CHAPTER 10

Reducibility

We examine the condition for the decomposition $P_{\mathbf{m}} = P_{\mathbf{m}'}P_{\mathbf{m}''}$ of universal operators with or without fixing the characteristic exponents (cf. Theorem 4.19 i)), which implies the reducibility of the equation $P_{\mathbf{m}}u = 0$. Note that the irreducibility of a Fuchsian differential equation equals the irreducibility of the monodromy of the equation and that it is kept under our reduction of the equation. In §10.2 we study the value of spectral parameters which makes the equation reducible and obtain Theorem 10.10. In particular we have a necessary and sufficient condition on characteristic exponents so that the monodromy of the solutions of the equation $P_{\mathbf{m}}u = 0$ with a rigid spectral type $\mathbf{m}$ is irreducible, which is given in Theorem 10.13.

10.1. Direct decompositions

For a realizable $(p+1)$-tuple $\mathbf{m} \in \mathcal{P}_{p+1}^{(n)}$, Theorem 6.14 gives the universal Fuchsian differential operator $P_{\mathbf{m}}(\lambda_{j,\nu}, g_i)$ with the Riemann scheme (4.15). Here $g_1, \dots, g_N$ are accessory parameters and $N = \operatorname{Ridx}\mathbf{m}$.

First suppose $\mathbf{m}$ is basic. Choose positive numbers n', n'', $m'_{j,1}$ and $m''_{j,1}$ such that

$$\begin{gathered} n = n' + n'', \quad 0 < m'_{j,1} \le n', \quad 0 < m''_{j,1} \le n'', \\ m'_{0,1} + \cdots + m'_{p,1} \le (p-1)n', \quad m''_{0,1} + \cdots + m''_{p,1} \le (p-1)n''. \end{gathered} \tag{10.1}$$

We choose other positive integers $m'_{j,\nu}$ and $m''_{j,\nu}$ so that $\mathbf{m}' = \bigl(m'_{j,\nu}\bigr)$ and $\mathbf{m}'' = \bigl(m''_{j,\nu}\bigr)$ are monotone tuples of partitions of n' and n'', respectively, and moreover

$$\mathbf{m} = \mathbf{m}' + \mathbf{m}''. \tag{10.2}$$

Theorem 6.6 shows that $\mathbf{m}'$ and $\mathbf{m}''$ are realizable. If $\{\lambda_{j,\nu}\}$ satisfies the Fuchs relation

$$\sum_{j=0}^{p}\sum_{\nu=1}^{n_j} m'_{j,\nu}\lambda_{j,\nu} = n' - \frac{\operatorname{idx}\mathbf{m}'}{2} \tag{10.3}$$

for the Riemann scheme $\bigl\{[\lambda_{j,\nu}]_{(m'_{j,\nu})}\bigr\}$, Theorem 4.19 shows that the operators

$$P_{\mathbf{m}''}(\lambda_{j,\nu} + m'_{j,\nu} - \delta_{j,0}(p-1)n', g''_i) \cdot P_{\mathbf{m}'}(\lambda_{j,\nu}, g'_i) \tag{10.4}$$

has the Riemann scheme $\{[\lambda_{j,\nu}]_{(m_{j,\nu})}\}$. Hence the equation $P_{\mathbf{m}}(\lambda_{j,\nu}, g_i)u = 0$ is not irreducible when the parameters take the values corresponding to (10.4).

In this section, we study the condition

$$\operatorname{Ridx}\mathbf{m} = \operatorname{Ridx}\mathbf{m}' + \operatorname{Ridx}\mathbf{m}'' \tag{10.5}$$

for realizable tuples $\mathbf{m}'$ and $\mathbf{m}''$ with $\mathbf{m} = \mathbf{m}' + \mathbf{m}''$. Under this condition the Fuchs relation (10.3) assures that the universal operator is reducible for any values of accessory parameters.

DEFINITION 10.1 (direct decomposition). If realizable tuples $\mathbf{m}$, $\mathbf{m}'$ and $\mathbf{m}''$ satisfy (10.2) and (10.5), we define that $\mathbf{m}$ is the *direct sum* of $\mathbf{m}'$ and $\mathbf{m}''$ and call $\mathbf{m} = \mathbf{m}' + \mathbf{m}''$ a *direct decomposition* of $\mathbf{m}$ and express it as follows.

$$\mathbf{m} = \mathbf{m}' \oplus \mathbf{m}''. \tag{10.6}$$

THEOREM 10.2. *Let* (10.6) *be a direct decomposition of a realizable tuple* $\mathbf{m}$*.*

i) *Suppose* $\mathbf{m}$ *is irreducibly realizable and* $\operatorname{idx}\mathbf{m}'' > 0$*. Put* $\overline{\mathbf{m}}' = \gcd(\mathbf{m}')^{-1}\mathbf{m}'$*. If* $\mathbf{m}'$ *is indivisible or* $\operatorname{idx}\mathbf{m} \le 0$*, then*

$$\alpha_{\mathbf{m}} = \alpha_{\mathbf{m}'} - 2\frac{(\alpha_{\overline{\mathbf{m}}''}|\alpha_{\mathbf{m}'})}{(\alpha_{\overline{\mathbf{m}}''}|\alpha_{\overline{\mathbf{m}}''})}\alpha_{\overline{\mathbf{m}}''} \tag{10.7}$$

or $\mathbf{m} = \mathbf{m}' \oplus \mathbf{m}''$ *is isomorphic to one of the decompositions*

$$\begin{aligned}
32,32,32,221 &= 22,22,22,220 \oplus 10,10,10,10,001\\
322,322,2221 &= 222,222,2220 \oplus 100,100,0001\\
54,3222,22221 &= 44,2222,22220 \oplus 10,1000,00001\\
76,544,2222221 &= 66,444,2222220 \oplus 10,100,0000001
\end{aligned} \tag{10.8}$$

under the action of $\widetilde{W}_\infty$*.*

ii) *Suppose* $\operatorname{idx}\mathbf{m} \le 0$ *and* $\operatorname{idx}\mathbf{m}' \le 0$ *and* $\operatorname{idx}\mathbf{m}'' \le 0$*. Then* $\mathbf{m} = \mathbf{m}' \oplus \mathbf{m}''$ *or* $\mathbf{m} = \mathbf{m}'' \oplus \mathbf{m}'$ *is transformed into one of the decompositions*

$$\begin{gathered}
\Sigma = 11,11,11,11 \quad 111,111,111 \quad 22,1^4,1^4 \quad 33,222,1^6\\
m\Sigma = k\Sigma \oplus \ell\Sigma\\
mm,mm,mm,m(m-1)1 = kk,kk,kk,k(k-1)1 \oplus \ell\ell,\ell\ell,\ell\ell,\ell\ell 0\\
mmm,mmm,mm(m-1)1 = kkk,kkk,kkk,kk(k-1)1 \oplus \ell\ell\ell,\ell\ell\ell,\ell\ell\ell 0\\
(2m)^2,m^4,mmm(m-1)1 = (2k)^2,k^4,k^4,kkk(k-1)1 \oplus (2\ell)^2,\ell^4,\ell^4 0\\
(3m)^2,(2m)^3,m^5(m-1)1 = (3k)^2,(2k)^3,k^5(k-1)1 \oplus (3\ell)^2,(2\ell)^3,\ell^6 0
\end{gathered} \tag{10.9}$$

under the action of $\widetilde{W}_\infty$*. Here* m*,* k *and* ℓ *are positive integers satisfying* $m = k+\ell$*. These are expressed by*

$$\begin{aligned}
&m\tilde{D}_4 = k\tilde{D}_4 \oplus \ell\tilde{D}_4, \quad m\tilde{E}_j = k\tilde{E}_j \oplus \ell\tilde{E}_j \quad (j = 6,7,8),\\
&D_4^{(m)} = D_4^{(k)} \oplus \ell\tilde{D}_4, \quad E_j^{(m)} = E_j^{(k)} \oplus \ell\tilde{E}_j \quad (j = 6,7,8).
\end{aligned} \tag{10.10}$$

PROOF. Put $\mathbf{m}' = k\overline{\mathbf{m}}'$ and $\mathbf{m}'' = \ell\overline{\mathbf{m}}''$ with indivisible $\overline{\mathbf{m}}'$ and $\overline{\mathbf{m}}''$. First note that

$$(\alpha_{\mathbf{m}}|\alpha_{\mathbf{m}}) = (\alpha_{\mathbf{m}'}|\alpha_{\mathbf{m}'}) + 2(\alpha_{\mathbf{m}'}|\alpha_{\mathbf{m}''}) + (\alpha_{\mathbf{m}''}|\alpha_{\mathbf{m}''}). \tag{10.11}$$

ii) Using Lemma 10.3, we will prove the theorem. If $\operatorname{idx}\mathbf{m} = 0$, then (10.11) and (10.12) show $0 = (\alpha_{\mathbf{m}'}|\alpha_{\mathbf{m}''}) = k\ell(\alpha_{\overline{\mathbf{m}}'}|\alpha_{\overline{\mathbf{m}}''})$, Lemma 10.3 proves $\operatorname{idx}\mathbf{m}' = 0$ and $\overline{\mathbf{m}}' = \overline{\mathbf{m}}''$ and we have the theorem.

Suppose $\operatorname{idx}\mathbf{m} < 0$.

If $\operatorname{idx}\mathbf{m}' < 0$ and $\operatorname{idx}\mathbf{m}'' < 0$, we have $\operatorname{Pidx}\mathbf{m} = \operatorname{Pidx}\mathbf{m}' + \operatorname{Pidx}\mathbf{m}''$, which implies $(\alpha_{\mathbf{m}'}|\alpha_{\mathbf{m}''}) = -1$ and contradicts to Lemma 10.3.

Hence we may assume $\operatorname{idx}\mathbf{m}'' = 0$.

<u>Case: $\operatorname{idx}\mathbf{m}' < 0$</u>. It follows from (10.11) that $2 - 2\operatorname{Ridx}\mathbf{m} = 2 - 2\operatorname{Ridx}\mathbf{m}' + 2\ell(\mathbf{m},\overline{\mathbf{m}})$. Since $\operatorname{Ridx}\mathbf{m} = \operatorname{Ridx}\mathbf{m}' + \ell$, we have $(\alpha_{\mathbf{m}}|\alpha_{\overline{\mathbf{m}}'}) = -1$ and the theorem follows from Lemma 10.3.

<u>Case: $\operatorname{idx}\mathbf{m}' = 0$</u>. It follows from (10.11) that $2 - 2\operatorname{Ridx}\mathbf{m} = 2k\ell(\alpha_{\overline{\mathbf{m}}'}|\alpha_{\overline{\mathbf{m}}''})$. Since the condition $\operatorname{Ridx}\mathbf{m} = k+\ell$ shows $(\alpha_{\overline{\mathbf{m}}'}|\alpha_{\overline{\mathbf{m}}''}) = \frac{1}{k\ell} - \frac{1}{k} - \frac{1}{\ell}$ and we have $(\alpha_{\overline{\mathbf{m}}'}|\alpha_{\overline{\mathbf{m}}''}) = -1$. Hence the theorem also follows from Lemma 10.3.

i) First suppose $\operatorname{idx}\mathbf{m}' \neq 0$. Note that $\mathbf{m}$ and $\mathbf{m}'$ are rigid if $\operatorname{idx}\mathbf{m}' > 0$. We have $\operatorname{idx}\mathbf{m} = \operatorname{idx}\mathbf{m}'$ and $\operatorname{idx}\mathbf{m} = (\alpha_{\mathbf{m}'} + \ell\alpha_{\overline{\mathbf{m}}''}|\alpha_{\mathbf{m}'} + \ell\alpha_{\overline{\mathbf{m}}''}) = \operatorname{idx}\mathbf{m}' + 2\ell(\alpha_{\mathbf{m}}|\alpha_{\overline{\mathbf{m}}''}) + 2\ell^2$, which implies (10.7).

Thus we may assume $\operatorname{idx}\mathbf{m} < 0$ and $\operatorname{idx}\mathbf{m}' = 0$. If $k = 1$, $\operatorname{idx}\mathbf{m} = \operatorname{idx}\mathbf{m}' = 0$ and we have (10.7) as above. Hence we may moreover assume $k \geq 2$. Then (10.11) and the assumption imply $2 - 2k = 2k\ell(\alpha_{\overline{\mathbf{m}}'}|\alpha_{\overline{\mathbf{m}}''}) + 2\ell^2$, which means

$$-(\alpha_{\overline{\mathbf{m}}'}|\alpha_{\overline{\mathbf{m}}''}) = \frac{k - 1 + \ell^2}{k\ell}.$$

Here k and ℓ are mutually prime and hence there exists a positive integer m with $k = m\ell + 1$ and

$$-(\alpha_{\overline{\mathbf{m}}'}|\alpha_{\overline{\mathbf{m}}''}) = \frac{m + \ell}{m\ell + 1} = \frac{1}{\ell + \frac{1}{m}} + \frac{1}{m + \frac{1}{\ell}} < 2.$$

Thus we have $m = \ell = 1$, $k = 2$ and $(\alpha_{\overline{\mathbf{m}}'}|\alpha_{\overline{\mathbf{m}}''}) = -1$. By the transformation of an element of $\widetilde{W}_\infty$, we may assume $\overline{\mathbf{m}}' \in \mathcal{P}_{p+1}$ is a tuple in (10.16). Since $(\alpha_{\overline{\mathbf{m}}'}|\alpha_{\overline{\mathbf{m}}''}) = -1$ and $\alpha_{\overline{\mathbf{m}}''}$ is a positive real root, we have the theorem by a similar argument as in the proof of Lemma 10.3. Namely, $m'_{p,n'_p} = 2$ and $m'_{p,n'_p+1} = 0$ and we may assume $m''_{j,n'_j+1} = 0$ for $j = 0, \ldots, p-1$ and $m''_{p,n'_p+1} + m''_{p,n'_p+2} + \cdots = 1$, which proves the theorem in view of $\alpha_{\mathbf{m}''} \in \Delta^{re}_+$. $\square$

Lemma 10.3. *Suppose* $\mathbf{m}$ *and* $\mathbf{m}'$ *are realizable and* $\operatorname{idx}\mathbf{m} \leq 0$ *and* $\operatorname{idx}\mathbf{m}' \leq 0$. *Then*

$$(\alpha_{\mathbf{m}}|\alpha_{\mathbf{m}'}) \leq 0. \tag{10.12}$$

If $\mathbf{m}$ *and* $\mathbf{m}'$ *are basic and monotone,*

$$(\alpha_{\mathbf{m}}|w\alpha_{\mathbf{m}'}) \leq (\alpha_{\mathbf{m}}|\alpha_{\mathbf{m}'}) \qquad (\forall w \in W_\infty). \tag{10.13}$$

If $(\alpha_{\mathbf{m}}|\alpha_{\mathbf{m}'}) = 0$ *and* $\mathbf{m}$ *and* $\mathbf{m}'$ *are indivisible, then* $\operatorname{idx}\mathbf{m} = 0$ *and* $\mathbf{m} = \mathbf{m}'$.

If $(\alpha_{\mathbf{m}}|\alpha_{\mathbf{m}'}) = -1$, *then the pair is isomorphic to one of the pairs*

$$\begin{aligned}
&(D_4^{(k)}, \tilde{D}_4) : \big((kk, kk, kk, k(k-1)1), && (11, 11, 11, 110)\big)\\
&(E_6^{(k)}, \tilde{E}_6) : \big((kkk, kkk, k(k-1)1), && (111, 111, 1110)\big)\\
&(E_7^{(k)}, \tilde{E}_7) : \big(((2k)^2, kkkk, kkk(k-1)1), && (22, 1111, 11110)\big)\\
&(E_8^{(k)}, \tilde{E}_8) : \big(((3k)^2, (2k)^3, kkkkk(k-1)1), && (33, 222, 1111110)\big)
\end{aligned} \tag{10.14}$$

under the action of $\widetilde{W}_\infty$.

Proof. We may assume that $\mathbf{m}$ and $\mathbf{m}'$ are indivisible. Under the transformation of the Weyl group, we may assume that $\mathbf{m}$ is a basic monotone tuple in $\mathcal{P}_{p+1}$, namely, $(\alpha_{\mathbf{m}}|\alpha_0) \leq 0$ and $(\alpha_{\mathbf{m}}|\alpha_{j,\nu}) \leq 0$.

If $\mathbf{m}'$ is basic and monotone, $w\alpha_{\mathbf{m}'} - \alpha_{\mathbf{m}'}$ is a sum of positive real roots, which proves (10.13).

Put $\alpha_{\mathbf{m}} = n\alpha_0 + \sum n_{j,\nu}\alpha_{j,\nu}$ and $\mathbf{m}' = n'_0\alpha_0 + \sum n'_{j,\nu}\alpha_{j,\nu}$. Then

$$\begin{gathered}
(\alpha_{\mathbf{m}}|\alpha_{\mathbf{m}'}) = n'_0(\alpha_{\mathbf{m}}|\alpha_0) + \sum n'_{j,\nu}(\alpha_{\mathbf{m}}|\alpha_{j,\nu}),\\
(\alpha_{\mathbf{m}}|\alpha) \leq 0 \quad (\forall \alpha \in \operatorname{supp}\alpha_{\mathbf{m}}).
\end{gathered} \tag{10.15}$$

Let k_j be the maximal positive integer satisfying $m_{j,k_j} = m_{j,1}$ and put $\Pi_0 = \{\alpha_0, \alpha_{j,\nu}\,;\, 1 \leq \nu < k_j,\ j = 0, \ldots, p\}$. Note that Π_0 defines a classical root system if $\operatorname{idx}\mathbf{m} < 0$ (cf. Remark 7.12).

Suppose $(\alpha_{\mathbf{m}}|\alpha_{\mathbf{m}'}) = 0$ and $\mathbf{m} \in \mathcal{P}_{p+1}$. Then $m_{0,1} + \cdots + m_{p,1} = (p-1)\operatorname{ord}\mathbf{m}$ and $\operatorname{supp}\alpha_{\mathbf{m}'} \subset \Pi_0$ because $(\alpha_{\mathbf{m}}|\alpha) = 0$ for $\alpha \in \operatorname{supp}\alpha_{\mathbf{m}'}$. Hence it follows from

$\operatorname{idx}\mathbf{m}' \le 0$ that $\operatorname{idx}\mathbf{m} = 0$ and we may assume that $\mathbf{m}$ is one of the tuples (10.16). Since $\operatorname{supp}\alpha_{\mathbf{m}'} \subset \operatorname{supp}\alpha_{\mathbf{m}}$ and $\operatorname{idx}\mathbf{m}' \le 0$, we conclude that $\mathbf{m}' = \mathbf{m}$.

Lastly suppose $(\alpha_{\mathbf{m}}|\alpha_{\mathbf{m}'}) = -1$.

Case: $\operatorname{idx}\mathbf{m} = \operatorname{idx}\mathbf{m}' = 0$. If $\mathbf{m}'$ is basic and monotone and $\mathbf{m}' \ne \mathbf{m}$, then it is easy to see that $(\alpha_{\mathbf{m}}|\alpha_{\mathbf{m}'}) < -1$ (cf. Remark 7.1). Hence (10.13) assures $\mathbf{m}' = w\mathbf{m}$ with a certain $w \in W_\infty$ and therefore $\operatorname{supp}\mathbf{m} \subsetneq \operatorname{supp}\mathbf{m}'$. Moreover there exists j_0 and $L \ge k_{j_0}$ such that $\operatorname{supp} m' = \operatorname{supp} m \cup \{\alpha_{j_0,k_{j_0}}, \alpha_{j_0,k_j+1}, \dots, \alpha_{j_0,L}\}$ and $m_{j_0,k_{j_0}} = 1$ and $m'_{j_0,k_{j_0}+1} = 1$. Then by a transformation of an element of the Weyl group, we may assume $L = k_{j_0}$ and $\mathbf{m}' = r_{i_N}\cdots r_{i_1} r_{(j_0,k_{j_0})}\mathbf{m}$ with suitable i_ν satisfying $\alpha_{i_\nu} \in \operatorname{supp}\mathbf{m}$ for $\nu = 1, \dots, N$. Applying $r_{i_1}\cdots r_{i_N}$ to the pair $(\mathbf{m}, \mathbf{m}')$, we may assume $\mathbf{m}' = r_{(j_0,k_{j_0})}\mathbf{m}$. Hence the pair $(\mathbf{m}, \mathbf{m}')$ is isomorphic to one of the pairs in the list (10.14) with $k = 1$.

Case: $\operatorname{idx}\mathbf{m} < 0$ and $\operatorname{idx}\mathbf{m}' \le 0$. There exists j_0 such that $\operatorname{supp}\alpha_{\mathbf{m}'} \ni \alpha_{j_0,k_j}$. Then the fact $\operatorname{idx}(\mathbf{m}, \mathbf{m}') = -1$ implies $n'_{j_0,k_0} = 1$ and $n'_{j,k_j} = 0$ for $j \ne j_0$. Let L be the maximal positive integer with $n'_{j_0,L} \ne 0$. Since $(\alpha_{\mathbf{m}}|\alpha_{j_0,\nu}) = 0$ for $k_0 + 1 \le \nu \le L$, we may assume $L = k_0$ by the transformation $r_{(j_0,k_0+1)} \circ \cdots \circ r_{(j_0,L)}$ if $L > k_0$. Since the Dynkin diagram corresponding to $\Pi_0 \cup \{\alpha_{j_0,k_{j_0}}\}$ is classical or affine and $\operatorname{supp}\mathbf{m}'$ is contained in this set, $\operatorname{idx}\mathbf{m}' = 0$ and $\mathbf{m}'$ is basic and we may assume that $\mathbf{m}'$ is one of the tuples

$$11,11,11,11 \quad 111,111,111 \quad 22,1111,1111 \quad 33,222,111111 \tag{10.16}$$

and $j_0 = p$. In particular $m'_{p,1} = \cdots = m'_{p,k_p} = 1$ and $m'_{p,k_p+1} = 0$. It follows from $(\alpha_{\mathbf{m}}|\alpha_{p,k_p}) = -1$ that there exists an integer $L' \ge k_p + 1$ satisfying $\operatorname{supp}\mathbf{m} = \operatorname{supp}\mathbf{m}' \cup \{\alpha_{p,\nu}\,;\, k_p \le \nu < L'\}$ and $m_{p,k_p} = m_{p,k_p-1} - 1$. In particular, $m_{j,\nu} = m_{j,1}$ for $\nu = 1, \dots, k_j - \delta_{j,p}$ and $j = 0, \dots, p$. Since $\sum_{j=0}^{p} m_{j,1} = (p-1)\operatorname{ord}\mathbf{m}$, there exists a positive integer k such that

$$m_{j,\nu} = \begin{cases} km'_{j,1} & (j = 0, \dots, p,\ \nu = 1, \dots, k_j - \delta_{j,p}), \\ km'_{p,1} - 1 & (j = p,\ \nu = k_p). \end{cases}$$

Hence $m_{p,k_p+1} = 1$ and $L' = k_p + 1$ and the pair $(\mathbf{m}, \mathbf{m}')$ is one of the pairs in the list (10.14) with $k > 1$. $\square$

REMARK 10.4. Let k be an integer with $k \ge 2$ and let P be a differential operator with the spectral type $D_4^{(k)}$, $E_6^{(k)}$, $E_7^{(k)}$ or $E_8^{(k)}$. It follows from Theorem 4.19 and Theorem 6.14 that P is reducible for any values of accessory parameters when the characteristic exponents satisfy Fuchs relation with respect to the subtuple given in (10.14). For example, the Fuchsian differential operator P with the Riemann scheme

$$\begin{Bmatrix} [\lambda_{0,1}]_{(k)} & [\lambda_{1,1}]_{(k)} & [\lambda_{2,1}]_{(k)} & [\lambda_{3,1}]_{(k)} \\ [\lambda_{0,2}]_{(k)} & [\lambda_{1,2}]_{(k)} & [\lambda_{2,2}]_{(k)} & [\lambda_{3,2}]_{(k-1)} \\ & & & \lambda_{3,2} + 2k - 2 \end{Bmatrix}$$

is reducible.

EXAMPLE 10.5. i) (generalized Jordan-Pochhammer) If $\mathbf{m} = k\mathbf{m}' \oplus \ell\mathbf{m}''$ with a rigid tuples $\mathbf{m}$, $\mathbf{m}'$ and $\mathbf{m}''$ and positive integers k and ℓ satisfying $1 \le k \le \ell$, we have

$$(\alpha_{\mathbf{m}'}|\alpha_{\mathbf{m}''}) = -\frac{k^2 + \ell^2 - 1}{k\ell} \in \mathbb{Z}. \tag{10.17}$$

For positive integers k and ℓ satisfying $1 \le k \le \ell$ and

$$p := \frac{k^2 + \ell^2 - 1}{k\ell} + 1 \in \mathbb{Z}, \tag{10.18}$$

we have an example of direct decompositions

$$\begin{aligned}\overbrace{\ell k, \ell k, \ldots, \ell k}^{p+1 \text{ partitions}} &= 0k, 0k, \ldots, 0k \oplus \ell 0, \ell 0, \ldots, \ell 0 \\ &= ((p-1)k-\ell)k, ((p-1)k-\ell)k, \ldots, ((p-1)k-\ell)k \\ &\quad \oplus (2\ell-(p-1)k)0, (2\ell-(p-1)k)0, \ldots, (2\ell-(p-1)k)0.\end{aligned} \tag{10.19}$$

Here $p = 3 + \frac{(k-\ell)^2-1}{k\ell} \geq 2$ and the condition $p = 2$ implies $k = \ell = 1$ and the condition $p = 3$ implies $\ell = k+1$. If $k = 1$, then $(\alpha_{\mathbf{m}'}|\alpha_{\mathbf{m}''}) = -\ell$ and we have an example corresponding to Jordan-Pochhammer equation:

$$\overbrace{\ell 1, \cdots, \ell 1}^{\ell+2 \text{ partitions}} = 01, \cdots, 01 \oplus \ell 0, \cdots, \ell 0. \tag{10.20}$$

When $\ell = k+1$, we have $(\alpha_{\mathbf{m}'}|\alpha_{\mathbf{m}''}) = -2k$ and an example

$$\begin{aligned}&(k+1)k, (k+1)k, (k+1)k, (k+1)k \\ &\quad = 0k, 0k, 0k, 0k \oplus (k+1)0, (k+1)0, (k+1)0, (k+1)0 \\ &\quad = (k-1)k, (k-1)k, (k-1)k, (k-1)k \oplus 20, 20, 20, 20.\end{aligned} \tag{10.21}$$

We have another example

$$\begin{aligned}83, 83, 83, 83, 83 &= 03, 03, 03, 03, 03 \oplus 80, 80, 80, 80, 80 \\ &= 13, 13, 13, 13, 13 \oplus 70, 70, 70, 70, 70\end{aligned} \tag{10.22}$$

in the case $(k, \ell) = (3, 8)$, which is a special case where $\ell = k^2 - 1$, $p = k+1$ and $(\alpha_{\mathbf{m}'}|\alpha_{\mathbf{m}''}) = -k$.

When p is odd, the equation (10.18) is equal to the Pell equation

$$y^2 - (m^2-1)x^2 = 1 \tag{10.23}$$

by putting $p - 1 = 2m$, $x = \ell$ and $y = m\ell - k$ and hence the reduction of the tuple of partition (10.19) by $\partial_{\max}$ and its inverse give all the integer solutions of this Pell equation.

The tuple of partitions $\ell k, \ell k, \ldots, \ell k \in \mathcal{P}_{p+1}^{(\ell+k)}$ with (10.18) is called a *generalized Jordan-Pochhammer* tuple and denoted by $P_{p+1,\ell+k}$. In particular, $P_{n+1,n}$ is simply denoted by P_n.

ii) We give an example of direct decompositions of a rigid tuple:

$$\begin{aligned}3322, 532, 532 &= 0022, 202, 202 \oplus 3300, 330, 330 : 1 \\ &= 1122, 312, 312 \oplus 2200, 220, 220 : 1 \\ &= 0322, 232, 232 \oplus 3000, 300, 300 : 2 \\ &= 3302, 332, 332 \oplus 0020, 200, 200 : 2 \\ &= 1212, 321, 321 \oplus 2110, 211, 211 : 4 \\ &= 2211, 321, 312 \oplus 1111, 211, 220 : 2 \\ &= 2212, 421, 322 \oplus 1110, 111, 210 : 4 \\ &= 2222, 431, 422 \oplus 1100, 101, 110 : 2 \\ &= 2312, 422, 422 \oplus 1010, 110, 110 : 4 \\ &= 2322, 522, 432 \oplus 1000, 010, 100 : 4.\end{aligned}$$

They are all the direct decompositions of the tuple 3322, 532, 532 modulo obvious symmetries. Here we indicate the number of the decompositions of the same type.

Corollary 10.6. *Let* $\mathbf{m} \in \mathcal{P}$ *be realizable. Put* $\mathbf{m} = \gcd(\mathbf{m})\overline{\mathbf{m}}$. *Then* $\mathbf{m}$ *has no direct decomposition* (10.6) *if and only if*

$$\operatorname{ord} \mathbf{m} = 1 \tag{10.24}$$

or

$$\operatorname{idx}\mathbf{m} = 0 \textit{ and basic} \tag{10.25}$$

or

$$\begin{aligned}&\operatorname{idx}\mathbf{m} < 0 \textit{ and } \overline{\mathbf{m}} \textit{ is basic and } \mathbf{m} \textit{ is not isomorphic to any one of tuples}\\ &\textit{in } \text{Example 7.14 } \textit{with } m > 1.\end{aligned} \tag{10.26}$$

Moreover we have the following result.

Proposition 10.7. *The direct decomposition* $\mathbf{m} = \mathbf{m}' \oplus \mathbf{m}''$ *is called rigid decomposition if* $\mathbf{m}$, $\mathbf{m}'$ *and* $\mathbf{m}''$ *are rigid. If* $\mathbf{m} \in \mathcal{P}$ *is rigid and* $\operatorname{ord}\mathbf{m} > 1$, *there exists a rigid decomposition.*

Proof. We may assume that $\mathbf{m}$ is monotone and there exist a non-negative integer p such that $m_{j,2} \neq 0$ if and only if $0 \le j < p+1$. If $\operatorname{ord}\partial\mathbf{m} = 1$, then we may assume $\mathbf{m} = (p-1)1, (p-1)1, \dots, (p-1)1 \in \mathcal{P}^{(p)}_{p+1}$ and there exists a decomposition

$$(p-1)1, (p-1)1, \dots, (p-1)1 = 01, 10, \dots, 10 \oplus (p-1)0, (p-2)1, \dots, (p-2)1.$$

Suppose $\operatorname{ord}\partial\mathbf{m} > 1$. Put $d = \operatorname{idx}(\mathbf{m}, \mathbf{1}) = m_{0,1} + \cdots + m_{p,1} - (p-1)\cdot\operatorname{ord}\mathbf{m} > 0$.

The induction hypothesis assures the existence of a decomposition $\partial\mathbf{m} = \bar{\mathbf{m}}' \oplus \bar{\mathbf{m}}''$ such that $\bar{\mathbf{m}}'$ and $\bar{\mathbf{m}}''$ are rigid. If $\partial\bar{\mathbf{m}}'$ and $\partial\bar{\mathbf{m}}''$ are well-defined, we have the decomposition $\mathbf{m} = \partial^2\mathbf{m} = \partial\bar{\mathbf{m}}' \oplus \partial\bar{\mathbf{m}}''$ and the proposition.

If $\operatorname{ord}\bar{\mathbf{m}}' > 1$, $\partial\bar{\mathbf{m}}'$ is well-defined. Suppose $\bar{\mathbf{m}}' = \big(\delta_{\nu,\ell_j}\big)_{\substack{j=0,\dots,p\\ \nu=1,2,\dots}}$. Then

$$\begin{aligned}\operatorname{idx}(\partial\mathbf{m}, \mathbf{1}) - \operatorname{idx}(\partial\mathbf{m}, \bar{\mathbf{m}}') &= \sum_{j=0}^{p}\big((m_{j,1} - d - (m_{j,\ell_j} - d\delta_{\ell_j,1})\big)\\ &\ge -d\#\{j\,;\,\ell_j > 1,\ 0 \le j \le p\}.\end{aligned}$$

Since $\operatorname{idx}(\partial\mathbf{m}, \mathbf{1}) = -d$ and $\operatorname{idx}(\partial\mathbf{m}, \bar{\mathbf{m}}') = 1$, we have $d\#\{j\,;\,\ell_j > 1,\ 0 \le j \le p\} \ge d+1$ and therefore $\#\{j\,;\,\ell_j > 1,\ 0 \le j \le p\} \ge 2$. Hence $\partial\bar{\mathbf{m}}'$ is well-defined. □

Remark 10.8. The author's original construction of a differential operator with a given rigid Riemann scheme doesn't use the middle convolutions and additions but uses Proposition 10.7.

Example 10.9. We give direct decompositions of a rigid tuple:

$$\begin{aligned}721, 3331, 22222 &= 200, 2000, 20000 \oplus 521, 1331, 02222 : 15\\ &= 210, 1110, 11100 \oplus 511, 2221, 11122 : 10\\ &= 310, 1111, 11110 \oplus 411, 2220, 11112 : \ \ 5\end{aligned} \tag{10.27}$$

The following irreducibly realizable tuple has only two direct decompositions:

$$\begin{aligned}44, 311111, 311111 &= 20, 200000, 200000 \oplus 24, 111111, 111111\\ &= 02, 200000, 200000 \oplus 42, 111111, 111111\end{aligned} \tag{10.28}$$

But it cannot be a direct sum of two irreducibly realizable tuples.

10.2. Reduction of reducibility

We give a necessary and sufficient condition so that a Fuchsian differential equation is irreducible, which follows from [**Kz**] and [**DR, DR2**]. Note that a Fuchsian differential equation is irreducible if and only if its monodromy is irreducible.

THEOREM 10.10. *Retain the notation in* §10.1. *Suppose* $\mathbf{m}$ *is monotone, realizable and* $\partial_{max}\mathbf{m}$ *is well-defined and*

$$d := m_{0,1} + \cdots + m_{p,1} - (p-1)\operatorname{ord}\mathbf{m} \geq 0. \tag{10.29}$$

Put $P = P_{\mathbf{m}}$ (cf. (6.25)) *and*

$$\mu := \lambda_{0,1} + \lambda_{1,1} + \cdots + \lambda_{p,1} - 1, \tag{10.30}$$

$$Q := \partial_{max}P, \tag{10.31}$$

$$P^o := P|_{\lambda_{j,\nu}=\lambda_{j,\nu}^o,\ g_i=g_i^o}, \quad Q^o := Q|_{\lambda_{j,\nu}=\lambda_{j,\nu}^o,\ g_i=g_i^o} \tag{10.32}$$

with some complex numbers $\lambda_{j,\nu}^o$ *and* g_i^o *satisfying the Fuchs relation* $|\{\lambda_{\mathbf{m}}^o\}| = 0$.

i) *The Riemann scheme* $\{\tilde\lambda_{\tilde{\mathbf{m}}}\}$ *of* Q *is given by*

$$\begin{cases} \tilde m_{j,\nu} = m_{j,\nu} - d\delta_{\nu,1}, \\ \tilde\lambda_{j,\nu} = \lambda_{j,\nu} + \big((-1)^{\delta_{j,0}} - \delta_{\nu,1}\big)\mu. \end{cases} \tag{10.33}$$

ii) *Assume that the equation* $P^o u = 0$ *is irreducible. If* $d > 0$, *then* $\mu \notin \mathbb{Z}$. *If the parameters given by* $\lambda_{j,\nu}^o$ *and* g_i^o *are locally non-degenerate, the equation* $Q^o v = 0$ *is irreducible and the parameters are locally non-degenerate.*

iii) *Assume that the equation* $Q^o v = 0$ *is irreducible and the parameters given by* $\lambda_{j,\nu}^o$ *and* g_i^o *are locally non-degenerate. Then the equation* $P^o v = 0$ *is irreducible if and only if*

$$\sum_{j=0}^{p} \lambda_{j,1+\delta_{j,j_o}(\nu_o-1)}^o \notin \mathbb{Z} \ \ \textit{for any}\ \ (j_o,\nu_o)\ \ \textit{satisfying}\ \ m_{j_o,\nu_o} > m_{j_o,1} - d. \tag{10.34}$$

If the equation $P^o v = 0$ *is irreducible, the parameters are locally non-degenerate.*

iv) *Put* $\mathbf{m}(k) := \partial_{max}^k\mathbf{m}$ *and* $P(k) = \partial_{max}^k P$. *Let* K *be a non-negative integer such that* $\operatorname{ord}\mathbf{m}(0) > \operatorname{ord}\mathbf{m}(1) > \cdots > \operatorname{ord}\mathbf{m}(K)$ *and* $\mathbf{m}(K)$ *is fundamental. The operator* $P(k)$ *is essentially the universal operator of type* $\mathbf{m}(k)$ *but parametrized by* $\lambda_{j,\nu}$ *and* g_i. *Put* $P(k)^o = P(k)|_{\lambda_{j,\nu}=\lambda_{j,\nu}^o}$.

If the equation $P^o u = 0$ *is irreducible and the parameters are locally non-degenerate, so are* $P(k)^o u = 0$ *for* $k = 1, \ldots, K$.

If the equation $P^o u = 0$ *is irreducible and locally non-degenerate, so is the equation* $P(K)^o u = 0$.

Suppose the equation $P(K)^o u = 0$ *is irreducible and locally non-degenerate, which is always valid when* $\mathbf{m}$ *is rigid. Then the equation* $P^o u = 0$ *is irreducible if and only if the equation* $P(k)^o u = 0$ *satisfy the condition* (10.34) *for* $k = 0, \ldots, K-1$. *If the equation* $P^o u = 0$ *is irreducible, it is locally non-degenerate.*

PROOF. The claim i) follows from Theorem 5.2 and the claims ii) and iii) follow from Lemma 5.3 and Corollary 9.10, which implies the claim iv). □

REMARK 10.11. i) In the preceding theorem the equation $P^o u = 0$ may not be locally non-degenerate even if it is irreducible. For example the equation satisfied by ${}_3F_2$ is contained in the universal operator of type $111, 111, 111$.

ii) It is also proved as follows that the irreducible differential equation with a rigid spectral type is locally non-degenerate.

The monodromy generators M_j of the equation with the Riemann scheme at $x = c_j$ satisfy

$$\operatorname{rank}(M_j' - e^{2\pi\sqrt{-1}\lambda_{j,1}}) \cdots (M_j' - e^{2\pi\sqrt{-1}\lambda_{j,k}}) \leq m_{j,k+1} + \cdots + m_{j,n_j} \quad (k = 1, \ldots, n_j)$$

for $j = 0, \ldots, p$. The equality in the above is clear when $\lambda_{j,\nu} - \lambda_{j,\nu'} \notin \mathbb{Z}$ for $1 \leq \nu < \nu' \leq n_j$ and hence the above is proved by the continuity for general $\lambda_{j,\nu}$. The rigidity index of $\mathbf{M}$ is calculated by the dimension of the centralizer of M_j

and it should be 2 if $\mathbf{M}$ is irreducible and rigid, the equality in the above is valid (cf. [**Kz**], [**O6**]), which means the equation is locally non-degenerate.

iii) The same results as in Theorem 10.10 are also valid in the case of the Fuchsian system of Schlesinger canonical form (9.1) since the same proof works. A similar result is given by a different proof (cf. [**CB**]).

iv) Let $(M_0, \dots, M_p)$ be a tuple of matrices in $GL(n, \mathbb{C})$ with $M_p M_{p-1} \cdots M_0 = I_n$. Then $(M_0, \dots, M_p)$ is called *rigid* if for any $g_0, \dots, g_p \in GL(n, \mathbb{C})$ satisfying $g_p M_p g_p^{-1} \cdot g_{p-1} M_{p-1} g_{p-1}^{-1} \cdots g_0 M_0 g_0^{-1} = I_n$, there exists $g \in GL(n, \mathbb{C})$ such that $g_i M_i g_i^{-1} = g M_i g^{-1}$ for $i = 0, \dots, p$. The tuple $(M_0, \dots, M_p)$ is called *irreducible* if no subspace V of $\mathbb{C}^n$ satisfies $\{0\} \subsetneqq V \subsetneqq \mathbb{C}^n$ and $M_i V \subset V$ for $i = 0, \dots, p$. Choose $\mathbf{m} \in \mathcal{P}_{p+1}^{(n)}$ and $\{\mu_{j,\nu}\}$ such that $L(\mathbf{m}; \mu_{j,1}, \dots, \mu_{j,n_j})$ are in the conjugacy classes containing M_j, respectively. Suppose $(M_0, \dots, M_p)$ is irreducible and rigid. Then Katz [**Kz**] shows that $\mathbf{m}$ is rigid and gives a construction of irreducible and rigid $(M_0, \dots, M_p)$ for any rigid $\mathbf{m}$ (cf. Remark 9.4 ii)). It is an open problem given by Katz [**Kz**] whether the monodromy generators M_j are realized by solutions of a single Fuchsian differential equations without an apparent singularity, whose affirmative answer is given by the following corollary.

Corollary 10.12. *Let* $\mathbf{m} = \big(m_{j,\nu}\big)_{\substack{0 \le j \le p \\ 1 \le \nu \le n_j}}$ *be a rigid monotone* $(p+1)$*-tuple of partitions with* $\operatorname{ord} \mathbf{m} > 1$. *Retain the notation in* Definition 5.12.

i) *Fix complex numbers* $\lambda_{j,\nu}$ *for* $0 \le j \le p$ *and* $1 \le \nu \le n_j$ *satisfying the Fuchs relation* (4.32). *The universal operator* $P_{\mathbf{m}}(\lambda) u = 0$ *with the Riemann scheme* (0.11) *is irreducible if and only if the condition*

$$\sum_{j=0}^{p} \lambda(k)_{j,\ell(k)_j + \delta_{j,j_o}(\nu_o - \ell(k)_j)} \notin \mathbb{Z} \qquad \text{for any } (j_o, \nu_o) \text{ satisfying } m(k)_{j_o,\nu_o} > m(k)_{j_o,\ell(k)_{j_o}} - d(k) \tag{10.35}$$

is satisfied for $k = 0, \dots, K-1$.

ii) *Fix complex numbers* $\mu_{j,\nu}$ *for* $0 \le j \le p$ *and* $1 \le \nu \le n_j$ *and define* $\tilde{\mu}(k)$ *and* $\mu(k)_{j,\nu}$ *for* $k = 0, \dots, K$ *by*

$$\mu(0)_{j,\nu} = \mu_{j,\nu} \quad (j = 0, \dots, p,\ \nu = 1, \dots, n_j), \tag{10.36}$$

$$\tilde{\mu}(k) = \prod_{j=0}^{p} \mu(k)_{j,\ell(k)_j}, \tag{10.37}$$

$$\mu(k+1)_{j,\nu} = \mu(k)_{j,\nu} \cdot \tilde{\mu}(k)^{(-1)^{\delta_{j,0}} - \delta_{\nu,1}}. \tag{10.38}$$

Then there exists an irreducible tuple $(M_0, \dots, M_p)$ *of matrices satisfying*

$$\begin{gathered} M_p \cdots M_0 = I_n, \\ M_j \sim L(m_{j,1}, \dots, m_{j,n_j}; \mu_{j,1}, \dots, \mu_{j,n_j}) \quad (j = 0, \dots, p) \end{gathered} \tag{10.39}$$

under the notation (4.33) *if and only if*

$$\prod_{j=0}^{p} \prod_{\nu=1}^{n_j} \mu_{j,\nu}^{m_{j,\nu}} = 1 \tag{10.40}$$

and moreover the condition

$$\prod_{j=0}^{p} \mu(k)_{j,\ell(k)_j + \delta_{j,j_o}(\nu_o - \ell(k)_j)} \neq 1 \qquad \text{for any } (j_o, \nu_o) \text{ satisfying } m(k)_{j_o,\nu_o} > m(k)_{j_o,\ell(k)_{j_o}} - d(k) \tag{10.41}$$

is satisfied for $k = 0, \dots, K-1$.

iii) *Let* $(M_0,\dots,M_p)$ *be an irreducible tuple of matrices satisfying* (10.39)*. Then there uniquely exists a Fuchsian differential equation* $Pu=0$ *with* $p+1$ *singular points* $c_0,\dots,c_p$ *and its local independent solutions* $u_1,\dots,u_{\mathrm{ord}\,\mathbf{m}}$ *in a neighborhood of a non-singular point* q *such that the monodromy generators around the points* c_j *with respect to the solutions equal* M_j*, respectively, for* $j=0,\dots,p$ (cf. (9.25)).

PROOF. The clam i) is a direct consequence of Theorem 10.10 and the claim ii) is proved by Theorem 9.3 and Lemma 9.11 as in the case of the proof of Theorem 10.10 (cf. Remark 9.4 ii)).

iii) Since $\gcd\mathbf{m}=1$, we can choose $\lambda_{j,\nu}\in\mathbb{C}$ such that $e^{2\pi\sqrt{-1}\lambda_{j,\nu}}=\mu_{j,\nu}$ and $\sum_{j,\nu}m_{j,\nu}\lambda_{j,\nu}=\mathrm{ord}\,\mathbf{m}-1$. Then we have a universal operator $P_{\mathbf{m}}(\lambda_{j,\nu})u=0$ with the Riemann scheme (0.11). The irreducibility of $(M_p,\dots,M_0)$, Remark 10.11 ii) (or Theorem 9.6) and Theorem 6.14 assure the claim. $\square$

Now we state the condition (10.35) using the terminology of the Kac-Moody root system. Suppose $\mathbf{m}\in\mathcal{P}$ is monotone and irreducibly realizable. Let $\{\lambda_{\mathbf{m}}\}$ be the Riemann scheme of the universal operator $P_{\mathbf{m}}$. According to Remark 5.9 iii) we may relax the definition of $\ell_{max}(\mathbf{m})$ as is given by (5.43) and then we may assume

$$v_ks_0\cdots v_1s_0\Lambda(\lambda)\in W'_\infty\Lambda\big(\lambda(k)\big)\qquad(k=1,\dots,K) \tag{10.42}$$

under the notation in Definition 5.12 and (7.31). Then we have the following theorem.

THEOREM 10.13. *Let* $\mathbf{m}=\big(m_{j,\nu}\big)_{\substack{0\le j\le p\\1\le\nu\le n_j}}$ *be an irreducibly realizable monotone tuple of partition in* $\mathcal{P}$*. Under the notation in* Corollary 10.12 *and* §7.1*, there uniquely exists a bijection*

$$\begin{aligned}\varpi:\Delta(\mathbf{m})\xrightarrow{\sim}\big\{&(k,j_0,\nu_0)\,;\,0\le k<K,\ 0\le j_0\le p,\ 1\le\nu_0\le n_{j_0},\\&\nu_0\ne\ell(k)_{j_0}\ \ \textit{and}\ \ m(k)_{j_0,\nu_0}>m(k)_{j_0,\ell(k)_{j_0}}-d(k)\big\}\\&\cup\big\{(k,0,\ell(k)_0)\,;\,0\le k<K\big\}\end{aligned} \tag{10.43}$$

such that

$$(\Lambda(\lambda)|\alpha)=\sum_{j=0}^p\lambda(k)_{j,\ell(k)_j+\delta_{j,j_o}(\nu_o-\ell(k)_j)}\quad\textit{when}\quad\varpi(\alpha)=(k,j_0,\nu_0). \tag{10.44}$$

Moreover we have

$$\begin{aligned}(\alpha|\alpha_{\mathbf{m}})=m(k)_{j_0,\nu_0}-m(k)_{j_0,\ell(k)_{j_0}}+d(k)&\\(\alpha\in\Delta(\mathbf{m}),\ (k,j_0,\nu_0)=\varpi(\alpha))&\end{aligned} \tag{10.45}$$

and if the universal equation $P_{\mathbf{m}}(\lambda)u=0$ *is irreducible, we have*

$$(\Lambda(\lambda)|\alpha)\notin\mathbb{Z}\quad\textit{for any}\ \ \alpha\in\Delta(\mathbf{m}). \tag{10.46}$$

In particular, if $\mathbf{m}$ *is rigid and* (10.46) *is valid, the universal equation is irreducible.*

PROOF. Assume $\mathrm{ord}\,\mathbf{m}>1$ and use the notation in Theorem 10.10. Since $\tilde{\mathbf{m}}$ may not be monotone, we consider the monotone tuple $\mathbf{m}'=s\tilde{\mathbf{m}}$ in $S'_\infty\tilde{\mathbf{m}}$ (cf. Definition 4.11). First note that

$$d-m_{j,1}+m_{j,\nu}=(\alpha_0+\alpha_{j,1}+\cdots+\alpha_{j,\nu-1}|\alpha_{\mathbf{m}}).$$

Let $\bar\nu_j$ be the positive integers defined by

$$m_{j,\bar\nu_j+1}\le m_{j,1}-d<m_{j,\bar\nu_j}$$

for $j=0,\dots,p$. Then

$$\alpha_{\mathbf{m}'}=v^{-1}\alpha_{\tilde{\mathbf{m}}}\ \ \text{with}\ \ v:=\prod_{j=0}^p\big(s_{j,1}\cdots s_{j,\bar\nu_j-1}\big)$$

and $w(\mathbf{m}) = s_0 v s_{\alpha_{\bar{\mathbf{m}}}}$ and

$$\Delta(\mathbf{m}) = \Xi \cup s_0 v \Delta(\mathbf{m}'),$$
$$\Xi := \{\alpha_0\} \cup \bigcup_{\substack{0\le j\le p\\ \nu_j\ne 1}} \{\alpha_0 + \alpha_{j,1} + \cdots + \alpha_{j,\nu}\,;\, \nu = 1,\ldots,\bar{\nu}_j - 1\}.$$

Note that $\ell(0) = (1,\ldots,1)$ and the condition $m_{j_0,\nu_0} > m_{j_0,1} - d(0)$ is valid if and only if $\nu_0 \in \{1,\ldots,\bar{\nu}_{j_0}\}$. Since

$$\sum_{j=0}^{p} \lambda(0)_{j,1+\delta_{j,j_0}(\nu_0-1)} = (\Lambda(\lambda)|\alpha_0 + \alpha_{j_0,1} + \cdots + \alpha_{j_0,\nu_0-1}) + 1,$$

we have

$$L(0) = \big\{(\Lambda(\lambda)|\alpha) + 1\,;\, \alpha \in \Xi\big\}$$

by denoting

$$L(k) := \Big\{\sum_{j=0}^{p} \lambda(k)_{j,\ell(k)_j+\delta_{j,j_o}(\nu_o-\ell(k)_j)}\,;\, m(k)_{j_o,\nu_o} > m(k)_{j_o,\ell(k)_{j_o}} - d(k)\Big\}.$$

Applying $v^{-1}s_0$ to $\mathbf{m}$ and $\{\lambda_{\mathbf{m}}\}$, they changes into $\mathbf{m}'$ and $\{\lambda'_{\mathbf{m}'}\}$, respectively, such that $\Lambda(\lambda') - v^{-1}s_0\Lambda(\lambda) \in \mathbb{C}\Lambda_0$. Hence we obtain the theorem by the induction as in the proof of Corollary 10.12. $\square$

Remark 10.14. Let $\mathbf{m}$ be an irreducibly realizable monotone tuple in $\mathcal{P}$. Fix $\alpha \in \Delta(\mathbf{m})$. We have $\alpha = \alpha_{\mathbf{m}'}$ with a rigid tuple $\mathbf{m}' \in \mathcal{P}$ and

$$|\{\lambda_{\mathbf{m}'}\}| = (\Lambda(\lambda)|\alpha). \tag{10.47}$$

Definition 10.15. Define an *index* $\operatorname{idx}_{\mathbf{m}}\big(\ell(\lambda)\big)$ of the non-zero linear form $\ell(\lambda) = \sum_{j=0}^{p}\sum_{\nu=1}^{n_j} k_{j,\nu}\lambda_{j,\nu}$ of with $k_{j,\nu} \in \mathbb{Z}_{\ge 0}$ as the positive integer d_i such that

$$\Big\{\sum_{j=0}^{p}\sum_{\nu=1}^{n_j} k_{j,\nu}\epsilon_{j,\nu}\,;\, \epsilon_{j,\nu} \in \mathbb{Z} \text{ and } \sum_{j=0}^{p}\sum_{\nu=1}^{n_j} m_{j,\nu}\epsilon_{j,\nu} = 0\Big\} = \mathbb{Z}d_i. \tag{10.48}$$

Proposition 10.16. *For a rigid tuple* $\mathbf{m}$ *in* Corollary 10.12, *define rigid tuples* $\mathbf{m}^{(1)},\ldots,\mathbf{m}^{(N)}$ *with a non-negative integer* N *so that* $\Delta(\mathbf{m}) = \{\mathbf{m}^{(1)},\ldots,\mathbf{m}^{(N)}\}$ *and put*

$$\ell_i(\lambda) := \sum_{j=0}^{p}\sum_{\nu=1}^{n_j} m^{(i)}_{j,\nu}\lambda_{j,\nu} \qquad (i = 1,\ldots,N). \tag{10.49}$$

Here we note that Theorem 10.13 *implies that* $P_{\mathbf{m}}(\lambda)$ *is irreducible if and only if* $\ell_i(\lambda) \notin \mathbb{Z}$ *for* $i = 1,\ldots,n$.

Fix a function $\ell(\lambda)$ *of* $\lambda_{j,\nu}$ *such that* $\ell(\lambda) = \ell_i(\lambda) - r$ *with* $i \in \{1,\ldots,N\}$ *and* $r \in \mathbb{Z}$. *Moreover fix generic complex numbers* $\lambda_{j,\nu} \in \mathbb{C}$ *under the condition* $\ell(\lambda) = |\{\lambda_{\mathbf{m}}\}| = 0$ *and a decomposition* $P_{\mathbf{m}}(\lambda) = P''P'$ *such that* P', $P'' \in W(x)$, $0 < n' := \operatorname{ord} P' < n$ *and the differential equation* $P'v = 0$ *is irreducible. Then there exists an irreducibly realizable subtuple* $\mathbf{m}'$ *of* $\mathbf{m}$ *compatible to* $\ell(\lambda)$ *such that the monodromy generators* M'_j *of the equation* $P'u = 0$ *satisfies*

$$\operatorname{rank}(M_j - e^{2\pi\sqrt{-1}\lambda_{j,1}})\cdots(M_j - e^{2\pi\sqrt{-1}\lambda_{j,k}}) \le m'_{j,k+1} + \cdots + m'_{j,n_j} \quad (k = 1,\ldots,n_j)$$

for $j = 0,\ldots,p$. *Here we define that the decomposition*

$$\mathbf{m} = \mathbf{m}' + \mathbf{m}'' \quad (\mathbf{m}' \in \mathcal{P}^{(n')}_{p+1},\ \mathbf{m}'' \in \mathcal{P}^{(n'')}_{p+1},\ 0 < n' < n) \tag{10.50}$$

is compatible to $\ell(\lambda)$ and that $\mathbf{m}'$ is a subtuple of $\mathbf{m}$ compatible to $\ell(\lambda)$ if the following conditions are valid

$$|\{\lambda_{\mathbf{m}'}\}| \in \mathbb{Z}_{\le 0} \quad \text{and} \quad |\{\lambda_{\mathbf{m}''}\}| \in \mathbb{Z}, \tag{10.51}$$

$$\mathbf{m}' \text{ is realizable if there exists } (j,\nu) \text{ such that } m''_{j,\nu} = m_{j,\nu} > 0, \tag{10.52}$$

$$\mathbf{m}'' \text{ is realizable if there exists } (j,\nu) \text{ such that } m'_{j,\nu} = m_{j,\nu} > 0. \tag{10.53}$$

Here we note $|\{\lambda_{\mathbf{m}'}\}| + |\{\lambda_{\mathbf{m}''}\}| = 1$ if $\mathbf{m}'$ and $\mathbf{m}''$ are rigid.

Proof. The equation $P_{\mathbf{m}}(\lambda)u = 0$ is reducible since $\ell(\lambda) = 0$. We may assume $\lambda_{j,\nu} - \lambda_{j,\nu'} \neq 0$ for $1 \le \nu < \nu' \le n_j$ and $j = 0, \dots, p$. The solutions of the equation define the map $\mathcal{F}$ given by (2.15) and the reducibility implies the existence of an irreducible submap $\mathcal{F}'$ such that $\mathcal{F}'(U) \subset \mathcal{F}(U)$ and $0 < n' := \dim \mathcal{F}'(U) < n$. Then $\mathcal{F}'$ defines a irreducible Fuchsian differential equation $P'v = 0$ which has regular singularities at $x = c_0 = \infty, c_1, \dots, c_p$ and may have other apparent singularities $c'_1, \dots, c'_q$. Then the characteristic exponents of P' at the singular points are as follows.

There exists a decomposition $\mathbf{m} = \mathbf{m}' + \mathbf{m}''$ such that $\mathbf{m}' \in \mathcal{P}^{(n')}$ and $\mathbf{m}'' \in \mathcal{P}^{(n'')}$ with $n'' := n - n'$. The sets of characteristic exponents of P' at $x = c_j$ are $\{\lambda'_{j,\nu,i}\,;\, i = 1, \dots, m'_{j,\nu},\ \nu = 1, \dots, n\}$ which satisfy

$$\lambda'_{j,\nu,i} - \lambda_{j,\nu} \in \{0, 1, \dots, m_{j,\nu} - 1\} \quad \text{and} \quad \lambda'_{j,\nu,1} < \lambda'_{j,\nu,2} < \cdots < \lambda'_{j,\nu,m'_{j,\nu}}$$

for $j = 0, \dots, p$. The sets of characteristic exponents at $x = c'_j$ are $\{\mu_{j,1}, \dots, \mu_{j,n'}\}$, which satisfy $\mu_{j,i} \in \mathbb{Z}$ and $0 \le \mu_{j,1} < \cdots < \mu_{j,n'}$ for $j = 1, \dots, q$. Then Remark 4.17 ii) says that the Fuchs relation of the equation $P'v = 0$ implies $|\{\lambda_{\mathbf{m}'}\}| \in \mathbb{Z}_{\le 0}$.

Note that there exists a Fuchsian differential operator $P'' \in W(x)$ such that $P = P''P'$. If there exists j_o and ν_o such that $m'_{j_o,n_o} = 0$, namely, $m''_{j_o,\nu_o} = m_{j_o,\nu_o} > 0$, the exponents of the monodromy generators of the solution $P'v = 0$ are generic and hence $\mathbf{m}'$ should be realizable. The same claim is also true for the tuple $\mathbf{m}''$. Hence we have the proposition. □

Example 10.17. i) The reduction of the universal operator with the spectral type $11, 11, 11$ which is given by Theorem 10.10 is

$$\begin{aligned} &\begin{Bmatrix} x = \infty & 0 & 1 \\ \lambda_{0,1} & \lambda_{1,1} & \lambda_{2,1} \\ \lambda_{0,2} & \lambda_{1,2} & \lambda_{2,2} \end{Bmatrix} \quad \left(\sum \lambda_{j,\nu} = 1\right) \\ &\longrightarrow \begin{Bmatrix} x = \infty & 0 & 1 \\ 2\lambda_{0,2} + \lambda_{1,1} + \lambda_{2,1} & -\lambda_{0,2} - \lambda_{2,2} & -\lambda_{0,2} - \lambda_{1,2} \end{Bmatrix} \end{aligned} \tag{10.54}$$

because $\mu = \lambda_{0,1} + \lambda_{1,1} + \lambda_{2,1} - 1 = -\lambda_{0,2} - \lambda_{1,2} - \lambda_{2,2}$. Hence the necessary and sufficient condition for the irreducibility of the universal operator given by (10.34) is

$$\begin{cases} \lambda_{0,1} + \lambda_{1,1} + \lambda_{2,1} \notin \mathbb{Z}, \\ \lambda_{0,2} + \lambda_{1,1} + \lambda_{2,1} \notin \mathbb{Z}, \\ \lambda_{0,1} + \lambda_{1,2} + \lambda_{2,1} \notin \mathbb{Z}, \\ \lambda_{0,1} + \lambda_{1,1} + \lambda_{2,2} \notin \mathbb{Z}, \end{cases}$$

which is equivalent to

$$\lambda_{0,i} + \lambda_{1,1} + \lambda_{2,j} \notin \mathbb{Z} \quad \text{for} \quad i = 1, 2 \text{ and } j = 1, 2. \tag{10.55}$$

The rigid tuple $\mathbf{m} = 11, 11, 11$ corresponds to the real root $\alpha_{\mathbf{m}} = 2\alpha_0 + \alpha_{0,1} + \alpha_{1,1} + \alpha_{2,1}$ under the notation in §7.1. Then $\Delta(\mathbf{m}) = \{\alpha_0, \alpha_0 + \alpha_{j,1}\,;\, j = 0, 1, 2\}$ and $(\Lambda|\alpha_0) = \lambda_{0,1} + \lambda_{1,1} + \lambda_{2,1}$ and $(\Lambda|\alpha_0 + \alpha_{0,1}) = \lambda_{0,2} + \lambda_{1,1} + \lambda_{2,1}$, etc. under the notation in Theorem 10.13.

The Riemann scheme for the Gauss hypergeometric series ${}_2F_1(a,b,c;z)$ is given by $\begin{Bmatrix} x=\infty & 0 & 1 \\ a & 0 & 0 \\ b & 1-c & c-a-b \end{Bmatrix}$ and therefore the condition for the irreducibility is

$$a \notin \mathbb{Z},\ b \notin \mathbb{Z},\ c-b \notin \mathbb{Z} \ \text{ and } \ c-a \notin \mathbb{Z}. \tag{10.56}$$

ii) The reduction of the Riemann scheme for the equation corresponding to ${}_3F_2(\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2;x)$ is

$$\begin{aligned} &\begin{Bmatrix} x=\infty & 0 & 1 \\ \alpha_1 & 0 & [0]_{(2)} \\ \alpha_2 & 1-\beta_1 & -\beta_3 \\ \alpha_3 & 1-\beta_2 & \end{Bmatrix} \qquad (\sum_{i=1}^{3}\alpha_i = \sum_{i=1}^{3}\beta_i) \\ &\longrightarrow \begin{Bmatrix} x=\infty & 0 & 1 \\ \alpha_2-\alpha_1+1 & \alpha_1-\beta_1 & 0 \\ \alpha_3-\alpha_1+1 & \alpha_1-\beta_2 & \alpha_1-\beta_3-1 \end{Bmatrix} \end{aligned} \tag{10.57}$$

with $\mu=\alpha_1-1$. Hence Theorem 10.10 says that the condition for the irreducibility equals

$$\begin{cases} \alpha_i \notin \mathbb{Z} & (i=1,2,3), \\ \alpha_1-\beta_j \notin \mathbb{Z} & (j=1,2) \end{cases}$$

together with

$$\alpha_i-\beta_j \notin \mathbb{Z} \qquad (i=2,3,\ j=1,2).$$

Here the second condition follows from i). Hence the condition for the irreducibility is

$$\alpha_i \notin \mathbb{Z} \ \text{ and } \ \alpha_i-\beta_j \notin \mathbb{Z} \quad (i=1,2,3,\ j=1,2). \tag{10.58}$$

iii) The reduction of the even family is as follows:

$$\begin{aligned} &\begin{Bmatrix} x=\infty & 0 & 1 \\ \alpha_1 & [0]_{(2)} & [0]_{(2)} \\ \alpha_2 & 1-\beta_1 & [-\beta_3]_{(2)} \\ \alpha_3 & 1-\beta_2 & \\ \alpha_4 & & \end{Bmatrix} \longrightarrow \begin{Bmatrix} x=\infty & 0 & 1 \\ \alpha_2-\alpha_1+1 & 0 & 0 \\ \alpha_3-\alpha_1+1 & \alpha_1-\beta_1 & [\alpha_1-\beta_3-1]_{(2)} \\ \alpha_4-\alpha_1+1 & \alpha_1-\beta_2 & \end{Bmatrix} \\ &\xrightarrow{(x-1)^{-\alpha_1+\beta_3+1}} \begin{Bmatrix} x=\infty & 0 & 1 \\ \alpha_2-\beta_3 & 0 & -\alpha_1+\beta_3+1 \\ \alpha_3-\beta_3 & \alpha_1-\beta_1 & [0]_{(2)} \\ \alpha_4-\beta_3 & \alpha_1-\beta_2 & \end{Bmatrix}. \end{aligned}$$

Hence the condition for the irreducibility is

$$\begin{cases} \alpha_i \notin \mathbb{Z} & (i=1,2,3,4), \\ \alpha_1-\beta_3 \notin \mathbb{Z} & \end{cases}$$

together with

$$\begin{cases} \alpha_i-\beta_3 \notin \mathbb{Z} & (i=2,3,4). \\ \alpha_1+\alpha_i-\beta_j-\beta_3 \notin \mathbb{Z} & (i=2,3,4,\ j=1,2) \end{cases}$$

by the result in ii). Thus the condition is

$$\begin{gathered} \alpha_i \notin \mathbb{Z},\ \alpha_i-\beta_3 \notin \mathbb{Z} \text{ and } \alpha_1+\alpha_k-\beta_j-\beta_3 \notin \mathbb{Z} \\ (i=1,2,3,4,\ j=1,2,\ k=2,3,4). \end{gathered} \tag{10.59}$$

Hence the condition for the irreducibility for the equation with the Riemann scheme

$$\begin{Bmatrix} \lambda_{0,1} & [\lambda_{1,1}]_{(2)} & [\lambda_{2,1}]_{(2)} \\ \lambda_{0,2} & \lambda_{1,2} & [\lambda_{2,2}]_{(2)} \\ \lambda_{0,3} & \lambda_{1,3} & \\ \lambda_{0,4} & & \end{Bmatrix} \tag{10.60}$$

of type $1111, 211, 22$ is

$$\begin{cases} \lambda_{0,\nu}+\lambda_{1,1}+\lambda_{2,k}\notin\mathbb{Z} & (\nu=1,2,3,4,\ k=1,2) \\ \lambda_{0,\nu}+\lambda_{0,\nu'}+\lambda_{1,1}+\lambda_{1,2}+\lambda_{2,1}+\lambda_{2,2}\notin\mathbb{Z} & (1\le\nu<\nu'\le 4). \end{cases} \tag{10.61}$$

This condition corresponds to the rigid decompositions

$$1^4, 21^2, 2^2 = 1, 10, 1 \oplus 1^3, 11^2, 21 = 1^2, 11, 1^2 \oplus 1^2, 11, 1^2, \tag{10.62}$$

which are also important in the connection formula.

iv) (generalized Jordan-Pochhammer) The reduction of the universal operator of the rigid spectral type $32, 32, 32, 32$ is as follows:

$$\begin{Bmatrix} [\lambda_{0,1}]_{(3)} & [\lambda_{1,1}]_{(3)} & [\lambda_{2,1}]_{(3)} & [\lambda_{3,1}]_{(3)} \\ [\lambda_{0,2}]_{(2)} & [\lambda_{1,2}]_{(2)} & [\lambda_{2,2}]_{(2)} & [\lambda_{3,2}]_{(2)} \end{Bmatrix} \quad (3\sum_{j=0}^{3}\lambda_{j,1}+2\sum_{j=0}^{3}\lambda_{j,2}=4)$$

$$\longrightarrow \begin{Bmatrix} \lambda_{0,1}-2\mu & \lambda_{1,1} & \lambda_{2,1} & \lambda_{3,1} \\ [\lambda_{0,2}-\mu]_{(2)} & [\lambda_{1,2}+\mu]_{(2)} & [\lambda_{2,2}+\mu]_{(2)} & [\lambda_{3,2}+\mu]_{(2)} \end{Bmatrix}$$

with $\mu=\lambda_{0,1}+\lambda_{1,1}+\lambda_{2,1}+\lambda_{3,1}-1$. Hence the condition for the irreducibility is

$$\begin{cases} \sum_{j=0}^{3}\lambda_{j,1+\delta_{j,k}}\notin\mathbb{Z} & (k=0,1,2,3,4), \\ \sum_{j=0}^{3}(1+\delta_{j,k})\lambda_{j,1}+\sum_{j=0}^{3}(1-\delta_{j,k})\lambda_{j,2}\notin\mathbb{Z} & (k=0,1,2,3,4). \end{cases} \tag{10.63}$$

Note that under the notation defined by Definition 10.15 we have

$$\operatorname{idx}_{\mathbf{m}}\big(\lambda_{0,1}+\lambda_{1,1}+\lambda_{2,1}+\lambda_{3,1}\big)=2 \tag{10.64}$$

and the index of any other linear form in (10.63) is 1.

In general, the universal operator with the Riemann scheme

$$\begin{gathered} \begin{Bmatrix} [\lambda_{0,1}]_{(k)} & [\lambda_{1,1}]_{(k)} & [\lambda_{2,1}]_{(k)} & [\lambda_{3,1}]_{(k)} \\ [\lambda_{0,2}]_{(k-1)} & [\lambda_{1,2}]_{(k-1)} & [\lambda_{2,2}]_{(k-1)} & [\lambda_{3,2}]_{(k-1)} \end{Bmatrix} \\ (k\sum_{j=0}^{3}\lambda_{j,1}+(k-1)\sum_{j=0}^{3}\lambda_{j,2}=2k) \end{gathered} \tag{10.65}$$

is irreducible if and only if

$$\begin{cases} \sum_{j=0}^{3}(\nu-\delta_{j,k})\lambda_{j,1}+\sum_{j=0}^{3}(\nu-1+\delta_{j,k})\lambda_{j,1}\notin\mathbb{Z} & (k=0,1,2,3,4), \\ \sum_{j=0}^{3}(\nu'+\delta_{j,k})\lambda_{j,1}+\sum_{j=0}^{3}(\nu'-\delta_{j,k})\lambda_{j,2}\notin\mathbb{Z} & (k=0,1,2,3,4) \end{cases} \tag{10.66}$$

for any integers ν and ν' satisfying $1\le 2\nu\le k$ and $1\le 2\nu'\le k-1$.

The rigid decomposition

$$65, 65, 65, 65 = 12, 21, 21, 21 \oplus 53, 44, 44, 44 \tag{10.67}$$

gives an example of the decomposition $\mathbf{m}=\mathbf{m}'\oplus\mathbf{m}''$ with $\operatorname{supp}\alpha_{\mathbf{m}}=\operatorname{supp}\alpha_{\mathbf{m}'}=\operatorname{supp}\alpha_{\mathbf{m}''}$.

v) The rigid Fuchsian differential equation with the Riemann scheme

$$\begin{Bmatrix} x=0 & 1 & c_3 & c_4 & \infty \\ [0]_{(9)} & [0]_{(9)} & [0]_{(9)} & [0]_{(9)} & [e_0]_{(8)} \\ [a]_{(3)} & [b]_{(3)} & [c]_{(3)} & [d]_{(3)} & [e_1]_{(3)} \\ & & & & e_2 \end{Bmatrix}$$

is reducible when

$$a + b + c + d + 3e_0 + e_1 \in \mathbb{Z},$$

which is equivalent to $\frac{1}{3}(e_0 - e_2 - 1) \in \mathbb{Z}$ under the Fuchs relation. At the generic point of this reducible condition, the spectral types of the decomposition in the Grothendieck group of the monodromy is

$$93, 93, 93, 93, 831 = 31, 31, 31, 31, 211 + 31, 31, 31, 31, 310 + 31, 31, 31, 31, 310.$$

Note that the following reduction of the spectral types

$$\begin{array}{ccccc} 93, 93, 93, 93, 831 & \to & 13, 13, 13, 13, 031 & \to & 10, 10, 10, 10, 001 \\ 31, 31, 31, 31, 211 & \to & 11, 11, 11, 11, 011 & & \\ 31, 31, 31, 31, 310 & \to & 01, 01, 01, 01, 010 & & \end{array}$$

and $\mathrm{idx}(31, 31, 31, 31, 211) = -2$.