}$$

which is easily verified examining the standard model (3.3.2) then one can choose smooth vectors $z_1(\rho)$, $h_j(\rho)$, $j = 1, 2$ defined near a reference point $\bar{\rho} \in \Sigma$ so that

$$\operatorname{Ker} F_p(\rho) \cap \operatorname{Im} F_p^3(\rho) = \langle z_1(\rho)\rangle, \quad \rho \in \Sigma,$$
$$\operatorname{Ker} F_p^2(\rho) \cap \operatorname{Im} F_p^2(\rho) = \langle h_1(\rho), h_2(\rho)\rangle, \quad \rho \in \Sigma.$$

Lemma 3.3.2 *There are smooth $z_1(\rho)$ and $z_2(\rho)$ defined near the reference point such that*

$$\operatorname{Ker} F_p^2(\rho) \cap \operatorname{Im} F_p^2(\rho) = \langle z_1(\rho), z_2(\rho)\rangle,$$
$$F_p(\rho) z_1(\rho) = 0, \quad F_p(\rho) z_2(\rho) \neq 0.$$

Proof: Let $\operatorname{Ker} F_p^2(\rho) \cap \operatorname{Im} F_p^2(\rho) = \langle h_1(\rho), h_2(\rho)\rangle$. Since $F_p(\rho)h_j(\rho)$, $j = 1, 2$ are in $\operatorname{Ker} F_p(\rho) \cap \operatorname{Im} F_p^3(\rho)$ there exist smooth $\alpha(\rho)$, $\beta(\rho)$ such that

$$\alpha(\rho)F_p(\rho)h_1(\rho) + \beta(\rho)F_p(\rho)h_2(\rho) = 0.$$

Then it is enough to choose

$$z_1(\rho) = \alpha(\rho)h_1(\rho) + \beta(\rho)h_2(\rho),$$
$$z_2(\rho) = \beta(\rho)h_1(\rho) - \alpha(\rho)h_2(\rho).$$

□

Note that, in the canonical model (3.3.2) it is easy to see that

$$\text{Ker}\, F_p^2(\rho) \cap \text{Im}\, F_p^2(\rho) = \langle H_{\xi_0}, H_{x_1} \rangle \tag{3.3.4}$$

and

$$z_2(\rho) = aH_{\xi_0} + bH_{x_1}, \quad b \neq 0. \tag{3.3.5}$$

Lemma 3.3.3 *There exists a smooth $S(x, \xi)$ defined near the reference point vanishing on Σ such that*

$$H_S(\rho) = z_2(\rho), \quad \rho \in \Sigma.$$

Proof: Note that from (3.2.5) it follows that

$$F_p(\rho)v = \sum_{j=0}^{r} \epsilon_j \sigma(v, H_{\phi_j}(\rho)) H_{\phi_j}(\rho), \quad \epsilon_0 = -1,\ \epsilon_j = 1,\ \ j \geq 1 \tag{3.3.6}$$

and hence

$$F_p^2(\rho)v = \sum_{k=0}^{r} \epsilon_k \sigma(v, H_{\phi_k}(\rho)) \left[\sum_{j=0}^{r} \epsilon_j \sigma(H_{\phi_k}(\rho), H_{\phi_j}(\rho)) H_{\phi_j}(\rho) \right]. \tag{3.3.7}$$

This shows that

$$\text{Im}\, F_p^2(\rho) = \langle \sum_{j=0}^{r} \epsilon_j \sigma(H_{\phi_k}(\rho), H_{\phi_j}(\rho)) H_{\phi_j}(\rho); k = 0, ..., r \rangle$$

and with $A(\rho) = (a_{kj}(\rho)) = (\{\phi_k, \phi_j\}(\rho))$ we have $\text{Im}\, F_p^2(\rho) = \langle f_1(\rho), ..., f_r(\rho) \rangle$ where $f(\rho) = A(\rho) H_\phi(\rho)$, $H_\phi = {}^t(-H_{\phi_0}, ..., H_{\phi_r})$. Since the rank of $A(\rho)$ is constant there exists $\beta_{ik}(\rho)$ such that with

$$g_i(\rho) = \sum_{k=0}^{r} \beta_{ik}(\rho) f_k(\rho), \quad i = 1, ..., s$$

we have

$$\text{Im}\, F_p^2(\rho) = \langle g_1(\rho), ..., g_s(\rho) \rangle.$$

Since $z_2(\rho) \in \text{Im}\, F_p^2(\rho)$ one can write

$$z_2(\rho) = \sum_{k=1}^{s} \alpha_k(\rho) g_k(\rho)$$

with smooth $\alpha_k(\rho)$. Then

$$z_2(\rho) = \sum_{k=1}^{s} \alpha_k(\rho) \sum_{j=0}^{r} \beta_{kj}(\rho) f_j(\rho) = \sum_{k=1}^{s} \sum_{j=0}^{r} \sum_{\ell=0}^{r} \alpha_k(\rho) \beta_{kj}(\rho) a_{j\ell}(\rho) H_{\phi_\ell(\rho)}.$$

Let us define

$$S = \sum_{k=1}^{s}\sum_{j=0}^{r}\sum_{\ell=0}^{r} \tilde{\alpha}_k \tilde{\beta}_{kj} \tilde{a}_{j\ell}\phi_\ell$$

where $\tilde{\alpha}_k$, $\tilde{\beta}_{kj}$ and $\tilde{a}_{j\ell}$ are smooth extensions outside Σ of α_k, β_{kj} and $a_{j\ell}$. This is a desired one. □

Lemma 3.3.4 *There exists a smooth $\Lambda(x,\xi)$ defined near the reference point vanishing on Σ such that*

$$H_\Lambda(\rho) = z_1(\rho), \quad \rho \in \Sigma.$$

Proof: Repeat the same arguments as in the proof of Lemma 3.3.3. □

Lemma 3.3.5 *In a neighborhood of the reference point we have*

$$\forall w \in \langle z_1(\rho)\rangle^\sigma \Longrightarrow \sigma(w, F_p(\rho)w) \geq 0.$$

Proof: Choose a symplectic coordinates on which p_ρ takes the form (3.3.2). Then it is easy to see that

$$\langle z_1(\rho)\rangle = \langle H_{\xi_0}\rangle$$

and hence if $w \in \langle z_1(\rho)\rangle^\sigma$ then

$$\sigma(w, F_p(\rho)w) = Q(w) = x_1^2/\sqrt{2} + \sum_{j=2}^{k} \mu_j(x_j^2 + \xi_j^2) + \sum_{j=k+1}^{k+\ell} \xi_j^2 \geq 0$$

which is the assertion. □

We summarize what we have proved in

Proposition 3.3.1 *Assume that p satisfies* (3.2.1), (3.2.2) *and* (3.3.1). *Then there exist smooth vectors $z_1(\rho)$, $z_2(\rho)$, $\rho \in \Sigma$ defined near the reference point such that*

(3.3.8) $$z_1(\rho) \in \operatorname{Ker} F_p(\rho) \cap \operatorname{Im} F_p^3(\rho), \quad \rho \in \Sigma,$$

(3.3.9) $$z_2(\rho) \in \operatorname{Ker} F_p^2(\rho) \cap \operatorname{Im} F_p^2(\rho), \quad \rho \in \Sigma,$$

(3.3.10) $$w \in \langle z_1(\rho)\rangle^\sigma \Longrightarrow \sigma(w, F_p(\rho)w) \geq 0.$$

Since $F_p(\rho)z_2(\rho)$ is proportional to $z_1(\rho)$, $\rho \in \Sigma$ we may assume, without restrictions, that

(3.3.11) $$F_p(\rho)z_2(\rho) = -z_1(\rho), \quad \rho \in \Sigma.$$

We have

Proposition 3.3.2 *One can write p, near every $\rho \in \Sigma$, as*

$$\begin{aligned} p &= -\xi_0^2 + \sum_{j=1}^{r} \phi_j(x,\xi')^2 \\ &= -(\xi_0 + \phi_1(x,\xi'))(\xi_0 - \phi_1(x,\xi')) + \sum_{j=2}^{r} \phi_j(x,\xi')^2 \end{aligned}$$

where Σ is given by $\{\xi_0 = 0, \phi_1 = \cdots = \phi_r = 0\}$ and

$$\{\xi_0 - \phi_1, \phi_j\} = 0, \;\; j = 1, ..., r, \quad \{\phi_1, \phi_2\} \neq 0 \quad \textit{on} \quad \Sigma. \tag{3.3.12}$$

Proof: Let $\Lambda(x,\xi)$ be a smooth function vanishing on Σ such that $H_\Lambda(\rho)$ is proportional to $z_1(\rho)$ of which existence is assured by Lemma 3.3.4. Since $\sigma(z_1, H_{x_0}) \neq 0$ by (3.3.10), without restrictions, we may assume that

$$\Lambda = \xi_0 - \lambda, \quad \lambda = \sum_{j=1}^{r} \gamma_j(x,\xi')\phi_j$$

where ϕ_j are those in (3.2.4). Writing

$$p = -(\xi_0 - \lambda)(\xi_0 + \lambda) + \sum_{j=1}^{r} \phi_j^2 - (\sum_{j=1}^{r} \gamma_j \phi_j)^2$$

one obtains

$$\begin{aligned} \sigma(v, F_p v) &= -2\sigma(v, H_\Lambda)\sigma(v, H_{\xi_0+\lambda}) \\ &+ \sum_{j=1}^{r} \sigma(v, H_{\phi_j})^2 - (\sum_{j=1}^{r} \gamma_j(\rho)\sigma(v, H_{\phi_j}))^2. \end{aligned}$$

Because of (3.3.10) we have

$$\sum_{j=1}^{r} \sigma(v, H_{\phi_j})^2 - (\sum_{j=1}^{r} \gamma_j(\rho)\sigma(v, H_{\phi_j}))^2 \geq 0 \tag{3.3.13}$$

if $v \in \langle H_\Lambda(\rho)\rangle^\sigma$. As observed in Section 4.2, the mapping

$$\langle H_\Lambda(\rho)\rangle^\sigma / T_\rho\Sigma \ni v \mapsto (\sigma(v, H_{\phi_j}))_{j=1,\ldots,r} \in \mathbb{R}^r \tag{3.3.14}$$

is surjective and hence (3.3.13) shows that

$$\sum_{j=1}^{r} \gamma_j(\rho)^2 = |\gamma(\rho)|^2 \leq 1.$$

We now show that

$$|\gamma(\rho)| = 1, \quad \rho \in \Sigma. \tag{3.3.15}$$

We first note that $\sigma(z_2, F_p z_2) = \sigma(z_1, z_2) = \sigma(F_p^3 w, z_2) = -\sigma(w, F_p^3 z_2) = 0$ because $z_1 = F_p^3 w$ with some w and $z_2 \in \operatorname{Ker} F_p^2$. Since $\sigma(z_2, z_1) = \sigma(z_2, H_\Lambda) = 0$ we have

$$0 = \sigma(z_2, F_p z_2) = \sum_{j=1}^{r} \sigma(z_2, H_{\phi_j})^2 - (\sum_{j=1}^{r} \gamma_j(\rho)\sigma(z_2, H_{\phi_j}))^2.$$

If $\sigma(z_2, H_{\phi_j}) = 0$ for $j = 1, ..., r$ then $z_2 \in \langle H_\Lambda, H_{\phi_1}, ..., H_{\phi_r}\rangle^\sigma = \operatorname{Ker} F_p$ which contradicts to $F_p z_2 = -z_1$. This proves that $\sigma(z_2(\rho), H_{\phi_j}(\rho))_{1\le j\le r}$ is different from zero and hence one get (3.3.15) because

$$\sum_{j=1}^{r} \sigma(z_2, H_{\phi_j})^2 = (\sum_{j=1}^{r} \gamma_j \sigma(z_2, H_{\phi_j}))^2 \le |\gamma|^2 \sum_{j=1}^{r} \sigma(z_2, H_{\phi_j})^2.$$

We still denote by $\gamma(x, \xi')$ an extension of $\gamma(\rho)$ outside Σ such that $|\gamma(x, \xi')| = 1$. Thus we can write

$$p(x, \xi) = -(\xi_0 + \langle\gamma, \phi\rangle)(\xi_0 - \langle\gamma, \phi\rangle) + |\phi|^2 - \langle\gamma, \phi\rangle^2$$

where $\{\xi_0 - \langle\gamma, \phi\rangle, \phi_j\} = 0$, $j = 1, ..., r$ on Σ since $H_{\xi_0 + \langle\gamma,\phi\rangle} \in \operatorname{Im} F_p$. Let us set $\psi_1(x, \xi') = \sum_{j=1}^r \gamma_j(x, \xi')\phi_j(x, \xi')$ and taking a smooth orthonormal basis

$$\gamma(x, \xi'), e_2(x, \xi'), ..., e_r(x, \xi'), \;\; e_j = (e_{j1}, ..., e_{jr})$$

and define

$$\psi_j(x, \xi') = \sum_{h=1}^{r} e_{jh}(x, \xi')\phi_h(x, \xi')$$

so that $\sum_{j=1}^r \psi_j(x, \xi')^2 = \sum_{j=1}^r \phi_j(x, \xi')^2$. Switching the notation to $\{\phi_j\}$ we can thus write

$$p(x, \xi) = -(\xi_0 + \phi_1(x, \xi'))(\xi_0 - \phi_1(x, \xi')) + \sum_{j=2}^{r} \phi_j(x, \xi')^2$$

where $\{\xi_0 - \phi_1, \phi_j\} = 0$ on Σ for $j = 1, ..., r$. We finally check that $\{\phi_1, \phi_k\} \neq 0$ for some k. Indeed if otherwise we would have $\{\xi_0, \phi_j\} = 0$, $j = 1, ..., r$ and this would contradict (3.3.1). In fact if this would happen then we have

$$p_\rho = -\xi_0^2 + \sum_{j=1}^{r} \ell_j^2, \;\; \{\xi_0, \ell_j\} = 0, \;\; j = 1, ..., r.$$

Since $\sum_{j=1}^r \ell_j^2$ is a non negative definite quadratic form, in a suitable symplectic basis, p_ρ takes the form (1) of Theorem 2.3.1. Renumbering the coordinates so that $k = 2$ we have the assertion. □

Remark: From Proposition 3.3.2 one can write $\{\xi_0 - \phi_1, \phi_j\} = \sum_{k=1}^r c_{jk}\phi_k$ but if $c_{j1} \neq 0$ then $\{\xi_0 - \phi_1, \phi_j\}$ could not controlled by $\sum_{j=2}^r \phi_j^2$.

3.4 Vector field H_S

Let $S(x,\xi)$ be a smooth real function defined on $T^*\Omega$, homogeneous of degree 0, such that

$$S(x,\xi) = 0, \quad (x,\xi) \in \Sigma \tag{3.4.1}$$

and we have on Σ

$$H_S(\rho) \in \operatorname{Ker} F_p^2(\rho) \cap \operatorname{Im} F_p^2(\rho), \;\; F_p(\rho)H_S(\rho) \neq 0. \tag{3.4.2}$$

We first remark that it is possible to choose S independent of ξ_0. In fact from Lemma 3.3.4 one can take $\Lambda(x,\xi)$ so that

$$\Lambda(\rho) = 0, \;\; H_\Lambda(\rho) = z_1(\rho), \quad \rho \in \Sigma.$$

Since $\sigma(H_{x_0}, F_p(\rho)H_{x_0}) = -1$ it follows that $\sigma(H_{x_0}, H_\Lambda(\rho)) \neq 0$, $\rho \in \Sigma$ due to (3.3.10). This proves that one can write, without restrictions,

$$\Lambda(x,\xi) = \xi_0 - \lambda(x,\xi').$$

Let us write $S(x,\xi) = \alpha\xi_0 + f(x,\xi')$ and put

$$\tilde{S}(x,\xi') = S(x,\xi) - \alpha\Lambda(x,\xi).$$

Then it is clear that $\tilde{S}(x,\xi')$ verifies (3.4.1) and (3.4.2) for $H_\Lambda(\rho) \in \operatorname{Ker} F_p(\rho) \cap \operatorname{Im} F_p^3(\rho)$.

Recall that $p(x,\xi)$ takes the form

$$p(x,\xi) = -\xi_0^2 + q(x,\xi'), \quad q(x,\xi') \geq 0. \tag{3.4.3}$$

Lemma 3.4.1 *Assume that p admits an elementary decomposition such that $p = -M\Lambda + Q$. Then $H_\Lambda(\rho)$ is proportional to $z_1(\rho)$, $\rho \in \Sigma$.*

Proof: Let $\Lambda = \xi_0 - \lambda$. It is obvious that $q = Q + \lambda^2 \geq 0$ and hence λ and Q vanishes on Σ at least of order 1 and 2 respectively. Then it is clear that $H_\Lambda(\rho) \in \operatorname{Im} F_p(\rho)$. Recall

$$F_p H_\Lambda = -\sigma(H_\Lambda, H_M)H_\Lambda + F_Q H_\Lambda.$$

It is clear that $\sigma(H_\Lambda, H_M) = \{\Lambda, M\} = 0$ and from (3.1.1) we have $F_Q H_\Lambda = H_{\{Q,\Lambda\}} = 0$ on Σ. This shows that $F_p H_\Lambda = 0$ and hence H_Λ is in $\operatorname{Im} F_p \cap \operatorname{Ker} F_p$ on Σ.

Let S be a smooth function verifying (3.4.1) and (3.4.2). Since $H_S \in \operatorname{Im} F_p$ then $\sigma(H_\Lambda, H_S) = \{\Lambda, S\} = 0$ on Σ. Thus one has

$$F_p H_S = -(1/2)\sigma(H_S, H_M)H_\Lambda + F_Q H_S$$

which gives $\sigma(H_S, F_p H_S) = \sigma(H_S, F_Q H_S) = 0$ because $F_p H_S \in \operatorname{Ker} F_p$ and $H_S \in \operatorname{Im} F_p$. This proves

$$F_Q H_S = 0 \quad \text{on} \quad \Sigma \tag{3.4.4}$$

because $\sigma(H_S, F_Q H_S) = Q_\rho(H_S)$ and Q_ρ is non negative definite. Thus we have

$$F_p H_S = -\frac{1}{2}\sigma(H_S, H_M)H_\Lambda.$$

By definition of S we have $F_p H_S \neq 0$ and this proves the assertion. □

Lemma 3.4.2 *Assume that p admits a decomposition*

$$p(x,\xi) = -(\xi_0 + \lambda)(\xi_0 - \lambda) + Q(x,\xi') = -M\Lambda + Q$$

with $Q(x,\xi') \geq 0$. If $F_Q H_\Lambda = 0$ on Σ and F_p has no non zero real eigenvalues then (3.1.2) *holds.*

Proof: We first note that

$$\Sigma = \{\Lambda = 0, M = 0, Q = 0\}.$$

Since $F_p H_\Lambda = -(1/2)\sigma(H_\Lambda, H_M)H_\Lambda$ by $F_Q H_\Lambda = 0$. If $\sigma(H_\Lambda, H_M) \neq 0$ then F_p would have a non zero real eigenvalue which contradicts the assumption. Hence

$$\sigma(H_\Lambda, H_M) = \{\Lambda, M\} = 0 \quad \text{on} \quad \Sigma.$$

Then one can write

$$\{M, \Lambda\} = \sum_{j=1}^{r} c_j \psi_j \tag{3.4.5}$$

where

$$q = Q + \lambda^2 = \sum_{j=1}^{r} \psi_j^2$$

because $\{M, \Lambda\}$ is independent of ξ_0. The assertion follows from (3.4.5). □

We now show

Proposition 3.4.1 ([6]) *Let S_1, S_2 be two smooth functions verifying* (3.4.1) *and* (3.4.2). *Then there exists $C \neq 0$ such that*

$$H_{S_1}^3 p\big|_\Sigma = C H_{S_2}^3 p\big|_\Sigma.$$

We first show

Lemma 3.4.3 *Assume that p admits a decomposition $p = -M\Lambda + Q$ with $\Lambda = \xi_0 - \lambda$, $M = \xi_0 + \lambda$, $Q \geq 0$ such that H_Λ is proportional to $z_1(\rho)$ for $\rho \in \Sigma$. Let S be a smooth function verifying* (3.4.1) *and* (3.4.2). *Then we have*

$$H_S^3 Q = 0 \quad \text{on} \quad \Sigma.$$

Proof: Let ϕ_j be as in (3.2.4). It is clear that $\Sigma = \{\xi_0 = 0, \lambda = 0, Q = 0\}$ and hence one can write

$$\Lambda = \xi_0 - \sum_{j=1}^{r} \gamma_j \phi_j, \quad Q = |\phi|^2 - \langle \gamma, \phi \rangle^2.$$

It is also clear that $|\gamma(x, \xi')| \le 1$ near Σ because $Q \ge 0$ by assumption. Repeating the same arguments in the proof of Proposition 3.3.2 we conclude that

$$|\gamma(\rho)| = 1 \quad \rho \in \Sigma$$

and $\gamma(\rho)$ is proportional to $\sigma(H_S(\rho), H_\phi(\rho))$

$$H_S\phi(\rho) = \sigma(H_S(\rho), H_\phi(\rho)) = \alpha(\rho)\gamma(\rho), \quad \rho \in \Sigma \tag{3.4.6}$$

where we have denoted $\sigma(H_S, H_\phi) = (\sigma(H_S, \phi_1), ..., \sigma(H_S, H_{\phi_r}))$. As shown in the proof of Lemma 3.4.1 we have

$$0 = \sigma(H_S, F_p H_S) = \sigma(H_S, F_Q H_S)$$

and hence $F_Q H_S = 0$ on Σ because $Q \ge 0$. We now study $H_S^3(|\phi|^2 - \langle \gamma, \phi \rangle^2)$. It is clear that $H_S^3\langle \phi, \phi \rangle = 6\langle H_S^2\phi, H_S\phi \rangle$ on Σ and hence

$$H_S^3\langle \phi, \phi \rangle = 6\alpha\langle H_S^2\phi, \gamma \rangle \quad \text{on} \quad \Sigma. \tag{3.4.7}$$

On the other hand one obtains

$$\begin{aligned} H_S^3\langle \gamma, \phi \rangle^2 &= 4(\langle H_S\gamma, \phi \rangle + \langle \gamma, H_S\phi \rangle) \\ &\quad \times (2\langle H_S\gamma, H_S\phi \rangle + \langle \gamma, H_S^2\phi \rangle) \\ +2\langle \gamma, H_S\phi \rangle(\langle H_S^2\gamma, \phi \rangle &+ 2\langle H_S\gamma, H_S\phi \rangle + \langle \gamma, H_S^2\phi \rangle). \end{aligned}$$

On Σ this becomes

$$6\alpha\langle \gamma, H_S^2\phi \rangle + 12\alpha^2\langle H_S\gamma, \gamma \rangle. \tag{3.4.8}$$

Since $1 - |\gamma|^2 \ge 0$ near Σ and $1 - |\gamma|^2 = 0$ on Σ it follows that

$$H_S(1 - |\gamma|^2) = -H_S\langle \gamma, \gamma \rangle = -2\langle H_S\gamma, \gamma \rangle = 0 \ \text{ on } \ \Sigma.$$

Thus (3.4.8) is equal to $6\alpha\langle \gamma, H_S^2\phi \rangle$ and hence the assertion. □

Proof of Proposition 3.4.1: Let S_1, S_2 be two functions verifying our assumptions. From Proposition 3.3.2 we can write

$$p = -M\Lambda + Q, \quad Q \ge 0$$

where H_Λ is proportional to $z_1(\rho)$ and $\{\Lambda, Q\}$ vanishes of second order on Σ. By (3.4.2) one can write $F_p H_{S_j} = c_j H_\Lambda$ with $c_j \ne 0$, $j = 1, 2$. Now

$$\begin{aligned} H_{S_j}^3 p &= \{S_j, \{S_j, \{S_j, -\Lambda M + Q\}\}\} \\ &= -3\{S_j, M\}\{S_j, \{S_j, \Lambda\}\} \end{aligned}$$

on Σ because $\{S_j, \Lambda\} = 0$ and $H_{S_j}^3 Q = 0$ on Σ by Lemma 3.4.3. Since one can write

$$H_{S_j} = \theta_j z_2(\rho) + H_{f_j}(\rho), \quad \rho \in \Sigma,\ j = 1, 2$$

with $H_{f_j} \in \operatorname{Ker} F_p \cap \operatorname{Im} F_p^3$ where f_j vanishes on Σ then we obtain that

$$H_{S_1}(\rho) = \frac{\theta_1}{\theta_2} H_{S_2}(\rho) + H_f(\rho)$$

where $H_f(\rho) \in \operatorname{Ker} F_p$ and f vanishes on Σ. Let us set

$$-3\{S_j, M\} = \alpha_j, \quad j = 1, 2$$

which is different from zero. Indeed if $\{S_j, M\} = 0$ then we would have $\{S_j, \xi_0\} = \sigma(H_{S_j}, H_{\xi_0}) = 0$ and hence $F_p H_{S_j} = \sum \sigma(H_{S_j}, H_{\phi_k}) H_{\phi_k}$ which is not proportional to H_Λ. Then we have

$$\begin{aligned} H_{S_1}^3 p = \alpha_1 \{S_1, \{S_1, \Lambda\}\} &= \alpha_1 \{\frac{\theta_1}{\theta_2} S_2 + f, \{\frac{\theta_1}{\theta_2} S_2 + f, \Lambda\}\} \\ &= \alpha_1 \Big[(\frac{\theta_1}{\theta_2})^2 \{S_2, \{S_2, \Lambda\}\} + \frac{\theta_1}{\theta_2} \{S_2, \{f, \Lambda\}\} \\ &\qquad + \frac{\theta_1}{\theta_2} \{f, \{S_2, \Lambda\}\} + \{f, \{f, \Lambda\}\} \Big]. \end{aligned}$$

Since $\{S_j, \Lambda\} = 0$, $\{f, \Lambda\} = 0$ on Σ and hence

$$\{f, \{S_2, \Lambda\}\} = 0, \quad \{f, \{f, \Lambda\}\} = 0, \quad \text{on} \quad \Sigma.$$

This shows that the third and fourth terms in the above formula vanish on Σ. Taking into account the Jacobi identity

$$\{S_2, \{f, \Lambda\}\} = -\{f, \{\Lambda, S_2\}\} - \{\Lambda, \{S_2, f\}\}$$

we see that the second term also vanishes on Σ because $H_f \in \operatorname{Im} F_p \cap \operatorname{Ker} F_p$. Hence one has

$$H_{S_1}^3 p\big|_\Sigma = \frac{\alpha_1}{\alpha_2} (\frac{\theta_1}{\theta_2})^2 H_{S_2}^3 p\big|_\Sigma.$$

This is the desired assertion. $\square$

3.5 Elementary decomposition revisited

Recall that we are assuming (3.2.1) and (3.2.2) throughout this chapter. The next result was proved in [44] under some restrictions on the double characteristic manifold and in [6] in full generality removing the previous restrictions.

Theorem 3.5.1 ([6], [44]) *Let S be a smooth function verifying* (3.4.1) *and* (3.4.2). *Then the following assertions are equivalent.*

(i) $H_S^3\, p(\rho) = 0,\ \rho \in \Sigma,$

(ii) *p admits an elementary decomposition at every $\rho \in \Sigma$.*

Proof: We start by proving that (ii)$\Longrightarrow$(i). From Lemma 3.4.1 we see that H_Λ is proportional to $z_1(\rho)$. Then due to Lemma 3.4.3 one has $H_S^3 Q = 0$ on Σ and hence

$$H_S^3\, p\big|_\Sigma = -3\{S, M\}\{S, \{S, \Lambda\}\}\big|_\Sigma. \tag{3.5.1}$$

From (3.5.1) it suffices to show

$$\sigma(H_S, H_{\{S,\Lambda\}}) = 0$$

on Σ. Thanks to (3.1.1) we have $\operatorname{Ker} F_Q \subset \operatorname{Ker} F_{\{\Lambda,Q\}}$. This together with (3.4.4) shows that

$$H_{\{S,\{\Lambda,Q\}\}} = -F_{\{\Lambda,Q\}} H_S = 0$$

on Σ. Recall the Jacobi identity

$$\{Q, \{S, \Lambda\}\} + \{S, \{\Lambda, Q\}\} + \{\Lambda, \{Q, S\}\} = 0. \tag{3.5.2}$$

Considering the Hamilton vector field of (3.5.2) we obtain

$$F_Q H_{\{S,\Lambda\}} + H_{\{\Lambda,\{Q,S\}\}} = 0 \quad \text{on} \quad \Sigma. \tag{3.5.3}$$

Let us study the second term in (3.5.3)

$$H_{\{\Lambda,\{Q,S\}\}} = [H_\Lambda, H_{\{Q,S\}}].$$

Since $H_{\{Q,S\}}|_\Sigma = F_Q H_S|_\Sigma = 0$ and $H_\Lambda \in T_\rho\Sigma = \operatorname{Ker} F_p,\ \rho \in \Sigma$ it follows that $[H_\Lambda, H_{\{Q,S\}}] = 0$. This gives

$$F_Q H_{\{S,\Lambda\}} = 0. \tag{3.5.4}$$

Then we have $F_p H_{\{\Lambda,S\}} = -(1/2)\sigma(H_{\{\Lambda,S\}}, H_M) H_\Lambda$ because $\sigma(H_{\{\Lambda,S\}}, H_\Lambda) = 0$ which follows from $\{S, \Lambda\} = 0$ on Σ. From Lemma 3.4.1, H_Λ is proportional to z_1 and then $F_p H_S$ is so

$$H_\Lambda = \alpha(\rho) F_p H_S. \tag{3.5.5}$$

This gives that

$$H_{\{\Lambda,S\}} + \frac{1}{2}\alpha(\rho)\sigma(H_{\{\Lambda,S\}}, H_M) H_S \in \operatorname{Ker} F_p(\rho)$$

which proves clearly

$$\sigma(H_S, H_{\{\Lambda,S\}}) = 0$$

and thus we have proved (ii)$\Longrightarrow$(i).

The implication (i)$\Longrightarrow$ (ii) follows immediately from the following result which will be key observations in this chapter. To make the statement of the following proposition to be clear, using $\tilde{\phi}_j$ instead of ϕ_j, assume that p is written as

$$p = -\xi_0^2 + \sum_{j=1}^{r} \tilde{\phi}_j^2$$

near ρ.

Proposition 3.5.1 *Assume* (3.3.1)*. Let S be a smooth function verifying* (3.4.1) *and* (3.4.2) *and assume that*

$$H_S^3 p = 0$$

near ρ on Σ. Then near ρ we can rewrite p as

$$p = -(\xi_0 + \lambda)(\xi_0 - \lambda) + Q$$

with

$$\lambda = \phi_1 + L(\phi')\phi_1 + \gamma\phi_1^3|\xi'|^{-2},$$

$$Q = \sum_{j=2}^{r} \phi_j^2 + a(\phi)\phi_1^4|\xi'|^{-2} + b(\phi')L(\phi')\phi_1^2 \geq c(|\phi'|^2 + \phi_1^4|\xi'|^{-2})$$

with some $c > 0$ where ϕ_j are linear combinations of $\tilde{\phi}_j$, $j = 1, ..., r$ and $\phi = (\phi_1, \phi_2, ..., \phi_r)$, $\phi' = (\phi_2, ..., \phi_r)$. Here $\xi_0 - \lambda$ and ϕ_j satisfy

$$|\{\xi_0 - \lambda, Q\}| \leq C(|\phi'|^2 + \phi_1^4|\xi'|^{-2}), \tag{3.5.6}$$

$$\{\xi_0 - \lambda, \phi_j\} = O(|\phi|), \quad j = 1, ..., r, \tag{3.5.7}$$

$$\{\phi_1, \phi_j\} = O(|\phi|), \ \ j \geq 3, \tag{3.5.8}$$

$$\{\phi_1, \phi_2\} > 0 \tag{3.5.9}$$

near ρ. Here $L(\phi') = O(|\phi'||\xi'|^{-1})$ and γ is a real constant.

Proof: Denote $\tilde{\phi}_j$ by ϕ_j. Let p be as in (3.4.3). From Proposition 3.3.2 we can write

$$p(x, \xi) = -(\xi_0 + \phi_1(x, \xi'))(\xi_0 - \phi_1(x, \xi')) + |\phi'(x, \xi')|^2 \tag{3.5.10}$$

where

$$\{\xi_0 - \phi_1, \phi_j\}\big|_{\Sigma} = 0, \ \ j = 1, ..., r, \quad \{\phi_1, \phi_2\}(\rho) \neq 0. \tag{3.5.11}$$

Recall that $H_{\xi_0 - \phi_1}$ is proportional to $z_1(\rho)$ on Σ near ρ.

Let us consider

$$\tilde{\phi}_j = \sum_{k=2}^{r} O_{jk}\phi_k, \ \ j = 2, ..., r.$$

where $O = (O_{jk})$ is an orthogonal matrix which is smooth near ρ. Choosing O suitably and switching the notation $\{\tilde{\phi}_j\}$ to $\{\phi_j\}$ again we can assume that

$$\{\phi_1, \phi_2\}(\rho) \neq 0, \ \ \{\phi_1, \phi_j\} = 0 \ \text{ near } \rho \text{ on } \Sigma, \ \ j = 3, ..., r.$$

We may assume $\{\phi_1, \phi_2\} > 0$ without restrictions. Thus the assertion (3.5.9) are proved.

We now determine $L(\phi') = \langle \beta', \phi' \rangle$ where $\beta' = (\beta_2, ..., \beta_r)$ and β_j are smooth functions of (x, ξ'), homogeneous of degree -1 in ξ', following the arguments in [6]. We rewrite (3.5.10) as

$$\begin{aligned} p(x,\xi) &= -(\xi_0 + \phi_1 + L(\phi')\phi_1 + \gamma\hat{\phi}_1^3|\xi'|^{-2}) \\ &\times(\xi_0 - \phi_1 - L(\phi')\phi_1 - \gamma\phi_1^3|\xi'|^{-2}) + |\phi'|^2 - L(\phi')^2\phi_1^2 \\ &-\gamma^2\phi_1^6|\xi'|^{-4} - 2\phi_1^2 L(\phi') - 2\gamma\phi_1^4|\xi'|^{-2} - 2\gamma L(\phi')\phi_1^4|\xi'|^{-2} \\ &= -(\xi_0 + \phi_1 + L(\phi')\phi_1 + \gamma\phi_1^3|\xi'|^{-2}) \\ &\times(\xi_0 - \phi_1 - L(\phi')\phi_1 - \gamma\phi_1^3|\xi'|^{-2}) \\ &+|\phi'|^2 - 2\gamma\big(1 + L(\phi') + \gamma\phi_1^2|\xi'|^{-2}/2\big)\phi_1^4|\xi'|^{-2} \\ &-2L(\phi')\big(1 + L(\phi')/2\big)\phi_1^2 = -(\xi_0 + \lambda)(\xi_0 - \lambda) + Q \end{aligned} \tag{3.5.12}$$

where

$$\begin{aligned} &\lambda = \phi_1 + L(\phi')\phi_1 + \gamma\phi_1^3|\xi'|^{-2}, \\ &Q = |\phi'|^2 - 2\gamma(1 + L(\phi') + \gamma\phi_1^2|\xi'|^{-2}/2)\phi_1^4|\xi'|^{-2} - 2L(\phi')(1 + L(\phi')/2)\phi_1^2. \end{aligned}$$

Now the assertion (3.5.7) follows from (3.5.11) immediately. Taking γ negative large enough it is clear that

$$Q \geq c(|\phi'|^2 + \phi_1^4|\xi'|^{-2}) \tag{3.5.13}$$

with some $c > 0$. We prove that we can choose β' so that (3.5.6) holds. Note that

$$\begin{aligned} \{\xi_0 - \lambda, Q\} &= \{\xi_0 - \phi_1, |\phi'|^2 - 2L(\phi')(1 + L(\phi')/2)\phi_1^2\} \\ &-\{L(\phi')\phi_1, |\phi'|^2\} + O(Q) \end{aligned} \tag{3.5.14}$$

where one can write

$$\{\xi_0 - \phi_1, \phi_j\} = \sum_{k=1}^{r} \alpha_{jk}\phi_k, \quad j = 1, ..., r \tag{3.5.15}$$

with smooth α_{jk}. Using (3.5.15) and (3.5.13), (3.5.14) reads as

$$
\begin{aligned}
\{\xi_0-\lambda, Q\} &= 2\sum_{\ell=2}^{r}\phi_\ell\sum_{k=1}^{r}\alpha_{\ell k}\phi_k \\
&-2\phi_1^2\sum_{\ell=2}^{r}\beta_\ell\sum_{k=1}^{r}\alpha_{\ell k}\phi_k(1+L(\phi')/2) \\
&-2\phi_1\sum_{\ell=2}^{r}\phi_\ell\sum_{k=2}^{r}\beta_k\{\phi_k,\phi_\ell\}+O(Q).
\end{aligned}
\tag{3.5.16}
$$

Distinguishing the role of ϕ_1 from that of ϕ', we can write

$$
\begin{aligned}
\{\xi_0-\lambda, Q\} = 2\sum_{\ell=2}^{r}\alpha_{\ell 1}\phi_\ell\phi_1 - 2\phi_1\sum_{\ell=2}^{r}\phi_\ell\sum_{k=2}^{r}\beta_k\{\phi_k,\phi_\ell\} \\
-2\phi_1^3\sum_{\ell=2}^{r}\beta_\ell\alpha_{\ell 1}+O(Q).
\end{aligned}
\tag{3.5.17}
$$

Put $\alpha_1' = (\alpha_{21}, ..., \alpha_{r1})$ then (3.5.17) becomes

$$
\begin{aligned}
\{\xi_0-\lambda, Q\} = 2\big(\langle\alpha_1',\phi'\rangle + \langle\{\phi',\phi'\}\beta',\phi'\rangle\big)\phi_1 \\
-2\phi_1^3\langle\alpha_1',\beta'\rangle + O(Q).
\end{aligned}
\tag{3.5.18}
$$

We show that we can choose $\beta' = (\beta_2, ..., \beta_r)$ such that

$$
\{\phi',\phi'\}\beta' + \alpha_1' = 0, \quad \langle\alpha_1',\beta'\rangle = 0 \tag{3.5.19}
$$

on Σ so that the right-hand side of (3.5.18) is $O(Q)$.

Lemma 3.5.1 *We have*

$$
\langle\alpha_1', v\rangle = 0
$$

for any v satisfying $\{\phi',\phi'\}v = 0$.

Proof: We first make a closer look at our assumption $H_S^3 p = 0$. Since S vanishes on Σ and one can assume that S is independent of ξ_0 then we can write

$$
S(x,\xi') = \sum_{j=1}^{r} c_j(x,\xi')\phi_j(x,\xi'). \tag{3.5.20}
$$

Since $H_{\xi_0-\phi_1}$ is proportional to $z_1(\rho)$ on Σ then F_pH_S is also proportional to $H_{\xi_0-\phi_1}$ on Σ. Thanks to Proposition 3.4.1, multiplying S by a non zero function if necessary, we may assume that

$$
F_pH_S = -H_{\xi_0-\phi_1} \quad \text{on} \quad \Sigma. \tag{3.5.21}
$$

We study the identity (3.5.21). Plugging (3.5.20) into (3.5.21) to get

$$F_pH_S(\rho) = -\frac{1}{2}\{S, \xi_0 + \phi_1\}H_{\xi_0-\phi_1} + \sum_{j=2}^{r}\{S, \phi_j\}H_{\phi_j}$$

$$= -\frac{1}{2}\sum_{h=1}^{r} c_h\{\phi_h, \xi_0 + \phi_1\}H_{\xi_0-\phi_1} + \sum_{j=2}^{r}\sum_{h=1}^{r} c_h\{\phi_h, \phi_j\}H_{\phi_j}$$

$$= -H_{\xi_0-\phi_1}$$

on Σ because $\{S, \xi_0 - \phi_1\} = 0$. Hence we have on Σ

$$\frac{1}{2}\sum_{h=1}^{r} c_h\{\phi_h, \xi_0 + \phi_1\} = 1,$$

(3.5.22) $$c_1\{\phi_1, \phi_j\} + \sum_{h=2}^{r} c_h\{\phi_h, \phi_j\} = 0, \quad j = 2, ..., r$$

and, taking $\{\phi_h, \xi_0 + \phi_1\} = \{\phi_h, \xi_0 - \phi_1\} + 2\{\phi_h, \phi_1\}$ into account, we have

(3.5.23) $$c_2\{\phi_2, \phi_1\} = 1$$

because $\{\phi_j, \phi_1\} = 0$ for $j \geq 3$. We multiply (3.5.22) by c_j and sum up over $j = 2, ..., r$ which yields

$$-c_1 + \sum_{h=2}^{r}\sum_{j=2}^{r} c_jc_h\{\phi_h, \phi_j\} = 0.$$

The second term in the left-hand side vanishes because $(\{\phi_k, \phi_h\})$ is anti symmetric and thus we get $c_1 = 0$ and (3.5.22) gives

(3.5.24) $$\{S, \phi_j\} = 0, \quad j = 2, ..., r, \quad S = \sum_{h=2}^{r} c_h\phi_h$$

near ρ on Σ where $c_2 = \{\phi_2, \phi_1\}^{-1} \neq 0$.

By Lemma 3.4.3 one obtains

$$H_S^3 p = -3\{S, \xi_0 + \phi_1\}\{S, \{S, \xi_0 - \phi_1\}\} = c\{S, \{S, \xi_0 - \phi_1\}\}$$

with some $c \neq 0$ which is examined in the proof of Lemma 3.4.3. Take (3.5.23) and (3.5.24) into account we see that $H_S^3 p = 0$ on Σ implies that

(3.5.25) $$\{S, \xi_0 - \phi_1\} = O(|\phi'| + \phi_1^2).$$

Since $\{S, \phi_1\} = 1$ then from (3.5.24) it follows that $\alpha_{j1} = \{S, \{\xi_0 - \phi_1, \phi_j\}\}$. Thanks to the Jacobi identity we get for $j \geq 2$

$$\alpha_{j1} = -\{\xi_0 - \phi_1, \{\phi_j, S\}\} - \{\phi_j, \{S, \xi_0 - \phi_1\}\}$$

$$= -\{\phi_j, \{S, \xi_0 - \phi_1\}\}$$

on Σ because of (3.5.24). Thus from (3.5.25) we can write

$$\alpha_{j1} = \sum_{k=2}^{r} w_k \{\phi_j, \phi_k\}$$

with some w_k. Then one has

$$\sum_{j=2}^{r} v_j \alpha_{j1} = \sum_{k=2}^{r} w_k \sum_{j=2}^{r} \{\phi_j, \phi_k\} v_j = 0$$

which is the desired assertion. □

Thanks to Lemma 3.5.1 it follows that the equation

$$\{\phi', \phi'\}\beta' = -\alpha_1'$$

has a smooth solution β'. Finally we note that $\langle \alpha_1', \beta' \rangle = 0$ holds since $\{\phi', \phi'\}$ is anti-symmetric. Thus we have proved the assertion (3.5.6). □