Introductory expositions on projective representations of groups

T. Hirai, A. Hora and E. Hirai

Abstract. This paper gives introductory expositions on the theory of projective (or spin) representations of groups, together with plotting historical milestones of the theory, starting from Schur's trilogy in 1904, 1907 and 1911, Cartan's work in 1913, Pauli's introduction of spin quantum number in 1925, and Dirac's relativistic equation of electron in 1928, and so on. We pick up many situations where multi-valued representation of groups appear naturally, and thus explain how the projective representations are indispensable and worth to study. We discuss rather in detail the case of Weil representations of the symplectic groups $Sp(2n, \boldsymbol{R})$, and the case of various actions of the symmetric group $\mathfrak{S}_n$ on the full matrix algebra $M(2^k, \boldsymbol{C})$ of degree 2^k with $k = [n/2]$.[2]

0 Introduction

In this paper [E], we give several introductory expositions for readers who are not so familiar with the theory of projective representations (so to say, spin theory). The first part of the next paper [I] in this volume can also work as an introduction to spin theory of group representations.

In the first four sections, §§1–4, we explain how and why projective representations occur naturally in representations of groups and algebras. We note in particular that a work of A. Clifford says, as is summarized in **E-5**, that if a representation of a finite group G is restricted on a normal subgroup N, then usually it gives rise to projective representations, and a special but essential case is given as Theorem 3.1 in **E-5**. Moreover, we can deduce from this theorem a general method of constructing all irreducible representations (=IRs) of a finite

[2] *2010 Mathematical Subject Classification*: Primary 01-02; Secondary 20B99, 20C25, 20E22, 20F55

Keywords: projective representation, spin representation, spin character, positive definite function, spin representation of symmetric group, Weil representation of $Sp(2n, \boldsymbol{R})$

semidirect groups $G = U \rtimes S$ from those of U and S, as is given in Theorem 4.1 in **E-9**, where projective representations of U and S should appear naturally. This method (which we call as *classical method*) will be able to be applied in many occasions, in particular, in the present papers [I] and [II] (see e.g. [I, §17] and [II, §3]). Note that this classical method contains Mackey's method for constructing IRs of semidirect product groups, in which projective representations (or multi-valued representations) do not appear.

In the 5th section, we show, as an important example, the situation of Weil representations of the symplectic group $Sp(2n, \boldsymbol{R})$, which appeared to be double-valued representations, and are usual linear representations of its double covering group, the metaplectic group $Mp(2n, \boldsymbol{R})$.

In the 6th section, we give an interesting case of the full matrix algebra $M(2^k, \boldsymbol{R})$ of degree 2^k and the symmetric groups $\mathfrak{S}_{2k}$ and $\mathfrak{S}_{2k+1}$ acting on it. This subject has intimate relations with the paper [II]. Here is the origin of Schur's spin representation 'Hauptdarstellung' of the symmetric group.

1 How projective representations appear naturally

1.1 Schur's fundamental trilogy [Sch1] – [Sch3]

The theory of projective (or spin) representations of a finite group is initiated by J. Schur (= I. Schur) who begun his first paper [Sch1, 1904] of the trilogy on this subject with saying (quoted from [Sch1], p.20, from the top of Introduction till 11th line)

> Das Problem der Bestimmung aller endlichen Gruppen lineare Substitutionen bei gegebener Variabelnzahl n ($n > 1$) gehört zu den schwierigsten Problemen der Algebra und hat bis jetzt nur für die binären und ternären Substitutionsgruppen seine vollständige Lösung gefunden. Für den allgemeinen Fall ist nur bekannt, daß die Anzahl der in Betracht kommenden Typen von Gruppen eine endliche ist; dagegen fehlt noch jede Übersicht über die charakteristischen Eigenschaften dieser Gruppen.
>
> Die Umkehrung dieses Problems bildet in einem gewissen Sinne die Aufgabe: alle Gruppen von höchstens h ganzen oder gebrochenen linearen Substitutionen zu finden, die einer gegebenen endlichen Gruppe $\mathfrak{H}$ der Ordnung h ein- order mehrstufig isomorph sind, oder auch, wie man sagt, alle Darstellungen der Gruppe $\mathfrak{H}$ durch lineare Substitutionen zu bestimmen.

Schur says that Darstellung durch ganze lineare Substitutionen (representations through linear transformations) of finite groups has begun with *T. Molien* and *F. Frobenius*[3], and he himself will study Darstellung durch gebrochene lineare Substitutionen, that is, representation through linear fractional transformations. For an element A of a given group $\mathfrak{H}$, assign a regular matrix $(C) = \big(a_{ij}\big)_{1\leqslant i,j\leqslant n}$ which gives a projective transformation $\{A\}: \ (y_i)_{1\leqslant i\leqslant n-1} \mapsto (x_i)_{1\leqslant i\leqslant n-1}$ by

$$x_\nu = \frac{a_{\nu 1}y_1 + \cdots + a_{\nu,n-1}y_{n-1} + a_{\nu n}}{a_{n1}y_1 + \cdots + a_{n,n-1}y_{n-1} + a_{nn}} \quad (\nu = 1, 2, \ldots, n-1),$$

and ask to satisfy $\{A\}\{B\} = \{AB\}$ $(A, B \in \mathfrak{H})$. Then there exists a non-zero scalar $r_{A,B} \in \boldsymbol{C}^\times$ such that

$$(1.1) \qquad (C)(B) = r_{A,B}\,(AB) \qquad (A, B \in \mathfrak{H}).$$

The associativity $(PQ)R = P(QR)$ $(P, Q, R \in \mathfrak{H})$ of $\mathfrak{H}$ is represented as

$$\big(\{P\}\{Q\}\big)\{R\} = \{P\}\big(\{Q\}\{R\}\big)$$

and this gives us

$$(1.2) \qquad r_{P,Q}\,r_{PQ,R} = r_{P,QR}\,r_{Q,R} \qquad (P, Q, R \in \mathfrak{H}).$$

On the other hand, Schur constructed, for any $\boldsymbol{C}^\times$-valued function $r_{P,Q}$ on $\mathfrak{H}\times\mathfrak{H}$ satisfying the relation (1.2), an assignment $\mathfrak{H} \ni A \mapsto (C)$ satisfying (1.1). This is done by appropriately twisting the regular representation of $\mathfrak{H}$.

Now let us come into the mode of the present day expression for the sake of familiarity. As seen above, in Introduction, Schur indicated the problem of determining all the representations of a finite group $\mathfrak{H}$ into some projective general linear group $PGL(n, \boldsymbol{C})$. A $\boldsymbol{C}^\times$-valued function $r_{P,Q}$ on $\mathfrak{H}\times\mathfrak{H}$ satisfying the relation (1.2) is called a *($\boldsymbol{C}^\times$-valued) cocycle* of degree 2 on $\mathfrak{H}$, and $\rho(C) := (C)$ $(A \in \mathfrak{H})$ a *projective representation* of $\mathfrak{H}$, and $r_{A,B}$ in (1.1) the *factor set* associated to ρ. The product of two cocycles is defined naturally.

If ρ is replaced by ρ' given as $\rho'(C) := c_A\,\rho(A)$ $(c_A \in \boldsymbol{C}^\times)$, then the associated linear fractional transformation remains unchanged but the associated factor set $r_{P,Q}$ is replaced by

$$(1.3) \qquad r'_{P,Q} = \frac{c_P\,c_Q}{c_{PQ}}\,r_{P,Q} \qquad (P, Q \in \mathfrak{H}).$$

In turn two cocycle $r_{A,B}$ and $r'_{A,B}$ are defined to be mutually *equivalent* if they satisfy the relation (1.3). The quotient of the abelian group of cocycles by this equivalence relation is denoted by $H^2(\mathfrak{H}, \boldsymbol{C}^\times)$ and is called *Schur multiplier* $\mathfrak{M}$ of $\mathfrak{H}$.

Let $\mathfrak{H}'$ be a central extension of $\mathfrak{H}$ by a central subgroup $\mathfrak{Z}$ as

$$(1.4) \qquad 1 \longrightarrow \mathfrak{Z} \longrightarrow \mathfrak{H}' \xrightarrow{\Phi} \mathfrak{H} \longrightarrow 1 \quad \text{(exact)},$$

[3] cf. [Mol]; cf. [Fro1], [Fro2].

where Φ denotes the canonical homomorphism of $\mathfrak{H}'$ onto $\mathfrak{H}$. Take a section $\Psi : \mathfrak{H} \to \mathfrak{H}'$ so that $\Phi \cdot \Psi$ is the identity map on $\mathfrak{H}$, then for any A and B in $\mathfrak{H}$, there exists a $C_{A,B} \in \mathfrak{Z}$ such that $\Psi(A)\Psi(B) = C_{A,B}\Psi(AB)$. For an irreducible linear representation ρ' of $\mathfrak{H}'$, put $\rho(A) := \rho'\big(\Psi(A)\big)$ $(A \in \mathfrak{H})$, then it is a projective representation of $\mathfrak{H}$ with a factor set $r_{A,B} \in \boldsymbol{C}^\times$ given by $\rho'(C_{A,B}) = r_{A,B}I$, $I =$ the identity transformation. In fact, since ρ' is assumed to be irreducible, the operator $\rho'(C_{A,B})$ should be a scalar operator by Schur's lemma. Seeing in the reverse way from $\mathfrak{H}$ upwards to $\mathfrak{H}'$, we say that ρ is linearized to ρ' by lifting up (from the level of $\mathfrak{H}$) to $\mathfrak{H}'$. We call a central extension of $\mathfrak{H}$ also as a *covering group* of $\mathfrak{H}$.

•• In the first paper [Sch1, 1904] of the trilogy, Schur proved among others the following:

(S1) *For any finite group $\mathfrak{H}$, there exists a covering group $\mathfrak{H}'$ such that any projective representation ρ can be linearized by lifting it up to $\mathfrak{H}'$.*

Among such covering groups $\mathfrak{H}'$, we call anyone with the minimum order a *representation group* of $\mathfrak{H}$.

(S2) *For any representation group $\mathfrak{H}'$ of a finite group $\mathfrak{H}$, the central subgroup $\mathfrak{Z}$ in (1.4) is isomorphic to the Schur multiplier $\mathfrak{M} = H^2(\mathfrak{H}, \boldsymbol{C}^\times)$. The number of representation groups, modulo isomorphisms, is finite.*

The theory of linear representations for any representation group of $\mathfrak{H}$ is mutually equivalent, and the problem of projective representations of a finite group $\mathfrak{H}$ is reduced to the following:

(a) Construct a representation group of $\mathfrak{H}$ $\big($denote it by $R(\mathfrak{H})\big)$;

(b) Study linear representations of $R(\mathfrak{H})$ and their characters.

•• In the second paper [Sch2, 1907], he first gave in Introduction a theorem characterizing a representation group, which says (cf. Theorem 1.1 in Part I, §1 in [II])

> *a covering group $\mathfrak{H}'$ of $\mathfrak{H}$ is a representation group of $\mathfrak{H}$ if and only if $|\mathfrak{Z}| = \big|H^2(\mathfrak{H}, \boldsymbol{C}^\times)\big|$, and $[\mathfrak{H}', \mathfrak{H}'] \supset \mathfrak{Z}$, where $\mathfrak{Z}$ is as in (1.4).*

Then he studied

(1) the number of different representation groups of $\mathfrak{H}$,

(2) method of calculating Schur multipliers, and

(3) explicit examples of constructing representation groups and calculating spin (and non-spin) characters for $SL(2, K)$, $PSL(2, K)$, $GL(2, K)$, and $PGL(2, K)$ for a finite field $K = GF[p^n]$.

•• In the third paper [Sch3, 1911], he first constructed representation groups for the symmetric group $\mathfrak{S}_n$ and the alternating group $\mathfrak{A}_n$, for $n \geq 4$. He gave two representation groups $\mathfrak{T}_n$ and $\mathfrak{T}'_n$ of $\mathfrak{S}_n$ as abstract groups by giving pairs of the set of generators and the set of fundamental relations. They are isomorphic to each other only when $n = 6$, and he utilizes the first one $\mathfrak{T}_n$ in his studies in [Sch3]. It is given as follows:

Theorem 1.1 ([Sch3, §3])**.** *For $n \geq 4$, define a group $\mathfrak{T}_n$ by giving*

- *generators :* $\{J, T_1, T_2, \ldots, T_{n-1}\}$;
- *fundamental relations :*

$$\begin{cases} J^2 = E\,,\ T_\alpha{}^2 = J, & (\alpha = 1, 2, \ldots, n-1)\ ; \\ (T_\beta T_{\beta+1})^3 = J, & (\beta = 1, 2, \ldots, n-2)\ ; \\ T_\gamma T_\delta = J\, T_\delta T_\gamma, & (\gamma = 1, 2, \ldots, n-3,\ \delta = \gamma+2, \ldots, n-1)\,, \end{cases}$$

where E denotes the identity element. Then

$$1 \longrightarrow \mathfrak{Z} = \{E, J\} \longrightarrow \mathfrak{T}_n \longrightarrow \mathfrak{S}_n \longrightarrow 1 \quad \text{(exact)},$$

with the canonical homomorphism $\mathfrak{T}_n \ni T_i \mapsto s_i = (i\ \ i{+}1) \in \mathfrak{S}_n$, and $\mathfrak{T}_n$ gives a representation group of $\mathfrak{S}_n$.

Here $\mathfrak{Z} = H^2(\mathfrak{S}_n, \boldsymbol{C}^\times) \cong \boldsymbol{Z}_2$. For a similar presentation of the second representation group $\mathfrak{T}'_n$, see Theorem 1.2 in Part I in [II]. With reasons stated in Remark 1.1 in Part I (2), §1, loc. cit., we prefer to use the second group $\mathfrak{T}'_n$ for our present study and denote it by $\widetilde{\mathfrak{S}}_n$.

Succeedingly, Schur constructed so called "Hauptdarstellung" Δ_n of $\mathfrak{T}_n$, and used it as the fundamental ingredient to give spin irreducible representations (=IRs) of $\mathfrak{T}_n$. In Part III, §15 in [II], we rewrite Δ_n as a fundamental spin representation Δ'_n of $\widetilde{\mathfrak{S}}_n = \mathfrak{T}'_n$. Schur constructed induced representations from 'Young type' subgroups of $\mathfrak{T}_n$, and then succeeded to classify all the spin IRs of the symmetric groups $\mathfrak{S}_n$ and also of the alternating group $\mathfrak{A}_n$ by giving irreducible spin characters as integral linear combinations of induced characters above.

Remark 1.1 Schur multipliers of finite groups have in particular certain intimate relations with the problem of classification of simple finite groups, cf. a book [Kar, 1985] by G. Karpilovski.

In connection to the paper [II], Schur multipliers of Weyl groups were calculated by S. Ihara and T. Yokonuma [IhYo, 1965], and those for generalized symmetric groups by J.W. Davies and A.O. Morris [DaMo, 1974], and finally those for complex reflection groups $G(m, p, n)$, $p|m$, by E.W. Read [Rea1, 1976].

1.2 Work of Élie Cartan in 1913

1.2.1. Nowadays we know that the rotation group $SO(n)$, $n \geq 3$, is not simply connected, and its universal covering group is $\mathrm{Spin}(n)$ for $n \geq 3, \neq 4$, which in turn is a special central extension of $SO(n)$ as

$$\{e\} \longrightarrow \{e, z\} \longrightarrow \mathrm{Spin}(n) \longrightarrow SO(n) \longrightarrow \{e\} \quad \text{(exact)},$$

where e denotes the identity element and z a central element generating $\boldsymbol{Z}_2 = \{e, z\}$. An irreducible linear representation ρ of $\mathrm{Spin}(n)$ which cannot be reduced to $SO(n)$, or equivalently such that $\rho(z) = -I$, is called a *spin* representation of $SO(n)$. É. Cartan first discovered the existence of spin irreducible representations of the rotation groups and classified them by means of their highest weights in [Car, 1913], on which S.S. Chern and C. Chevalley commented in the report [CC] as is quoted below:

From p.219 :

> ······ Once the structures of all simple groups were known, it became possible to look for all possible realizations of any one of these structures by transformations of a specified nature, and in particular, for their realizations as groups of linear transformations. This is the problem of the determination of the representations of a given group; it was solved completely by Cartan for simple groups. The solution led in particular to the discovery, as early as 1913, of the spinors, which were to be re-discovered later in a special case by the physicists.

From pp.223-224 :

> The general theory of linear representations is the object of the paper [**5**] [4]. ....
> .... <*almost a page omitted here*> ....
> .... In the process of classifying all possible linear representations, Cartan discovered the spin representations of the orthogonal Lie algebras, which later played such an important rôle in physics. In a book published later (*Leçon sur la théorie des spineurs,* Hermann, Paris, 1938), Cartan developed the theory of spinors from a geometric point of view.

In general, for a connected Lie group G, its universal covering group $\widetilde{G}$ exists uniquely (up to isomorphism) and corresponds to the representation group $R(\mathfrak{H})$ in the case of a finite group $\mathfrak{H}$. Any (continuous) projective representation of G can be linearized by lifting up to $\widetilde{G}$. We call a projective representation also as a spin representation (cf. Part I, §1, **1.4** in [II]).

1.2.2. Let us explain a little more in detail in the case of $n = 3$ or of the rotation group $SO(3)$ in the Euclidean three-dimensional space E^3. In this case its universal covering group can be realized as $SU(2)$ and the canonical homomorphism $\Phi : SU(2) \to SO(3) \cong SU(2)/Z,\ Z = \{E_2, -E_2\}$, is given as follows. Let $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ be a system of unit coordinate vectors mutually orthogonal,

[4] [**5**] = [Car]

and a point $\boldsymbol{x} = x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2 + x_3\boldsymbol{e}_3 \in E^3$, expressed also by a column vector ${}^t(x_1, x_2, x_3)$, is mapped to a 2×2 Hermitian matrix as

$$\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \longrightarrow X = \begin{pmatrix} x_3 & x_1 + ix_2 \\ x_1 - ix_2 & -x_3 \end{pmatrix} \quad (i = \sqrt{-1}). \tag{1.5}$$

For an element $u \in SU(2)$, put $g = \Phi(u) \in SO(3)$, then $\boldsymbol{x} \mapsto g\boldsymbol{x}$ is given by $X \mapsto uXu^{-1}$.

On the other hand, the Lie algebra of $SU(2)$ is the matrix Lie algebra $\mathfrak{su}(2)$ consisting of 2×2 skew-Hermitian matrices of trace 0, whose complexification is $\mathfrak{sl}(2, \boldsymbol{C})$. Put

$$a := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad b := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad c := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{1.6}$$

$$A_1 := ia, \quad A_2 = -ib, \quad A_3 := ic. \tag{1.7}$$

Then the commutation relations are $[a, b] = 2ic$, $[b, c] = 2ia$, $[c, a] = 2ib$, and $\{A_1, A_2, A_3\}$ is a basis of $\mathfrak{su}(2)$ over $\boldsymbol{R}$ with $[A_i, A_j] = 2A_k$ for cyclically permuted $(i\ j\ k)$ of $(1\ 2\ 3)$. Put $H := c$, $X_\pm := \frac{1}{2}(a \pm ib)$, then they give a basis of $\mathfrak{sl}(2, \boldsymbol{C})$ with commutation relations

$$[H, X_+] = 2X_+, \quad [H, X_-] = -2X_-, \quad [X_+, X_-] = H.$$

The covering map Φ is given as follows: put $e^{tA_j} = \exp(tA_j)$ and let $g_j(\varphi)$ $(j = 1, 2, 3)$ be the matrix representing a rotation of angle φ around the x_j-axis. Then,

$$\Phi\big(e^{tA_j}\big) = g_j(2t) \quad \text{for} \quad j = 1, 2, 3\,; \tag{1.8}$$

$$e^{tA_1} = \begin{pmatrix} \cos t & i\sin t \\ i\sin t & \cos t \end{pmatrix}, \; e^{tA_2} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}, \; e^{tA_3} = \begin{pmatrix} e^{ti} & 0 \\ 0 & e^{-ti} \end{pmatrix};$$

$$g_1(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos t & -\sin t \\ 0 & \sin t & \cos t \end{pmatrix}, \quad g_2(t) = \begin{pmatrix} \cos t & 0 & \sin t \\ 0 & 1 & 0 \\ -\sin t & 0 & \cos t \end{pmatrix},$$

$$g_3(t) = \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Returning to the rotation group $SO(3)$, we have a natural basis of its Lie algebra $\mathfrak{so}(3)$ given by $B_j := \frac{d}{dt}g_j(t)|_{t=0}$, and with the commutation relation $[B_i, B_j] = B_k$ for (i, j, k) cyclically permuted of $(1, 2, 3)$. From (1.8), the differential $d\Phi$ of Φ is given by $d\Phi(A_j) = 2B_j$ $(1 \le j \le 3)$. Put $H' := iB_3$, $X'_\pm := iB_1 \mp B_2$, then $[H', X'_\pm] = \pm X'_\pm$, $[X'_+, X'_-] = 2H'$. The natural isomorphism $d\Phi : \mathfrak{su}(2) \to \mathfrak{so}(3)$ maps $H \mapsto 2H'$, $X_\pm \mapsto X'_\pm$.

Actually the set of all equivalence classes of irreducible representations (=IRs) of the covering group $SU(2)$ corresponds to the following set of IRs of Lie algebra $\mathfrak{su}(2)$ or rather of $\mathfrak{sl}(2, \boldsymbol{C})$. Let $\ell \geq 0$ be an integer or a half-integer (half an odd integer), and put $\Omega_\ell := \{-\ell, -\ell+1, \ldots, \ell-1, \ell\}$, and prepare an orthonormal basis $\{v_k\ (k \in \Omega_\ell)\}$ of a Hilbert space of dimension $2\ell+1$. Then the differential of a unitary IR π_ℓ of $SU(2)$ is given as follows, described from the base level of $\mathfrak{so}(3)$ (or of $SO(3)$) and denoted as $d\pi'_\ell = d\pi_\ell \circ d\Phi^{-1}$,

$$
\text{(1.9)} \qquad \left\{ \begin{array}{rcl} d\pi'_\ell(H')v_k & = & k\, v_k\,, \\ d\pi'_\ell(X'_+)v_k & = & a_{\ell,k}\, v_{k+1}\,, \\ d\pi'_\ell(X'_-)v_{k+1} & = & a_{\ell,k}\, v_k\,, \end{array} \right. \qquad (k \in \Omega_\ell),
$$

$$
\text{with} \qquad a_{\ell,k} = \sqrt{\frac{(\ell+k+1)(\ell-k)}{2}}\,.
$$

Here the eigenvalues k of $d\pi'_\ell(H')$ are called *weights*, and the *highest weight* ℓ characterizes the equivalence class of IR π_ℓ.

A global realization of IR π_ℓ of $SU(2)$ is given as follows. For $\ell = \frac{1}{2}$, take the identical mapping $u \mapsto u$ of $SU(2)$, then this gives a spin (or double-valued) IR $\pi_{\frac{1}{2}}$ of $SO(3)$ on the two-dimensional vector space $V_{\frac{1}{2}} := \boldsymbol{C}^2$. For $\ell = (2\ell)\cdot\frac{1}{2}$, IR π_ℓ is given by restricting the (2ℓ)-times tensor product of $\pi_{\frac{1}{2}}$ on the (2ℓ)-times symmetric tensor product space V_ℓ of $V_{\frac{1}{2}}$. We can see that π_ℓ with a half-integer ℓ is spin, and π_ℓ with an integer ℓ is non-spin (cf. the character formula (1.10) below).

Every element in $SU(2)$ is conjugate to an element $\exp(tic) = \mathrm{diag}(e^{it}, e^{-it})$ in the Cartan subgroup $C := \exp(\boldsymbol{R}ic)$, and the character χ_{π_ℓ} is uniquely determined by its value on C. On the other hand, with respect to the basis in (1.9), the matrix $\pi_\ell\big(\exp(tic)\big)$ takes the diagonal form as $\mathrm{diag}(e^{2i\ell t}, e^{2i(\ell-1)t}, \ldots, e^{-2i\ell t})$, and so the character is given by the formula

$$
\text{(1.10)} \qquad \chi_{\pi_\ell}\big(\exp(tic)\big) = \frac{e^{(2\ell+1)it} - e^{-(2\ell+1)it}}{e^{it} - e^{-it}} = \frac{\sin(2\ell+1)t}{\sin t}\,.
$$

Note that, if we see from the level of $SO(3)$, formulas (1.9) and (1.10) give double-valued representations and characters, in case ℓ is odd.

Remark 1.2. Let $\boldsymbol{q} = \alpha + \beta\boldsymbol{i} + \gamma\boldsymbol{j} + \delta\boldsymbol{k} \in \boldsymbol{H}$ $(\alpha, \beta, \gamma, \delta \in \boldsymbol{R})$ be a quaternion which is found by W.R. Hamilton [Ham] in 1843. Put $\|\boldsymbol{q}\| = \big(\alpha^2+\beta^2+\gamma^2+\delta^2\big)^{1/2}$, and let $\boldsymbol{H}_-$ be the set of all pure quaternions $\boldsymbol{x}' = x_1\boldsymbol{i} + x_2\boldsymbol{j} + x_3\boldsymbol{k}$. We know that the universal covering group of $SO(3)$ is also realized as the group of unit quaternions $\boldsymbol{H}_1 := \{\boldsymbol{q} \in \boldsymbol{H};\, \|\boldsymbol{q}\| = 1\}$ as follows:

Identify E^3 with $\boldsymbol{H}_-$ through their coordinates, and for each $\boldsymbol{q} \in \boldsymbol{H}_1$ consider the transformation $R(\boldsymbol{q})$ on $\boldsymbol{H}_-$ given by $\boldsymbol{x}' \mapsto \boldsymbol{q}\boldsymbol{x}'\boldsymbol{q}^{-1}$. Then $R(\boldsymbol{q})$ belongs to $SO(3)$, and $\boldsymbol{H}_1 \ni \boldsymbol{q} \mapsto R(\boldsymbol{q}) \in SO(3)$ gives a covering map.

In 1838, preceding to the discovery by Hamilton, O. Rodrigues described this covering map in [Rod] from the geometric point of view in a form of a variant of spherical trigonometry but actually using the quaternion multiplication rule. Its formula is called Rodrigues' formula, and see also [Alt, 1989] in this connection.

Remark 1.3. Let $O(p,q)$, $p \geq q \geq 0$, be the real orthogonal group leaving the (indefinite) quadratic form $x_1^2+\cdots+x_p^2-x_{p+1}^2-\cdots-x_{p+q}^2$ in $\boldsymbol{R}^{p+q}$ invariant. Then it is not connected, and it has two connected components if $q=0$, and four if $q>0$. Moreover it has 4 covering groups if $q=0$, and 8 otherwise. At most two of them admit Clifford algebras as representations, and are denoted by $\mathrm{Pin}(p,q)$ and $\mathrm{Pin}(q,p)$ (different from each other if $p \neq q$), as is given in [ABS].

On the intimate connection of Clifford algebra and IRs of the covering group $\widetilde{D}_n$ of the n-times direct product D_n of the cyclic group $\boldsymbol{Z}_m$, see Part II, §§5–6 in [II].

Apart from the first motivation of Schur " Determination of all finite subgroups of $GL(n,\boldsymbol{C})$ or of $PGL(n,\boldsymbol{C})$", and Cartan's result in [Car, 1913], we have various natural sources where projective representations of groups should come out naturally. We simply list up them without detailed explanations and proofs hereafter in §§2–4, numbering them as **E-1** $\sim$ **E-10**, and these sections are devoted to introduce the theory of projective representations and explain how such projective objects occur naturally and why they are so interesting to be studied.

2 Pauli's spin quantum number and Dirac's equation

E-1. Spin angular momentum of electron. In 1925, W. Pauli introduced, in his paper [Pau1], a new quantum degree of freedom, or a fourth quantum number, for an electron, and discovered a general principle which asserts that, with four quantum numbers containing the new one, electrons of an atom can be identified. We quote from the original paper the related parts in the following:

[Pau1, 1925], p.775, lines 24–28
(eine weitere Quantenzahl m_1):

> $\cdots\cdots\cdots$. Wir sind dort, auf die Möklichkeit der Aufrechterhaltung der Permanenz der Quantenzahlen gestützt, dazu gelangt, jedes Elektron im Atom außer durch die Hauptquantenzahl n durch die beiden Nebenquantenzahlen k_1 und k_2 zu charakterisieren. In starken Magnetfeldern kam noch eine Inpulsquantenzahl m_1 für jedes Elektron hinzu und $\cdots\cdots$

[Pau1, 1925], p.776, lines 5–10
(allgmeine Regel über das Vorkommen von äquivalenten Elektronen in Atom):

> Es kann niemals zwei oder mehrere äquivalente Elektronen im Atom geben, für welche in starken Feldern die Werte aller Quantenzahlen n, k_1, k_2, m_1 (oder, was dasselbe ist, n, k_1, m_1, m_2) übereinstimmen. Ist ein Elektron im Atom vorhanden, für das diese Quantenzahlen (im äußeren Felde) bestimmte Werte haben, so ist dieser Zustand „besetzt".

P.M.A. Dirac called in [Dir1, 1926] this principle as Pauli's exclusion principle, and quoted as "*not more than one electron can be in any given orbit*". The discovery of this principle leads Pauli to his Nobel prize in 1945.

After Pauli's discovery of the forth quantum number in 1925, to explain this new quantum number of an electron, R. Kronig and then S.A. Goudsmit and G.E. Uhlenbeck presented ideas of electron spin, that is, a virtual rotation of electron around a fixed axis.

In 1927, Pauli succeeded to formulate these ideas in a clear form by introducing the so-called Pauli matrices to treat 'die Drehimpulskomponenten' of an electron. The principal idea was the following, quoted from the introduction of [Pau2, 1927], p.601, lines 1–9:

> Es wird gezeigt, wie man zu einer Formulierung der Quantenmechanik des magnetischen Elektrons nach der Schrödingerschen Methode der Eigenfunktionen ohne Verwendung zweideutiger Funktionen gelangen kann, indem man, gestützt auf die allgemeine Dirac-Jordansche Transformationstheorie, neben den Ortskoordinaten jedes Elektrons, um seinen rotatorischen Freiheitsgeraden Rechnung zu tragen, die Komponente seines Eigenimpulsmomentes in einer festen Richtung als weitere unabhängige Veränderliche einführt. Im Gegensatz zur klassischen Mechanik kann diese Variable jedoch, $\cdots\cdots\cdots\cdot$, nur die Werte $+\frac{1}{2}\frac{h}{2\pi}$ und $-\frac{1}{2}\frac{h}{2\pi}$ annehmen.

Mathematically speaking, this means that electrons do not live in the usual three-dimensional Euclidean space E^3 where the rotation group $SO(3)$ acts, but they live in the space $V_{\frac{1}{2}} = \boldsymbol{C}^2$ (real four-dimensional), where the covering group $SU(2)$ acts (cf. **1.2.2**).

In connection to spin representations of $SO(3)$, let us try to describe Pauli's result in today's framework of group representations. To do so, we first introduce a mathematical formalism for a group action on certain function spaces.

Mathematical formalism for a group action:

Let G be a group, and Y a set on which G acts as $Y \ni y \mapsto gy \in Y$ $(g \in G)$. Take a vector space V on which G acts linearly as $V \ni v \mapsto \pi(g)v \in V$ $(g \in G)$, and consider a space $\mathcal{F}(Y;V)$ of V-valued functions on Y. Then G acts on $\mathcal{F}(Y;V)$ by the following formula: for $g \in G$ and $f \in \mathcal{F}(Y;V)$,

$$\big(T_\pi(g)f\big)(y) := \pi(g)\big(f(g^{-1}y)\big) \quad (y \in Y), \tag{2.1}$$

where $\pi(g)$ acts on the value $v = f(g^{-1}y) \in V$. In fact, it is not difficult to prove $T_\pi(g_1g_2) = T_\pi(g_1)T_\pi(g_2)$ $(g_1, g_2 \in G)$, and $T_\pi(e) = I$ for the identity element $e \in G$, where I denotes the identity map on $\mathcal{F}(Y; V)$.

In this formalism, (π, V) is a linear representation of G, for which V is also denoted by $V(\pi)$. Here we may also take as π a projective representation, that is, multiple-valued representation of G, then we get a multiple-valued representation T_π of G on the function space $\mathcal{F}(Y; V)$.

In Pauli's case in [Pau2], we take as Y the Euclidean space E^3 (since the theory is non-relativistic here), and as (π, V) the irreducible representation $\pi_{\frac{1}{2}}$ on $V_{\frac{1}{2}} = \boldsymbol{C}^2$ with the highest weight $\frac{1}{2}$, and as a function f a wave function ψ of an electron which is a pair of two scalar-valued functions, written with respect to the canonical coordinates in $V_{\frac{1}{2}}$, as

$$\psi(\boldsymbol{x}) = \begin{pmatrix} \psi_1(\boldsymbol{x}) \\ \psi_2(\boldsymbol{x}) \end{pmatrix} \quad (\boldsymbol{x} \in E^3).$$

Note that, for canonical coordinates in $V_{\frac{1}{2}} = \boldsymbol{C}^2$, the differential $d\pi_{\frac{1}{2}}(H) = \frac{1}{2}c$ has eigenvalues $+\frac{1}{2}$ and $-\frac{1}{2}$ respectively.

The formula (2.1) can be understood in two ways. On the one hand, if we take $G = SO(3)$, as might be the case of Pauli, then $\pi = \pi_{\frac{1}{2}}$ is double-valued seeing from $SO(3)$. On the other hand, if we take $G = SU(2)$, the covering group of $SO(3)$, since $\pi_{\frac{1}{2}}(u) = u \ \big(u \in SU(2)\big)$, the formula (2.1) is written as

$$\big(T_\pi(u)\psi\big)(\boldsymbol{x}) := u \begin{pmatrix} \psi_1(g^{-1}\boldsymbol{x}) \\ \psi_2(g^{-1}\boldsymbol{x}) \end{pmatrix} \quad (u \in SU(2),\ \boldsymbol{x} \in E^3), \tag{2.2}$$

where $g = \Phi(u) \in SO(3)$ is the image of u under the covering map $\Phi : SU(2) \to SO(3)$. The natural norm in $V_{\frac{1}{2}}$ is given for $\boldsymbol{v} = \binom{v_1}{v_2}$ as $\|\boldsymbol{v}\|_V = \sqrt{|v_1|^2 + |v_2|^2}$, which is $SU(2)$-invariant. We put

$$\|\psi\|^2 := \int_{E^3} \|\psi(\boldsymbol{x})\|_V^2 \, d\boldsymbol{x}, \qquad d\boldsymbol{x} := dx_1 dx_2 dx_3.$$

A wave function ψ is in the Hilbert space $L^2(E^3, V; \mu)$ of V-valued L^2-functions on E^3, and the norm $\|\psi\|$ is $SU(2)$-invariant as $\|T_\pi(u)\psi\| = \|\psi\| \ \big(u \in SU(2)\big)$. For a ψ normalized as $\|\psi\| = 1$, the integral $\int_{\mathcal{D}} \|\psi(\boldsymbol{x})\|^2 \, d\mu(\boldsymbol{x})$ over a domain $\mathcal{D}$ in E^3 gives the probability for that the electron, described by ψ, exists in $\mathcal{D}$.

Now rewrite the coordinates (x_1, x_2, x_3) as (x, y, z). Then Pauli [Pau2] proposed to consider "die Drehimpulskomponenten s_x, s_y, s_z" of an electron as variables which can take values only $\pm c'$, as seen on p.605, lines 17–20, as

> Indessen kann man das Auftreten solcher Zweiseitigkeiten, wie überhaupt die explizite Verwendung irgendwelcher Polarwinkel dadurch vermeiden, daß man an Stelle von φ die Impulskomponente s_z als unabhängige Variable in die Eigenfunktion einführt. $\cdots\cdots\cdots\cdots$

Then he represented s_x, s_y, s_z by operators $\boldsymbol{s}_x, \boldsymbol{s}_y, \boldsymbol{s}_z$ acting on a wave function ψ as

$$\boldsymbol{s}_x(\psi) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \psi\,; \quad \boldsymbol{s}_y(\psi) = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \psi\,; \quad \boldsymbol{s}_z(\psi) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \psi\,. \tag{2.3}$$

Here these three matrices are exactly a, b, c in (1.6) in §1.2. They are called Pauli matrices, and in today's notation

$$\sigma_1 = \sigma_x := a, \quad \sigma_2 = \sigma_y := b, \quad \sigma_3 = \sigma_z := c, \tag{2.4}$$

and the commutation relations are expressed as

$$[\sigma_i, \sigma_j] = 2i\varepsilon_{ijk}\,\sigma_k, \quad \{\sigma_j, \sigma_j\} = 2\delta_{ij}E_2\,, \tag{2.5}$$

where ε_{ijk} is the Levi-Civita symbol.

Pauli discussed the invariance (or covariance) under a coordinates change $(x, y, z) \mapsto (x', y', z')$. The invariance of Pauli's equation for a free electron under the $SU(2)$-action (2.2) is explained in detail in [HiYa, Chap. 4]. Note that, purely in the framework of the theory of group representations, the infinitesimal action on ψ corresponding to the "Drehimpulskomponente" s_z is calculated as

$$\begin{aligned} \big(dT_\pi(c)\psi\big) &= \frac{1}{i}\frac{d}{dt}\Big(T_\pi(e^{tA_3})\psi(x)\Big)\Big|_{t=0} \\ &= \boldsymbol{s}_z(\psi) + 2\Big(-x_2\frac{\partial}{\partial x_1} + x_1\frac{\partial}{\partial x_2}\Big)\psi\,. \end{aligned}$$

E-2. Dirac equation and spin representations of Lorentz group.

In 1928, P.A.M. Dirac gave in [Dir2] a relativistic wave equation for an electron (called Dirac equation), which comes out of Klein-Gordon equation. He stated at the top of Introduction (loc. cit., p.610, lines 1–10) as

> The new quantum mechanics, when applied to the problem of the structure of the atom with point-charge electrons, does not give results in agreement with experiment. The discrepancies consist of "duplexity" phenomena, the observed number of stationary states for an electron in an atom being twice the number given by the theory. To meet the difficulty, Goudsmit and Uhlenbeck have introduced the idea of an electron with a spin angular momentum of half a quantum and a magnetic moment of one Bohr magneton. This model for the electron has been fitted into the new mechanics by Pauli,* and Dawin,† working with an equivalent theory, has shown that it gives results in agreement with experiment for hydrogen-like spectra to the first order of accuracy.

Then, at the end of §1. *Previous Relativity Treatments*,

> In the present paper we shall be concerned with the removal of the first of these two difficulties. The resulting theory is therefore still only an approximation, but it appears to be good enough to account for all the duplexity phenomena without arbitrary assumptions.

Minkowski space and Lorentz groups : To describe the things correctly, we introduce the Minkowski space M^4 consisting of $\boldsymbol{x} = {}^t(x_1, x_2, x_3, x_4)$, a column vector, with ${}^t(x_1, x_2, x_3) \in E^3$, $x_4 = ct \in \boldsymbol{R}$, and equipped with the indefinite quadratic form

$$\langle \boldsymbol{x}, \boldsymbol{x} \rangle_{3,1} := x_1^2 + x_2^2 + x_3^2 - x_4^2 = {}^t x \, J_{3,1} x \,,$$

where $J_{3,1} := \mathrm{diag}(1,1,1,-1)$ is a diagonal matrix with entries $1,1,1,-1$. The group $O(3,1)$ consists of $g \in GL(4,\boldsymbol{R})$ satisfying $\langle g\boldsymbol{x}, g\boldsymbol{x} \rangle_{3,1} = \langle \boldsymbol{x}, \boldsymbol{x} \rangle_{3,1}$ ($\boldsymbol{x} \in M^4$) or ${}^t g J_{3,1} g = J_{3,1}$, and has 4 connected components. The one containing the identity element $e = E_4$ is called (*proper*) Lorentz group and is given as

$$SO_0(3,1) := \big\{ g = (g_{ij})_{1 \le i,j \le 4} \in O(3,1) \; ; \; \det(g) = 1, \; g_{44} \ge 1 \big\},$$

which is denoted also as $\mathcal{L}_4$.

Dirac equation : Dirac's wave function $\psi(\boldsymbol{x})$ is a function on M^4, valued in $V = \boldsymbol{C}^4$, and denoted as $\psi(\boldsymbol{x}) = \big(\psi_j(\boldsymbol{x})\big)_{1 \le j \le 4}$ in the form of column vector. He proposed in [Dir2] a relativistic wave equation for an electron. Changing the notation in [loc. cit.] appropriately, we put

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \sigma_4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{2.6}$$

and let $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ be 4×4 matrices given as

$$\gamma_j = \begin{pmatrix} 0_2 & -i\sigma_j \\ i\sigma_j & 0_2 \end{pmatrix} \quad (1 \le j \le 3), \qquad \gamma_4 = \begin{pmatrix} -i\sigma_4 & 0_2 \\ 0_2 & i\sigma_4 \end{pmatrix}, \tag{2.7}$$

with 0_2 the zero matrix of order 2. Then the Dirac equation is expressed as follows: in the case of no electromagnetic field

$$\big(D + \kappa\big)\psi = 0, \quad D := \sum_{1 \le j \le 4} \gamma_j \partial_{x_j}, \;\; \partial_{x_j} := \frac{\partial}{\partial x_j}, \;\; \kappa := \frac{mc}{h}. \tag{2.8}$$

Note that the Klein-Gordon equation is factored as

$$(\Box - \kappa^2)\psi = (D - \kappa)(D + \kappa)\psi = 0, \qquad \Box := \sum_{1 \le j \le 3} \partial_{x_j}^2 - \partial_{x_4}^2 \,.$$

Dirac's proof of Lorentz-invariance:

Mathematically speaking, this invariance means that, under a *natural* transformation of ψ in the formula below, with a certain representation ϖ of $G = \mathcal{L}_4$ on V,

$$(2.9) \qquad \big(T_\varpi(g)\psi\big)(\boldsymbol{x}) = \varpi(g)\big(\psi(g^{-1}\boldsymbol{x})\big) \quad (g \in G),$$

the differential operator D is invariant, that is,

$$D\,T_\varpi(g)\psi = T_\varpi(g)\,D\psi \quad (g \in G),$$

or $T_\varpi(g)^{-1} \cdot D \cdot T_\varpi(g) = D$. Here the representation ϖ is left to be determined.

Dirac's proof of the Lorentz invariance proceeds as follows. The discussions have a strong similarity as those in §§3 – 4 and also in §§7–10 in [II]. For $\boldsymbol{x} \in M^4$ and $g = (g_{ij})_{1\le i,j\le 4} \in G$, put $\boldsymbol{x}' = g^{-1}\boldsymbol{x}$, then $\boldsymbol{x} = g\boldsymbol{x}'$ and so

$$(2.10) \qquad \boldsymbol{\partial}' = \boldsymbol{\partial} g, \quad \boldsymbol{\partial}' := (\partial_{x'_1}, \partial_{x'_2}, \partial_{x'_3}, \partial_{x'_4}),\ \boldsymbol{\partial} := (\partial_{x_1}, \partial_{x_2}, \partial_{x_3}, \partial_{x_4}),$$

where the matrix multiplication rule is applied. Consider $\boldsymbol{\gamma} := (\gamma_j)_{1\le j\le 4}$ as a row vector (of matrix entries), then we have

$$D = \boldsymbol{\gamma} \cdot {}^t\boldsymbol{\partial} = \boldsymbol{\gamma} \cdot {}^t(\boldsymbol{\partial}' g^{-1}) = \boldsymbol{\gamma}' \cdot {}^t\boldsymbol{\partial}', \quad \boldsymbol{\gamma}' := \boldsymbol{\gamma}\,{}^t g^{-1}.$$

On the other hand, the matrices γ_j's satisfy

$$(2.11) \qquad \begin{cases} \gamma_j^{\,2} = E_4 \ (1 \le j \le 3), \quad \gamma_4^{\,2} = -E_4\,, \\ \gamma_j\gamma_k + \gamma_k\gamma_j = 0 \quad (1 \le j,k \le 4,\ j \ne k), \end{cases}$$

and this is symbolically written as ${}^t\boldsymbol{\gamma}\,\boldsymbol{\gamma} + {}^t({}^t\boldsymbol{\gamma}\,\boldsymbol{\gamma}) = 2J_{3,1}$, where ${}^t\boldsymbol{\gamma}\,\boldsymbol{\gamma} := (\gamma_i\gamma_j)_{1\le i,j\le 4}$, and the entries 1 and 0 in $J_{3,1}$ is replaced by E_4 and 0_4 respectively. Then we have

$${}^t\boldsymbol{\gamma}'\,\boldsymbol{\gamma}' + {}^t({}^t\boldsymbol{\gamma}'\,\boldsymbol{\gamma}') = g^{-1}\big\{({}^t\boldsymbol{\gamma}\,\boldsymbol{\gamma}) + {}^t({}^t\boldsymbol{\gamma}\,\boldsymbol{\gamma})\big\}{}^t g^{-1} = g^{-1} 2J_{3,1}{}^t g^{-1} = 2J_{3,1}\,,$$

that is, $\boldsymbol{\gamma}'$ satisfies the analogous relations as (2.11).

Then Dirac proved [loc. cit., pp.616–617] essentially that " by a canonical transformation, γ'_j's can be brought into the form of γ_j's." Mathematical expression of this fact is that, for $g \in G$, there exists a linear transform S_g on V such that

$$\gamma'_k = S_g^{-1}\gamma_k S_g \quad (1 \le k \le 4), \qquad \text{for } \gamma' = g^{-1}\boldsymbol{\gamma},$$

and we should identify the *canonical* transformation S_g, which will be $\varpi(g)$ in (2.9). The answer is well-known, and here we give a short and mathematically clear proof for it (for more details, see [HiYa, Chap. 5]).

Lemma 2.1. *For* $\boldsymbol{\varepsilon} = (\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)$, $\varepsilon_k = 0, 1$, *put* $\boldsymbol{\gamma}^{\boldsymbol{\varepsilon}} := {\gamma_1}^{\varepsilon_1}{\gamma_2}^{\varepsilon_2}{\gamma_3}^{\varepsilon_3}{\gamma_4}^{\varepsilon_4}$, *where* $\boldsymbol{\gamma}^{\mathbf{0}} = E_4$ *for* $\boldsymbol{\varepsilon} = \mathbf{0} = (0,0,0,0)$. *Then, the set of* $\boldsymbol{\gamma}^{\boldsymbol{\varepsilon}}$ *over all* $\boldsymbol{\varepsilon}$ *gives a linear basis of the full matrix algebra* $M(4, \boldsymbol{C})$.

Proof. First note that, for $\boldsymbol{\varepsilon} \neq \mathbf{0}$, we have $\mathrm{tr}\big(\boldsymbol{\gamma}^{\boldsymbol{\varepsilon}}\big) = 0$. Then, take a linear relation as $\sum_{\boldsymbol{\varepsilon}} a_{\boldsymbol{\varepsilon}} \boldsymbol{\gamma}^{\boldsymbol{\varepsilon}} = 0$ ($a_{\boldsymbol{\varepsilon}} \in \boldsymbol{C}$). Multiply the both sides by $\boldsymbol{\gamma}^{\boldsymbol{\varepsilon}^0}$ with $\boldsymbol{\varepsilon}^0 = (\varepsilon_1^0, \varepsilon_2^0, \varepsilon_3^0, \varepsilon_4^0)$, and take the trace of both sides. Then, since $\boldsymbol{\gamma}^{\boldsymbol{\varepsilon}}\boldsymbol{\gamma}^{\boldsymbol{\varepsilon}^0} = \pm \boldsymbol{\gamma}^{\boldsymbol{\varepsilon}-\varepsilon^0}$, we obtain $\pm 4 a_{\boldsymbol{\varepsilon}^0} = 0$. This proves the linear independence of $\boldsymbol{\gamma}^{\boldsymbol{\varepsilon}}$'s.

Since the number of $\boldsymbol{\gamma}^{\boldsymbol{\varepsilon}}$'s is $2^4 = 16$, the assertion is thus proved. $\square$

Now suppose that $S_g^{-1}\gamma_k S_g = {S'_g}^{-1}\gamma_k S'_g$ $(1 \leq j \leq 4)$ for two matrices S_g and S'_g. Then, since γ_k's generate $M(4, \boldsymbol{C})$ and so we have $S_g^{-1} A S_g = {S'_g}^{-1} A S'_g$ for all $A \in M(4, \boldsymbol{C})$, whence $S'_g = \lambda_g S_g$ with $\lambda_g \in \boldsymbol{C}^\times$. This implies that the correspondence $g \mapsto S_g$ gives a projective representation of G. The point is to identify what is this representation and to verify if it is really projective or not. Before that, we prepare spin representations of the Lorentz group G.

Spin representations of Lorentz group : Lorentz group $G := SO_0(3,1)$ is not simply connected and its universal covering group $\widetilde{G}$ is realized by $H = SL(2, \boldsymbol{C})$ with the following double covering map Φ, an extension of the covering map in §1.2.2 from $SU(2)$ onto $SO(3)$ a maximal compact subgroup of G. Let $\mathcal{H}_2$ be the space of all 2×2 Hermitian matrices, then M^4 is expressed by $\mathcal{H}_2$ by the map $M^4 \ni \boldsymbol{x} = {}^t(x_1, x_2, x_3, x_4) \mapsto \boldsymbol{X} \in \mathcal{H}_2$ as

$$\boldsymbol{X} = \begin{pmatrix} x_3 + x_4 & x_1 + i x_2 \\ x_1 - i x_2 & -x_3 + x_4 \end{pmatrix} = \sum_{j=1}^{4} \sigma_j x_j \,,$$

then $\langle \boldsymbol{x}, \boldsymbol{x} \rangle_{3,1} = -\det \boldsymbol{X}$. Note that σ_j's gives a basis of $\mathcal{H}_2$, and put $\boldsymbol{\sigma} := (\sigma_1, \sigma_2, \sigma_3, \sigma_4)$ a row vector of matrix entries. Then $\boldsymbol{X} = \boldsymbol{\sigma} \cdot \boldsymbol{x}$ in the matrix multiplication rule. The group H acts on $\mathcal{H}_2$ through $\boldsymbol{X} \mapsto h\boldsymbol{X}h^*$ $(h \in H)$, and this gives us a linear transformation $\boldsymbol{x} \mapsto g\boldsymbol{x}$. As an action on the basis $\boldsymbol{\sigma}$, this is expressed as

$$(2.12) \qquad (h\sigma_1 h^*, h\sigma_2 h^*, h\sigma_3 h^*, h\sigma_4 h^*) = (\sigma_1, \sigma_2, \sigma_3, \sigma_4) g = \boldsymbol{\sigma} g \,.$$

Since $\det\big(h\boldsymbol{X}h^*\big) = \det \boldsymbol{X}$, the element g belongs to $O(3,1)$ and moreover to $SO_0(3,1)$. Thus we get a holomorphic map $\Phi : H \ni h \mapsto g \in G$. Put $A_4 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) = -A_3 \in \mathfrak{sl}(2, \boldsymbol{C})$ and consider a one-parameter subgroup $e^{tA_4} \in SL(2, \boldsymbol{C})$, then $\Phi(e^{tA_4}) = g_4(2t) \in SO_0(3,1)$, where

$$e^{tA_4} = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}, \qquad g_4(t) := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cosh t & \sinh t \\ 0 & 0 & \sinh t & \cosh t \end{pmatrix}.$$

From this, together with $\Phi\big(e^{tA_j}\big) = g_j(2t)$ $(1 \le j \le 3)$ for $SU(2) \to SO(3)$ $\big($cf. (1.7)$\big)$, we see that Φ is actually surjective, and we have

$$1 \longrightarrow \{\pm E_2\} \longrightarrow SL(2,\boldsymbol{C}) \xrightarrow{\Phi} SO_0(3,1) \longrightarrow 1 \quad \text{(exact)} \tag{2.13}$$

Through a section Ψ of this covering map Φ, an irreducible representation (= IR) π of $H = SL(2,\boldsymbol{C})$ is a spin (projective) representation of Lorentz group $G = SO_0(3,1)$ if $\pi(-E_2) = -I$, and is a (non-spin) linear representation of G if $\pi(-E_2) = I$.

Consider H as a complex Lie group. Then $\pi_{1,0}(h) := h$ $(h \in H)$ is a holomorphic IR, and $\pi_{0,1}(h) := \overline{h}$ (complex conjugate) is an anti-holomorphic IR. The tensor product $\pi_{p,q} := (\otimes^p \pi_{1,0}) \otimes (\otimes^q \pi_{0,1})$ is spin or non-spin according as $p+q$ is odd or even. Note that $\pi_{0,1}$ is unitary equivalent to the representation $\pi'_{0,1}(h) := (h^*)^{-1}$ $(h \in H)$, since $(h^*)^{-1} = {}^t\overline{h}^{-1} = w\,\overline{h}\,w^{-1}$ $(h \in H)$ with $w = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. We define a representation $\varpi' \cong \pi_{1,0} \oplus \pi_{0,1}$ by putting

$$\varpi'(h) := \pi_{1,0}(h) \oplus \pi'_{0,1}(h) = \begin{pmatrix} h & 0_2 \\ 0_2 & (h^*)^{-1} \end{pmatrix} \quad \big(h \in H = SL(2,\boldsymbol{C})\big).$$

Then, on the subgroup $SU(2)$, we have $\varpi'(u) = u \oplus u$ $\big(u \in SU(2)\big)$.

Identification of the representation ϖ coming from $g \mapsto S_g$: Consider a bijective map from $V = \boldsymbol{C}^4$ to $V' := V(\varpi') = \boldsymbol{C}^4$ given as

$$M_U : \; V \ni v \longrightarrow v' = Uv \in V', \qquad U = \frac{1}{\sqrt{2}} \begin{pmatrix} E_2 & -E_2 \\ E_2 & E_2 \end{pmatrix},$$

and $\widetilde{M_U} : \; \psi \mapsto \phi$ from a V-valued function $\psi(\boldsymbol{x}) = \big(\psi_j(\boldsymbol{x})\big)_{1 \le j \le 4}$ on M^4 to a V'-valued function $\phi(\boldsymbol{x}) := U\big(\psi(\boldsymbol{x})\big)$. Then the Dirac equation on ψ is transformed to that on ϕ as follows:

$$\begin{aligned} &\big(D^U + \kappa\big)\phi = 0, \quad \text{with} \quad D^U := UDU^{-1} = \boldsymbol{\Gamma} \cdot {}^t\boldsymbol{\partial}\,, \\ &\boldsymbol{\Gamma} := (\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4), \;\; \Gamma_j := U\gamma_j U^{-1} \; (1 \le j \le 4), \end{aligned} \tag{2.14}$$

where Γ_j are calculated as

$$\Gamma_j = \begin{pmatrix} 0_2 & -i\sigma_j \\ i\sigma_j & 0_2 \end{pmatrix} \;\; (1 \le j \le 3), \quad \Gamma_4 = \begin{pmatrix} 0_2 & -i\sigma_4 \\ -i\sigma_4 & 0_2 \end{pmatrix}.$$

Decompose $\phi = (\phi_j)_{1 \le j \le 4}$ into two components as $\phi = \boldsymbol{\phi}_+ \oplus \boldsymbol{\phi}_-$ with $\boldsymbol{\phi}_+ := {}^t(\phi_1, \phi_2)$, $\boldsymbol{\phi}_- := {}^t(\phi_3, \phi_4)$. Then, for $h \in H$, with $g = \Phi(h) \in G$,

$$T_{\varpi'}(h)\big(\boldsymbol{\phi}_+ \oplus \boldsymbol{\phi}_-\big)(\boldsymbol{x}) \;=\; h\,\boldsymbol{\phi}_+(g^{-1}\boldsymbol{x}) \oplus (h^*)^{-1}\boldsymbol{\phi}_-(g^{-1}\boldsymbol{x}), \tag{2.15}$$

and Dirac equation takes the form

$$\left\{ \begin{aligned} -i\big(\sigma_1\partial_{x_1} + \sigma_2\partial_{x_2} + \sigma_3\partial_{x_3} + \sigma_4\partial_{x_4}\big)\boldsymbol{\phi}_- \;+\; \kappa\boldsymbol{\phi}_+ \;&=\; 0, \\ i\big(\sigma_1\partial_{x_1} + \sigma_2\partial_{x_2} + \sigma_3\partial_{x_3} - \sigma_4\partial_{x_4}\big)\boldsymbol{\phi}_+ \;+\; \kappa\boldsymbol{\phi}_- \;&=\; 0. \end{aligned} \right. \tag{2.16}$$

Under the variable change $\boldsymbol{x}' = g^{-1}\boldsymbol{x}$, we have $\boldsymbol{\partial} = \boldsymbol{\partial}' g^{-1}$ as in (2.10), and so $D^U = \boldsymbol{\Gamma}' \cdot {}^t\boldsymbol{\partial}'$ with $\boldsymbol{\Gamma}' = \boldsymbol{\Gamma}\, {}^t g^{-1}$. Moreover, from the simple forms of Γ_j's, we see easily that for $h \in H$,

$$\varpi'(h)^{-1}\Gamma_j\varpi'(h) = \begin{pmatrix} 0_2 & -ih^{-1}\sigma_j(h^*)^{-1} \\ ih^*\sigma_j h & 0_2 \end{pmatrix} \quad (1 \le j \le 3),$$
$$\varpi'(h)^{-1}\Gamma_4\varpi'(h) = \begin{pmatrix} 0_2 & -ih^{-1}\sigma_4(h^*)^{-1} \\ -ih^*\sigma_4 h & 0_2 \end{pmatrix}.$$

Here we apply the formula (2.12) and the following one, obtained from (2.12) by multiplying $J_{3,1}$ from the right to the both sides and using $gJ_{3,1} = J_{3,1}\, {}^t g^{-1}$:

$$(h\sigma_1 h^*, h\sigma_2 h^*, h\sigma_3 h^*, -h\sigma_4 h^*) = (\sigma_1, \sigma_2, \sigma_3, -\sigma_4)\, {}^t g^{-1}. \tag{2.17}$$

Then we get

$$\big(\varpi'(h)^{-1}\Gamma_j\varpi'(h)\big)_{1\le j\le 4} = \boldsymbol{\Gamma}\, {}^t g^{-1} = \boldsymbol{\Gamma}'. \tag{2.18}$$

Theorem 2.2. *The map $g \mapsto S_g$ gives a spin representation ϖ of Lorentz group $G = SO_0(3,1)$ which is equivalent to $\varpi' = \pi_{1,0} \oplus \pi'_{1,0} \cong \pi_{1,0} \oplus \pi_{1,0}$.*

Under the isomorphic map M_U from $V = \boldsymbol{C}^4$ onto $V' = V(\varpi')$, transform a V-valued wave function ψ to V'-valued one $\phi(\boldsymbol{x}) = U\big(\psi(\boldsymbol{x})\big)$. Then Dirac equation for ψ is transformed to

$$(D^U + \kappa)\phi = 0, \qquad D^U = \boldsymbol{\Gamma} \cdot {}^t\boldsymbol{\partial} = \sum_{1\le j\le 4} \Gamma_j \frac{\partial}{\partial x_j},$$

and this is Lorentz-invariant in the sense that the operator $D^U + \kappa$ is invariant under the transform $T_{\varpi'}(h)\phi(\boldsymbol{x}) = \varpi'(h)\big(\phi(g^{-1}\boldsymbol{x})\big)$ $(\boldsymbol{x} \in M^4)$ for $h \in H = SL(2, \boldsymbol{C})$ and $g = \Phi(h) \in G = SO_0(3,1)$.

Note 2.1. For this kind of schemes, where spin (projective) representations of a group G are naturally born, see also **E-5** and **E-8** ∼ **E-10** below.

Remark 2.1. The term 'Spinor' was probably coined by P. Ehrenfest, as is seen in the introduction of van der Waerden's paper [Waer, 1929], quoted below, 1st – 9th lines of its introduction:

> „Nennen wir die neuartigen Gößen, die neben den Vektoren und Tensoren in der Quantenmechanik des Spinning Electron aufgetreten sind, und die sich bei der Lorentzgruppe ganz anders transformieren wie Tensoren, kurz Spinoren. Gibt es keine Spinoranalyse, die jeder Physiker lernen kann wie Tensoranalyse, und mit deren Hilfe man erstens alle möglichen Spinoren, zweitens alle invarianten Gleichungen, in denen Spinoren auftreten, bilden kann ?“ So fragte mich Herr Ehrenfest, und die Antwort soll im folgenden gegeben werden.

3 Sources where projective representations occur (1)

E-3. Ray representations of groups. In 1927, J. von Neumann, gave a rigorous mathematical foundation of quantum mechanics, and H. Weyl pursued it as seen in [Wey1]. He observed that, for a wave function ψ and any fixed number $\lambda \in \boldsymbol{C}$, $|\lambda| = 1$, ψ and $\lambda\psi$ describe identical quantum states, so one should employ *ray* (or projective) representations rather than ordinary linear representations. We quote from [Wey1] as

> (Quotation from [Wey1], §16)
>
> In der Quantentheorie finden die Darstellungen im Systemraum statt; dieser ist aber nicht als Vektor-, sondern als Strahlenkörper zu verstehen, weil der einzelne reine Fall nicht durch den Vektor, sondern durch den Strahl repräsentiert wird. Zwei unitäre Transformationen U und εU, die sich um einen Zahlfaktor ε vom absoluten Betrage **1** unterscheiden, sind hier einander gleich, $U \simeq \varepsilon U$, da sie dieselbe Drehung im Strahlenkörper bewirken. In der „Strahldarstellung", welche jedem Element s der abstrakten Gruppe $\mathfrak{g}$ eine unitäre Drehung $U(s)$ im Strahlenkörper des n-dimensionalen Darstellungsraumes entsprechen läßt, werde der Eichfaktor ε für jede unitäre Matrix $U(s)$ in willkürlicher Weise festgelekt; doch so, daß $U(s)$ stetig von s abhängt, wenn $\mathfrak{g}$ eine kontinuierliche Gruppe ist. Die Darstellungsbedingung verlangt jetzt nur
>
> $$U(s)U(t) \simeq U(st)$$
>
> d. i.
>
> $$U(s)U(t) = \delta(s,t)U(st),$$
>
> $\delta(s,t)$ ist ein von s und t abhängiger Zahlfaktor vom absoluten Betrage **1**. $\cdots\cdots\cdots\cdots\cdots\cdots\cdots$
>
> (English translation by H.P. Robertson):
>
> In quantum theory the representations take place in system space; but this is to be considered as a ray rather than a vector space, for a pure state is represented by a ray rather than a vector. Two unitary representations U and εU which differ only by a numerical factor ε of absolute magnitude 1 are consequently to be considered as the same, $U \simeq \varepsilon U$, for they determine the same rotation of the ray field. In a "*ray representation*" $\cdots\cdots\cdots\cdots\cdots\cdots\cdots$

Before publishing this book, he has worked out, in 1925–'26, calculation of characters of all irreducible representations of classical groups, containing spin ones for orthogonal groups. These results were collected together in another book [Wey2]. However at this stage all representations considered are still finite-dimensional.

E-4. Projective representations of finite abelian groups.
Schur studies in [Sch2, 1907], the so-called Schur multiplier $\mathfrak{M} := H^2(G, \boldsymbol{C}^\times)$ of a finite group G, and also the number of non-isomorphic representation groups. In §4 [loc. cit.] he determined explicitly $H^2(G, \boldsymbol{C}^\times)$ for several interesting examples. Here we comment only for the case of abelian groups.

(S3) *Let G be a direct product of cyclic groups of orders $m_1, m_2, \ldots, m_r$. Then $H^2(G, \boldsymbol{C}^\times)$ is a direct product of $r(r-1)/2$ number of cyclic groups of orders (m_j, m_k) $(j < k)$, where (m_j, m_k) denotes GCD of m_j and m_k.*

In fact, first we present the group G as an abstract group by giving

$$\begin{array}{rcl} \text{set of generators} & : & \{y_1, y_2, \ldots, y_r\}, \\ \text{set of fundamental equations} & : & \begin{cases} y_j^{\,m_j} = e & (1 \le j \le r), \\ y_j y_k = y_k y_j & (j \ne k). \end{cases} \end{array}$$

Let π be a projective representation of G and put $S_j = \pi(y_j)$. Then, as a reflection of fundamental relations, we have, with $\lambda_j, \lambda_{jk} \in \boldsymbol{C}^\times$,

$$S_j^{\,m_j} = \lambda_j I \ (1 \le j \le r), \quad S_j S_k = \lambda_{jk} S_k S_j \ (j \ne k).$$

First we have $\lambda_{kj} = \lambda_{jk}^{\,-1}$. Replace each S_j by a scalar multiple $S_j' = \mu_j S_j$. Then the corresponding constants λ_j', λ_{jk}' are given as

$$\lambda_j' = \lambda_j / \mu_j^{\,m_j} \ (1 \le j \le r), \quad \lambda_{jk}' = \lambda_{kj} \ (j \ne k).$$

Hence we may assume that $\lambda_j = 1$ $(1 \le j \le r)$ from the beginning. Then $S_j = S_j (S_k)^{m_k} = \lambda_{jk}^{\,m_k} (S_k)^{m_k} S_j = \lambda_{jk}^{\,m_k} S_j$, whence $\lambda_{jk}^{\,m_k} = 1$ $(j \ne k)$. Therefore $\lambda_{jk}^{\,m_k} = \lambda_{jk}^{\,m_j} = 1$ $(j \ne k)$ and so

$$\lambda_{jk}^{\,(m_j, m_k)} = 1 \qquad (j \ne k). \tag{3.1}$$

We see that no further normalization of the set of constants $\boldsymbol{\lambda} := \{\lambda_{jk}\,;\, 1 \le j, k \le r,\ j \ne k\}$ is possible. This means that, if $\boldsymbol{\lambda} \ne \boldsymbol{\lambda}'$, the factor sets corresponding to them (if exist) are not mutually equivalent.

Conversely, for a given $\boldsymbol{\lambda}$ satisfying (3.1), to prove the existence of a factor set corresponding to it, we can check that the cocycle condition (1.2) is satisfied. Or, it is enough to give explicitly a projective representation π corresponding to $\boldsymbol{\lambda}$.

Example 3.1. Direct product of cyclic groups of the same order.
Let $\boldsymbol{Z}_m$ be a cyclic group of order m and denote the product by multiplication. We treat in Part I in [II] the case of the group $D_n := T^n$, the n-times direct product of $T = \boldsymbol{Z}_m$. The Schur multiplier of D_n is $n(n-1)/2$-times direct product of $\boldsymbol{Z}_m$, and so a representation group, denoted by $R(D_n)$ here, is of order $m^{n(n+1)/2}$, very big.

In Part I and the subsequents in [II], D_n appears as a normal subgroup of a generalized symmetric group $G(m,1,n) = D_n \rtimes \mathfrak{S}_n$ (cf. §4 or §2 for definition). In a representation group $R\big(G(m,1,n)\big)$ of $G(m,1,n)$, there appears only the double covering group $\widetilde{D}_n$ which is a quotient group of $R(D_n)$. In case m is odd, the order of $R(D_n)$ is also odd, and so there exists no double cover of D_n. This situation reflects to Theorem 2.2 (case m odd) and Theorem 2.3 (case m even) on the structure of the representation group $R\big(G(m,1,n)\big)$.

Remark 3.1. For further account on the representation groups of finite abelian groups and their projective representation, see e.g. [Fruc, 1955], [Mor2, 1973] etc.

E-5. A work of Clifford. A.H. Clifford studied in [Clif, 1937] the restriction $\pi|_N$ of an IR π of a group G onto its normal subgroup N. We restate some of his results in the case of a finite group G, and explain that projective representations of a certain subgroup appear here naturally. Assume that π is defined over any field P and the restriction $\pi|_N$ is reducible.

(C1) *Let $\rho^{(1)}, \rho^{(2)}, \ldots, \rho^{(m)}$ be the complete set of mutually non-equivalent IRs of N appearing in $\pi|_N$. Then, they are all mutually conjugate under G, and has the same dimension (say) n, and each of them appears in $\pi|_N$ with the same multiplicity (say) ℓ. We have $\dim \pi = \ell mn$, and denoting the multiplicity by $[\ell]$,*

$$\pi|_N \cong \sum_{1\le i\le m}^{\oplus} [\ell]\cdot \rho^{(i)}. \tag{3.2}$$

(C2) *Let V_i be the subspace of the representation space V of π on which the irreducible components of $\pi|_N$ are equivalent to $\rho^{(i)}$. Then V is a direct sum of V_i's. Moreover they are permuted bodily among themselves by $\pi(g)$ for any $g \in G$ (in Clifford's terminology, $V_1, V_2, \ldots, V_m$ constitute a system of imprimitivity of π). Put, for $1 \le i \le m$,*

$$H_i := \{g \in G\,;\ \pi(g)V_i = V_i\},$$

and let $\tau^{(i)}$ be the representation of H_i induced from π on the subspace V_i. Then, each $\tau^{(i)}$ is irreducible, and the triplet $(H_i, \tau^{(i)}, V_i)$ is conjugate to $(H_1, \tau^{(1)}, V_1)$ by an element $g_i \in G$ such that $\pi(g_i)V_i = V_1$.

(C3) *For any $1 \le i \le m$, $\tau^{(i)}|_N \cong [\ell]\cdot\rho^{(i)}$ and $\pi \cong \mathrm{Ind}_{H_i}^G \tau^{(i)}$.*

Thus, through (C2) – (C3), the structure of π can be analysed by studying $(H,\tau) := (H_1, \tau^{(1)})$, and we have the following theorem, which will be applied in **E-9** below.

Theorem 3.1 (cf. §§2–3 in [Clif]). *Let N be a normal subgroup of a finite group H. For an irreducible representation τ of H, assume that all irreducible* *component of* $\tau|_N$ *are mutually equivalent, or* $\tau|_N \cong [\ell]\cdot\rho$, *a multiple of an IR* ρ *of* N, *and that the ground-field* P *is algebraically closed.*

(i) *The IR* τ *is equivalent to the tensor product of two matrix irreducible projective representations* C *and* Γ *of* H*:*

$$\tau(h) \cong C(h) \otimes \Gamma(h), \tag{3.3}$$

where $\dim C = \dim\rho,\ \rho(h^{-1}uh) = C(h)^{-1}\rho(u)C(h)\ (h \in H, u \in N),\ \dim\Gamma = \ell.$

(ii) *It can be normalized in such a way that, for* $h \in H$ *and* $u \in N$,

$$C(hu) = C(h)\rho(u),\ C(u) = \rho(u); \quad \Gamma(hu) = \Gamma(h),\ \Gamma(u) = E_\ell.$$

Then $h \mapsto \Gamma(h)$ *is actually a projective representation of* H/N, *and the factor sets associated to* C *and* Γ, *mutually inverse to the other, are actually for the quotient group* H/N.

E-6. Unitary representations of Lorentz groups and so on.

In [Wig, 1936], E. Wigner studied unitary representations of the *inhomogeneous* Lorentz group, starting from the point of view of quantum mechanics. This group is the semidirect product group

$$\mathcal{P}_+^\uparrow := N \rtimes \mathcal{L}_4,\ \mathcal{L}_4 = SO_0(3,1),\ N = \boldsymbol{R}^4, \tag{3.4}$$

where N is the translation group acting on the Minkowski space M^4 as

$$\text{for } \boldsymbol{y} \in N : \qquad M^4 \ni \boldsymbol{x} \longrightarrow \boldsymbol{x} + \boldsymbol{y} \in M^4.$$

Naturally (cf. **E-3**) the representations considered are projective ones as he stated at the end of 1. ORIGIN AND CHARACTERIZATION OF THE PROBLEM as

> We shall endeavor, in the ensuing sections, to determine all the continuous unitary representations up to a factor of the inhomogeneous Lorentz group, i.e., all continuous systems of linear, unitary operators satisfying (3a).[5] $\cdots\cdots\cdots\cdots$

The universal covering group of $\mathcal{P}_+^\uparrow$ is realized as the double cover $\widetilde{\mathcal{P}}_+^\uparrow = N \rtimes \mathcal{S}$ with $\mathcal{S} = SL(2,\boldsymbol{C})$ as the universal covering group of $\mathcal{L}_4$. Wigner's theory to construct irreducible unitary representations (=IURs) of $\widetilde{\mathcal{P}}_+^\uparrow$ is the following, which is a protocol of Lie group version of Theorem 3.1 for finite groups:

(W-1) Take an element $\chi \in \widehat{N}$, the unitary dual of N, and its stationary subgroup in $\mathcal{S}$ as $\mathcal{S}(\chi) := \{s \in \mathcal{S}\,;\, s(\chi) = \chi\}$.

---

[5] The equation (3a) quoted here is the following:

For a Lorentz transformation L, there corresponds a unitary operator $D(L)$, and for two such transformations L_1 and L_2,

$$D(L_1)D(L_2) = \omega\, D(L_1L_2), \tag{3a}$$

where ω is a number of modulus 1 and can depend on L_2 and L_1.

(W-2) Take an IUR π^1 of $\mathcal{S}(\chi)$. Then $(\chi\cdot\pi^1)(\boldsymbol{x},s) := \chi(\boldsymbol{x})\,\pi^1(s)$ is an IUR of $N \rtimes \mathcal{S}(\chi)$.

(W-3) One obtains an IUR of $N \rtimes \mathcal{S}$ by inducing up as $\mathrm{Ind}_{N\rtimes\mathcal{S}(\chi)}^{N\rtimes\mathcal{S}}(\chi\cdot\pi^1)$.

For $N = \boldsymbol{R}^4$, its dual is $\widehat{N} = \boldsymbol{R}^4$, and for $\chi \in \widehat{N}$ there corresponds a unique $\boldsymbol{\xi} = (\xi_j)_{1\le j\le 4} \in \boldsymbol{R}^4$ as $\chi(\boldsymbol{x}) := \exp\big(i\langle \boldsymbol{x},\boldsymbol{\xi}\rangle\big)$ with $\langle \boldsymbol{x},\boldsymbol{\xi}\rangle = \sum_j x_j\xi_j$. Stationary subgroups $\mathcal{S}(\chi)$ in the cases $\boldsymbol{\xi} \ne \boldsymbol{0}$ are isomorphic, modulo the center $\{\pm E_2\}$ of $\mathcal{S}$, to

(1) the 3-dimensional rotation group $SO(3)$,

(2) the motion group of Euclidean plane, or

(3) the low-dimensional Lorentz group $\mathcal{L}_3 := S_0(2,1)$;

and (4) the stationary subgroup in the case $\boldsymbol{\xi} = \boldsymbol{0}$ is $\mathcal{L}_4$ itself.

Note that the universal covering group of $\mathcal{L}_3$ is a covering of infinitely many times, since such one for the maximal compact subgroup $SO(2)$ is $\boldsymbol{R}$, and that the double covering group is given by $SL(2,\boldsymbol{R})$ with the restriction of $\Phi : SL(2,\boldsymbol{C}) \to \mathcal{L}_4$ in (2.13) as the covering map $SL(2,\boldsymbol{R}) \to \mathcal{L}_3$.

Determinations of spin or non-spin IURs of *homogeneous* Lorentz groups $\mathcal{L}_3$ and $\mathcal{L}_4$ in the case (3) and (4) are put aside in [loc. cit.], where $\mathcal{S}(\chi)$ are actually $\mathcal{S}_3 = SL(2,\boldsymbol{R})$ or $\mathcal{S}_4 = SL(2,\boldsymbol{C})$ respectively.

Thus the problems of determining all the IURs of these *homogeneous* Lorentz groups were left open.

After the world war II, in the same year 1947, there appear three papers on IURs of homogeneous Lorentz groups. The one [Bar, 1947] is by V. Bargmann, in which he constructed all the IURs of $\mathcal{S}_3 = SL(2,\boldsymbol{R})$ or all spin or non-spin IURs of the Lorentz group $\mathcal{L}_3$, and studied their properties. In particular, the continuous principal series, the continuous exceptional series, and also the holomorphic and the anti-holomorphic discrete series of IURs were discovered. From Introduction of that paper, we quote

> Plan of the investigation. We shall discuss both single- and double-valued representations and hence deal with the corresponding spinor groups $\mathcal{S}_3$ and $\mathcal{S}_4$ rather than with $\mathcal{L}_3$ and $\mathcal{L}_4$. $\cdots\cdots\cdots\cdots\cdots$

The other [GeNa, 1947] is by I.M. Gelfand and M.A. Naimark, in which all the IURs of $\mathcal{S}_4 = SL(2,\boldsymbol{C})$ (spin or non-spin of $\mathcal{L}_4$), were constructed, and in particular the principal series and the supplementary series of IURs were found. We quote from its Abstract (from the top till the 5th row)

> В работе нахадятся все унитарные неприводимые представления унимодулярной комплексной группы второго порятка, локалъно изоморфной группе Лоренца.
>
> Доказывается, что всякое унитарное представрение разлагается на найденные неприводимые.
>
> (English Translation in [GeNa])

> In this paper all unitary irreducible representations of the unimodular complex group of second order are determined; This group is locally isomorphic to the Lorentz group.
>
> It is proved also that an arbitrary unitary representation can be decomposed into these irreducible representations.

The last one [HC, 1947] is by Harish-Chandra, in which spin IURs and non-spin IURs of Lorentz group $\mathcal{L}_4$ were studied principally by an infinitesimal method. Almost at the end of Introduction, he wrote

> Further, as pointed out by Dirac (1945), it is possible to use expinors to describe the transformation properties of the wave function of a spinning particle. In a theory based on these expinors it is possible to make the charge density positive definite for particles of integral spin or the energy density positive definite for particles of half-integral spin, in contradistinction with the results of the existing theory (see Pauli 1940). This is made possible by the circumstance that infinite unitary representations of the Lorentz group exist for both integral and half-integral spins. $\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$

E-7. Projective representations of locally compact groups.
In 1958, G.W. Mackey begun to study unitary ray (or projective) representations [Mac], after finishing his fundamental works on unitary representations of locally compact groups, in particular on induced representations. Apart from mathematical interest, another important reason why ray representations should be studied comes from quantum electrodynamics. He laid the foundation of the general theory of projective representations for locally compact groups by giving several fundamental results, and gave some interesting examples. He also studied in particular multipliers, which correspond to factor sets in the case of finite groups.

4 Sources where projective representations occur (2)

E-8. Action of a group and related intertwining operators.
Suppose that a group S acts on a group G as its automorphisms, that is, for $\sigma \in S$ there is assigned an automorphism $G \ni g \mapsto \sigma(g) \in G$ of G satisfying $(\sigma\tau)(g) = \sigma\big(\tau(g)\big)$ $(\sigma, \tau \in S)$. Take a unitary representation π of G on a Hilbert space $\mathcal{H}$. Then S acts naturally on π as $\big({}^\sigma\pi\big)(g) := \pi\big(\sigma^{-1}(g)\big)$. Denote by $[\pi]$ the equivalence class of π and put

$$(4.1) \qquad S([\pi]) := \big\{\, \sigma \in S\,;\ \big[{}^\sigma\pi\big] = [\pi] \ \text{or}\ {}^\sigma\pi \cong \pi \,\big\}.$$

For $\sigma \in S([\pi])$, take an intertwining operator $J_\pi(\sigma)$ such that

$$J_\pi(\sigma)^{-1}\pi(g)J_\pi(\sigma) = \big({}^\sigma\pi\big)(g) \quad (g \in G).$$

Then $J_\pi(\sigma)$ is determined modulo non-zero scalar multiples, and $J_\pi(\tau)J_\pi(\sigma) = \lambda_{\tau,\sigma}\, J_\pi(\tau\sigma)$ $(\tau,\sigma \in S)$, $\lambda_{\tau,\sigma} \in \boldsymbol{C}^\times$, and so $S([\pi]) \ni \sigma \mapsto J_\pi(\sigma)$ gives a projective representation.

This situation is quite general, and here is one of natural sources where projective representations are born. For instance, we may assume that a group S acts directly on representations of G, not necessarily on G itself. Or, replacing a group algebra by an associative algebra, we may assume more generally a group S acts on an algebra $\mathcal{A}$. Taking a linear representation π of $\mathcal{A}$, we can discuss quite parallel as above.

From another point of view, at first, take a linear algebra $\mathcal{B}$ over $\boldsymbol{C}$, say the one generated by $\pi(G) = \{\pi(g); g \in G\}$ or $\pi(\mathcal{A})$, or else. Then consider the group $\mathcal{J}_\pi$ of all linear invertible operators J such that $b \mapsto JbJ^{-1}$ $(b \in \mathcal{B})$ leaves $\mathcal{B}$ invariant (thus giving an automorphism of $\mathcal{B}$). Then $\mathcal{J}_\pi$ has a property that $\lambda J \in \mathcal{J}$ for any $J \in \mathcal{J}_\pi$ and $\lambda \in \boldsymbol{C}^\times$. Now, *take a group* $\widetilde{S}$ *inside* $\mathcal{J}_\pi$ *and a subgroup* Z *contained in* $\widetilde{S} \cap \{\lambda I\,;\, \lambda \in \boldsymbol{C}^\times\}$*, and put* $S = \widetilde{S}/Z$. Then, naturally $\widetilde{S}$ is a covering group (or a central extension) of S as

$$1 \longrightarrow Z \longrightarrow \widetilde{S} \xrightarrow{\Phi} S \longrightarrow 1 \quad \text{(exact)}.$$

Then any section $\Psi : S \to \widetilde{S}$ gives a projective representation of S.

The above mentioned two cases, linear representations of a group G or of an algebra $\mathcal{A}$, on which S acts, suggest how to find such a system.

For similar situations, we refer **E-9** below. For more explicit examples, see Section 3 below where the algebra $\mathcal{A}_{2^k} = M(2^k, \boldsymbol{C})$ of all matrices of special degree 2^k and its several hidden symmetries are treated. This topic has very intimate relations with the contents in the main body of this paper.

E-9. Construction of IRs of semidirect product groups.

For simplicity, let G be a finite group and semidirect product of a normal subgroup U and a subgroup S as $G = U \rtimes S$, where the action of $s \in S$ on $u \in U$ is denoted by $s(u)$. Take an IR ρ of U and consider its equivalence class $[\rho]$. Take a stationary subgroup $S([\rho])$ of $[\rho]$ in S, and put $H := U \rtimes S([\rho])$. For $s \in S([\rho])$, we determine explicitly an intertwining operator $J_\rho(s)$:

$$\rho\big(s(u)\big) = J_\rho(s)\, \rho(u)\, J_\rho(s)^{-1} \quad (u \in U). \tag{4.2}$$

Then it is determined up to a non-zero scalar factor. Thus we have a projective representation $S([\rho]) \ni s \mapsto J_\rho(s)$. Let $\alpha_{s,t}$ be its factor set given as

$$J_\rho(s)J_\rho(t) = \alpha_{s,t}\, J_\rho(st) \quad \big(s,t \in S([\rho])\big).$$

By the result of Schur [Sch1], we know that, by going up to an appropriate quotient of a representation group $S([\rho])^\sim$ of $S([\rho])$, denoted as

$$\{e\} \longrightarrow Z' \longrightarrow S([\rho])^\sim \xrightarrow{\Phi_S} S([\rho]) \longrightarrow \{e\} \quad \text{(exact)},$$

J_ρ can be linearized to a usual representation $\widetilde{J}_\rho$ of $S([\rho])^\sim$. Here Φ_S denotes the canonical homomorphism. Put, for $\widetilde{H} := U \rtimes S([\rho])^\sim$ with $s'(u) := s(u)$, $s' \in S([\rho])^\sim$, $s = \Phi_S(s')$,

$$\pi^0\big((u, s')\big) = \rho(u) \cdot \widetilde{J}_\rho(s') \quad \big(u \in U,\, s' \in S([\rho])^\sim\big),$$

then $\pi^0 = \rho \cdot \widetilde{J}_\rho$ is an IR of $\widetilde{H}$. Take an IR π^1 of $S([\rho])^\sim$ and consider it as a representation of $\widetilde{H}$ through the homomorphism $\widetilde{H} \to S([\rho])^\sim \cong \widetilde{H}/U$, and consider inner tensor product $\pi := \pi^0 \boxdot \pi^1$ as an IR of $\widetilde{H}$. Let the factor set of π^1, viewed as a spin representation of the base group $S([\rho])$, be $\beta_{s,t}$, then that of π is $\beta_{s,t}\,\alpha_{s,t}$.

To get an IR of G, we should restrict ourselves to pick up π^1 with the factor set $\alpha_{s,t}^{-1}$, the inverse of $\alpha_{s,t}$. Then π is nothing but a linear representation of $H = U \rtimes S([\rho])$. Then we obtain an IR $\Pi(\pi^0, \pi^1)$ of G by inducing it up as

$$\Pi(\pi^0, \pi^1) := \mathrm{Ind}_H^G\, \pi = \mathrm{Ind}_H^G\, \pi^0 \boxdot \pi^1. \tag{4.3}$$

By applying the results of Clifford in **E-5**, in particular, Theorem 3.1, we obtain the following theorem, and we call the method given here *classical induced representation method* (of constructing all IRs) .

Theorem 4.1. *Let G be a finite semidirect product group as $G = U \rtimes S$. Then, the induced representation $\Pi(\pi^0, \pi^1)$ constructed above is irreducible, and conversely any IR of G is equivalent to $\Pi(\pi^0, \pi^1)$ for a certain (π^0, π^1).*

Moreover two IRs $\Pi(\pi^0, \pi^1)$ and $\Pi(\pi'^0, \pi'^1)$ are mutually equivalent if and only if the pairs (π^0, π^1) and (π'^0, π'^1) are mutually equivalent under conjugation of an element in G.

In the paper [II], we study, with Theorem 4.1 as a background, the cases of covering groups (quotients of a representation group) $\widetilde{G}(m, 1, n)$ of the generalized symmetric group $G(m, 1, n) = D(m, n) \rtimes \mathfrak{S}_n$, where $D(m, n) = \boldsymbol{Z}_m^n$ is the n-times direct product of the cyclic group $\boldsymbol{Z}_m$, and $\mathfrak{S}_n$ is the n-th symmetric group, with the action of $\sigma \in \mathfrak{S}_n$ by permuting components of $d = (t_1, t_2, \ldots, t_n) \in \boldsymbol{Z}_m^n$.

The setting in these cases can be translated in the above general scheme by taking G to $\widetilde{G}(m, 1, n)$, U to $\widetilde{D}(m, n)$ a certain covering group of $D(m, n)$, and S to $\mathfrak{S}_n$ or to a double covering $\widetilde{\mathfrak{S}}_n$ of $\mathfrak{S}_n$.

E-10. Projective representations of finite groups.

For our present interest, spin representations of finite groups, after a long vacancy from the ages of Schur, in 1962, Morris begun with recapturing from another point of view Schur's results on the spin theory for symmetric groups by noting that it is a subgroup of $\mathrm{Pin}_+(n)$ or of $\mathrm{Pin}_-(n)$. Since then, many followers of Morris and then Hoffmann, Humphreys, Kleshchev and Nazarov and so on are working heavily on this interesting subject.

5 Weil representations of symplectic groups

In this section, we explain about the famous Weil representations from the framework of **E-8** above. A. Weil treated in his important work [Weil, 1964] symplectic groups over a locally compact abelian group, auto-dual in the sense of Pontrjagin, but here we restrict ourselves to the case of the real field $\boldsymbol{R}$.

5.1 $Sp(2n, \boldsymbol{R})$ acts on Heisenberg group H_n

The $(2n+1)$-dimensional Heisenberg group H_n or $H(V)$ is defined as a central extension of $2n$-dimensional real symplectic vector space (V, ω), where ω is a non-degenerate skew symmetric bilinear form on V :

$$0 \longrightarrow Z = \boldsymbol{R} \longrightarrow H(V) \longrightarrow V \longrightarrow 0 \quad \text{(exact)}.$$

Here Heisenberg group $H(V)$ on (V, ω) is $V \times \boldsymbol{R}$ endowed with the group law

$$(v_1, t_1) \cdot (v_2, t_2) = \big(v_1 + v_2\,,\ t_1 + t_2 + \frac{1}{2}\,\omega(v_1, v_2)\big), \tag{5.1}$$

where $V = \{v = (x, y)\ ;\ x, y \in \boldsymbol{R}^n\} = \boldsymbol{R}^{2n}$, and for $v = (x, y),\ v' = (x', y')$,

$$\omega(v, v') = \langle x, y'\rangle - \langle x', y\rangle = {}^t v\, Q_n v', \quad Q_n := \begin{pmatrix} 0_n & E_n \\ -E_n & 0_n \end{pmatrix},$$

where $\langle x, y\rangle := \sum_{1\le i\le n} x_i y_i$ denotes the bilinear form on $\boldsymbol{R}^n \times \boldsymbol{R}^n$.

The group $Sp(2n, \boldsymbol{R})$ acts on H_n as $g\cdot(v,t) := (gv, t)$ $\big((v,t) \in H_n,\ g \in Sp(2n, \boldsymbol{R})\big)$, since $Sp(2n, \boldsymbol{R})$ is nothing but the orthogonal group for the indefinite quadratic form Q_n : ${}^t g\, Q_n g = Q_n$, and so $\omega(gv, gv') = \omega(v, v')$. Thus we have a special case of the general situation in **E-8**. A basis of the Lie algebra $\mathfrak{h}_n := \mathrm{Lie}(H_n)$ is given in such a manner that

$$\begin{cases} \{Z_0,\ I_1, I_2, \ldots, I_n\ ;\ J_1, J_2, \ldots, J_n\}, \quad Z_0 \ \text{central}, \\ [I_i, I_j] = 0, \quad [J_i, J_j] = 0, \quad [I_i, J_j] = \delta_{ij}\, Z_0\,. \end{cases} \tag{5.2}$$

An element $h(x, y\,; z) \in H_n$ is given as

$$h(x, y; z) := \exp\big(x \cdot \boldsymbol{I} + y \cdot \boldsymbol{J} + z Z_0\big)\,, \tag{5.3}$$

where $\boldsymbol{I} = (I_1, \ldots, I_n), \boldsymbol{J} = (J_1, \ldots, J_n)$ and $x \cdot \boldsymbol{I} := \sum_{1\le i\le n} x_i I_i$ etc. We denote $h(x, y; z)$ simply by $(x, y; z)$. The group H_n has one-dimensional center $Z = \boldsymbol{R}\, Z_0$.

It is known that H_n has a series of infinite-dimensional unitary IRs determined uniquely by their central characters. Let us review it shortly. Define subgroups of H_n as

$$X := \{\, (x, 0\,; 0)\ ;\ x \in \boldsymbol{R}^n\}, \qquad Y := \{\, (0, y\,; 0)\ ;\ y \in \boldsymbol{R}^n\},$$
$$N_n := \{\, (x, 0\,; z)\ ;\ x \in \boldsymbol{R}^n,\ z \in \boldsymbol{R}\,\}.$$

Then N_n is abelian and normal in H_n, and H_n is a semidirect product as $H_n \cong N_n \rtimes Y$ and the action of Y on N_n is given by

$$(0, y\,;0)\,(x, 0\,; z)\,(0, y\,;0)^{-1} = (x, 0\,;\; z - \langle x, y\rangle).$$

We can work in a similar framework as in **E-9**. The dual group $\widehat{N_n}$ consists of characters $\psi_{\xi,\zeta}\,:\,(x, 0\,;z) \mapsto \psi_\xi(x)\psi_\zeta(z)$, where $\psi_\xi(x) := e^{i\,{}^t\xi x}$, $\psi_\zeta(z) := e^{i\zeta z}$, for $(\xi\,;\zeta) \in \boldsymbol{R}^n \times \boldsymbol{R}$. The action of $(0, y\,;0) \in Y$ on $\psi_{\xi,\zeta} \in \widehat{N_n}$ is defined as $(x, 0\,;z) \mapsto \psi_{\xi,\zeta}\big((0, y\,;0)^{-1}(x, 0\,;z)\,(0, y\,;0)\big)$ and is described by

$$(\xi\,;\zeta) \longrightarrow (\xi + \zeta y\,;\,\zeta). \tag{5.4}$$

The orbits under this action of Y are

(1) for any $\zeta \neq 0$, $\mathcal{O}_\zeta := \{(\xi\,;\zeta); \xi \in \boldsymbol{R}^n\}$, and the stationary subgroup of $(0\,;\zeta) \in \mathcal{O}_\zeta$ in Y is trivial ;

(2) for $\zeta = 0$ and $\xi \in \boldsymbol{R}^n$, $\mathcal{O}_{\xi;0} := \{(\xi\,;0)\}$ (one point set), and its stationary subgroup in Y is Y itself.

Theorem 5.1. *The IR π_ζ associated to $(0\,;\zeta) \in \mathcal{O}_\zeta$ is realized as an induced representation on the space $L^2(Y)$ as $\pi_\zeta = \mathrm{Ind}_{N_n}^{H_n}\psi_{0,\zeta}$: for $y \in Y$,*

$$\begin{cases} \pi_\zeta\big((x_0, 0\,;0)\big)f(y) = \psi_\zeta(-\langle x_0, y\rangle)f(y) & \big((x_0, 0\,;0) \in X\big), \\ \pi_\zeta\big((0, y_0\,;0)\big)f(y) = f(y + y_0) & \big((0, y_0\,;0) \in Y\big), \\ \pi_\zeta\big((0, 0\,;z_0)\big) = \psi_\zeta(z_0)\,I & \big((0, 0\,;z_0) \in Z\big), \end{cases}$$

where I denotes the identity operator.

We explain how to construct π_ζ. Consider the space of continuous functions φ on H_n which have the homogeneity with respect to N_n as $\varphi(\xi h) = \psi_{(0,\zeta)}(\xi)\varphi(h)$ $(\xi \in N_n,\, h \in H_n)$, and action of $h_0 \in H_n$ on φ is defined by right translation as $\varphi(hh_0)$. By its homogeneity, φ is determined by its restriction on $Y \cong N_n\backslash H_n$, which is denoted by φ'. The space of Schwartz functions φ' is completed to get the Hilbert space $\mathcal{H} = L^2(Y)$ with respect to the measure $d\nu(y) := dy_1dy_2\cdots dy_n$, $y = (y_1, y_2, \cdots, y_n) \in Y$, where $(0, y; 0) \in Y$ are denoted by y for simplicity. The action of h_0 on φ' is given by taking into account the transformation of the measure $d\nu$ as follows: for $y \in Y$, decompose yh_0 as $yh_0 = \xi'y'$ $(\xi' \in N_n,\, y' \in Y)$ and denote y' by $y\overline{h_0}$, then

$$\big(\pi_\zeta(h_0)\varphi'\big)(y) := \Big(\frac{d\nu(y\overline{h_0}\,)}{d\nu(y)}\Big)^{1/2}\,\psi_{(0,\zeta)}(\xi')\,\varphi'\big(y\overline{h_0}\,\big) = \psi_{(0,\zeta)}(\xi')\,\varphi'\big(y\overline{h_0}\,\big).$$

For example, to get the formula for $\pi_\zeta\big((x_0, 0\,;0)\big)$, we calculate as

$$(0, y; 0)\,(x_0, 0\,;0) = (x_0, y; -\tfrac{1}{2}\langle x_0, y\rangle) = (x_0, 0\,; -\langle x_0, y\rangle)\,(0, y; 0).$$

Any automorphism of H_n, which is trivial on the center Z, fixes the equivalence class of any infinite-dimensional IR π_ζ, $\zeta \neq 0$, because the central character $Z \ni z \mapsto \exp(i\zeta z)$ determines its equivalence class. Since the action of $S = Sp(2n,\boldsymbol{R})$ on H_n leaves the center Z elementwise-invariant, we have $S([\pi_\zeta]) = S$. Then, according to the general scheme given in **E-8**, we obtain, for any infinite dimensional IR π_ζ of H_n, a projective representation $s \mapsto J_{\pi_\zeta}(s)$ of S by $({}^s\pi_\zeta)(h) = J_{\pi_\zeta}(s)^{-1}\pi_\zeta(h)J_{\pi_\zeta}(s)$ or $\pi_\zeta\big(s(h)\big) = J_{\pi_\zeta}(s)\pi_\zeta(h)J_{\pi_\zeta}(s)^{-1}$ $(h \in H_n)$.

Remark 5.1. A matrix expression $h'(x, y\,; z)$ of Heisenberg group H_n is given as follows: For $x, y \in \boldsymbol{R}^n$, $z \in \boldsymbol{R}$, let $h'(x,y\,;z) = \exp X(x,y\,;z)$ with

$$X(x,y\,;z) = \begin{pmatrix} 0 & 0 & 0 \\ y & 0_n & 0 \\ z & {}^t x & 0 \end{pmatrix}, \quad \text{then} \quad h'(x,y\,;z) = \begin{pmatrix} 1 & 0 & 0 \\ y & 1_n & 0 \\ \frac{1}{2}\langle x,y\rangle + z & {}^t x & 1 \end{pmatrix}.$$

5.2 Group of intertwining operators and $Sp(2n,\boldsymbol{R})$

We define certain elements of $G = Sp(2n,\boldsymbol{R})$. Let Sym_n (resp. Sym_{GL_n}) be the set of symmetric matrices in $M(n,\boldsymbol{R})$ $\big($resp. in $GL(n,\boldsymbol{R})\big)$, and put

$$d(\alpha) := \begin{pmatrix} \alpha & 0_n \\ 0_n & {}^t\alpha^{-1} \end{pmatrix}, \ b(\beta) := \begin{pmatrix} E_n & \beta \\ 0_n & E_n \end{pmatrix}, \ c(\gamma) := \begin{pmatrix} E_n & 0_n \\ \gamma & E_n \end{pmatrix}, \tag{5.5}$$

where $\alpha \in GL(n,\boldsymbol{R})$, and $\beta,\ \gamma \in \mathrm{Sym}_n$. Denote by D, B and C the subgroups of G consisting of $d(\alpha)$'s, $b(\beta)$'s and $c(\gamma)$'s respectively.

Lemma 5.2. (i) *An element $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$, with $a,b,c,d \in M(n,\boldsymbol{C})$ is uniquely decomposed as $g = b(\beta)\,d(\alpha)\,w\,b(\beta')$ if c is regular, where $\alpha = -{}^t c^{-1}$, $\beta = ac^{-1}$, $\beta' = c^{-1}d$ and $w := Q_n$.*

(ii) *$G = Sp(2n,\boldsymbol{R})$ is generated by w and the subgroup $B = \{b(\beta)\,;\, \beta \in \mathrm{Sym}_n\}$.*

(iii) *There hold the relations*

$$\begin{cases} w^2 = -E_{2n}\,, \quad b(\beta)b(\beta') = b(\beta+\beta') & (\beta,\beta' \in \mathrm{Sym}_n), \\ d(\alpha) = b(\alpha)\,w\,b(\alpha^{-1})\,w\,b(\alpha)\,w & (\alpha \in \mathrm{Sym}_{GL_n}), \\ d(\alpha)\,b(\beta)\,d(\alpha^{-1}) = b(\alpha\,\beta\,{}^t\alpha) & (\alpha \in GL(n,\boldsymbol{R}),\ \beta \in \mathrm{Sym}_n), \\ d(\alpha)\,d(\alpha') = d(\alpha\alpha'), \ \ w\,d(\alpha)\,w^{-1} = d({}^t\alpha^{-1}) & \big(\alpha,\ \alpha' \in GL(n,\boldsymbol{R})\big). \end{cases}$$

The relations listed here determine the product rule of G completely, by analytic continuation.

In fact, the assertions (i) and (iii) are proved by calculations. Then the assertion (ii) follows from them by the fact that the subset Sym_{GL_n} generates $GL(n,\boldsymbol{R})$ as is easily seen for $n = 2$. For more detailed assertion than the last paragraph in (iii) above, cf. Lemma 6 in [Weil, p.195].

Note moreover the following relations:

$$w\,b(\beta)\,w^{-1} = c(-\beta) \quad \text{for } \ \beta \in \mathrm{Sym}_n, \quad \text{and}$$

$$b(\beta)c(\gamma) = c(\gamma')d(\alpha')b(\beta'), \ \ \beta' := \alpha'^{-1}\beta, \ \ \gamma' := \gamma\alpha'^{-1},$$

if $\alpha' := E_n + \beta\gamma$ is regular for $\beta, \gamma \in \mathrm{Sym}_n$.

Lemma 5.3. (i) *A maximal compact subgroup K of $Sp(2n, \boldsymbol{R})$ is isomorphic to the unitary group $U(n)$, and a natural isomorphism Ψ_K is given as follows: express an element of $U(n)$ as $U = u + iv$ with $u, v \in M(n, \boldsymbol{R})$, then $u\,{}^t u + v\,{}^t v = E_n,\ u\,{}^t v - v\,{}^t u = 0_n$, and*

$$U(n) \ni U = u + iv \xrightarrow{\Psi_K} \begin{pmatrix} u & -v \\ v & u \end{pmatrix} \in K \subset Sp(2n, \boldsymbol{R}).$$

The element $w \in K$ corresponds to $iE_n \in U(n)$: $\Psi_K(iE_n) = w$, and $\Psi_K(-E_n) = w^2 = -E_{2n}$ is the unique non-trivial central element of G.

(ii) *A universal covering group of $U(n)$ is given as*

$$R\big(U(n)\big) := \boldsymbol{R} \times SU(n) \ni (\theta, h) \xrightarrow{\Phi_U} e^{i\theta}\, h \in U(n),$$

where Φ_U denotes the canonical homomorphism. Let $\kappa_0 := (\frac{2\pi}{n}, e^{-\frac{2\pi i}{n}} E_n) \in R\big(U(n)\big)$, then the kernel of Φ_U is $\mathcal{Z} := \langle \kappa_0 \rangle = \{\kappa_0^{\,q} = (\frac{2q\pi}{n}, e^{-\frac{2q\pi i}{n}} E_n)\,;\, q \in \boldsymbol{Z}\}$. For any $p \geq 2$, a p-times covering group $\widetilde{U}^p(n)$ is given by $R\big(U(n)\big)/\mathcal{Z}^p$, $\mathcal{Z}^p = \langle \kappa_0^p \rangle$, which has a section $[0, \frac{2\pi}{n}\, p) \times SU(n) \subset R\big(U(n)\big)$.

Lemma 5.4. *The group $G = Sp(2n, \boldsymbol{R})$ has an Iwasawa decomposition $G = K \times A \times N$, where*

$$A = \big\{a(t)\ ;\ a(t) = \mathrm{diag}(e^{t_1}, \ldots, e^{t_n}, e^{-t_1}, \ldots, e^{-t_n}),\ t = (t_1, t_2, \ldots, t_n) \in \boldsymbol{R}^n\big\}$$

and N is a maximal unipotent subgroup consisting of elements of the form

$$\begin{pmatrix} a & b \\ 0_n & {}^t a^{-1} \end{pmatrix}, \quad a \in GL(n, \boldsymbol{C}),\ b \in M(n, \boldsymbol{C}),$$

where a is unipotent upper triangular matrix. Then a p-times covering group $\widetilde{G}^p$ of G is given by $\widetilde{G}^p = \widetilde{K}^p \times A \times N$, $\widetilde{K}^p \cong \widetilde{U}^p(n)$, with the product rule analytically extended from $G = K \times A \times N$.

We calculate, for any infinite dimensional IR π_ζ of H_n, a projective representation $s \mapsto J_{\pi_\zeta}(s)$ of $S = Sp(2n, \boldsymbol{R})$ by $({}^s\pi_\zeta)(h) = J_{\pi_\zeta}(s)^{-1}\pi_\zeta(h)J_{\pi_\zeta}(s)$ $(h \in H_n)$. Then we obtain the following.

Theorem 5.5. *For elements $s = b(\beta)$, w and $d(\alpha)$ of $Sp(2n, \boldsymbol{R})$ the intertwining operators $J_{\pi_\zeta}(s)$ are given modulo scalar multiples of modulus 1 as*

$$\begin{aligned}
J_{\pi_\zeta}\big(b(\beta)\big)\varphi(y) &= \psi_\zeta(\tfrac{1}{2}\langle \beta y, y \rangle)\varphi(y) \qquad (\beta \in \mathrm{Sym}_n), \\
J_{\pi_\zeta}(w)\varphi(y) &= \mathcal{F}_\zeta \varphi(y)\,, \\
J_{\pi_\zeta}\big(d(\alpha)\big)\varphi(y) &= |\det \alpha|^{-1/2}\ \varphi({}^t\alpha^{-1} y) \qquad \big(\alpha \in GL(n, \boldsymbol{R})\big),
\end{aligned}$$

where $\varphi \in \mathcal{H} := L^2(Y)$, $y \in Y$, *and* $\mathcal{F}_\zeta$ *is a Fourier transform on* $\mathcal{H}$ *given as*

$$(5.6)\qquad \big(\mathcal{F}_\zeta\varphi\big)(x) := \int_Y \varphi(y)\psi_\zeta(\langle x, y\rangle)\, d\mu_\zeta(y) \quad (x \in Y).$$

Here the measure μ_ζ *on* $Y = \boldsymbol{R}^n$ *is chosen in such a way that* $\mathcal{F}_\zeta$ *defines a unitary operator on* $\mathcal{H}$.

Proof. **(1)** To prove the formula for $J_{\pi_\zeta}\big(b(\beta)\big)$, put $\pi' := {}^{b(\beta)}(\pi_\zeta)$ or $\pi'(h) := \pi_\zeta\big(b(\beta)^{-1}(h)\big)$. Since $b(\beta)^{-1}(x_0, y_0; z_0) = (x_0 - \beta y_0, y_0; z_0)$, we have for $y \in Y$,

$$\begin{cases} \pi'\big((x_0, 0\,; 0)\big)\varphi(y) = \psi_\zeta(-\langle x_0, y\rangle)\,\varphi(y), \\ \pi'\big((0, y_0; 0)\big)\varphi(y) = \psi_\zeta(\langle \beta y_0, y + \frac{1}{2}y_0\rangle)\,\varphi(y + y_0), \\ \pi'\big((0, 0\,; z_0)\big)\varphi(y) = \psi_\zeta(z_0)\,\varphi(y), \end{cases}$$

which come from the following calculations respectively:

- $(0, y; 0)\cdot b(\beta)^{-1}(x_0, 0\,; 0) = (0, y; 0)\cdot(x_0, 0\,; 0) = (x_0, y\,; -\frac{1}{2}\langle x_0, y\rangle) =$
  $= (x_0, 0\,;\, -\langle x_0, y\rangle)(0, y, 0)$;
- $(0, y; 0)\cdot b(\beta)^{-1}(0, y_0; 0) = (0, y; 0)\cdot(-\beta y_0, y_0; 0) =$
  $= (-\beta y_0, y + y_0, \frac{1}{2}\langle \beta y_0, y\rangle) = (-\beta y_0, 0\,; \langle \beta y_0, y + \frac{1}{2}y_0\rangle)(0, y + y_0; 0)$.

Then, noting that β is symmetric, we obtain an intertwining operator $J_{\pi_\zeta}\big(b(\beta)\big)$ between π_ζ and $\pi' = {}^{b(\beta)}\pi_\zeta$ as

$$(5.7)\qquad J_{\pi_\zeta}\big(b(\beta)\big)\varphi(y) := \psi_\zeta(\tfrac{1}{2}\langle \beta y, y\rangle)\varphi(y) \qquad (y \in Y).$$

In fact, for $A := J_{\pi_\zeta}\big(b(\beta)\big)$, we solve the equations $\pi'(h) = A^{-1}\,\pi_\zeta(h)\,A$ for $h = (x_0, 0\,; 0)$, $(0, y_0; 0)$, which are generators of H_n. In commutative diagrams, this is displayed for $h = (x_0, 0\,; 0)$ and $h = (0, y_0, 0)$ respectively as

$$\begin{array}{ccc} \varphi(y) & \xrightarrow{\pi'((x_0,0\,;0))} & \psi_\zeta(-\langle x_0, y\rangle)\,\varphi(y) \\ A\downarrow & \circlearrowleft & \uparrow A^{-1} \\ \varphi'(y) & \xrightarrow[\pi_\zeta((x_0,0\,;0))]{} & \psi_\zeta(-\langle x_0, y\rangle)\,\varphi'(y) \end{array}$$

$$\begin{array}{ccc} \varphi(x) & \xrightarrow{\pi'((0,y_0;0))} & \psi_\zeta(\langle \beta y_0, y+\frac{1}{2}y_0\rangle)\varphi(y+y_0) \\ A\downarrow & \circlearrowleft & \uparrow A^{-1} \\ \varphi'(y) & \xrightarrow[\pi_\zeta((0,y_0;0))]{} & \varphi'(y + y_0) \end{array}$$

Here $\varphi'(y) := (A\varphi)(y) = \psi_\zeta(\frac{1}{2}\langle \beta y, y\rangle)\varphi(y)$.

To show the second diagram, the calculation necessary is

$\psi_\zeta(\frac{1}{2}\langle\beta(y+y_0), y+y_0\rangle)/\psi_\zeta(\frac{1}{2}\langle\beta y, y\rangle) = \psi_\zeta\big(\langle\beta y_0, y+\frac{1}{2}y_0\rangle\big)$.

(2) To get the formula for $J_{\pi_\zeta}(w)$, note that $w^{-1}(x_0, y_0; z_0) = (-y_0, x_0; z_0)$, and put $\pi' = {}^w\pi_\zeta$ or $\pi'(h) := \pi_\zeta\big(w^{-1}(h)\big)$. Then,

$$\begin{cases} \pi'\big((x_0, 0\,; 0)\big)\varphi(y) = \varphi(y + x_0) & (x_0 \in X), \\ \pi'\big((0, y_0; 0)\big)\varphi(y) = \psi_\zeta(\langle y_0, y\rangle)\varphi(y) & (y_0 \in Y). \end{cases}$$

Then an intertwining operator $A = J_{\pi_\zeta}(w)$ is obtained from $\pi'(h) = A^{-1}\pi_\zeta(h)A$ $(h \in H_n)$ as $J_{\pi_\zeta}(w) := \lambda_w \mathcal{F}_\zeta$ with a scalar λ_w, $|\lambda_w| = 1$.

(3) For $d(\alpha)$, put $\pi' := {}^{d(\alpha)}\pi_\zeta$ or $\pi'(h) := \pi_\zeta\big(d(\alpha)^{-1}(h)\big)$ $(h \in H_n)$. Since $d(\alpha)^{-1}(x_0, y_0; z_0) = (\alpha^{-1}x_0, {}^t\alpha y_0; z_0)$, we have

$$\begin{cases} \pi'\big((x_0, 0\,; 0)\big)\varphi(y) = \psi_\zeta(-\langle\alpha^{-1}x_0, y\rangle)\,\varphi(y) & (x_0 \in X), \\ \pi'\big((0, y_0; 0)\big)\varphi(y) = \varphi(y + {}^t\alpha y_0) & (y_0 \in Y). \end{cases}$$

As an intertwining operator $J_{\pi_\zeta}\big(d(\alpha)\big)$ between π_ζ and π', we have

$$J_{\pi_\zeta}\big(d(\alpha)\big)\varphi(y) := \nu_\alpha\ |\det\alpha|^{-1/2}\ \varphi({}^t\alpha^{-1}y)\ (y \in Y),$$

with $\nu_\alpha \in \boldsymbol{C}$, $|\nu_\alpha| = 1$. This completes the proof of the theorem. □

Remark 5.2. Since $\psi_\zeta(t) = \psi_1(\zeta t)$ with $\psi_1(t) = e^{it}$ $(t \in \boldsymbol{R})$, the selfdual measure μ_ζ for Fourier transform $\mathcal{F}_\zeta$ is $d\mu_\zeta(y) = (|\zeta|/2\pi)^{n/2}dy_1dy_2\cdots dy_n$, $y = (y_1, y_2, \ldots, y_n) \in Y$. The relation $\mathcal{F}_\zeta$ and the standard Fourier transform $\mathcal{F}_1$ is $\mathcal{F}_\zeta = M_{|\zeta|^{1/2}} \cdot \mathcal{F}_1^{\,\varepsilon} \cdot M_{|\zeta|^{1/2}}^{-1}$, where $\varepsilon = \mathrm{sgn}(\zeta)$ and $M_a\varphi(y) = a^{n/2}\varphi(ay)$ $(y \in Y)$ for $a > 0$.

5.3 Obtained projective representations of $Sp(2n, \boldsymbol{R})$

Now we consider that the operators $J_{\pi_\zeta}(\cdot)$ are acting on the Hilbert space $L^2(W)$ of the abelian group $W = \boldsymbol{R}^n \cong Y$. We should check if the projective representation J_{π_ζ} is really projective. If so, among p-times covering groups for $2 \le p \le \infty$ of $Sp(2n, \boldsymbol{R})$, which one is enough to get a linearization of J_{π_ζ}?

Denote by Φ_2 the canonical homomorphism $\widetilde{G}^2 \to G$. The subgroup B of $G = Sp(2n\boldsymbol{R})$ is simply-connected and is imbedded into the double covering $\widetilde{G}^2 = Mp(2n, \boldsymbol{R}) = \widetilde{K}^2 \times A \times N$ as a subgroup B', then $\Phi_2^{-1}(B) = \{e, z'\} \times B'$ with the non-trivial central element $z' \in \widetilde{K}^2$ such that $\Phi_2(z') = e$. We denote by $b'(\beta) \in B'$ the preimage of $b(\beta) \in B$. According to Lemma 5.3, we have $\Psi_K(iE_n) = w$ and $\det(iE_n) = i^n$, and then take a preimage $w' = (\theta, h_w) \in R\big(U(n)\big)$ of $iE_n \in U(n)$ under Φ_U in the section $\big[0, \frac{2\pi}{n}\cdot 2\big) \times SU(n) \cong \widetilde{K}^2$ with the smallest θ. Then $(w')^8 = e'$, but $(w')^2$ and $(w')^4$ depend on the parity of n modulo 8, and we list up them in Table 5.1 below. Remark that

$$\begin{array}{ccccc} R\big(U(n)\big) = \boldsymbol{R}\times SU(n) & \xrightarrow{\Phi_U} & U(n) & \xrightarrow[\cong]{\Psi_K} & K \subset Sp(2n, \boldsymbol{R}) \\ w' = (\theta, h_w) & \longmapsto & e^{i\theta} h_w = iE_n & \longmapsto & w \end{array}$$

Table 5.1. List of w', $(w')^2$, $(w')^4$, depending on n modulo 8.
Put $h_n := e^{\frac{4\pi i}{n}} E_n \in SU(n)$, then $\kappa_0^2 = (\frac{4\pi}{n}, h_n^{-1})$.

n	$4k$	$8k+1$	$8k+2$	$8k+3$	$8k+5$	$8k+6$	$8k+7$
w'	$(0, iE_n)$	$(\frac{\pi}{2n}, h_n^k)$	$(\frac{\pi}{n}, h_n^k)$	$(\frac{3\pi}{2n}, h_n^k)$	$(\frac{5\pi}{2n}, h_n^k)$	$(\frac{3\pi}{n}, h_n^k)$	$(\frac{7\pi}{2n}, h_n^k)$
$(w')^2$	$(0, -E_n)$	$(\frac{\pi}{n}, h_n^{2k})$	$(\frac{2\pi}{n}, h_n^{2k})$	$(\frac{3\pi}{n}, h_n^{2k})$	$(\frac{\pi}{n}, h_n^{2k+1})$	$(\frac{2\pi}{n}, h_n^{2k+1})$	$(\frac{3\pi}{n}, h_n^{2k+1})$
$(w')^4$	e'	κ_0	e'	κ_0	κ_0	e'	κ_0

The group $\widetilde{G}^2$ is generated by B' and w', since B and w generate G. Put $D' := \Phi_2^{-1}(D)$, and choose a preimage $d'(\alpha) \in D'$ of $d(\alpha) \in D$, for $\alpha \in \mathrm{Sym}_{GL_n}$, according as the 2nd equality in Lemma 5.2 (iii), or $d(\alpha) = b(\alpha)\, w\, b(\alpha^{-1})\, w^{-1} b(\alpha)\, w^{-1}$, in such a way that

$$d'(\alpha) := b'(\alpha)\, w'\, b'(\alpha^{-1})\, w'^{\,-1} b'(\alpha)\, w'^{\,-1} \quad \big(\alpha \in \mathrm{Sym}_{GL_n}\big). \tag{5.8}$$

Then D' is generated by the set of $d'(\alpha)$, $\alpha \in \mathrm{Sym}_{GL_n}$.

The result on the present construction of (possibly) spin representations is the following. We give the symbol $J_\zeta(\cdot)$ to a linearization to $\widetilde{G}^2$ of $J_{\pi_\zeta}(\cdot)$, and denote by $m_+(\alpha)$ $\big($resp. $m_-(\alpha)\big)$ the number of positive (resp. negative) eigenvalues of $\alpha \in \mathrm{Sym}_{GL_n}$.

Theorem 5.6. *Let $\zeta \in \boldsymbol{R}$, $\neq 0$, be fixed. Put for $\varphi \in \mathcal{H}$,*

$$J_\zeta\big(b'(\beta)\big)\varphi(x) := \psi_\zeta\big(\tfrac{1}{2}\langle \beta x, x\rangle\big)\varphi(x) \quad (\beta \in \mathrm{Sym}_n),$$

$$J_\zeta(w')\varphi(x) := \lambda(w)\, \mathcal{F}_\zeta \varphi(x), \quad \lambda(w) = \exp\Big(\frac{n\pi i}{4}\,\mathrm{sgn}(\zeta)\Big),$$

Then they determine a unique representation J_ζ of $Mp(n, \boldsymbol{R})$, for which

$$J_\zeta\big(d'(\alpha)\big)\varphi(x) = \nu(\alpha)\ |\det\alpha|^{-1/2}\ \varphi({}^t\alpha^{-1}x) \quad (\alpha \in \mathrm{Sym}_{GL_n}), \tag{5.9}$$

with $\nu(\alpha) = \exp\Big(-\dfrac{\pi i}{2}\, m_-(\alpha)\,\mathrm{sgn}(\zeta)\Big)$, and $\nu(\alpha)^2 = \mathrm{sgn}(\det\alpha)$.

The representation J_ζ is double-valued seeing from the level of $Sp(2n, \boldsymbol{R})$, and so a spin one. The formulas for J_ζ should be essentially well-known but we cannot find an appropriate reference. The direct origin of the appearance of 8th root $e^{\pi i/4}$ of 1 is the following numerical integral: for $b > 0$ and $a \in \boldsymbol{R}^\times$, with a positive constant $C[a, b] > 0$,

$$(5.10)\qquad \int_{-\infty}^{\infty}\Big(\int_{-\infty}^{\infty} e^{-b(x-y)^2} e^{iay^2} dy\Big) dx = C[a,b]\cdot \exp\Big(\frac{\pi i}{4}\operatorname{sgn}(a)\Big).$$

Proof. Put $b = 1$, then

$$\int_{-\infty}^{\infty}\Big(\int_{-\infty}^{\infty} e^{-(x-y)^2} e^{iay^2} dy\Big) dx = \int_{-\infty}^{\infty}\Big(\int_{-\infty}^{\infty} e^{-(1-ia)(y-\frac{1}{1-ia}x)^2} dy\cdot e^{\frac{ia}{1-ia}x^2}\Big) dx$$

$$= c(a)\cdot I(a) I(a^{-1}) \quad \text{with} \quad c(a) > 0,\ \ I(a) := \int_{-\infty}^{\infty} e^{-(1-ia)y^2} dy.$$

On the other hand, $I(a)^2 = \frac{\pi}{1-ia}$, and

$I(a)I(a^{-1}) = c'(a)\cdot \exp\big(\frac{\pi i}{4}\operatorname{sgn}(a)\big)$ with $c'(a) > 0$. □

Note that the factor $\nu(\alpha)$ on the subset $\mathrm{Sym}_{GL_n} \subset GL(n, \boldsymbol{R})$ is not continuous but piecewise constant, whereas $\nu(\alpha)^2 = \operatorname{sgn}(\det\alpha)$ is continuous. It might be denoted as $\nu(\alpha)\ |\det\alpha|^{-1/2} = \sqrt{\det\alpha}\ |\det\alpha|^{-1}$, but with serious faults of the ambiguity $\sqrt{1} = \pm 1, \sqrt{-1} = \pm i$, and of the disappearance of the effect of $\operatorname{sgn}(\zeta)$.

Corollary 1. *As a representation of the double covering D' of $D \cong GL(n, \boldsymbol{R})$, the formula (5.9) for the operators $J_\zeta\big(d'(\alpha)\big)$ ($\alpha \in \mathrm{Sym}_{GL_n}$) defines a double-valued representation of $GL(n, \boldsymbol{R})$. In another point of view, it defines a double covering group of $GL(n, \boldsymbol{R})$ (but not a continuous one).*

Sketch of a proof of the theorem. A proof of the theorem is done by checking relations among the operators $J_{\pi_\zeta}(\cdot)$ corresponding to the product rule in Lemma 5.2 (iii). Since we cannot calculate completely the factor set $r(g_1, g_2)$ for $J_{\pi_\zeta}(g_1)J_{\pi_\zeta}(g_2) = r(g_1, g_2)J_{\pi_\zeta}(g_1g_2)$ $(g_1, g_2 \in G)$, we are forced to proceed as follows. Choose appropriately scalar factors $\lambda(\beta)$, $\lambda(w)$ and $\nu(\alpha)$ of modulus 1 and then check if $J_\zeta\big(b'(\beta)\big) := \lambda(\beta)J_{\pi_\zeta}\big(b(\beta)\big)$, $J_\zeta(w') := \lambda(w)J_{\pi_\zeta}(w)$ and $J_\zeta\big(d'(\alpha)\big) := \nu(\alpha)J_{\pi_\zeta}\big(d(\alpha)\big)$ actually give a linear representation of the double covering $\widetilde{G}^2$. At first, since the group $B' \cong B$ is simply-connected, we may and do put $\lambda(\beta) \equiv 1$.

Then, we should get a relation between $\lambda(w)$ and $\nu(\alpha)$ from (5.8). We calculate the product of operators corresponding to each side of $b'(-\alpha)d'(\alpha)\, w' = w'\, b'(\alpha^{-1})\, {w'}^{-1} b'(\alpha)$: for any φ in the Schwartz space $\mathcal{S}(W)$ on W, and $x \in W$,

$$J_{\pi_\zeta}\big(b(-\alpha)\big)\nu(\alpha)J_{\pi_\zeta}\big(d(\alpha)\big)\lambda(w)\mathcal{F}_\zeta\varphi(x) = \mathcal{F}_\zeta J_{\pi_\zeta}\big(b(\alpha^{-1})\big)\mathcal{F}_\zeta^{-1} J_{\pi_\zeta}\big(b(\alpha)\big)\varphi(x).$$

Take an $a \in GL(n, \boldsymbol{R})$ such that $\alpha' := {}^t a\,\alpha\, a = \operatorname{diag}(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$, $\varepsilon_j = \pm 1$. Then $|\det a| = |\det\alpha|^{-1/2}$. Put $x = 0 \in W$. Then the left hand side is equal to

$$\text{left} \ =\ \nu(\alpha)\lambda(w)\ |\det\alpha|^{-1/2}\int \varphi(y)\, d\mu_\zeta(y).$$

The right hand side is written as

$$\text{right} \ =\ \iint \varphi(\xi)\,\psi_\zeta\big(\tfrac{1}{2}\langle\alpha\xi, \xi\rangle\big)\,\psi_\zeta\big(-\langle\xi, y\rangle\big)\,\psi_\zeta\big(\tfrac{1}{2}\langle\alpha^{-1}y, y\rangle\big)\, d\mu_\zeta(\xi)\, d\mu_\zeta(y),$$

and put $\xi = a\xi'$, $y = {}^t a^{-1} y'$, then

$$\text{right} = \iint \varphi(a\xi')\, \psi_\zeta(\tfrac{1}{2}\langle \alpha'\xi', \xi'\rangle)\, \psi_\zeta(-\langle \xi', y'\rangle)\, \psi_\zeta(\tfrac{1}{2}\langle {\alpha'}^{-1} y', y'\rangle)\, d\mu_\zeta(\xi')\, d\mu_\zeta(y'),$$

where ${\alpha'}^{-1} = \alpha'$ and

$$\psi_\zeta(\tfrac{1}{2}\langle \alpha'\xi', \xi'\rangle)\, \psi_\zeta(-\langle \xi', y'\rangle)\, \psi_\zeta(\tfrac{1}{2}\langle {\alpha'}^{-1} y', y'\rangle) = \prod_{1\le j\le n} \exp\big(i\,\tfrac{1}{2}\,\zeta\varepsilon_j(\xi'_j - \varepsilon_j y'_j)^2\big).$$

Now, as a function φ, we take $\varphi(a\xi') = \prod_{1\le j\le n} e^{-{\xi'_j}^2}$, then $\int \varphi(y)\, d\mu_\zeta(y) > 0$ for the left hand side, and we apply the formula (5.10) for each j-th integral in the right hand side. Comparing argument parts of both hand sides, modulo positive constants, we obtain

$$\begin{aligned} \nu(\alpha)\lambda(w) &= \prod_{1\le j\le n} \exp\Big(\frac{\pi i}{4}\,\varepsilon_j\,\mathrm{sgn}(\zeta)\Big) = \exp\Big(\frac{\pi i}{4}\big(m_+(\alpha) - m_-(\alpha)\big)\,\mathrm{sgn}(\zeta)\Big) \\ &= \exp\Big(\frac{n\pi i}{4}\,\mathrm{sgn}(\zeta)\Big)\cdot \exp\Big(-\,\frac{\pi i}{2}\, m_-(\alpha)\,\mathrm{sgn}(\zeta)\Big). \end{aligned}$$

Thus our choice of $\lambda(w)$ and $\nu(\alpha)$ in the theorem is very natural.

On the other hand, since D' is generated by $d'(\alpha)$ $\big(\alpha \in \mathrm{Sym}_{GL_n}\big)$, the operators $J_\zeta\big(d'(\alpha)\big)$ generate a double-valued representation of D'. Therefore $J_\zeta(b')$ $\big(b' \in \Phi_2^{-1}(B)\big)$, $J_\zeta(w')$, $J_\zeta(d')$ $(d' \in D')$ and $J_\zeta(c') := J_\zeta(w')J_\zeta(b')J_\zeta(w')^{-1}$ with $b' = {w'}^{-1}c'w'$ $\big(c' \in \Phi_2^{-1}(C)\big)$ generate a double-valued representation of G, not more than triple-valued.

This is the end of the sketch of the proof. □

For the choice of a preimage $d'(\alpha) \in D'$ for $d(\alpha) \in D$ $\big(\alpha \in \mathrm{Sym}_{GL_n}\big)$, we remark the following. Let $\alpha, \alpha' \in \mathrm{Sym}_{GL_n}$ and assume that $\alpha\alpha' \in \mathrm{Sym}_{GL_n}$. To compute the exponent X for $d'(\alpha)d'(\alpha') = {z'}^X d'(\alpha\alpha')$, we can utilize the faithful realization $J_\zeta(d')$ of D' as follows. Since α and α' commute with each other, they can be diagonalized simultaneously by a $u \in O(n)$ as ${}^t u\alpha u = \mathrm{diag}(a_1, a_2, \ldots, a_n)$, ${}^t u\alpha' u = \mathrm{diag}(a'_1, a'_2, \ldots, a'_n)$, and so

$${}^t u(\alpha\alpha')u = \mathrm{diag}(a_1a'_1, a_2a'_2, \ldots, a_na'_n).$$

Then we get

$$\begin{aligned} & m_-(\alpha) = \sharp\{\, j\,;\, a_j < 0\},\ \ m_-(\alpha') = \sharp\{\, j\,;\, a'_j < 0\},\ \ m_-(\alpha\alpha') = \sharp\{\, j\,;\, a_ja'_j < 0\}, \\ & \therefore\quad m_-(\alpha) + m_-(\alpha') = m_-(\alpha\alpha') + 2\,\sharp\{\, j\,; a_j < 0,\, a'_j < 0\}, \\ & \therefore\quad \nu(\alpha)\,\nu(\alpha') = \nu(\alpha\alpha')\cdot(-1)^X,\ \ X = \sharp\{\, j\,; a_j < 0,\, a'_j < 0\}. \end{aligned}$$

Corollary 2. *Let $\alpha, \alpha' \in \mathrm{Sym}_{GL_n}$. If the product $\alpha\alpha'$ is again symmetric, then, with the exponent $X = \sharp\{\, j\,; a_j < 0,\, a'_j < 0\}$ just above, we have*

$$d'(\alpha)d'(\alpha') = {z'}^X d'(\alpha\alpha').$$

Usually, to study Weil type representations, one appeals to the results of Weil in [Weil], in particular, Corollaires 1 and 2, p.162, of Théorème 2, as is done in [Sait] and [Yos]. There the formulas for Fourier transform of so-called *characters of the second degree* are given. Its very special case is the formula (5.10) above.

6 Hidden symmetries in the algebra $M(2^k, \boldsymbol{C})$

The full matrix algebra $\mathcal{A}_n := M(n, \boldsymbol{C})$ is special at the point that any automorphism is inner, that is, $\mathrm{Aut}(\mathcal{A}_n) = \mathrm{Int}(\mathcal{A}_n)$. For the degree $n = 2^k,\ k \geq 1$, $\mathcal{A}_{2^k} = M(2^k, \boldsymbol{C})$ is again more special at the point that several finite groups act naturally on it, and it gives especially Schur's *Hauptdarstellung* of $\mathfrak{S}_{2k}$ and $\mathfrak{S}_{2k+1}$. We explain these things in this section, which have intimate relations with the main body of this paper. For $g \in GL(n, \boldsymbol{C})$, denote by $J(g)$ the inner automorphism $x \mapsto gxg^{-1}$ $(x \in \mathcal{A}_n)$ of $\mathcal{A}_n$.

6.1 Hidden symmetries inside $GL(2, \boldsymbol{C})$ and $M(2, \boldsymbol{C})$

Put as in (1.6) and (2.6),

$$a = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad b = \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad c = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{6.1}$$

and $\varepsilon = \mathrm{diag}(1, 1)$ the identity matrix. Then $\sigma_1\sigma_2 = i\sigma_3$ and the set

$$\{\sigma_1^{\epsilon_1}\,\sigma_2^{\epsilon_2}\,;\ \epsilon_j = 0, 1\ (j{=}1, 2)\} \qquad (\text{resp. } \{\varepsilon, \sigma_1, \sigma_2, \sigma_3\})$$

forms a linear basis of the full matrix algebra $\mathcal{A}_2 = M(2, \boldsymbol{C})$. When we take $\Omega := \{\sigma_1, \sigma_2, \sigma_3\}$ as a set of generators, the algebra $\mathcal{A}_2$ is presented by the following set of fundamental relations:

$$\begin{cases} \sigma_j^{\,2} = \varepsilon \qquad (1 \leq j \leq 3)\,; \\ \sigma_j\sigma_k = -\sigma_k\sigma_j \ \ (j \neq k,\ 1 \leq j, k \leq 3); \\ \sigma_1\sigma_2 = i\sigma_3,\ \sigma_2\sigma_3 = i\sigma_1,\ \sigma_3\sigma_1 = i\sigma_2\,. \end{cases} \tag{6.2}$$

6.1.1. Group action of $U = \boldsymbol{Z}_2^2$ as 'basis sign changes' :
The group generated by $\{\sigma_1, \sigma_2\}$ is of order 8 and is given as an abstract group $\widetilde{U}$ as

set of generators: $\{u_1',\, u_2',\, z'\}$,

set of fundamental relations: $\begin{cases} {z'}^{\,2} = e', \quad z'u_j' = u_j'z'\ (j = 1, 2), \\ {u_j'}^{\,2} = e' \qquad (j = 1, 2)\,, \\ u_j'\,u_k' = z'u_k'\,u_j'\ \ (j \neq k)\,, \end{cases}$

where e' denotes the identity element. On the other hand, let $U = \langle u_1\rangle \times \langle u_2\rangle$, with $u_1^2 = u_2^2 = e$, be the direct product of $\boldsymbol{Z}_2$ with itself, where $e =$ the identity element. Then the 'universal' covering group or the representation group of U is realized as the double covering group $\widetilde{U}$ with the canonical homomorphism as $u'_j \mapsto u_j$ $(j=1,2)$, $z' \mapsto e$. Here we show that U acts on $\mathcal{A}_2$ naturally, and it produces the double covering group $\widetilde{U}$ naturally through intertwining operators along with the general scheme in **E-8**.

Put $u_3 = u_1u_2 = u_2u_1$, then $U = \{e, u_1, u_2, u_3\}$. For $u_i \in U$, put

$$(6.3) \qquad u_i(\sigma_i) = \sigma_i, \quad u_i(\sigma_j) = -\sigma_j \ (j \neq i),$$

then this 'basis sign change' defines actually an action of U on $\mathcal{A}_2$. Put $\rho_0(u_i) := \sigma_i$ $(i = 1,2)$. Then the inner automorphism $J\big(\rho_0(u_i)\big)$ by $\rho_0(u_i)$ is just the above action of u_i for $i = 1,2$. By this property, the element $\rho_0(u_i)$ is uniquely determined modulo non-zero scalar multiples. The map $U \ni u \mapsto \rho_0(u)$, with $\rho_0(e) = \varepsilon$, gives a double-valued representation of U, since $\rho_0(u_1)\rho_0(u_2) = -\rho_0(u_2)\rho_0(u_1)$.

By this, we can find a faithful linear representation ρ'_0 of $\widetilde{U}$ by putting

$$(6.4) \qquad \rho'_0\big(u'_1\big) := \sigma_1\,, \quad \rho'_0\big(u'_2\big) := \sigma_2\,.$$

6.1.2. Action of $\mathfrak{S}_3$ as 'basis permutations' :

Let $\mathfrak{S}_3 = \langle s_1, s_2\rangle$ be the symmetric group of degree 3 with the standard generators $s_1 = (1\ 2)$, $s_2 = (2\ 3)$. Note that the Schur multiplier of $\mathfrak{S}_3$ is trivial and the representation group of $\mathfrak{S}_3$ is nothing but itself. As an abstract group, $\mathfrak{S}_3$ is presented by the set of generators $\{s_1, s_2\}$ and the set of fundamental relations as

$$(6.5) \qquad \mathfrak{S}_3 = \langle s_1, s_2\rangle, \quad {s_1}^2 = {s_2}^2 = e, \ \ (s_1s_2)^3 = e.$$

Actions on $\mathcal{A}_2$ of s_j $(j = 1,2)$ are given by T_j as 'basis permutations' as

$$(6.6) \qquad \begin{cases} T_1 \ : \ (\sigma_1,\sigma_2,\sigma_3) \ \mapsto \ (-\sigma_2, -\sigma_1, -\sigma_3), \\ T_2 \ : \ (\sigma_1,\sigma_2,\sigma_3) \ \mapsto \ (-\sigma_1, -\sigma_3, -\sigma_2). \end{cases}$$

To prove that T_j gives an automorphism of $\mathcal{A}_2$, it is sufficient to verify the set of fundamental relations (6.2) for $T_j\Omega = \{\sigma'_1, \sigma'_2, \sigma'_3\}$, instead of Ω.

Moreover the correspondence $s_j \mapsto T_j$ defines a linear representation of $\mathfrak{S}_3$. To confirm this, we should verify the fundamental relations (6.5): in fact, ${T_1}^2 = {T_2}^2 = I$ (the identity map on $\mathcal{A}_2$), and $\big(T_1T_2\big)^3 = I$.

Now look for elements $\tau_j \in \mathcal{A}_2$ $(j = 1,2)$ which realize T_j as inner automorphisms: $T_i = J(\tau_j)$. We omit the calculations and give directly the answer as

$$(6.7) \qquad \tau_1 = \frac{1}{\sqrt{2}}\,\big(a-b\big), \quad \tau_2 = \frac{1}{\sqrt{2}}\,\big(b-c\big).$$

Moreover we get ${\tau_1}^2 = {\tau_2}^2 = \varepsilon$, $(\tau_1\tau_2)^3 = \varepsilon$, and see that the correspondence $s_j \mapsto \tau_j$ $(j = 1,2)$ gives a linear representation π_0 of $\mathfrak{S}_3$ of dimension 2. Put

$\widetilde{\mathfrak{S}}_3 := \langle \tau_1, \tau_2 \rangle$, then $\widetilde{\mathfrak{S}}_3 \cong \mathfrak{S}_3$.

6.1.3. Semidirect product group $\widetilde{U} \rtimes \mathfrak{S}_3 \cong \widetilde{U} \rtimes \widetilde{\mathfrak{S}}_3$ **:**

Unifying the above two groups acting on $\mathcal{A}_2$, we get an action of the semidirect product group $\widetilde{U} \rtimes \widetilde{\mathfrak{S}}_3$ on the algebra. Since $\widetilde{U} = \{\pm\varepsilon, \pm\sigma_1, \pm\sigma_2, \pm i\sigma_3\}$, the action of generators τ_1 and τ_2 of $\widetilde{\mathfrak{S}}_3$ on $\widetilde{U}$ is given by T_1 and T_2 respectively:

$$\tau_j(u') = \tau_j u' \tau_j^{-1} := T_j u' \quad \big(j = 1, 2\,;\ u' \in \widetilde{U}\big). \tag{6.8}$$

Theorem 6.1. *The group $U = \boldsymbol{Z}_2^2$ acts on the algebra $\mathcal{A}_2 = M(2, \boldsymbol{C})$ as 'basis sign changes' as in (6.3). It induces through inner automorphisms of $\mathcal{A}_2$ an irreducible projective representation ρ_0 of U, unique up to equivalence, which comes from a faithful linear representation ρ'_0 in (6.4) of the universal covering group $\widetilde{U}$. The character of ρ'_0 is given as $\mathrm{tr}\big(\rho'_0(\pm\varepsilon)\big) = \pm 2$, $\mathrm{tr}\big(\rho'_0(u')\big) = 0$ otherwise.*

The group $\mathfrak{S}_3$ acts on $\mathcal{A}_2$ as 'basis permutations' as in (6.6). It gives a faithful, irreducible linear representation π_0 of $\mathfrak{S}_3$ by (6.7), unique up to equivalence.

These two groups $\widetilde{U}$ and $\mathfrak{S}_3$ produces a semidirect product group $\widetilde{U} \rtimes \mathfrak{S}_3$ in $GL(2, \boldsymbol{C})$ according to (6.8).

6.2 Hidden symmetries in $\mathcal{A}_{2^k} = M(2^k, \boldsymbol{C}),\ k \geq 2$

6.2.1. Action of $\mathfrak{S}_k$ as 'basis factor permutations' :

Let a, b, c, ε be as above, and let Ω_k be the set of elements of $\mathcal{A}_{2^k}$ given as

$$\boldsymbol{x} = x_1 \otimes x_2 \otimes \cdots \otimes x_k\,, \quad x_p \in \Omega_1 := \{a, b, c, \varepsilon\} \ \ (1 \leq p \leq k).$$

Then Ω_k gives a linear basis of the algebra $\mathcal{A}_{2^k}$. We can define a natural action of $\mathfrak{S}_k$ on $\mathcal{A}_{2^k}$ by permuting the factors x_p: for $\sigma \in \mathfrak{S}_k$,

$$\sigma(\boldsymbol{x}) := x_{\sigma^{-1}(1)} \otimes x_{\sigma^{-1}(2)} \otimes \cdots \otimes x_{\sigma^{-1}(k)}.$$

For the convenience of calculations, denote this action by $T(\sigma)$, then $T(\sigma_1)T(\sigma_2) = T(\sigma_1\sigma_2)$. Since $T(\sigma)$ can be realized as an inner automorphism $J(g_\sigma)$ by a $g_\sigma \in GL(2^k, \boldsymbol{C})$, we will determine g_σ for the transposition $\sigma = s_1 = (1\ \ 2)$. The calculation can be pursued inside the case $k = 2$. In that case, put $g_2 := g_{s_1}$, then

$$g_2\,(x \otimes y) = (y \otimes x)\,g_2 \qquad \big(x, y \in \Omega_1\big). \tag{6.9}$$

Under this equation, g_2 is determined uniquely modulo scalar multiples. An answer, satisfying $g_2^2 = \varepsilon \otimes \varepsilon$, is given by

$$g_2 := \frac{1}{2}\,\big(\varepsilon \otimes \varepsilon + a \otimes a + b \otimes b + c \otimes c\big) \ \in\ \mathcal{A}_{2^2}\,, \tag{6.10}$$

which is expressed by a 2×2 matrix as

$$g_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{under the rule } x \otimes y := \begin{pmatrix} x_{11}\, y & x_{12}\, y \\ x_{21}\, y & x_{22}\, y \end{pmatrix}, \tag{6.11}$$

for $x = (x_{ij})$ and y in $\mathcal{A}_2$. This answer can be confirmed by an explicit verification of (6.9).

Coming back to the general case of $k \geq 2$, we put

$$\begin{cases} \upsilon_1 := g_2 \otimes \varepsilon \otimes \cdots \otimes \varepsilon = g_2 \otimes \left(\varepsilon^{\otimes(k-2)}\right), \\ \upsilon_2 := \varepsilon \otimes g_2 \otimes \varepsilon \otimes \cdots \otimes \varepsilon = \varepsilon \otimes g_2 \otimes \left(\varepsilon^{\otimes(k-3)}\right), \\ \cdots \qquad \cdots\cdots \\ \cdots \qquad \cdots\cdots \\ \upsilon_{k-1} := \varepsilon \otimes \cdots \otimes \varepsilon \otimes g_2 = \left(\varepsilon^{\otimes(k-2)}\right) \otimes g_2, \end{cases} \tag{6.12}$$

where $\varepsilon^{\otimes q}$ denotes q-times tensor product of ε. Then $\upsilon_i \in GL(2^k, \boldsymbol{C})$ realize the action $T(s_i) : \boldsymbol{x} \mapsto s_i(\boldsymbol{x})$, as a 'basis factor permutation', for the simple reflection $s_i = (i\ \ i{+}1)$ as $s_i(\boldsymbol{x}) = \upsilon_i\, \boldsymbol{x}\, \upsilon_i^{-1}$.

From the general scheme in **E-8**, the correspondence $s_i \mapsto T(s_i)$ $(1 \leq i \leq k-1)$ gives a (possibly) projective representation $s_i \mapsto \upsilon_i$ of $\mathfrak{S}_k$. Let us check if this is really projective or multiple-valued, or is just a linear representation after certain adjustment of scalar multiples.

Theorem 6.2. (i) *There hold the following identities for 'basis factor transpositions': with $e = E_{2^k}$ the identity matrix in $\mathcal{A}_{2^k}$,*

$$\begin{cases} \upsilon_i^2 = e & (1 \leq i \leq k-1), \\ (\upsilon_i \upsilon_{i+1})^3 = e & (1 \leq i \leq k-2), \\ \upsilon_i \upsilon_j = \upsilon_j \upsilon_i & (|i-j| \geq 2). \end{cases}$$

(ii) *The correspondence $s_i \mapsto \upsilon_i$ $(1 \leq i \leq k-1)$ defines a linear representation $\pi^{(k)}$ of $\mathfrak{S}_k$ of dimension 2^k.*

(iii) *The trace of $\pi^{(k)}$ is given as follows: For a $\sigma \in \mathfrak{S}_k$, take a decomposition into disjoint cycles, and let $\ell_1, \ell_2, \ldots, \ell_s \geq 2$ be their lengths. Then*

$$\mathrm{tr}\big(\pi^{(k)}(g)\big) = 2^{k - \sum_{1 \leq j \leq s}(\ell_j - 1)}. \tag{6.13}$$

Proof. The assertion (ii) follows from (i), since the relations listed are just the fundamental relations for the set of simple reflections in $\mathfrak{S}_k$.

The relation $\upsilon_i^2 = e$ in (i) is proved by an easy calculation. For $(\upsilon_i \upsilon_{i+1})^3 = e$, we work in the case $k = 3,\ i = 1$, then

$4\, \upsilon_1 \upsilon_2 = 4(g_2 \otimes \varepsilon)(\varepsilon \otimes g_2) = X + iY$ with

$X = a \otimes \varepsilon \otimes a + \varepsilon \otimes a \otimes a + a \otimes a \otimes \varepsilon + b \otimes \varepsilon \otimes b + \varepsilon \otimes b \otimes b + b \otimes b \otimes \varepsilon +$

$$+\, c\otimes\varepsilon\otimes c + \varepsilon\otimes c\otimes c + c\otimes c\otimes\varepsilon + \varepsilon\otimes\varepsilon\otimes\varepsilon\,;$$

$$Y = -a\otimes b\otimes c - b\otimes c\otimes a - c\otimes a\otimes b + a\otimes c\otimes b + b\otimes a\otimes c + c\otimes b\otimes a\,;$$

$$\therefore\quad (X+iY)^2 = X^2 - Y^2 + i(XY+YX)\quad \text{with}$$

$$X^2 = 8\,\varepsilon\otimes\varepsilon\otimes\varepsilon + 2X,\quad Y^2 = 8\,\varepsilon\otimes\varepsilon\otimes\varepsilon - 2X,\quad XY = YX = -2Y\,;$$

$$\therefore\quad (\upsilon_1\upsilon_2)^2 = \tfrac{1}{4}(X-iY),\quad \upsilon_1\upsilon_2 = \tfrac{1}{4}(X+iY),$$

$$\therefore\quad (\upsilon_1\upsilon_2)^3 = \tfrac{1}{16}(X^2+Y^2) = \varepsilon\otimes\varepsilon\otimes\varepsilon.$$

The last relation in (i) is evidently true.

The assertion (iii) is proved by explicit calculations. The point is that $\mathrm{tr}(a) = \mathrm{tr}(b) = \mathrm{tr}(c) = 0$. We omit the details. □

Remark 6.1. The representation $\pi^{(k)}$ of $\mathfrak{S}_k$ is reducible and the first step to its irreducible decomposition is given as follows: Let $\boldsymbol{k} := (k_0, k_1, k_2, k_3)$, $k_0 + k_1 + k_2 + k_3 = k$, be a decomposition of k, and consider an element

$$\boldsymbol{x}[\boldsymbol{k}] := (\varepsilon^{\otimes k_0})\otimes(a^{\otimes k_1})\otimes(b^{\otimes k_3})\otimes(c^{\otimes k_3})\ \in\ \Omega_k$$

and its $\mathfrak{S}_k$-orbit $\Omega[\boldsymbol{k}] \subset \Omega_k$. Then, on the linear span of $\Omega[\boldsymbol{k}]$, $\pi^{(k)}$ induces a representation equivalent to the induced representation $\mathrm{Ind}_{\mathfrak{S}[\boldsymbol{k}]}^{\mathfrak{S}_k}\mathbf{1}_{\mathfrak{S}[\boldsymbol{k}]}$, where $\mathfrak{S}[\boldsymbol{k}] := \mathfrak{S}_{k_0}\times\mathfrak{S}_{k_1}\times\mathfrak{S}_{k_2}\times\mathfrak{S}_{k_3}$ and $\mathbf{1}_{\mathfrak{S}[\boldsymbol{k}]}$ denotes its trivial representation. Each of this induced representation contains exactly once the trivial representation $\mathbf{1}_{\mathfrak{S}_k}$ of $\mathfrak{S}_k$.

6.2.2. 'Generator permutations' as actions of $\mathfrak{S}_{2k}$, $\mathfrak{S}_{2k+1}$ and $\mathfrak{A}_{2k+1}$. The contents of this and the next terms are inspired by [Sch3, Abschnitt VI], and have intimate relations with Part II in [II], in particular with §§5 and 8 loc. cit.

The algebra $\mathcal{A}_{2^k} = M(2^k, \boldsymbol{C})$ has a very special structure with which it admits a 'standard' action of $\mathfrak{S}_{2k}$ and also of $\mathfrak{S}_{2k+1}$. Put $Y_1, Y_2, \ldots, Y_{2k+1} \in GL(2^k, \boldsymbol{C}) \subset \mathcal{A}_{2^k}$ as

$$\begin{array}{rcll}
Y_1 & = & a\otimes\varepsilon^{\otimes(k-1)} & (k\text{-times tensor product}),\\
Y_2 & = & b\otimes\varepsilon^{\otimes(k-1)}, & \\
Y_3 & = & c\otimes a\otimes\varepsilon^{\otimes(k-2)}, & \\
Y_4 & = & c\otimes b\otimes\varepsilon^{\otimes(k-2)}, & \\
\cdots & \cdots & \cdots & \\
\cdots & \cdots & \cdots & \\
Y_{2k-1} & = & c^{\otimes(k-1)}\otimes a\,, & \\
Y_{2k} & = & c^{\otimes(k-1)}\otimes b\,, & \\
Y_{2k+1} & = & c^{\otimes(k-1)}\otimes c\,. &
\end{array}$$

Then they have the following properties.

Lemma 6.3. (i) *The set* $\{Y_1, Y_2, \ldots, Y_{2k}\}$ *generates* $\mathcal{A}_{2^k}$*, and there hold*

$$
\left\{
\begin{array}{rcll}
Y_p^2 & = & E & (p \in \boldsymbol{I}_{2k+1}),\\
Y_pY_q & = & -Y_qY_p & (p \neq q,\ p, q \in \boldsymbol{I}_{2k+1}),\\
Y_1Y_2\cdots Y_{2k+1} & = & i^k\, E & \big(i = \sqrt{-1}\,\big),
\end{array}
\right.
\tag{6.14}
$$

where $E := E_{2^k}$ *is the identity matrix in* $\mathcal{A}_{2^k}$*, and* $\boldsymbol{I}_n := \{1, 2, \ldots, n\}$.

(ii) *Any non-trivial monomial product* $Y_{j_1}Y_{j_2}\cdots Y_{j_p}$, $1 \leq j_1 < j_2 < \ldots < j_p \leq 2k$*, has trace 0. Such a monomial product containing* Y_{2k+1} *has non-zero trace only when it is actually equal to* $Y_1Y_2\cdots Y_{2k}Y_{2k+1}$*. Or*

$$
\mathrm{tr}\big(Y_{j_1}Y_{j_2}\cdots Y_{j_p}Y_{2k+1}\big) = 0 \quad \text{for } \ 1 \leq j_1 < j_2 < \ldots < j_p \leq 2k,
$$

whenever one of $Y_j, j \leq 2k$*, is absent.*

(iii) *The set* $\mathcal{B}_{2k} := \big\{Y_1^{a_1}Y_2^{a_2}\ \cdots\ Y_{2k}^{a_{2k}}\ ;\ a_j = 0, 1\ (j \in \boldsymbol{I}_{2k})\big\}$ *gives a linear basis of the algebra* $\mathcal{A}_{2^k}$.

Proof. We prove here only (iii). Consider a linear relation

$$
\sum_{a_i = 0,1\ (j \in \boldsymbol{I}_{2k})} \lambda_{a_1,\ldots,a_{2k}}\, Y_1^{a_1}Y_2^{a_2}\ \cdots\ Y_{2k}^{a_{2k}} = O.
\tag{6.15}
$$

Multiply $Y_1^{a_1}Y_2^{a_2}\ \cdots\ Y_{2k}^{a_{2k}}$ from the left of this equation, and then take trace, then by (i) – (ii) we have

$$
\lambda_{a_1,\ldots,a_{2k}}\, \mathrm{tr}\Big(\big(Y_1^{a_1}Y_2^{a_2}\ \cdots\ Y_{2k}^{a_{2k}}\big)^2\Big) = \lambda_{a_1,\ldots,a_{2k}} \cdot 2^k = 0,
$$

whence $\lambda_{a_1,\ldots,a_{2k}} = 0$. □

The set $\mathcal{Y}_{2k} := \{Y_1, \ldots, Y_{2k}\}$ generates a group $\mathcal{G}_{2k}$ of order 2^{2k+1} given as

$$
\mathcal{G}_{2k} := \big\{\pm Y_1^{a_1}Y_2^{a_2}\ \cdots\ Y_{2k}^{a_{2k}}\ ;\ a_j = 0, 1\ (j \in \boldsymbol{I}_{2k})\big\}.
\tag{6.16}
$$

Define an abstract group $\mathcal{G}'_{2k}$ by giving a set of generators and a set of fundamental relations as follows:

set of generators: $\big\{z, \eta_1, \ldots, \eta_{2k}\big\}$,

set of fundamental relations:

$$
\left\{
\begin{array}{ll}
z^2 = e, \quad z\eta_j = \eta_j z\ (j \in \boldsymbol{I}_{2k}) & (z \text{ is a central element}),\\
\eta_j^2 = e & (j \in \boldsymbol{I}_{2k}),\\
\eta_i\eta_j = z\,\eta_j\eta_i & (i, j \in \boldsymbol{I}_{2k},\ i \neq j).
\end{array}
\right.
\tag{6.17}
$$

Theorem 6.4. (i) *The groups* $\mathcal{G}'_{2k}$ *and* $\mathcal{G}_{2k}$ *are mutually isomorphic under the correspondence* $z \mapsto -E$, $\eta_j \mapsto Y_j$ $(j \in \boldsymbol{I}_{2k})$*. Moreover this correspondence gives a faithful linear representation* π_{2k} *of the group* $\mathcal{G}'_{2k}$ *of dimension* 2^k.

(ii) *The algebra* $\mathcal{A}_{2^k}$ *is a quotient of the group algebra* $\boldsymbol{C}[\mathcal{G}'_{2k}]$ *of* $\mathcal{G}'_{2k}$ *as its homomorphic image, where* z *is mapped to* $-E$.

We see from the set of fundamental relations (6.17) that the symmetric group $\mathfrak{S}_{2k}$ acts on the group $\mathcal{G}'_{2k}$ naturally in two ways: for $\sigma \in \mathfrak{S}_{2k}$,

$$\sigma(\eta_j) := \eta_{\sigma(j)} \quad (j \in \boldsymbol{I}_{2k}); \tag{6.18}$$

and also, with the factor relative to $\mathrm{sgn}(\sigma)$,

$$\sigma(\eta_j) := z^{(1-\mathrm{sgn}(\sigma))/2} \cdot \eta_{\sigma(j)} \quad (j \in \boldsymbol{I}_{2k}). \tag{6.19}$$

In fact, the transform $\eta_j \mapsto \sigma(\eta_j)$, $z \mapsto z$, preserves (6.17). Then, through the representation π_{2k} these actions induce an action of the algebra $\mathcal{A}_{2^k}$ respectively given by $Y_j \mapsto Y_{\sigma(j)}$, and $Y_j \mapsto \mathrm{sgn}(\sigma)Y_{\sigma(j)}$, for $j \in \boldsymbol{I}_{2k}$. We call them *generator permutations*. Note that this is a special case of the general situation in **E-8**.

In connection to the contents in the later part of this paper, we call these actions as 'actions in CASE II' and denote them respectively as $\sigma^{\mathrm{II}}(Y_j) := Y_{\sigma(j)}$, and $\sigma^{\mathrm{II}-}(Y_j) := \mathrm{sgn}(\sigma)Y_{\sigma(j)}$, for $j \in \boldsymbol{I}_{2k}$.

Moreover we define another action, called CASE I, as follows. Put

$$Y'_j = (-1)^{j-1}Y_j \quad (j \in \boldsymbol{I}_{2k+1}).$$

Then, replace the role of Y_j's by Y'_j's, then we get similarly as above an action of $\mathfrak{S}_{2k}$ on $\mathcal{A}_{2^k}$. Thus we have three different actions on $\mathcal{A}_{2^k}$ listed below: for $j \in \boldsymbol{I}_{2k}$,

$$\begin{cases} \sigma^{\mathrm{I}}(Y'_j) & := & \mathrm{sgn}(\sigma)Y'_{\sigma(j)} & (\sigma \in \mathfrak{S}_{2k}); \\ \sigma^{\mathrm{II}-}(Y_j) & := & \mathrm{sgn}(\sigma)Y_{\sigma(j)} & (\sigma \in \mathfrak{S}_{2k}); \\ \sigma^{\mathrm{II}}(Y_j) & := & Y_{\sigma(j)} & (\sigma \in \mathfrak{S}_{2k}). \end{cases} \tag{6.20}$$

Let us now define actions of $\mathfrak{S}_{2k+1}$ on the algebra $\mathcal{A}_{2^k}$, extending those of $\mathfrak{S}_{2k}$. This is also a special case of the situation in **E-8**. Taking into account the relation in (6.14), we give an abstract group $\mathcal{G}'_{2k+1}$ as follows:

set of generators: $\big\{z', \eta_1, \eta_2, \ldots, \eta_{2k}, \eta_{2k+1}\big\}$,

set of fundamental relations:

$$\begin{cases} {z'}^4 = e, \quad z'\eta_j = \eta_j z' \ (j \in \boldsymbol{I}_{2k+1}), \\ \eta_j^2 = e \qquad\qquad (j \in \boldsymbol{I}_{2k+1}), \\ \eta_i\eta_j = {z'}^2\, \eta_j\eta_i \quad (i, j \in \boldsymbol{I}_{2k+1},\ i \neq j), \\ \eta_1\eta_2\cdots\eta_{2k}\eta_{2k+1} = {z'}^k. \end{cases} \tag{6.21}$$

Then, thanks to (6.14), we have a linear representation π_{2k+1} of the group $\mathcal{G}'_{2k+1}$ by the formula

$$\pi_{2k+1}(\eta_j) := Y_j \ (j \in \boldsymbol{I}_{2k+1}), \quad \pi_{2k+1}(z') := iE\,. \tag{6.22}$$

By this representation of dimension 2^k, the group algebra $\boldsymbol{C}[\mathcal{G}'_{2k+1}]$ is mapped homomorphically onto $\mathcal{A}_{2^k}$. Note that, when k is even, we can reduce (6.21) similarly as (6.17), by using $z = {z'}^2$ only. But for a general treatment, irrespective of the parity of k, we use (6.21) here.

From the symmetry of the fundamental relation (6.21), we see that the symmetric group $\mathfrak{S}_{2k+1}$ can acts on $\mathcal{G}'_{2k+1}$ as follows: for $\sigma \in \mathfrak{S}_{2k+1}$,

$$\sigma(\eta_j) = (z')^{1-\mathrm{sgn}(\sigma)}\, \eta_{\sigma(j)} \; (j \in \boldsymbol{I}_{2k+1}), \quad \sigma(z') = z' . \tag{6.23}$$

Through the representation π_{2k+1}, this action induces an action on $\mathcal{A}_{2^k}$ extending $\sigma^{\mathrm{II}-}$ for $\mathfrak{S}_{2k}$ above. Also, we can get another action σ^{I} by using $Y'_j := (-1)^{j-1} Y_j \; (j \in \boldsymbol{I}_{2k+1})$. Moreover we define an action of the alternating group $\mathfrak{A}_{2k+1}$, by $\sigma^{\mathrm{II}}(Y_j) := Y_{\sigma(j)} \; (j \in \boldsymbol{I}_{2k+1})$, taking into account the fourth relation $\eta_1 \cdots \eta_{2k+1} = {z'}^k$ in (6.21). Summarizing them, we have the following theorem.

Theorem 6.5. *On the full matrix algebra $\mathcal{A}_{2^k} = M(2^k, \boldsymbol{C})$, the symmetric group $\mathfrak{S}_{2k+1}$ and the alternating group $\mathfrak{A}_{2k+1}$ act according to the following formulas: for $j \in \boldsymbol{I}_{2k+1}$,*

$$\left\{ \begin{array}{rcll} \sigma^{\mathrm{I}}(Y'_j) & := & \mathrm{sgn}(\sigma) Y'_{\sigma(j)} & (\sigma \in \mathfrak{S}_{2k+1}); \\ \sigma^{\mathrm{II}-}(Y_j) & := & \mathrm{sgn}(\sigma) Y_{\sigma(j)} & (\sigma \in \mathfrak{S}_{2k+1}); \\ \sigma^{\mathrm{II}}(Y_j) & := & Y_{\sigma(j)} & (\sigma \in \mathfrak{A}_{2k+1}). \end{array} \right. \tag{6.24}$$

Remark 6.2. The action σ^{II} of the symmetric group $\mathfrak{S}_{2k}$ in (6.20) cannot be extended to the whole $\mathfrak{S}_{2k+1}$. Instead, its restriction on the alternating group $\mathfrak{A}_{2k}$ can be extended to $\mathfrak{A}_{2k+1}$. This situation reflects to a very delicate, complicated relations between spin representations of $\mathfrak{S}_{2k}$ and those of $\mathfrak{S}_{2k+1}$ or of $\mathfrak{A}_{2k+1}$, and so resulted to one of main reasons why theory of spin representations and spin characters in CASE II, in the paper [II], are so much complicated (see the related sections).

6.3 Intertwining operators and spin representations of symmetric groups

Since each of the isomorphism $\sigma^{\mathrm{I}}, \sigma^{\mathrm{II}-}$ and σ^{II} of $\mathcal{A}_{2^k}$ is inner, we look for elements in $\mathcal{A}_{2^k}$, modulo scalar multiples, which gives respectively these isomorphisms as inner automorphisms. In other words, we determine intertwining operators between π_{2k+1} and its transformed one under the action of σ. We do this for simple transpositions $s_j = (j\;\; j+1) \; (j \in \boldsymbol{I}_{2k})$ as generators of $\mathfrak{S}_{2k+1}$. After obtaining these elements explicitly, we can decide if the representation of $\mathfrak{S}_{2k+1}$ or of $\mathfrak{A}_{2k+1}$, obtained through intertwining operators, is actually a spin one or a non-spin ordinary linear representation.

Recall that $\mathfrak{S}_{2k+1}$ is given as an abstract group by the following set of fundamental relations:

$$\left\{ \begin{array}{l} s_j^2 = e \;\; (j \in \boldsymbol{I}_{2k}), \quad (s_j s_{j+1})^3 = e \;\; (j \in \boldsymbol{I}_{2k-1}), \\ s_i s_j = s_j s_i \quad (|i-j| \geq 2, \; i, j \in \boldsymbol{I}_{2k}). \end{array} \right.$$

For $\mathfrak{S}_n$, Schur proved in [Sch3] that, for $n = 2, 3$, its representation group is $\mathfrak{S}_n$ itself, and for $n \geq 4$, it has two representation groups $\mathfrak{T}_n$ and $\mathfrak{T}'_n$, which are mutually isomorphic only when $n = 6$. In this paper, we prefer to use $\mathfrak{T}'_n$ and denote it by $\widetilde{\mathfrak{S}}_n$. It is given as

$$
(6.25) \qquad \begin{cases} z_1^2 = e, \quad z_1 \text{ central}, \\ r_j^2 = e \ (j \in \boldsymbol{I}_{n-1}), \quad (r_j r_{j+1})^3 = e \ (j \in \boldsymbol{I}_{n-2}), \\ r_i r_j = z_1 r_j r_i \ (|i-j| \geq 2, \ i, j \in \boldsymbol{I}_{n-1}). \end{cases}
$$

$$
e \longrightarrow Z = \{e, z_1\} \longrightarrow \widetilde{\mathfrak{S}}_n \xrightarrow{\Phi_{\mathfrak{S}}} \mathfrak{S}_n \longrightarrow e \quad \text{(exact)},
$$

where the canonical homomorphism $\Phi_{\mathfrak{S}}$ is given by $\Phi_{\mathfrak{S}}(r_j) = s_j \ (j \in \boldsymbol{I}_{n-1})$ (cf. §1, Theorem 1.2 below). Note that the representation group $\mathfrak{B}_n$ of the alternating group $\mathfrak{A}_n$, $n \geq 4$, $n \neq 6, 7$, is given by the full inverse image $\widetilde{\mathfrak{A}}_n := \Phi_{\mathfrak{S}}^{-1}(\mathfrak{A}_n)$ (cf. [Sch3, §5]).

Now put $n = 2k$ or $n = 2k+1$. For the intertwining operators, we define, a priori assuming to treat spin things, for generators r_p $(p \in \boldsymbol{I}_{n-1})$ of $\widetilde{\mathfrak{S}}_n = \mathfrak{T}'_n$ or for generators $r_p r_q$ $(p, q \in \boldsymbol{I}_{n-1})$ of $\widetilde{\mathfrak{A}}_n$, invertible elements in the algebra $\mathcal{A}_{2^k}$ as follows.

Definition 6.1.

$$
\begin{aligned}
\nabla_n(r_p) &:= \frac{1}{\sqrt{2}}(Y'_p - Y'_{p+1}) = \frac{(-1)^{p-1}}{\sqrt{2}}(Y_p + Y_{p+1}) \quad (p \in \boldsymbol{I}_{n-1})\,; \\
\nabla'_n(r_p) &:= \frac{1}{\sqrt{2}}(Y_p - Y_{p+1}) \qquad (p \in \boldsymbol{I}_{n-1})\,; \\
\nabla''_n(r_p) &:= \nabla'_n(r_p) \cdot iY_{2k+1} \qquad (p \in \boldsymbol{I}_{n-1}, \ i = \sqrt{-1}\,)\,; \\
\mho_n(r_p r_q) &:= \nabla'_n(r_p)\nabla'_n(r_q) \qquad (p, q \in \boldsymbol{I}_{n-1}).
\end{aligned}
$$

Note that, for $p, q \in \boldsymbol{I}_{2k-1}$, we have $\mho_n(r_p r_q) = \nabla''_n(r_p)\nabla''_n(r_q)$ too.

By explicit calculations we can prove the following intertwining relations. On the other hand, these intertwining relations for Y'_j $(j \in \boldsymbol{I}_n)$ or Y_j $(j \in \boldsymbol{I}_n)$ determine the intertwining operators uniquely up to scalar multiples, since the latter generates the total algebra $\mathcal{A}_{2^k}$.

Theorem 6.6. *There hold the following intertwining relations.*
(i) *Let* $n = 2k+1$. *For generators* Y'_j $(j \in \boldsymbol{I}_n)$ *or* Y_j $(j \in \boldsymbol{I}_n)$ *of* $\mathcal{A}_{2^k}$,

$$
\begin{aligned}
\nabla_n(r_p)\, Y'_j\, \nabla_n(r_p)^{-1} &= -Y'_{s_p(j)} \quad (p \in \boldsymbol{I}_{n-1}), \\
\nabla'_n(r_p)\, Y_j\, \nabla'_n(r_p)^{-1} &= -Y_{s_p(j)} \quad (p \in \boldsymbol{I}_{n-1}), \\
\mho_n(r_p r_q)\, Y_j\, \mho_n(r_p r_q)^{-1} &= Y_{s_p s_q(j)} \quad (p, q \in \boldsymbol{I}_{n-1}).
\end{aligned}
$$

(ii) *Let* $n = 2k$. *For the generators* Y_j $(j \in \boldsymbol{I}_n)$ *of* $\mathcal{A}_{2^k}$,

$$
\nabla''_n(r_p) Y_j \nabla''_n(r_p)^{-1} = Y_{s_p(j)} \quad (p \in \boldsymbol{I}_{n-1}).
$$

By using explicit formulas in Definition 6.1, we can check if we get actually spin representations or non-spin ordinary linear representations of $\mathfrak{S}_n$ or of $\mathfrak{A}_n$. Then we get the following result.

Theorem 6.7. (i) *For $n = 2k+1$, ∇_n and ∇'_n define respectively spin representations of $\widetilde{\mathfrak{S}}_n$, and $\mho_n$ define a spin representation of $\widetilde{\mathfrak{A}}_n$.*

(ii) *For $n = 2k$ even, ∇''_n defines a spin representation of $\widetilde{\mathfrak{S}}_n$.*

Sketch of Proof. We check the fundamental relations in (6.25). For instance for ∇_n, we prove the following relations:

$$\begin{aligned}
&\nabla_n(r_p)^2 = E && (p \in \boldsymbol{I}_{n-1}),\\
&\big(\nabla_n(r_p)\nabla_n(r_{p+1})\big)^3 = E && (p \in \boldsymbol{I}_{n-2}),\\
&\nabla_n(r_p)\nabla_n(r_q) = -\nabla_n(r_q)\nabla_n(r_p) && (|p-q| \geq 2,\ p,q \in \boldsymbol{I}_{n-1}).
\end{aligned}$$

We omit here the details, but the readers, who are interested in the proof, can continue by imitating calculations in §§7 – 8 below. □

From these two theorems, we obtain the following.

Theorem 6.8. *For the representations ∇_n, ∇'_n, $\mho_n$ and ∇''_n, there hold the following intertwining relations: for elements Y in the algebra $\mathcal{A}_{2^k} = M(2^k, \boldsymbol{C})$,*

$$\begin{aligned}
&\nabla_n(\sigma')\,Y\,\nabla_n(\sigma')^{-1} = \sigma^{\mathrm{I}}(Y) && (\sigma' \in \widetilde{\mathfrak{S}}_n,\ n = 2k+1)\,;\\
&\nabla'_n(\sigma')\,Y\,\nabla'_n(\sigma')^{-1} = \sigma^{\mathrm{II},-}(Y) && (\sigma' \in \widetilde{\mathfrak{S}}_n,\ n = 2k+1)\,;\\
&\mho_n(\sigma')\,Y\,\mho_n(\sigma')^{-1} = \sigma^{\mathrm{II}}(Y) && (\sigma' \in \widetilde{\mathfrak{A}}_n,\ n = 2k+1)\,;\\
&\nabla''_n(\sigma')\,Y\,\nabla''_n(\sigma')^{-1} = \sigma^{\mathrm{II}}(Y) && (\sigma' \in \widetilde{\mathfrak{S}}_n,\ n = 2k)\,;
\end{aligned}$$

*where $\sigma = \Phi_{\mathfrak{S}}(\sigma')$. Thus the 'generator permutation' actions of $\mathfrak{S}_n$ or of $\mathfrak{A}_n$ on the algebra $\mathcal{A}_{2^k}$ are realized by **spin** representations of $\mathfrak{S}_n$ or $\mathfrak{A}_n$ respectively.*

Remark 6.3 (Relation to Schur's Hauptdarstellung Δ_n of $\mathfrak{T}_n$)**.**

In the case of $n = 2k$ even, the spin representations ∇_n and ∇'_n of $\widetilde{\mathfrak{S}}_n$ are of dimension 2^k and both are equivalent to the direct sum $\Delta'_n \oplus (\mathrm{sgn}\cdot\Delta'_n)$, where Δ'_n denotes 'Schur's Hauptdarstellung' rewritten for $\widetilde{\mathfrak{S}}_n = \mathfrak{T}'_n$ of dimension $2^{(n-1)/2} = 2^{k-1}$, and sgn denotes the one-dimensional character $\widetilde{\mathfrak{S}}_n \to \mathfrak{S}_n \xrightarrow{\mathrm{sgn}} \{\pm 1\}$.

On the other hand, in the case of $n = 2k+1$ odd, the spin representations ∇_n and ∇'_n of $\widetilde{\mathfrak{S}}_n$ are of dimension 2^k and both are equivalent to Δ'_n itself, where $\Delta'_n \cong \mathrm{sgn}\cdot\Delta'_n$.

These facts can be proved by calculating characters of them (see §15 in [II]).

References

[Alt] S.L. Altmann, Hamilton, Rodrigues, and the quaternion scandal, *What went wrong with one of the major mathematical discoveries of the nineteenth century,* Mathematics Magazine, **62**(1989), 291-308.

[Bar] V. Bargmann, Irreducible unitary representations of the Lorentz group, Ann. Math., **48**(1947), 568-640.

[Car] É. Cartan, Les groupes projectifs qui ne laissent invariante aucune multiplicité plane, Bull. Soc. Math. France, **41**(1913), 53-96.

[CC] S.-S. Chern and C. Chevalley, Élie Cartan and his mathematical work, Bull. Amer. Math. Soc., **58**(1952), 217-250.

[Clif] A.H. Clifford, Representations induced in an invariant subgroup, Ann. Math., **38**(1937), 533-550.

[DaMo] J.W. Davies and A.O. Morris, The Schur multiplier of the generalized symmetric group, J. London Math. Soc., **(2) 8**(1974), 615-620.

[Dir1] P.M.A. Dirac, On the theory of quantum mechanics, Proceedings of the Royal Society London A, **112**(1926), 661-677.

[Dir2] P.M.A. Dirac, The quantum theory of electron, ibid, **117**(1928), 610-624; and Part II, ibid., **118**(1928), 351-361.

[Fro1] F. Frobenius, Über Gruppencharaktere, Sitzungsberichte der Königlich Preußischen Akademie der Wissenschaften zu Berlin, 985–1021(1896).

[Fro2] F. Frobenius, Über die Primfactoren der Gruppendeterminante, ibid., 1343–1382(1896).

[Fruc] R. Frucht, Zur Darstellung endlicher Abelscher Gruppen durch Kollineationen, Math. Z., **63**(1955), 145-155.

[GeNa] I.M. Gelfand and M.I. Naimark, Unitary representations of Lorentz group (in Russian), Izvestia Akad. Nauk SSSR, **11**(1947), 411-504 [English Translation in Collected Works, Vol. 2, pp.41–123].

[Ham] W.R. Hamilton, On a new species of imaginary quantities connected with the theory of quaternions, Proc. Roy. Irish Acad., **2**(1843), 424-434.

[HC] Harish-Chandra, Infinite irreducible representations of the Lorentz group, Proceedings of the Royal Society London A, **189**(1947), 327-401.

[HHH3] T. Hirai, E. Hirai and A. Hora, Projective representations and spin characters of complex reflection groups $G(m,p,n)$ and $G(m,p,\infty)$, I, in this volume (referred as [I]).

[HHoH2] T. Hirai, A. Hora and E. Hirai, Projective representations and spin characters of complex reflection groups $G(m,p,n)$ and $G(m,p,\infty)$, II, Case of generalized symmetric groups, in this volume (referred as [II])

[HiYa] T. Hirai and H. Yamashita, Seminars on introduction to the theory of group representations — from concrete examples to the top of the theory —, Yûseisha, 2007 (in Japanese).

[IhYo] S. Ihara and T. Yokonuma, On the second cohomology groups (Shur multipliers) of finite reflexion groups, J. Fac. Sci. Univ. Tokyo, Ser. 1, **IX**(1965), 155-171.

[Kar] G. Karpilovski, *The Schur multiplier*, London Math. Soc. Monographs, New Series, vol.**2**, Oxford Science Publications, Clarendon Press, Oxford, 1987.

[Mac1] G.W. Mackey, Induced representations of locally compact groups I, Ann. Math., **55**(1952), 101-139.

[Mac2] G.W. Mackey, Unitary representations of group extensions, I, Acta Math., **99**(1958), 265-311.

[Mol] T. Molien, Eine Bemerkung zur Theorie der homogenen Substitutionensgruppe, Sitzungsberichte der Dorpater Naturforscher-Gesellschaft, **11**(1897), 259-274.

[Mor2] A.O. Morris, Projective representations of abelian groups, J. London Math. Soc., **(2) 7**(1973), 235-238.

[Pau1] W. Pauli, Über den Zusammenhang des Abschlusses der Elektronengruppen im Atom mit der Komplexstruktur der Spektren, Zeitschrift für Physik, **31**(1925), 765-783.

[Pau2] W. Pauli, Zur Quantenmechanik des magnetischen Elektrons, ibid., **43**(1927), 601-623.

[Rea1] E.W. Read, On the Schur multipliers of the finite imprimitive unitary reflexion groups $G(m,p,n)$, J. London Math. Soc., **(2), 13**(1976), 150-154.

[Rod] (Benjamin) Olinde Rodrigues, Des lois géométriques qui régissent les déplacements d'un système solide dans l'espace, et la variation des coordonnées provenant de ses déplacements considérés indépendamment des causes qui peuvent les produire, J. de Math. Pures et Appliquées, **5**(1840), 380-440.

[Sait] M. Saito, Représentations unitaires des groupes symplectiques, J. Math. Soc. Japan, **24**(1972), 232-251.

[Sch1] J. Schur, Über die Darstellung der endlichen Gruppen durch gebrochene lineare Substitutionen, J. für die reine und angewante Mathematik, **127**(1904), 20–50.

[Sch2] J. Schur, Untersuchungen über die Darstellung der endlichen Gruppen durch gebrochene lineare Substitutionen, ibid., **132**(1907), 85–137.

[Sch3] J. Schur, Über die Darstellung der symmetrischen und der alternierenden Gruppen durch gebrochene lineare Substitutionen, ibid., **139**(1911), 155–255.

[Waer] B.L. van der Waerden, Spinoranalyse, Nachrichten von der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse, **1929**, 100-109. http://gdz.sub.uni-goettingen.de/search-entry.shtml

[Weil] A. Weil, Sur certains groupes d'opérateurs unitaires, Acta Math., **111**(1964), 143-211.

[Wey1] H. Weyl, *Gruppentheorie und Quantenmechanik*, 1st edition 1928, 2nd edition 1931, Hirzel, Leipzig; English Translation of 2nd edition by H.P. Robertson, Dutton, New York, 1932.

[Wey2] H. Weyl, *Classical Groups, Their Invariants and Representations*, Princeton University Press, 1939.

[Wig] E. Wigner, On unitary representations of the inhomogeneous Lorentz group, Ann. Math., **40**(1939), 149-204.

[Yos] H. Yoshida, Remarks on metaplectic representations of $SL(2)$, J. Math. Soc. Japan, **44**(1992), 351-373.