CHAPTER 1

Surfaces

In this chapter we present some of the elements of the geometric theory of 2-dimensional (bounded) homology in an informal way. The main purpose of this chapter is to standardize definitions, to refresh the reader's mind about the relationship between 2-dimensional homology classes and maps of surfaces, and to compute the Gromov norm of a hyperbolic surface with boundary. All of this material is essentially elementary and many expositions are available; for example, [**10**] covers this material well.

We start off by discussing maps of surfaces into topological spaces. One way to study such maps is with linear algebra; this way leads to homology. The other way to study such maps is with group theory; this way leads to the fundamental group and the commutator calculus. These points of view are reconciled by Hopf's formula; a more systematic pursuit leads to rational homotopy theory.

1.1. Triangulating surfaces

A *surface* is a topological space (usually Hausdorff and paracompact) which is locally two dimensional. That is, every point has a neighborhood which is homeomorphic to the plane, usually denoted by $\mathbb{R}^2$.

1.1.1. The plane. It is unfortunate in some ways that the standard way to refer to the plane emphasizes its product structure. This product structure is topologically unnatural, since it is defined in a way which breaks the natural topological symmetries of the object in question. This fact is thrown more sharply into focus when one discusses more rigid topologies.

Example 1.1 (Zariski topology). The product topology on two copies of the affine line with its Zariski topology is *not* typically the same as the Zariski topology on the affine plane. A closed set in $\mathbb{R}^1$ with the Zariski topology is either all of $\mathbb{R}$, or a finite collection of points. A closed set in $\mathbb{R}^2$ with the product topology is therefore either all of $\mathbb{R}^2$, or a finite union of horizontal and vertical lines and isolated points. By contrast, closed sets in the Zariski topology in $\mathbb{R}^2$ include circles, ovals, and algebraic curves of every degree.

Part of the bias is biological in origin:

Example 1.2 (Primary visual cortex). The primary visual cortex of mammals (including humans), located at the posterior pole of the occipital cortex, contains neurons hardwired to fire when exposed to certain spatial and temporal patterns. Certain specific neurons are sensitive to stimulus along specific orientations, but in primates, more cortical machinery is devoted to representing vertical and horizontal than oblique orientations (see for example [**58**] for a discussion of this effect).

The correct way to discuss the plane is in terms of the separation properties of its 1-dimensional subsets. The foundation of many such results is the Jordan curve theorem, which says that there is essentially only one way to embed a circle in the plane, up to reparameterization and ambient homeomorphism. Moore [**158**] gave the first "natural" topological definition of the plane, in terms of separation properties of continua. Once this is understood, one is led to study the plane and other surfaces by cutting them up into simple pieces along 1-dimensional continua. The typical way to perform this subdivision is combinatorially, giving rise to triangulations.

1.1.2. Triangulations and homology. Every topological surface can be triangulated in an essentially unique way, up to subdivision (Radó [**176**]). Here by a *triangulation*, we mean a description of the surface as a simplicial complex built from countably many 2-dimensional simplices by identifying edges in pairs (note that a simplicial complex is topologized with the weak topology, so that every compact subset of a surface S meets only finitely many triangles).

Conversely, if we let $\coprod_i \Delta_i$ be a countable disjoint union of triangles, and glue the edges of the Δ_i in pairs, the result is a simplicial complex K. Every point in the interior of a face or an edge has a neighborhood homeomorphic to $\mathbb{R}^2$, by the gluing condition. Every vertex has a neighborhood homeomorphic to the open cone on its link. Each such link is a 1-manifold, and is therefore either homeomorphic to S^1 or to $\mathbb{R}$. It follows that the complex K is a surface if and only if the link of every vertex is *compact*.

If there are only finitely many triangles, every such identification gives rise to a surface. Otherwise, we need to impose the condition that each vertex in the quotient space is in the image of only finitely many triangles, so that the link of this vertex is compact.

Remark 1.3. It is worth looking more closely at the set of all possible ways in which a given surface can be triangulated. Any two triangulations τ, τ' (given up to isotopy) of a fixed surface S are related by a finite sequence of local moves and their inverses. These moves are of two kinds: the *1–3 move*, and the *2–2 move*, illustrated in Figure 1.1. Only the 1–3 move and its inverse change the number of vertices in a triangulation, and

Figure 1.1. The 1–3 and the 2–2 moves

therefore these moves cannot be dispensed with entirely. However, it is an important fact that any two triangulations τ, τ' of the same surface S with the *same* number of vertices are related by 2–2 moves *alone*.

In fact, somewhat more than this is true. Define a *cellulation* of a surface to be a decomposition of the surface into polygonal disks (each with at least 3 sides). Associated to a surface S and a discrete collection P of points in S there is a natural cell complex $A(S,P)$ with one cell for each cellulation of S whose vertex set is exactly P, and with the property that one cell is in the boundary of another if one cellulation is obtained from the other by adding extra edges as diagonals in some of the polygons. In $A(S,P)$, the vertices correspond to the triangulations of S with vertex set exactly P, and the edges correspond to pairs of triangulations related by 2–2 moves. Hatcher [**105**] proves not only that $A(S,P)$ is connected, but that it is *contractible*.

The combinatorial view of a surface as a union of triangles gives rise to a fundamental relationship between surfaces and 2-dimensional homology.

EXAMPLE 1.4 (integral cycles). Let X be a topological space, and let $\alpha \in H_2(X)$ be an integral homology class. The class α is represented (possibly in many different ways) by an integral 2-cycle A. By the definition of a 2-cycle, there is an expression

$$A = \sum_i n_i \sigma_i$$

where each $n_i \in \mathbb{Z}$, and each σ_i is a singular 2-simplex; i.e. a continuous map $\sigma_i : \Delta^2 \to X$ where Δ^2 is the standard 2-simplex. By allowing repetitions of the σ_i, we can assume that each n_i is ± 1.

Since A is a cycle, $\partial A = 0$. That is, for each σ_i and for each singular 1-simplex e which is a face of some σ_i, the signed sum of copies of e appearing in the expression $\sum_i n_i \partial \sigma_i$ is 0. It follows that each such e appears an even number of times with opposite signs. This lets us choose a pairing of the faces of the σ_i so that each pair of faces contributes 0 in the expression for ∂A.

Build a simplicial 2-complex K by taking one 2-simplex for each σ_i, and gluing the edges according to this pairing. Since the number of simplices is finite, and edges are glued in pairs, the result is a topological surface S (note that S need not be connected). Each simplex of K can be oriented compatibly with the sign of the coefficient of the corresponding singular simplex σ_i, so the result is an *oriented* surface. The maps σ_i induce a map from the simplices of K into X, and the definition of the gluing implies that these maps are compatible on the edges of the simplices. We obtain therefore an induced continuous map $f_A : S \to X$. Since S is closed and oriented, there is a fundamental class $[S] \in H_2(S)$, and by construction we have

$$(f_A)_*([S]) = [A] = \alpha$$

In words, *elements of $H_2(X)$ are represented by maps of closed oriented surfaces into X.*

REMARK 1.5. One can also consider homology with rational or real coefficients. Every rational chain has a finite multiple which is an integral chain, so if one is prepared to consider "weighted" surfaces mapping to X, the discussion above suffices. We think of $H_2(X;\mathbb{Q})$ as a subset of $H_2(X;\mathbb{R})$ by using the natural isomorphism $H_2(X;\mathbb{Q}) \otimes \mathbb{R} = H_2(X;\mathbb{R})$. Suppose $\alpha \in H_2(X;\mathbb{Q})$ is represented by a real 2-cycle $A = \sum r_i \sigma_i$. Then for any $\epsilon > 0$ there exists a *rational* 2-cycle $A' = \sum r_i' \sigma_i$ (i.e. with the same support as A) such that the following are true:

(1) The cycles A and A' are homologous (hence $[A'] = \alpha$)
(2) There is an inequality $\sum_i |r_i - r_i'| < \epsilon$

To see this, let V denote the abstract vector space with basis the σ_i. There is a natural map $\partial : V \to C_1(X) \otimes \mathbb{R}$. Since ∂ is defined over $\mathbb{Q}$, the kernel $\ker(\partial)$ is a *rational* subspace of V. There is a further map $h : \ker(\partial) \to H_2(X;\mathbb{R}) = H_2(X;\mathbb{Q}) \otimes \mathbb{R}$. This map is also defined over $\mathbb{Q}$, and therefore $h^{-1}(\alpha)$ is a rational subspace of V (and therefore rational points are dense in it). Since A is in $h^{-1}(\alpha)$, it can be approximated arbitrarily closely by a rational cycle A' also in $h^{-1}(\alpha)$.

1.1.3. Topological classification of surfaces. For simplicity, in this section we consider only connected surfaces.

Closed surfaces are classified by Euler characteristic and whether or not they are orientable. For each non-negative integer g, there is a unique (up to homeomorphism) closed orientable surface with Euler characteristic $2-2g$. The number g is called the *genus* of S, denoted $\text{genus}(S)$.

For each positive integer n, there is a unique (up to homeomorphism) closed non-orientable surface with Euler characteristic $2-n$.

EXAMPLE 1.6 (closed surfaces). The sphere is the unique closed surface with $\chi(\text{sphere})=2$ and the torus is the unique closed orientable surface with $\chi(\text{torus})=0$. The projective plane is the unique closed surface with $\chi(\text{projective plane})=1$. For the sake of notation, we abbreviate these surfaces by S^2, T, P. Every closed surface may be obtained from these by the connect sum operation, denoted #. This operation is commutative and associative, with unit S^2, and satisfies

$$T\#P = P\#P\#P$$

Moreover, every other relation for # is a consequence of this one.

Euler characteristic is subadditive under connect sum, and satisfies

$$\chi(S_1\#S_2) = \chi(S_1)+\chi(S_2)-2$$

A closed surface S is non-orientable if and only if P appears as a summand in some (and therefore any) expression of S as a sum of T and P terms.

If S is an oriented surface, we denote the same surface with opposite orientation by $\overline{S}$. We say that a topological surface is of *finite type* if it is homeomorphic to a closed surface minus finitely many points. If T is closed, and there is an inclusion $i: S\to T$ so that $T-i(S)$ is finite, then

$$\chi(S)=\chi(T)-\text{card}(T-i(S))$$

(here card denotes cardinality). Moreover, S is orientable if and only if T is.

1.1.4. Surfaces with boundary. A *surface with boundary* is a (Hausdorff, paracompact) topological space for which every point has a neighborhood which is either homeomorphic to $\mathbb{R}^2$ or to the closed half-space $\{(x,y)\in\mathbb{R}^2 \mid y\ge 0\}$. Points with neighborhoods homeomorphic to $\mathbb{R}^2$ are *interior points*, and the others are *boundary points*. Surfaces with boundary can be triangulated in such a way that the triangulation induces a triangulation (by 1-dimensional simplices) of the boundary. We denote the set of interior points of S by $\text{int}(S)$, and the set of boundary points by ∂S.

If S is a surface with boundary, the *double* of S, denoted DS, is the surface obtained from $S\coprod\overline{S}$ by identifying ∂S with $\partial\overline{S}$. Note that $\overline{S}$ is only distinguished from S if S is oriented, in which case the double is also oriented. We say S is of finite type if DS is. Note that in this case, DS may be obtained from a closed surface DT which is the double of a compact surface with boundary T by removing finitely many points. If this happens, we can always assume S is obtained from T by removing finitely many points. Note that some of these points may be contained in ∂T.

Genus is not a good measure of complexity for surfaces with boundary: $-\chi$ is better, in the sense that there are only finitely many homeomorphism types of connected compact surface for which $-\chi$ is less than or equal to any given value.

1.1.5. Fundamental group and commutators. Let S be an oriented surface of finite type. If S has genus g and $p > 0$ punctures, $\pi_1(S)$ is free of rank $2g + p - 1$, and similarly if S is compact with p boundary components.

If S is closed of genus g, then S can be obtained by gluing the edges of a $4g$-gon in pairs, and one obtains the "standard" presentation of π_1:

$$\pi_1(S) = \langle a_1, b_1, \ldots, a_g, b_g \mid [a_1, b_1] \cdots [a_g, b_g] \rangle$$

A closed surface is obtained from a surface with one boundary component by gluing on a disk. If S has genus g with one boundary component, $\pi_1(S)$ is free with generators $a_1, b_1, \ldots, a_g, b_g$ and ∂S represents the conjugacy class of the element $[a_1, b_1] \cdots [a_g, b_g]$.

Let X be a topological space, and let α_i, β_i be elements in $\pi_1(X)$ such that there is an identity

$$[\alpha_1, \beta_1] \cdots [\alpha_g, \beta_g] = \mathrm{id} \in \pi_1(X)$$

There is an induced map $\pi_1(S) \to \pi_1(X)$ sending each $a_i \to \alpha_i$ and $b_i \to \beta_i$. Thinking of S as the quotient of a polygon P with $4g$ sides glued together in pairs, this defines a map $\partial P \to X$ whose image is null homotopic in X, and therefore this map extends to a map $S \to X$. The homology class of the image of the fundamental class $[S]$ depends on the particular expression involving the α_i, β_i. Moreover, two different choices of the extension $\partial P \to X$ to P differ by a pair of maps of P which agree on the boundary; these maps sew together to define a map $S^2 \to X$ defining an element of $\pi_2(X)$. In words, *identities in the commutator subgroup of $\pi_1(X)$ correspond to homotopy classes of maps of closed orientable surfaces into X, up to elements of $\pi_2(X)$.*

In the relative case, let $\gamma \in \pi_1(X)$ be a conjugacy class represented by a loop $l_\gamma \subset X$. If γ has a representative in the commutator subgroup $[\pi_1(X), \pi_1(X)]$ then we can write

$$[\alpha_1, \beta_1] \cdots [\alpha_g, \beta_g] = \gamma \in \pi_1(X)$$

Let S be a genus g surface with one boundary component. S is obtained from a $(4g + 1)$-gon P by identifying sides in pairs. Choose loops in X representing the elements $\gamma, \alpha_i, \beta_i$ and let $f : \partial P \to X$ be defined by sending the edges of P to loops in X by $a_i \to \alpha_i, b_i \to \beta_i$, and the free edge to γ. By construction, f factors through the quotient map $\partial P \to S$ induced by gluing up all but one of the edges. Moreover, by hypothesis, $f(\partial P)$ is null-homotopic in X. Hence f can be extended to a map $f : S \to X$ sending ∂S to γ.

In words, *loops corresponding to elements of $[\pi_1(X), \pi_1(X)]$ bound maps of oriented surfaces into X.*

1.1.6. Hopf's formula. The two descriptions above of (relative) maps of surfaces, in terms of homology and in terms of fundamental group, are related by Hopf's formula.

Let X be a topological space. If $\pi_2(X)$ is nontrivial, we can attach 3-cells to X to kill π_2 while keeping π_1 fixed. If X' is the result, then $H_2(X'; \mathbb{Z})$ can be identified with the group homology $H_2(\pi_1(X); \mathbb{Z})$, by the relative Hurewicz theorem.

We let $G = \pi_1(X)$. Suppose we have a description of G as a quotient of a free group:

$$0 \to R \to F \to G \to 0$$

where F is free. Every map from a closed oriented surface S into X' is associated to a product of commutators $[a_1, b_1] \cdots [a_g, b_g]$ which is equal to 0 in G. A choice of word in F for each element a_i, b_i in such an expression determines an element of $R \cap [F, F]$. A substitution $a_i' = a_i r$ where $r \in R$ changes the result by an element of $[F, R]$, since

$$[ar, b] = [ara^{-1}, aba^{-1}][a, b]$$

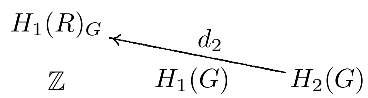
By the discussion above, there is a surjective homomorphism from the Abelian group $(R \cap [F, F])/[F, R]$ to $H_2(G)$.

Hopf's formula says this map is an isomorphism:

THEOREM 1.7 (Hopf's formula [**155**]). *Let G be a group written as a quotient $G = F/R$ where F is free. Then*

$$H_2(G) = (R \cap [F, F])/[F, R]$$

One quick way to see this is to use spectral sequences. This argument is short but a bit technical, and can be skipped by the novice, since the result will not be used elsewhere in this book. The extension $R \to F \to G$ defines a spectral sequence (the Hochschild–Serre spectral sequence [**110**]) whose $E^2_{n,0}$ term is $H_n(G)$ and whose $E^2_{0,1}$ term is $H_1(R)_G$, the quotient of $H_1(R)$ by the conjugation action of G. Since $H_1(R) = R/[R, R]$, we conclude that $H_1(R)_G$ is equal to $R/[F, R]$. Let $d_2 : E^2_{2,0} \to E^2_{0,1}$ be the differential connecting $H_2(G)$ to $R/[F, R]$:



Then there is an exact sequence

$$H_2(F) \to H_2(G) \to R/[F, R] \to H_1(F) \to H_1(G) \to 0$$

Since F is free, $H_2(F) = 0$ and therefore $H_2(G)$ is identified with the kernel of the map $R/[F, R] \to H_1(F)$. But the kernel of $F \to H_1(F)$ is exactly $[F, F]$, so we obtain Hopf's formula.

1.2. Hyperbolic surfaces

1.2.1. Conformal structures. A conformal structure on a surface is an atlas of charts for which the induced transition maps are angle-preserving. We do not require these maps to preserve the sense of the angles, so that non-orientable surfaces may still admit conformal structures. Orientable surfaces with conformal structures on them are synonymous with Riemann surfaces.

EXAMPLE 1.8 (conformal surfaces by cut-and-paste). A Euclidean polygon P inherits a natural conformal structure from the Euclidean plane, which we denote by $\mathbb{E}^2$. Isometries of $\mathbb{E}^2$ preserve the conformal structure, and therefore induce a natural conformal structure on any Euclidean surface. If S is obtained by gluing a locally finite collection of Euclidean polygons by isometries of the edges, the resulting surface is Euclidean away from the vertices, where there might be an angle deficit or surplus. If v is a vertex which has a cone angle of $r\pi$, we can develop the complement of v locally to the complement of the origin in $\mathbb{E}^2$. If we think of $\mathbb{E}^2$ as $\mathbb{C}$, and compose this developing map with the map $z \to z^{2/r}$ the result extends over v and defines a conformal chart near v which is compatible with the conformal charts on nearby points.

Let S be an arbitrary triangulated surface. By taking each triangle to be an equilateral Euclidean triangle with side length 1, and all gluing maps between edges to be isometries, we see that every surface can be given a conformal structure.

EXAMPLE 1.9 (Belyi's Theorem [**9**]). Belyi proved that a non-singular algebraic curve X is conformally equivalent to a surface obtained by gluing black and white equilateral triangles as above in a checkerboard pattern (i.e. so that no two triangles of the same color share an edge) if and only if X can be defined over an algebraic number field (i.e. a finite algebraic extension of $\mathbb{Q}$).

Such a description defines a map $X \to \mathbb{P}^1$, by taking the black triangles to the upper half space and the white triangles to the lower half space; this map is algebraic, and unramified except at $0, 1, \infty$. The preimage of the interval $[0, 1]$ is a bipartite graph on X, which Grothendieck called a "dessin d'enfant" (child's drawing; see [**100**]). The point of this construction is that the algebraic curve X can be recovered from the combinatorics and topology of the diagram. The Galois group $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on the set of all dessins, and gives unexpected topological insight into this fundamental algebraic object.

A conformal structure on S induces a tautological conformal structure on $\overline{S}$. We say that a conformal structure on S is *conformally finite* if it is conformally equivalent to a closed surface minus finitely many points. Every surface of finite type admits a conformal structure which is conformally finite.

The classical *Uniformization Theorem* for Riemann surfaces says that any surface S with a conformal structure admits a complete Riemannian metric of constant curvature in its conformal class, which is unique up to similarity (note that this theorem is also valid for conformal surfaces of *infinite* type).

1.2.2. Conformal structures on surfaces with boundary. Let S be a surface with boundary. We say that a conformal structure on S is given by a conformal structure on DS which induces the same conformal structures on the interiors of S and $\overline{S}$ by inclusion, after composing with the tautological identification of S and $\overline{S}$. A surface with boundary S is said to be of finite type if DS is of finite type, and a conformal structure on S is *conformally finite* if DS is conformally finite.

If S admits a conformally finite conformal structure, we *define*

$$\chi(S) = \frac{1}{2}\chi(DS)$$

Note that this may not be an integer, but always takes values in $\frac{1}{2}\mathbb{Z}$.

If T is compact with boundary, and there is an inclusion $i : S \to T$ so that $T - i(S)$ is finite, then

$$\chi(S) = \chi(T) - \text{card}(\text{int}(T) - i(\text{int}(S))) - \frac{1}{2}\text{card}(\partial T - i(\partial S))$$

1.2.3. Hyperbolic surfaces. A Riemannian metric on a surface S is said to be *hyperbolic* if it has curvature -1 everywhere. A conformally finite surface admits a unique compatible hyperbolic metric which is complete of finite area if and only if $\chi(S) < 0$. The *Gauss–Bonnet Theorem* says that for any closed Riemannian surface S there is an equality

$$\int_S K = 2\pi\chi(S)$$

where K is the sectional curvature on S.

If S is hyperbolic, we obtain an equality

$$\text{area}(S) = -2\pi\chi(S)$$

A conformally finite surface S with boundary admits a unique hyperbolic structure for which ∂S is totally geodesic if and only if $\chi(S) < 0$. For, by definition, $\chi(DS) < 0$ and therefore DS admits a unique complete finite area hyperbolic structure in its conformal class. If $i : DS \to DS$ is the involution which interchanges S and $\overline{S}$, then i preserves the conformal structure, and therefore it acts on DS as an isometry. It follows that the fixed point set, which can be identified with ∂S, is totally geodesic. Notice that with our definition of $\chi(S)$ the relation

$$\text{area}(S) = -2\pi\chi(S)$$

holds also for surfaces with boundary.

1.2.4. Straightening chains. Let Δ be a geodesic triangle in $\mathbb{H}^2$. The Gauss–Bonnet Theorem gives a straightforward relationship between the area of Δ and the sum of the interior angles:

$$\text{area}(\Delta) = \pi - \text{sum of interior angles of } \Delta$$

It follows that there is a fundamental inequality

$$\text{area}(\Delta) < \pi$$

A geodesic triangle in $\mathbb{H}^2$ is *semi-ideal* if some of its vertices lie at infinity, and *ideal* if all three vertices are at infinity. If we allow Δ to be semi-ideal above, the inequality becomes

$$\text{area}(\Delta) \leq \pi$$

with equality if and only if Δ is ideal.

Similar inequalities hold in every dimension; that is, for every dimension m there is a constant $c_m > 0$ such that every geodesic hyperbolic m-simplex has volume $\leq c_m$, with equality if and only if the simplex is ideal and regular (Haagerup and Munkholm [**101**]). Note that every ideal 2-simplex is regular.

A fundamental insight, due originally to Thurston, is that in a hyperbolic manifold M^m, a singular chain can be replaced by a (homotopic) chain whose simplices are all geodesic. Applying this observation to the fundamental class $[M]$ of M, and observing that there is an upper bound on the volume of a geodesic simplex in each dimension, we see that the complexity (in a suitable sense) of a chain representing $[M]$ can be *bounded from below* in terms of c_m and $\text{vol}(M)$. That is, one can use (hyperbolic) geometry to estimate the complexity of an *a priori* topological quantity. Technically, the right way to quantify the complexity of $[M]$ is to use bounded (co-)homology, which we will study in detail in Chapter 2.

Definition 1.10. Let M be a hyperbolic m-manifold, and let $\sigma : \Delta^n \to M$ be a singular n-simplex. Define the *straightening* σ_g of σ as follows. First, lift σ to a map from Δ^n to $\mathbb{H}^m$ which we denote by $\tilde{\sigma}$.

Let $v_0, \cdots, v_n$ denote the vertices of Δ^n. In the hyperboloid model of hyperbolic geometry, $\mathbb{H}^m$ is the positive sheet (i.e. the points where $x_{m+1} > 0$) of the hyperboloid $\|x\| = -1$ in $\mathbb{R}^{m+1}$ with the inner product

$$\|x\| = x_1^2 + x_2^2 + \cdots + x_m^2 - x_{m+1}^2$$

If $t_0, \cdots, t_n$ are barycentric co-ordinates on Δ^n, so that $v = \sum_i t_i v_i$ is a point in Δ^n, define

$$\tilde{\sigma}_g(v) = \frac{\sum_i t_i \tilde{\sigma}(v_i)}{-\| \sum_i t_i \tilde{\sigma}(v_i)\|}$$

and define σ_g to be the composition of $\tilde{\sigma}$ with projection $\mathbb{H}^m \to M$.

Since the isometry group of $\mathbb{H}^m$ acts on $\mathbb{R}^{m+1}$ linearly preserving the form $\|\cdot\|$, the straightening map $\sigma \to \sigma_g$ is well-defined, and independent of the choice of lift.

Let M be a hyperbolic manifold. Define

$$\text{str} : C_*(M) \to C_*(M)$$

by setting $\text{str}(\sigma) = \sigma_g$, and extending by linearity.

By composing a linear homotopy in $\mathbb{R}^{m+1}$ with radial projection to the hyperboloid, one sees that there is a chain homotopy between str and the identity map.

1.2.5. The Gromov norm. We now return to hyperbolic surfaces. Let S be conformally finite, possibly with boundary. If S is closed and oriented, the *fundamental class* of S, denoted $[S]$, is the generator of $H_2(S, \partial S)$ which induces the orientation on S.

Definition 1.11. Define the L^1 norm, also called the *Gromov norm* of S, as follows. Consider the homomorphism

$$i_* : H_2(S, \partial S; \mathbb{Z}) \to H_2(S, \partial S; \mathbb{R})$$

induced by inclusion $\mathbb{Z} \to \mathbb{R}$, and by abuse of notation, let $[S]$ denote the image of the fundamental class. Let $C = \sum_i r_i \sigma_i$ represent $[S]$, where the coefficients r_i are real, and denote

$$\|C\|_1 = \sum_i |r_i|$$

Then set

$$\|[S]\|_1 = \inf_C \|C\|_1$$

The following lemma, while elementary, is very useful in what follows.

Lemma 1.12. *Let S be an orientable surface with p boundary components. If $p > 1$ then for any integer $m > 1$ with m and $p-1$ coprime there is an m-fold cyclic cover S_m with p boundary components, each of which maps to the corresponding component of ∂S by an m-fold covering.*

Proof. The inclusion $\partial S \to S$ induces a homomorphism $H_1(\partial S) \to H_1(S)$ whose kernel is 1-dimensional, and generated by the homology class represented by the union ∂S. In particular, if $p > 1$, then we can take $p-1$ boundary components to be part of a basis for $H_1(S)$. Denote the images of the boundary components in $H_1(S)$ by $e_1, \cdots, e_p$, and let $e_1, \cdots, e_{p-1}$ be part of a basis for $H_1(S)$. If m and $p-1$ are coprime, let $\alpha \in H^1(S; \mathbb{Z}/m\mathbb{Z}) = \text{Hom}(H_1(S); \mathbb{Z}/m\mathbb{Z})$ satisfy $\alpha(e_i) = 1$ for $1 \leq i \leq p-1$. Then $\alpha(e_j)$ is primitive for all $1 \leq j \leq p$. The kernel of α defines a regular m-fold cover S_m with the desired properties. $\square$

Remark 1.13. A surface with exactly one boundary component has no regular (nontrivial) covers with exactly one boundary component. However *irregular* covers with this property do exist. For example, let S be a genus one surface with one boundary component, so $\pi_1(S)$ is free on two generators a, b. Let $\phi : \pi_1(S) \to S_3$ be the permutation representation defined by $\phi(a) = (12)$ and $\phi(b) = (23)$. Then $\phi([a, b]) = (312)$, which is a 3-cycle. This representation determines a 3-sheeted (irregular) cover of S with one boundary component.

It is straightforward to generalize this example to show that every connected oriented surface with $\chi \leq 0$ admits a connected cover of arbitrarily large degree with the same number of boundary components.

Theorem 1.14 (Gromov norm of a hyperbolic surface). *Let S be a compact orientable surface with $\chi(S) < 0$, possibly with boundary. Then*

$$\|[S]\|_1 = -2\chi(S)$$

Proof. Let S be a surface of genus g with p boundary components, so that

$$\chi(S) = 2 - 2g - p$$

The surface S admits a triangulation with one vertex on each boundary component, and no other vertices. Any such triangulation has $4g + 3p - 4$ triangles. Figure 1.2 exhibits the case $g = 1$, $p = 2$. By Lemma 1.12, there is an m-fold cover S_m of S with p boundary components.

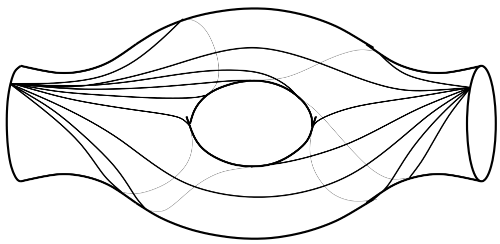


Figure 1.2. A triangulation of a surface with $g = 1, p = 2$ by 6 triangles

Since χ is multiplicative under covers, $\chi(S_m) = 2m - 2gm - mp$ and it can be triangulated with $p + m(4g + 2p - 4)$ triangles. Projecting this triangulation under the covering map $S_m \to S$ gives an integral chain representing $m[S]$ with L^1 norm equal to $p + m(4g + 2p - 4)$. Dividing coefficients by m and taking the limit as $m \to \infty$, we get

$$\|[S]\|_1 \leq -2\chi(S)$$

To obtain the other inequality, let C be any chain representing $[S]$. Then $\text{str}(C)$ has L^1 norm no greater than that of C, and also represents $[S]$. On the other hand, since every geodesic triangle has area $\leq \pi$, and $\text{area}(S) = -2\pi\chi(S)$, we obtain

$$\|[S]\|_1 \geq -2\chi(S)$$

□

Remark 1.15. If $\chi(S) \geq 0$ then S admits a proper self map $f : S \to S$ of any degree. By pushing forward a chain under this map and dividing coefficients, one sees that $\|[S]\|_1 = 0$.

If X is any topological space and α is a class in $H_2(X; \mathbb{R})$ the Gromov norm of α, denoted $\|\alpha\|_1$, is the infimum of the L^1 norm over all (real valued) 2-cycles representing the homology class α. If α is rational, any real 2-cycle representing

α can be approximated in L^1 by a rational 2-cycle representing α. By multiplying through to clear denominators, some multiple $n\alpha$ can be represented by a map of a surface $S \to X$. For such a surface S, let $\chi^-(S)$ denote the Euler characteristic of the union of the non-spherical components of S. Then Theorem 1.14 implies that

$$\|\alpha\|_1 = \inf_S \frac{-2\chi^-(S)}{n(S)}$$

where the image of the fundamental class of S under the map $S \to X$ represents $n(S)\alpha$ in homology, and the infimum is taken over all maps of (possibly disconnected) closed oriented surfaces into X.