CHAPTER 6

The Case of Complex Codimension One

The Godbillon–Vey class of *real* codimension-one foliations is deeply studied. One of the most significant results is due to Duminy.

THEOREM 6.1 (Duminy [27]). *Let $\mathcal{F}$ be a real codimension-one foliation of a closed manifold M. Then,* $\mathrm{GV}_1(\mathcal{F})$ *is non-trivial only if $\mathcal{F}$ admits a resilient leaf.*

A leaf of $\mathcal{F}$ is called *resilient* if the leaf admits a holonomy by which the leaf accumulates on itself. It is known that $\mathcal{F}$ admits a minimal set other than a closed leaf if there is a resilient leaf, and $\mathcal{F}$ has a leaf of exponential growth. This is shown by carefully studying minimal sets.

If the codimension of $\mathcal{F}$ is greater than one, then there is a following analogue shown by Hurder.

THEOREM 6.2 (Hurder [47], [48]). *Let $\mathcal{F}$ be a real codimension-q foliation of a closed manifold M. Suppose that* $\mathrm{GV}_q(\mathcal{F})$ *is non-trivial, then $\mathcal{F}$ admits a leaf of exponential growth.*

This theorem is shown by regarding h_1 as a measure and studying its support.

Compared with the real codimension-one case, study of minimal sets is much more difficult even if $\mathcal{F}$ is of complex codimension-one. There are however useful notions related to complex one-dimensional dynamical systems. For example, there are Julia sets of complex dynamical systems and limit sets of Kleinian groups. These two notions are considered to be of the same nature (Sullivan's dictionary). Roughly speaking, dynamics are complicated on Julia sets or limit sets, and simple on the complement which are called Fatou sets or domains of discontinuity. One can expect that there is also a similar decomposition, which we call a Fatou–Julia decomposition, of transversely holomorphic foliations of complex codimension one, and that Julia sets play a role of minimal sets. Such a decomposition is firstly introduced by Ghys, Gomez-Mont and Saludes in [30]. We will call it the GGS-decomposition, and here introduce another decomposition after [11].

Let $\mathcal{F}$ be a transversally holomorphic foliation of a closed manifold M. If we choose a complete transversal T, which is possibly disconnected, then we can associate the holonomy pseudogroup Γ of $\mathcal{F}$ acting on T. Thus obtained pseudogroups are compactly generated.

DEFINITION 6.3 (Haefliger [37]). A pseudogroup (Γ, T) is *compactly generated* if there are a relatively compact open set U in T which meets every orbit of Γ, and a finite collection of elements $\{\gamma_1, \dots, \gamma_r\}$ of Γ of which the domains and the ranges are contained in U such that

1) $\{\gamma_1, \dots, \gamma_r\}$ generates $\Gamma|_U$,
2) each γ_i is the restriction of an element of Γ defined on a neighborhood of the closure of the source of γ_i.

$(\Gamma|_U, U)$ is called a *reduction* of (Γ, T). A reduction of (Γ, T) will always be denoted by (Γ', T').

DEFINITION 6.4. A subset X of T is called *Γ-connected* if X satisfies the following condition: if $X = \coprod_{\lambda \in \Lambda} X_\lambda$ is the decomposition of X into its connected components, then for any $\lambda, \lambda' \in \Lambda$, there exists a sequence $\lambda_0 = \lambda, \lambda_1, \dots, \lambda_r = \lambda'$ such that $\Gamma(X_{\lambda_i}) \cap X_{\lambda_{i+1}} \neq \varnothing$ holds for $i = 0, \dots, r-1$.

DEFINITION 6.5. Let (Γ, T) be a compactly generated pseudogroup and let (Γ', T') be a reduction.

1) A connected open subset U of T' is called a *Fatou neighborhood* if the following conditions are satisfied:

(a) The germ of any element of Γ'_x, $x \in U$, extends to an element of Γ defined on the whole U.

(b) If we set

$$\begin{aligned}\Gamma_U &= \left\{\gamma \in \Gamma \,\middle|\, \begin{array}{l}\operatorname{dom}\gamma = U, \text{ and } \gamma \text{ is the extension of the germ of} \\ \text{an element of } \Gamma' \text{ as above}\end{array}\right\} \\ &= \{\gamma \in \Gamma \mid \operatorname{dom}\gamma = U \text{ and } \gamma(x) \in T' \text{ for some } x \in U\},\end{aligned}$$

then Γ_U is a normal family.

2) The union of Fatou neighborhoods is called the *Fatou set* of (Γ', T') and denoted by $F(\Gamma')$. The complement of the Fatou set is called the *Julia set* of (Γ', T') and denoted by $J(\Gamma')$.

3) The *Fatou set* of (Γ, T) is the Γ-orbit of $F(\Gamma')$, namely, $F(\Gamma) = \Gamma(F(\Gamma'))$. The *Julia set* of (Γ, T) is the complement of $F(\Gamma)$ and denoted by $J(\Gamma)$.

4) Γ-connected components of $F(\Gamma)$ and $J(\Gamma)$ are called the *Fatou components* and *Julia components*, respectively.

If $x \in F(\Gamma)$, then any Fatou neighborhood $U \subset F(\Gamma')$ which contains x is called a Fatou neighborhood of x, where (Γ', T') is a reduction of (Γ, T) such that $x \in T'$.

Remark 6.6.

1) $F(\Gamma)$ is open and Γ-invariant. $J(\Gamma)$ is closed and Γ-invariant.

2) It is known that we may assume that T' is a disjoint union of finite number of discs in $\mathbb{C}$. Then, the condition (b) in 1) is always satisfied by virtue of Montel's theorem. On the other hand, it is necessary to fix a domain of definition in order to speak of normal families. This leads to the condition (a) in 1) of Definition 6.5.

3) $J(\Gamma) = \Gamma(J(\Gamma'))$.

The following lemmata are some of the fundamental properties of the Fatou–Julia decomposition. The proofs are not quite difficult but we omit them.

Lemma 6.7. *The Fatou set of Γ is well-defined, namely, the decomposition $T = F(\Gamma) \amalg J(\Gamma)$ is independent of the choice of the reduction (Γ', T').*

LEMMA 6.8. *Let (Γ, T) and (Δ, S) be compactly generated pseudogroups. If Φ is an equivalence from (Γ, T) to (Δ, S), then $\Phi(F(\Gamma)) = F(\Delta)$.*

Lemma 6.8 implies that the Fatou–Julia decomposition has naturality, and it justifies the following definition.

DEFINITION 6.9. The *Fatou set* of $\mathcal{F}$ is the saturation of $F(\Gamma) \subset T \subset M$, and denoted by $F(\mathcal{F})$. The *Julia set* is the complement of $F(\mathcal{F})$ and denoted by $J(\mathcal{F})$. The connected components of the Fatou set and the Julia set are called the *Fatou components* and the *Julia components*, respectively.

One of the fundamental properties of the Fatou set is the following.

THEOREM 6.10 ([11]). *There is a Γ-invariant Hermitian metric on $F(\Gamma)$, and $\mathcal{F}|_{F(\mathcal{F})}$ is transversely Hermitian.*

An invariant metric in Theorem 6.10 is constructed as follows. For simplicity we will explain the construction of a norm instead of a metric. First we fix a Hermitian metric on $F(\Gamma)$, which is considered as an open subset of $\mathbb{C}$. Let $|\cdot|$ denote the norm with respect to the fixed metric. For $v \in T_x F(\Gamma')$, set $\|v\| = \sup_{\gamma} |\gamma_* v|$, where γ runs through elements of Γ' which is defined on a neighborhood of x. Thus constructed $\|\cdot\|$ is not necessarily continuous, however, by carefully choosing the initial metric, we can obtain a metric which is locally Lipschitz continuous. The Lipschitz continuity implies that elements of Γ' acting on $F(\Gamma')$ satisfies an ordinary differential equation, and completely determined by their 1-jets. One can consider a kind of the closure, denoted by $\overline{\Gamma'}$, of Γ' by collecting uniform limits of elements of Γ'. We can now apply classical arguments of H. Cartan [21] to show that $\overline{\Gamma'}$ is a local Lie transformation group. Finally, by using the local group, we can classify Fatou components and find a smooth invariant metric.

A version of the residue of $\xi_1(\mathcal{F})$ can be introduced by using Theorem 6.10. Let U be an arbitrarily small neighborhood of $J(\mathcal{F})$ and let V be a neighborhood of

$M \setminus U$, which is disjoint from $J(\mathcal{F})$. Recall that we have to choose a Bott connection ∇^b and a unitary connection ∇^u in order to define $\xi_1(\mathcal{F})$. By Theorem 6.10, we may choose ∇^u so that $\nabla^u = \nabla^b$ holds on V. It follows that $\widetilde{u}_1 = 0$ on V, and the support of $\xi_1(\nabla^b, \nabla^u)$ is contained in $M \setminus V$. It can be shown that the class in $H^3_c(V; \mathbb{R})$ determined by $\xi_1(\nabla^b, \nabla^u)$ is independent of the choices. Thus obtained class is called the residue. One can also induce a measure theoretic localization of the Bott class to $J(\mathcal{F})$ like the Godbillon measure [45]. A weak version of Duminy's theorem is obtained via residues and localizations.

THEOREM 6.11 ([8], [11]. See also [30]). *Let $\mathcal{F}$ be a transversely holomorphic foliation of a closed manifold M. If $\xi_1(\mathcal{F}) \neq 0$, then $J(\mathcal{F}) \neq \varnothing$. If moreover* $\mathrm{GV}_2(\mathcal{F}) \neq 0$, *then the Godbillon measure is supported on $J(\mathcal{F})$.*

We do not know if the support of the Godbillon measure coincides with $J(\mathcal{F})$.

REMARK 6.12. We denote by $J_{\mathrm{GGS}}(\mathcal{F})$ the GGS-Julia set. Then, it is known that $J_{\mathrm{GGS}}(\mathcal{F}) \subset J(\mathcal{F})$. Hence Theorem 6.11 is also valid for $J_{\mathrm{GGS}}(\mathcal{F})$.

The following result suggests that dynamics on $J(\mathcal{F})$ will be complicated as expected.

THEOREM 6.13. *If L is a leaf contained in $J(\mathcal{F})$, then either*

1) *L admits a hyperbolic holonomy, or*
2) *there is a leaf which admits a hyperbolic holonomy and which accumulates on L.*

Theorem 6.13 is much weaker than existence of resilient leaves. However, together with Theorem 6.11, it gives certainly an analogue.

We refer to [11] for examples of decompositions.

REMARK 6.14. Deroin and Kleptsyn [25] recently studied asymptotic behavior of leaves under the assumption that the foliation is transversely conformal and that the foliation does not admit any holonomy invariant measure.