Part 3A. The canonical S-operator and the canonical class of linear differential equations of second order on algebraic function field L of one variable over $\mathbf{C}$, and their algebraic characterizations when L is "arithmetic".

The S-operators.

§37. The symbol $\langle \eta, \xi \rangle$. Let L be any field, let $D(L)$ be a *one*-dimensional vector space over L, and let $d : L \to D(L)$ be a map satisfying $d(x+y) = dx + dy$, $d(xy) = xdy + ydx$ for all $x, y \in L$. For each positive integer h, denote by $D^h(L)$ the tensor product $D(L) \otimes \cdots \otimes D(L)$ (h copies) over L (so that $\dim_L D^h(L) = 1$), and call the elements of $D^h(L)$ *differentials of degree* h (in L). Put $D(L)^\times = D(L) \setminus \{0\}$. Then if ξ is any fixed element of $D(L)^\times$, the elements of $D^h(L)$ are expressed uniquely in the form $a \cdot \xi^h$ ($a \in L$). Here, ξ^h will always denote $\xi \otimes \cdots \otimes \xi$ (h copies). For any $\xi \in D(L)^\times$ and $\eta \in D(L)$, the number $a \in L$ with $\eta = a\xi$ will be denoted by η/ξ. Finally, we shall denote by k the constant field, i.e., $k = \{x \in L \mid dx = 0\}$. It is clear that k is a subfield of L.

Now for each $\xi, \eta \in D(L)^\times$, an element $\langle \eta, \xi \rangle$ of $D^2(L)$ is defined in the following way. Put $w_1 = \eta/\xi$, $w_{i+1} = dw_i/\xi$ ($i \geq 1$). Then

Definition.

$$\langle \eta, \xi \rangle = \frac{2w_1w_3 - 3w_2^2}{w_1^2}\xi^2.$$

In particular, if $x, y \in L \backslash k$, then we have

$$\langle dy, dx \rangle = \frac{2\left(\frac{dy}{dx}\right)\left(\frac{d^3y}{dx^3}\right) - 3\left(\frac{d^2y}{dx^2}\right)^2}{\left(\frac{dy}{dx}\right)^2}(dx)^2, \tag{76}$$

where $\frac{d^i}{dx^i} = \left(\frac{d}{dx}\right)^i$ ($i \geq 1$). Thus $\langle \eta, \xi \rangle$ is, so to speak, the "algebraic Schwarzian derivative". The following Proposition is classically well-known for the analytic Schwarzian derivative.

Proposition 7. (i) *For any* $\xi, \eta, \zeta \in D(L)^\times$, *we have*

$$\langle \eta, \zeta \rangle - \langle \xi, \zeta \rangle = \langle \eta, \xi \rangle. \tag{77}$$

(ii) *Let* $\eta \in D(L)^\times$ *and* $x \in L \backslash k$. *Then* $\langle \eta, dx \rangle = 0$ *if and only if* η *is of the form* $\eta = dx_1$ *with* $x_1 = \frac{ax+b}{cx+d}$, $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL_2(k)$.[20]

[20] Here, the same notation d is used for the map $d : L \to D(L)$ and for the $(2,2)$-element of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. I hope that this will not confuse the readers.

Corollary. *We have*

$$(78)\qquad \begin{cases} \langle \xi, \xi \rangle = 0 \\ \langle \xi, \eta \rangle = -\langle \eta, \xi \rangle \end{cases}$$

for any $\xi, \eta \in D(L)^{\times}$.

The Corollary follows immediately from (i) by putting $\zeta = \eta = \xi$, and $\zeta = \eta$.

Proof. (i) is obtained by a straightforward computation.

(ii) Put $\eta = zdx$ $(z \in L)$, so that $\langle \eta, dx \rangle = 0$ is equivalent to ($\sharp$) $2zz_{xx} = 3(z_x)^2$, where the suffix x denotes the effect of the derivation $\frac{d}{dx}$. First, let $ch(L) = 2$. Then $\langle \eta, dx \rangle = 0 \iff z_x = 0 \iff z \in k \iff \eta = dx_1$ with $x_1 = ax, a \in k^{\times}$. On the other hand, $d\left(\frac{ax+b}{cx+d}\right) = (ad - bc)(cx + d)^{-2}dx$, and since $ch(L) = 2$, all square elements of L are contained in k. This settles the case of $ch(L) = 2$. Now let $ch(L) \neq 2$ and put $z = y^{-1}$. Then the equation ($\sharp$) is equivalent to (b) $y_x^2 = 2yy_{xx}$. By applying $\frac{d}{dx}$ on (b), we obtain $2yy_{xxx} = 0$; hence $y_{xxx} = 0$; hence y is a quadratic polynomial of x over k. From this follows easily that the general solution of (b) is $y = a(bx + c)^2$ $(a, b, c \in k)$. Therefore, $\langle \eta, dx \rangle = 0$ if and only if z is of the form $a^{-1}(bx + c)^{-2}$; which settles (ii). □

§38. The S-operators. Let the notations be as in §37. A map $S : D(L)^{\times} \to D^2(L)$ will be called an *S-operator* (on L) if

$$(79)\qquad S\langle \eta \rangle - S\langle \xi \rangle = \langle \eta, \xi \rangle$$

holds for all $\xi, \eta \in D(L)^{\times}$. Thus by Proposition 7 (i), if ζ is any fixed element of $D(L)^{\times}$, the map S_{ζ} defined by $S_{\zeta}\langle \xi \rangle = \langle \xi, \zeta \rangle$ gives an S-operator (an inner S-operator), and it is clear that all other S-operators are given by $S\langle \xi \rangle = S_{\zeta}\langle \xi \rangle + C$, where C is an arbitrary constant in $D^2(L)$. In general, not all S-operators are inner (or equivalently, not all elements of $D^2(L)$ are of the form $\langle \zeta, \zeta' \rangle$ $(\zeta, \zeta' \in D(L)^{\times})$), and as is shown later, a certain *outer* S-operator plays a central role in our problems.

The canonical S-operator on algebraic function field of one variable over C, and its algebraic characterization in ample (arithmetic) cases.

§39. The canonical S-operator on the field of automorphic functions. Let X be any Riemann surface, compact or open. Let L_X be the field of meromorphic functions on X, and let $D(L_X)$ be the space of all meromorphic differential forms on X (of degree one), considered as a vector space over L_X. Let $d : L_X \to D(L_X)$ be the ordinary differentiation. Then the sympol $\langle \eta, \xi \rangle$ for this situation is essentially [21] the same as the classical Schwarzian derivative. If σ is any automorphism of X, then σ acts on $D^n(L_X)$ as $\omega \to \omega^{\sigma} = \omega \circ \sigma$, and it is clear that $\langle \eta, \xi \rangle^{\sigma} = \langle \eta^{\sigma}, \xi^{\sigma} \rangle$.

[21] I.e., up to a slight modification of the definition.

Now let $\mathfrak{H} = \{\tau \in \mathbf{C} \mid \operatorname{Im}\tau > 0\}$. We consider τ as a function on $\mathfrak{H}$;

$$\tau \in L_{\mathfrak{H}}. \tag{80}$$

Put $G_{\mathbf{R}} = \mathrm{PSL}_2(\mathbf{R}) = \mathrm{Aut}(\mathfrak{H})$ (by the usual action; see Chap.1, §3). Let $\sigma \in G_{\mathbf{R}}$ and $f(\tau) \in L_{\mathfrak{H}}$ with $f(\tau) \neq 0$. Then since τ^σ is a linear fractional transform of τ, Proposition 7 shows that $\langle f(\tau)d\tau, d\tau\rangle^\sigma = \langle (f(\tau)d\tau)^\sigma, d\tau^\sigma\rangle = \langle (f(\tau)d\tau)^\sigma, d\tau\rangle$; hence *if $f(\tau)d\tau$ is invariant by σ, so is $\langle f(\tau)d\tau, d\tau\rangle$.*

Let Δ be a fuchsian group, i.e., a discrete subgroup of $G_{\mathbf{R}}$ with finite-volume quotient. Let $(\Delta\backslash\mathfrak{H})^*$ denote the compact Riemann surface obtained by compactification and normalization of the quotient $\Delta\backslash\mathfrak{H}$, so that $L_{(\Delta\backslash\mathfrak{H})^*}$ is nothing but the field of automorphic functions with respect to Δ. Consider $L_{(\Delta\backslash\mathfrak{H})^*}$ and $D^h(L_{(\Delta\backslash\mathfrak{H})^*})$ as a subfield and a subspace of $L_{\mathfrak{H}}$ and $D^h(L_{\mathfrak{H}})$ respectively. Then $f(\tau)(d\tau)^h \in D^h(L_{\mathfrak{H}})$ belongs to $D^h(L_{(\Delta\backslash\mathfrak{H})^*})$ if and only if $f(\tau)$ is a meromorphic automorphic form of weight $2h$ with respect to Δ. Now consider the inner S-operator

$$D(L_{\mathfrak{H}})^\times \ni f(\tau)d\tau \to \langle f(\tau)d\tau, d\tau\rangle \in D^2(L_{\mathfrak{H}}) \tag{81}$$

on $L_{\mathfrak{H}}$. We shall show that (81) induces an (outer) [22] S-operator on $L_{(\Delta\backslash\mathfrak{H})^*}$ by restriction. It is enough to check that if $f(\tau)d\tau \in D(L_{(\Delta\backslash\mathfrak{H})^*})^\times$, then $\langle f(\tau)d\tau, d\tau\rangle \in D^2(L_{(\Delta\backslash\mathfrak{H})^*})$. Put $\langle f(\tau)d\tau, d\tau\rangle = \varphi(\tau)(d\tau)^2$. Then $\varphi(\tau) = \frac{2f(\tau)f''(\tau)-3f'(\tau)^2}{f(\tau)^2}$, where $'$ denotes the derivative with respect to τ. Since $f(\tau)d\tau$ is Δ-invariant, $\langle f(\tau)d\tau, d\tau\rangle$ is also Δ-invariant; hence $\varphi(\delta\tau) = \varphi(\tau)(c\tau+d)^4$ holds for all $\delta = \begin{pmatrix} a & b \\ c & d\end{pmatrix} \in \Delta$. Moreover, by a simple estimation of $|\varphi(\tau)|$ as each cusp of Δ, it follows easily that $\varphi(\tau)$ is a meromorphic automorphic form of weight 4 with respect to Δ. Therefore, $\langle f(\tau)d\tau, d\tau\rangle$ belongs to $D^2(L_{(\Delta\backslash\mathfrak{H})^*})$. So we have proved:

Proposition 8. *Let Δ be a fuchsian group. Then if $f(\tau)d\tau \in D(L_{(\Delta\backslash\mathfrak{H})^*})^\times$, we have $\langle f(\tau)d\tau, d\tau\rangle \in D^2(L_{(\Delta\backslash\mathfrak{H})^*})$. In other words, the inner S-operator $f(\tau)d\tau \to \langle f(\tau)d\tau, d\tau\rangle$ on $L_{\mathfrak{H}}$ induces an outer S-operator on $L_{(\Delta\backslash\mathfrak{H})^*}$.*

As we have seen above, this is equivalent to the classically known fact that if $f(\tau)$ is a (meromorphic) automorphic form of weight 2, then $\varphi(\tau) = \frac{2f(\tau)f''(\tau)-3f'(\tau)^2}{f(\tau)^2}$ is a (meromorphic) automorphic form of weight 4.

Definition . This special S-operator on $L_{(\Delta\backslash\mathfrak{H})^*}$ will be called the *canonical S-operator* on $L_{(\Delta\backslash\mathfrak{H})^*}$, and denoted by S^Δ.

Remark 1. Let $\sigma \in G_{\mathbf{R}}$, $\Delta' = \sigma^{-1}\Delta\sigma$, and put $L = L_{(\Delta\backslash\mathfrak{H})^*}$, $L' = L_{(\Delta'\backslash\mathfrak{H})^*}$. Let ι_σ be the isomorphism $L \to L'$ defines by $f(\tau) \to f(\sigma\tau)$, and let $\iota_{\sigma h}$ ($h \geq 1$) be the map $D^h(L) \to D^h(L')$ induced by ι_σ. Then,

$$\iota_{\sigma 2} \circ S^\Delta = S^{\Delta'} \circ \iota_{\sigma 1}. \tag{82}$$

This follows immediately by using Proposition 7 (ii).

[22] That this is outer on $L_{(\Delta\backslash\mathfrak{H})^*}$ is obvious by Proposition 7 (ii).

REMARK 2. Let Δ' be a subgroup of Δ with finite index. Then $L_{(\Delta\backslash\mathfrak{H})^*} \subset L_{(\Delta'\backslash\mathfrak{H})^*}$, and the restriction of $S^{\Delta'}$ to $D(L_{(\Delta\backslash\mathfrak{H})^*})^\times$ gives S^Δ. This is obvious by the definition of S^Δ.

§40. The canonical S-operator on algebraic function field over C (First formulation).

[1]. In this section, $\{L, e\}$ will denote the following pair:

- L is a finitely generated one-dimensional algebraic function field over $\mathbf{C}$ ($\mathbf{C}$: the field of complex numbers);
- $e = e(P)$ is a $\{1, 2, 3, \ldots; \infty\}$-valued function defined on the set of all prime divisors P of L, and satisfies:
  - (i) $e(P) = 1$ for almost all P,
  - (ii) the quantity

$$V\{L, e\} = 2g - 2 + \sum_P \left(1 - \frac{1}{e(P)}\right) \tag{83}$$

is positive, g being the genus of L.

Then, as is well-known, $\{L, e\}$ are in one-to-one correspondence with the fuchsian groups Δ, where $\{L, e\}$ are counted up to isomorphisms;[23] and Δ, up to conjugacy in $G_{\mathbf{R}}$;

$$\{L, e\} \underset{1:1}{\longleftrightarrow} \Delta. \tag{84}$$

Starting from Δ, this correspondence is defined as follows: Take L to be the field of automorphic functions $L_{(\Delta\backslash\mathfrak{H})^*}$. (So, the prime divisors of $L_{(\Delta\backslash\mathfrak{H})^*}$ are identified with the points on $(\Delta\backslash\mathfrak{H})^*$.) Define the function $e = e_\Delta$ by

$$e_\Delta(P) = \begin{cases} \infty & \cdots \quad P \text{ is a cusp of } \Delta, \\ e_0 & \cdots \quad \begin{array}{l} P \text{ is an elliptic fixed point} \\ \text{of } \Delta \text{ of order } e_0 > 1, \end{array} \\ 1 & \cdots \quad \text{otherwise.} \end{cases} \tag{85}$$

Then $\Delta \to \{L_{(\Delta\backslash\mathfrak{H})^*}, e_\Delta\}$ defines the above one-to-one correspondence.

REMARK 1. The automorphism group of $\{L_{(\Delta\backslash\mathfrak{H})^*}, e_\Delta\}$ is naturally identified with $N(\Delta)/\Delta$, where $N(\Delta)$ is the normalizer of Δ in $G_{\mathbf{R}}$.

REMARK 2. As is well-known,

$$V\{L_{(\Delta\backslash\mathfrak{H})^*}, e_\Delta\} = \frac{1}{2\pi} \int_{\Delta\backslash\mathfrak{H}} \frac{dxdy}{y^2} \quad (\tau = x + yi). \tag{86}$$

EXAMPLE. Let $g = 0$; $e(P) = 2, n, \infty$ for three P and $= 1$ for all other P, where $n \geq 3$. Then $v\{L, e\} = \frac{1}{2} - \frac{1}{n} > 0$, and Δ is the Hecke's group generated by $\begin{pmatrix} 1 & \lambda_n \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, where $\lambda_n = 2\cos\frac{\pi}{n}$. If $n = 3$, $\Delta = PSL_2(\mathbf{Z})$; and in general, Δ is commensurable with $PSL_2(\mathbf{Z})$ if and only if $n = 3, 4, 6, \infty$.

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[23] $\{L, e\} \simeq \{L', e'\}$ if there exists an isomorphism $\iota : L \simeq L'$, identical on $\mathbf{C}$, satisfying $e(P) = e'(\iota(P))$ for all P.

[2]. Now let $\{L, e\}$ be given. Let $\{\sigma^{-1}\Delta\sigma \mid \sigma \in G_{\mathbf{R}}\}$ be the corresponding $G_{\mathbf{R}}$-conjugacy class of fuchsian groups, and take a representative Δ. Let ι_Δ be any isomorphism ι_Δ : $\{L, e\} \simeq \{L_{(\Delta\backslash\mathfrak{H})^\bullet}, e_\Delta\}$. Let S^Δ be the canonical S-operator on $L_{(\Delta\backslash\mathfrak{H})^\bullet}$ and put $S^{\{L,e\}} = \iota_{\Delta 2}^{-1} \circ S^\Delta \circ \iota_{\Delta 1}$, where $\iota_{\Delta h}$ $(h \geq 1)$ is the map $D^h(L) \to D^h(L_{(\Delta\backslash\mathfrak{H})^\bullet})$ induced by ι_Δ :

$$(87)\qquad \begin{array}{ccc} D(L)^\times & \xrightarrow{S^{\{L,e\}}} & D^2(L) \\ \downarrow \iota_{\Delta 1} & & \downarrow \iota_{\Delta 2} \\ D(L_{(\Delta\backslash\mathfrak{H})^\bullet})^\times & \xrightarrow{S^\Delta} & D^2(L_{(\Delta\backslash\mathfrak{H})^\bullet}) \end{array} \qquad \text{(commutative).}$$

Then by the Remark 1 of §39 and Remark 1 of §40, $S^{\{L,e\}}$ is well-defined by $\{L, e\}$ and is independent of the choice of representative Δ of $\{\sigma^{-1}\Delta\sigma \mid \sigma \in G_{\mathbf{R}}\}$. By this, it is also clear that $S^{\{L,e\}}$ commutes with every automorphism ε of $\{L, e\}$; i.e., $S^{\{L,e\}} = \varepsilon_2^{-1} S^{\{L,e\}} \varepsilon_1$, where ε_h is defined from ε in the same manner as above.

Definition. We shall call this special S-operator $S^{\{L,e\}}$ on L *the canonical S-operator attached to* $\{L, e\}$.

[3]. Thus, *the notion of S-operators on algebraic function field L is algebraic, and the canonical S-operator $S^{\{L,e\}}$ is one of them defined analytically.* Since all S-operators on L are of the form $S\langle\xi\rangle = \langle\xi, \zeta\rangle + C$, where ζ is any fixed element of $D(L)^\times$ and C is an arbitrary constant in $D^2(L)$, S-operators are determined by its special value $C = S\langle\zeta\rangle$. Thus, we meet an interesting problem to find out (an algebraic formula for) $S^{\{L,e\}}\langle\zeta\rangle$, when $\{L, e\}$ and ζ are explicitly given *algebraically*. However, for the general $\{L, e\}$, this problem seems to be quite difficult! For example, to my best knowledge, the following is an open problem:

Problem. Let $L = \mathbf{C}(x, y)$, $y^2 = (x - \alpha_1)\cdots(x - \alpha_n)$, where $n \geq 5$ and $\alpha_1, \cdots, \alpha_n$ are distinct (hence $g \geq 2$). Let $e(P) = 1$ for all P, and let $S = S^{\{L,e\}}$ be the canonical S-operator attached to $\{L, e\}$. Then, what is $S\langle dx\rangle$?

For the special types of $\{L, e\}$ however, there are some principles for determining (or characterizing) $S^{\{L,e\}}$ algebraically. In fact, there are two such principles, of which the second is more important:

[4] The first principle. This is based on the following Propositions 9, 10:

Proposition 9. *Let $\xi \in D(L)^\times$ and put $S^{\{L,e\}}\langle\xi\rangle = -4\beta$. Let P be any prime divisor of L. Then,*

(i)[24] $\mathrm{ord}_P\beta \geq -2$;
(ii) *Put $e = e(P)$, $n = \mathrm{ord}_P\xi$; let t be a prime element of P (in the completion of L at P), and put*

$$(88)\qquad \xi = ct^n(1 + c_1 t + \cdots)dt,\ c \neq 0,\ c_1, c_2, \cdots \in \mathbf{C}.$$

[24] The definition of $\mathrm{ord}_P\,\omega$ for $\omega \in D^h(L)$, $\omega \neq 0$ is obvious; if $\omega \in L$ or $\in D(L)$, $\mathrm{ord}_P\,\omega$ is the ordinary "order" of ω at P; and for $\omega = a\xi^h$ $(a \in L,\ \xi \in D(L)^\times)$, $\mathrm{ord}_P\,\omega = \mathrm{ord}_P\,a + h\ \mathrm{ord}_P\,\xi$.

Then we have

$$\beta = \left\{\frac{\beta_0}{t^2} + \frac{\beta_1}{t} + \beta_2 + \cdots\right\}(dt)^2, \tag{89}$$

with

$$\begin{cases} \beta_0 = \frac{1}{4}\left\{(n+1)^2 - \frac{1}{e^2}\right\} & \cdots \ \textit{at any } P; \\ \beta_1 = \frac{1}{2}nc_1 & \cdots \ \textit{if } e(P) = 1. \end{cases} \tag{90}$$

The proof may be obtained directly, but an indirect proof will be given in §42 [5]. There, it is also shown that if S is any S-operator on L and if we put $S\langle\xi\rangle = -4\beta$, then (i) (ii) hold for *all* ξ if and only if they hold for *one* ξ (thus, (i) (ii) are conditions on S). The meaning of these conditions will also become clear there.

Definition. $\{L', e'\}$ is called an *admissible extension* of $\{L, e\}$ if

(i) L' is a finite extension of L, and

(ii) $e(P) = e'(P')e(P'/P)$ holds for all prime divisors P' of L', where P is the restriction of P' to L, and $e(P'/P)$ is the ramification index of P'/P.

It is clear that if Δ is the fuchsian group corresponding to $\{L, e\}$, then the admissible extensions of $\{L, e\}$ are those pairs $\{L', e'\}$ that correspond to the subgroups Δ' of Δ with finite indices. From this, and from the definition of $S^{\{L,e\}}$, we obtain immediately:

Proposition 10. *Let $\{L', e'\}$ be an admissible extension of $\{L, e\}$. Then $S^{\{L,e\}}$ is the restriction of $S^{\{L',e'\}}$ to $D(L)^\times$, and $S^{\{L',e'\}}$ is the unique S-operator on L' with this property.*

The second point is obvious since S-operator is determined by its special value.

Now, Proposition 9 determines $S^{\{L,e\}}$ up to $(3g-3+\sum_{P;e(P)>1} 1)$- dimensional subspace of $D^2(L)$. In fact, fix ξ and put $\beta_1 = \beta+\mu$ (μ: a variable in $D^2(L)$). Then β_1 also satisfies the conditions (i) (ii) of Proposition 9 if and only if μ is a multiple of $\prod_{e(P)>1} P^{-1}$. Therefore, if we put $W = (\xi)$ (the divisor of ξ), the dimension of μ is given by $\ell(W^{-2}\prod_{e(P)>1} P^{-1}) = 3g-3+\sum_{e(P)>1} 1$.

Remark 3. This number $3g-3+\sum_{e(P)>1} 1$ is equal to the dimension of the connected moduli variety of $\{L, e\}$. But we do not know why.

So, Proposition 9 determines $S^{\{L,e\}}$ uniquely only when $3g-3+\sum_{e(P)>1} 1 = 0$; i.e., only when $g = 0$ and $\sum_{e(P)>1} 1 = 3$ (called the triangular case).[25] In this case, we can determine $S^{\{L,e\}}$ easily by a direct application of Proposition 9. We have:

Proposition 11. *Let $L = \mathbf{C}(x)$ (the rational function field), and let $e(P) = 1$ except at three points P. We may assume that these three points are given by $x \equiv 0, 1, \infty \pmod P$ respectively. Call them P_0, P_1, P_∞, and put $e(P_i) = e_i$ $(i = 0, 1, \infty)$ (so that $\frac{1}{e_0}+\frac{1}{e_1}+\frac{1}{e_\infty} < 1$). Then the canonical S-operator $S^{\{L,e\}}$ is given by*

$$S^{\{L,e\}}\langle\xi\rangle = \langle\xi, dx\rangle + \frac{ax^2+bx+c}{x^2(x-1)^2} \quad (\xi \in D(L)^\times), \tag{91}$$

[25] If $g = 1$ and $e(P) = 1$ for all P, then $\{L, e\}$ does not satisfy the condition (ii) of §40.

where

$$a = \frac{1}{e_\infty^2} - 1,\ b = 1 + \frac{1}{e_1^2} - \frac{1}{e_0^2} - \frac{1}{e_\infty^2},\ c = \frac{1}{e_0^2} - 1. \tag{92}$$

Now, call the two $\{L, e\}$ and $\{L', e'\}$ *commensurable* if they have a common admissible extension. (Clearly, this is equivalent to the commensurability of the corresponding fuchsian groups.) Then by Propositions 10, 11, we conclude that *if* $\{L, e\}$ *is commensurable with the triangular pair, then* $S^{\{L,e\}}$ *is determined algebraically.*

[5] The second principle. This is based on the following very simple fact:

Proposition 12. *Consider the following situation:*

$$\begin{tikzcd}[column sep=small]
 & \{L,e\} \arrow[dl, dash, "\text{admissible}"'] \arrow[dr, dash, "\text{admissible}"] & \\
\{L_1,e_1\} \arrow[dr, dash] & & \{L_2,e_2\} \arrow[dl, dash] \\
 & \mathbf{C} = L_1 \bigcap L_2 &
\end{tikzcd} \tag{93}$$

Then $S = S^{\{L,e\}}$ *is the unique* S*-operator on* $\{L, e\}$ *satisfying*

$$S\langle D(L_i)^\times\rangle \subset D^2(L_i) \tag{94}$$

for $i = 1, 2$ *(both).*

That $S^{\{L,e\}}$ satisfies (94) is an immediate consequence of Proposition 10. To see how the uniqueness follows, let $S' = S^{\{L,e\}} + C$ ($C \in D^2(L)$) be another S-operator satisfying (94). Then C must be contained in $D^2(L_1) \cap D^2(L_2)$. But by the Corollary of Lemma 14 given in §42, we have $D^h(L_1) \cap D^h(L_2) = \{0\}$ ($h \geq 1$). Hence $C = 0$; hence the uniqueness!

Corollary . *The situation being as in Proposition 12, let* ξ_1, ξ_2 *be any element of* $D(L_1)^\times, D(L_2)^\times$ *respectively. Then* $\langle \xi_1, \xi_2\rangle$ *has a unique decomposition of the form*

$$\langle \xi_1, \xi_2\rangle = \omega_1 - \omega_2; \quad \begin{cases} \omega_1 \in D^2(L_1), \\ \omega_2 \in D^2(L_2). \end{cases} \tag{95}$$

Moreover, these ω_1, ω_2 *are given by* $\omega_1 = S^{\{L,e\}}\langle \xi_1\rangle$, $\omega_2 = S^{\{L,e\}}\langle \xi_2\rangle$.

That (95) holds for $\omega_i = S^{\{L,e\}}\langle \xi_i\rangle$ ($i = 1, 2$) is obvious. Uniqueness is an immediate consequence of $D^2(L_1) \cap D^2(L_2) = \{0\}$.

The importance of this simple principle lies on the fact that *if* $\{L, e\}$ *is such that the corresponding fuchsian group* Δ *is arithmetically defined,* or more generally, if the commensurability group of Δ in $G_{\mathbf{R}}$ is dense in $G_{\mathbf{R}}$, *then* $\{L, e\}$ *is always commensurable with a situation* (93). Call such a commensurability family of $\{L, e\}$ *ample* or *arithmetic*. Then, we conclude by Proposition 12 that $S^{\{L,e\}}$ *can be characterized algebraically if* $\{L, e\}$ *belongs to an ample (arithmetic) commensurability family.* Now we shall proceed to obtain a better formulation of this than Proposition 12: for although Proposition 12 is convenient for the understanding of this principle in the simplest form, it is not convenient for applications or generalizations.

REMARK 4. By a result of Každan [19], the commensurability group of the Hecke's group $\Delta = \Delta_n$ for $n \neq 3, 4, 6, \infty$ is Δ itself; hence the commensurability family of the triangular $\{L, e\}$ with $e(P_i) = 2, n, \infty$ ($n \neq 3, 4, 6, \infty$) is not ample.

§41. The canonical S-operator on algebraic function field over C (second formulation), and its algebraic characterization in ample (arithmetic) cases.

[1]. In this section, L will denote any one-dimensional extension of $\mathbf{C}$ not assumed to be finitely generated over $\mathbf{C}$, but assumed to satisfy the following conditions (L1), (L2):

(L1) *Let $\mathcal{L}_0$ be the set of all finitely generated extensions $L_0/\mathbf{C}$ contained in L such that L/L_0 is normally algebraic. Then $\mathcal{L}_0$ is non-empty.*

(L2) *For each $L_0 \in \mathcal{L}_0$ and a prime divisor P_0 of L_0, denote by $e_0(P_0)$ the ramification index of P_0 in L/L_0. Then $e_0(P_0) = 1$ for almost all P_0, and the quantity*

$$V(L_0) = 2g_0 - 2 + \sum_{P_0} \left(1 - \frac{1}{e_0(P_0)}\right) \quad (g_0 : \text{ the genus of } L_0) \tag{96}$$

is positive; in short, $\{L_0, e_0\}$ satisfies the conditions (i) (ii) of §40.

REMARK 1. For any $L_0, L_0' \in \mathcal{L}_0$, we have

$$V(L_0L_0') = V(L_0)[L_0L_0' : L_0] = V(L_0')[L_0L_0' : L_0'] \tag{97}$$

by Hurwitz' formula; hence the condition (L2) is satisfied for all $L_0 \in \mathcal{L}_0$ if it is satisfied for one L_0.

[2]. Now consider $\mathcal{L}_0$ as an ordered set by the inclusion relation $\supset$. Then if $L_0, L_0' \in \mathcal{L}_0$ with $L_0 \subset L_0'$ (L_0: *smaller* than L_0'), we have $V(L_0) = \frac{1}{[L_0' : L_0]} V(L_0')$ by (97); but on the other hand, it is well-known (and easily checked) that $V\{L_0, e_0\} \geq \frac{1}{42}$ for any pair $\{L_0, e_0\}$ satisfying (i) (ii) of §40. Therefore, the ordered set $\mathcal{L}_0$ is inductive (i.e., any linearly ordered subset contains a minimal element). Hence $\mathcal{L}_0$ *contains at least one minimal element.*

DEFINITION . We shall call L "*simple*" if $\mathcal{L}_0$ contains only one minimal element, and "*ample*" (or "*arithmetic*") if otherwise.

REMARK 2. If $L/\mathbf{C}$ is finitely generated, i.e., if $L \in \mathcal{L}_0$, then L is simple. If fact, since $V(L) > 0$, the genus of L is greater than one; hence $\mathrm{Aut}_{\mathbf{C}} L$ is finite. Therefore, the fixed field of $\mathrm{Aut}_{\mathbf{C}} L$ is the unique minimal element of $\mathcal{L}_0$; hence L is simple.

PROPOSITION 13. (i) *If L is simple and L_{00} is the unique minimal element of $\mathcal{L}_0$, then*

$$\mathcal{L}_0 = \{L_0 \mid L_{00} \subset L_0 \subset L,\ [L_0 : L_{00}] < \infty\}. \tag{98}$$

(ii) *If L is ample and L_0, L_0' are two distinct minimal elements, then*

$$L_0 \cap L_0' = \mathbf{C}. \tag{99}$$

PROOF. (i) is obvious. (ii) If $L_0 \cap L_0' \neq \mathbf{C}$, then $L_0 \cap L_0' \in \mathcal{L}_0$, which is a contradiction. □

[3]. Now let $\mathrm{Aut}_{\mathbf{C}} L$ be the group of all automorphisms of the field L over $\mathbf{C}$. Topologize $\mathrm{Aut}_{\mathbf{C}} L$ by taking $\mathrm{Aut}_{L_0} L$ ($L_0 \in \mathcal{L}_0$) as basis of neighborhoods of identity. Then the induced topology of $\mathrm{Aut}_{L_0} L$ coincides with the Krull topology; hence $\mathrm{Aut}_{\mathbf{C}} L$ is locally compact. It is clear that a closed subgroup of $\mathrm{Aut}_{\mathbf{C}} L$ is non-compact if and only if its fixed field is $\mathbf{C}$.

PROPOSITION 14. (i) *If L is simple, and L_{00} is the unique minimal element of $\mathcal{L}_0$, then*

$$\mathrm{Aut}_{\mathbf{C}} L = \mathrm{Aut}_{L_0} L \ (= \textit{compact}). \tag{100}$$

(ii) *If L is ample,* $\mathrm{Aut}_{\mathbf{C}} L$ *is non-compact, and its fixed field is* $\mathbf{C}$.

PROOF. (i) Let $\sigma \in \mathrm{Aut}_{\mathbf{C}} L$. Then L_{00}^{σ} is also a minimal element of $\mathcal{L}_0$; hence $L_{00}^{\sigma} = L_{00}$. Moreover, it is clear that $e_{00}(P_{00}^{\sigma}) = e_{00}(P_{00})$, where P_{00} is any prime divisor of L_{00} and e_{00} is the ramification index in L/L_{00}. Therefore, there is a homomorphism $\mathrm{Aut}_{\mathbf{C}} L \to \mathrm{Aut}\{L_{00}, e_{00}\}$ with the kernel $\mathrm{Aut}_{L_{00}} L$. But $\mathrm{Aut}\{L_{00}, e_{00}\}$ is finite by Remark 1 §40; hence $(\mathrm{Aut}_{\mathbf{C}} L : \mathrm{Aut}_{L_{00}} L)$ is finite. Therefore, if L_{00}' is the fixed field of $\mathrm{Aut}_{\mathbf{C}} L$, then $L_{00}' \in \mathcal{L}_0$ and $L_{00}' \subset L_{00}$; hence $L_{00}' = L_{00}$; hence $\mathrm{Aut}_{\mathbf{C}} L = \mathrm{Aut}_{L_{00}} L$.

(ii) is obvious by Proposition 13 (ii). □

EXAMPLE . Let L be a G_p-field over $\mathbf{C}$ (see §1). Then L satisfies (L1), (L2), and L is ample.

[4]. Now with these preparations, *we shall define the canonical S-operator on L, and characterize it algebraically when L is ample.* First, we must define $D(L)$ and d. Let $L_0 \in \mathcal{L}_0$ and let $D(L_0)$ be the space of all differentials of $L_0/\mathbf{C}$ in the usual sense (in the theory of algebraic functions of one variable). Let $d_0 : L_0 \to D(L_0)$ be the differentiation. Then if $L_0' \in \mathcal{L}_0$ with $L_0 \subset L_0'$, there is a natural injection $D(L_0) \subset D(L_0')$ compatible with the differentiation. Now, $D(L)$ and d are defined to be the injective limits of $D(L_0)$ and d_0.

Now take any $L_0 \in \mathcal{L}_0$ and let $S^{\{L_0,e_0\}}$ be the canonical S-operator attached to $\{L_0, e_0\}$. For each $\xi \in D(L)^{\times}$ put $S^{L_0}\langle\xi\rangle = S^{\{L_0,e_0\}}\langle\xi_0\rangle + \langle\xi, \xi_0\rangle$, where ξ_0 is any element of $D(L_0)^{\times}$. Then since $S^{\{L_0,e_0\}}$ is an S-operator on L_0, this expression is independent of ξ_0, and since, $\xi \to \langle\xi, \xi_0\rangle$ is an S-operator on L, S^{L_0} is also an S-operator on L. Moreover, S^{L_0} is independent of L_0. In fact, if $L_0' \in \mathcal{L}_0$, then $L_0 L_0' \in \mathcal{L}_0$; hence it is enough to check $S^{L_0} = S^{L_0'}$ when $L_0 \subset L_0'$. But this is an immediate consequence of Remark 2 (§39) and the definition of $S^{\{L_0,e_0\}}$. Since S^{L_0} is independent of L_0, we shall denote it by

$$S^L, \tag{101}$$

and call it *the canonical S-operator on L.*

REMARK 3. Thus the restriction of S^L to each L_0 ($L_0 \in \mathcal{L}_0$) is nothing but $S^{\{L_0,e_0\}}$.

[5]. Now we shall define the action of $\mathrm{Aut}_{\mathbf{C}}\, L$ on the set of all S-operators on L by $S^{\sigma}\langle\xi\rangle = S\langle\xi^{\sigma^{-1}}\rangle^{\sigma}$ ($\sigma \in \mathrm{Aut}_{\mathbf{C}}\, L$). Then we have:

THEOREM 9. (i) *The canonical S-operator S^L is invariant by* $\mathrm{Aut}_{\mathbf{C}}\, L$.

(ii) *If L is ample, S^L is the unique* $\mathrm{Aut}_{\mathbf{C}}\, L$*-invariant S-operator on L. More strongly, if Φ is any closed non-compact subgroup of* $\mathrm{Aut}_{\mathbf{C}}\, L$, *$S^L$ is already characterized by Φ-invariance.*

[6]. For this proof, we need the following lemma, which will be proved in §42.

LEMMA 14. *Let Φ be any closed non-compact subgroup of* $\mathrm{Aut}_{\mathbf{C}}\, L$, *and let $h \geq 1$. Then the only Φ-invariant element of $D^h(L)$ is* 0.

[7]. Proof of Theorem 9. (i) Let $L_0 \in \mathcal{L}_0$ and put $V = \mathrm{Aut}_{L_0} L$. Let ξ_0 be any fixed element of $D(L_0)^{\times}$ and let $\xi \in D(L)^{\times}$. Then by definition, $S^L\langle\xi\rangle = S^{\{L_0,e_0\}}\langle\xi_0\rangle + \langle\xi,\xi_0\rangle$. So, for any $\sigma \in V$, we have $(S^L)^{\sigma}\langle\xi\rangle = S^{\{L_0,e_0\}}\langle\xi_0\rangle^{\sigma} + \langle\xi^{\sigma^{-1}},\xi_0\rangle^{\sigma} = S^{\{L_0,e_0\}}\langle\xi_0\rangle + \langle\xi,\xi_0^{\sigma}\rangle = S^L\langle\xi\rangle$. Hence S^L is V-invariant. If L is simple, take L_0 to be the unique minimal element of $\mathcal{L}_0$. Then $V = \mathrm{Aut}_{\mathbf{C}}\, L$; hence S^L is $\mathrm{Aut}_{\mathbf{C}}\, L$-invariant. Now let L be ample, and let G_0 be the subgroup of $\mathrm{Aut}_{\mathbf{C}}\, L$ generated by all groups of the form $\mathrm{Aut}_{L_0} L$ ($L_0 \in \mathcal{L}_0$). Then S^L is G_0-invariant, and moreover, G_0 is open (hence also closed) and non-compact (by Proposition 13 (ii)). Hence by Lemma 14, the only G_0-invariant element of $D^2(L)$ is 0. Suppose that S' is another G_0-invariant S-operator on L, and put $S' - S^L = C$ (C: a constant in $D^2(L)$). Then C must also be G_0- invariant; hence $C = 0$; hence $S' = S^L$. Therefore, S^L is the unique G_0-invariant S-operator . On the other hand, since any element of $\mathrm{Aut}_{\mathbf{C}}\, L$ leaves the set $\mathcal{L}_0$ invariant (as a whole), G_0 is a normal subgroup of $\mathrm{Aut}_{\mathbf{C}}\, L$. Therefore, for any $\sigma \in \mathrm{Aut}_{\mathbf{C}}\, L$, $(S^L)^{\sigma}$ is again G_0-invariant; hence $(S^L)^{\sigma} = S^L$. Therefore, S^L is $\mathrm{Aut}_{\mathbf{C}}\, L$-invariant. This settles (i).

(ii) Suppose that S' is a Φ-invariant S-operator , and put $S^L - S' = C$ (C: a constant in $D^2(L)$). Then C is Φ-invariant; hence by Lemma 14, $C = 0$; hence $S' = S^L$. This settles (ii). □

§42. Proof of Lemma 14, and its Corollary. Let L be as in §41. For each open compact subgroup V of $\mathrm{Aut}_{\mathbf{C}}\, L$ let L_V denote its fixed field in L, and for each prime divisor P of L_V let $e_V(P)$ denote its ramification index in L/L_V (so that $L_V \in \mathcal{L}_0$, and $\{L_V, e_V\}$ satisfies the conditions (i), (ii) of §40). Assume now that L is ample. Then there exists a discrete subgroup $\tilde{\Gamma}$ of $\tilde{G} = G_R \times \mathrm{Aut}_{\mathbf{C}}\, L$ with finite volume quotient, unique up to conjugacy in $\tilde{G}$, satisfying the following conditions:

(i) The projection of $\tilde{\Gamma}$ to each component of $\tilde{G}$ is injective, and its image is dense in that component;

(ii) For each open compact subgroup V of $\mathrm{Aut}_{\mathbf{C}}\, L$, put $\Delta = \mathrm{proj}_{\mathbf{R}}\{\tilde{\Gamma} \cap (G_R \times V)\}$, so that Δ is a fuchsian group depending on V. Put $\{L'_V, e'_V\} = \{L_{(\Delta\backslash\mathfrak{H})^*}, e_{\Delta}\}$[26] and $L' = \bigcup_V L'_V$. Then there is an isomorphism $\iota : L \to L'$ such that (a): $\iota|_{L_V}$ gives an isomorphism of

[26]See §40 for the symbol $\{L_{(\Delta\backslash\mathfrak{H})^*}, e_{\Delta}\}$.

$\{L_V, e_V\}$ onto $\{L'_V, e'_V\}$ for each V, and that (b): for each $\tilde{\gamma} = \gamma_{\mathbf{R}} \times \gamma \in \tilde{\Gamma}$, the action of γ on L corresponds to the action $f(\tau) \to f(\gamma_{\mathbf{R}}\tau)$ of $\gamma_{\mathbf{R}}$ on L' (by ι).

This can be proved exactly in the same manner as Theorem 1 (Part 1). Now let Φ be any closed non-compact subgroup of $\mathrm{Aut}_{\mathbf{C}} L$, let $h \geq 1$, and let ω be a Φ-invariant differential in L of degree h. Since $\omega \in D^h(L_V)$ for some V, ω is also invariant by V; hence we may further assume that Φ contains V. Put $G = G_{\mathbf{R}} \times \Phi$, $\Gamma = \tilde{\Gamma} \cap G$, and let Γ_R be the projection of Γ to $G_{\mathbf{R}}$. Then since Φ is non-compact, $(\Phi : V) = \infty$; hence $(\Gamma_{\mathbf{R}} : \Delta) = \infty$; hence $\Gamma_{\mathbf{R}}$ is dense in $G_{\mathbf{R}}$. Now put $\iota(\omega) = f(\tau)(d\tau)^h$; τ being as in §39. Then since ω is Φ-invariant, $f(\tau)$ is a meromorphic function on $\mathfrak{H}$ and satisfies

$$f\left(\frac{a\tau + b}{c\tau + d}\right)(c\tau + d)^{-2h} = f(\tau) \tag{102}$$

for all $\gamma_{\mathbf{R}} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_{\mathbf{R}}$. But since $\Gamma_{\mathbf{R}}$ is dense in $G_{\mathbf{R}}$, (102) holds for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_{\mathbf{R}}$. In particular, we have $f(\tau + \lambda) = f(\tau)$ for all $\lambda \in \mathbf{R}$; hence $f(\tau)$ is a function of $\mathrm{Im}\,(\tau)$. But since $f(\tau)$ is meromorphic, this implies that $f(\tau)$ must be a constant. But then, since $h \geq 1$, it is clear by (102) that $f(\tau) \equiv 0$; hence $\omega = 0$. This proves Lemma 14. □

As a Corollary of Lemma 14, we shall prove the following assertion, which is used in §40, §43:

Corollary . *Let $\{L, e\}, \{L_1, e_1\}, \{L_2, e_2\}$ be as in Proposition 12. Then*

$$D^h(L_1) \cap D^h(L_2) = \{0\} \quad (h \geq 1). \tag{103}$$

Proof. Rewrite $\{L, e\} = \{L_0, e_0\}$ (we shall use the notation L for some other field). Let Δ_0 be the fuchsian group corresponding to $\{L_0, e_0\}$, and for each subgroup $\Delta' \subset \Delta_0$ with finite index, let $\{L', e'\}$ denote the corresponding admissible extension of $\{L_0, e_0\}$. Put $L = \bigcup_{\Delta'} L'$, where Δ' runs over all subgroups of Δ_0 with finite indices. Then clearly for each prime divisor P_0 of L_0, its ramification index in L/L_0 divides $e_0(P_0)$; but moreover, it is well-known (and easily proved[27]) that the ramification index *coincides* with $e_0(P_0)$. Therefore, L satisfies the conditions (L1) (L2) of §41. Put $V_i = \mathrm{Aut}_{L_i} L$ $(i = 1, 2)$, and let Φ be the subgroup of $\mathrm{Aut}_{\mathbf{C}} L$ generated by V_1 and V_2. Then Φ is open (hence also closed), and since $L_1 \cap L_2 = \mathbf{C}$, Φ is non-compact. Now let $\omega \in D^h(L_1) \cap D^h(L_2)$. Then ω is Φ-invariant; hence by Lemma 14, $\omega = 0$. □

[27] See Supplement §5.

The canonical class of linear differential equations of second order on algebraic function fields over C, and its algebraic characterization in ample (arithmetic) cases.

§43. The first formulation.

[1]. Let $\{L, e\}$ be as in §40. Let $\xi \in D(L)^\times$ and let $\mathbb{D}_\xi$ denote the derivation of L defined by $L \ni y \to \frac{dy}{\xi} \in L$. By a (differential) equation $\Theta = [\xi; A, B]$ $(A, B \in L)$, we will mean the following linear differential equation:

$$(104) \qquad (\mathbb{D}_\xi^2 + A \cdot \mathbb{D}_\xi + B)u = 0.$$

Let $\eta \in D(L)^\times$ and put $w_1 = \eta/\xi$, $w_{i+1} = dw_i/\xi$ $(i \geq 1)$, so that $\mathbb{D}_\xi = w_1\mathbb{D}_\eta$, $\mathbb{D}_\xi^2 = w_1^2\mathbb{D}_\eta^2 + w_2\mathbb{D}_\eta$; hence the equation Θ may be rewritten as:

$$(104') \qquad \begin{cases} (\mathbb{D}_\eta^2 + A_1\mathbb{D}_\eta + B_1)u = 0, \quad \text{with} \\ A_1 = Aw_1^{-1} + w_2w_1^{-2}, \; B_1 = Bw_1^{-2}. \end{cases}$$

We shall always identify two such equations (consider as different expressions of the same equation);

$$(105) \qquad [\xi, A, B] = [\eta; Aw_1^{-1} + w_2w_1^{-2}, Bw_1^{-2}] \quad (\xi, \eta \in D(L)^\times).$$

Since $B_1 \cdot \eta^2 = B \cdot \xi^2$ and $A_1 \cdot \eta - A \cdot \xi = d\log w_1$, the quantities $B \cdot \xi^2$, $\mathrm{Res}_P(A\xi) - \mathrm{ord}_P\,\xi$, $A \cdot \xi \pmod{d\log L^\times}$ are independent of the expressions of Θ.

Let $\Theta = [\xi; A, B]$ and $C \in L^\times$. By $\sqrt{C^{-1}}\Theta$, we shall mean the equation obtained by substituting u by $\sqrt{C}u$ in (104).[28] Thus, by definition,

$$(106) \qquad [\xi; A, B] = \sqrt{C^{-1}}[\xi; A', B'] \Leftrightarrow \begin{cases} A' = A + \frac{\mathbb{D}_\xi(C)}{C}, \\ B' = B + \frac{\mathbb{D}_\xi(C)}{2C}A + \frac{2C\mathbb{D}_\xi^2(C) - (\mathbb{D}_\xi(C))^2}{4C^2}. \end{cases}$$

The two equations Θ, Θ' are called *equivalent* (or belong to the same *class*) if $\Theta' = \sqrt{C^{-1}}\Theta$ holds for some $C \in L^\times$. It is clear that this is an equivalence relation.

[2].

PROPOSITION 15. *Let S be an S-operator on L, let $\xi \in D(L)^\times$, and put $S\langle\xi\rangle = -4B_\xi \cdot \xi^2$ $(B_\xi = B_\xi^S \in L)$. Then the class of the equation $[\xi; 0, B_\xi]$ depends only on S, and is independent of ξ.*

PROOF. It is enough to check

$$(107) \qquad [\eta; 0, B_\eta] = \sqrt{\eta/\xi}[\xi; 0, B_\xi] \quad (\xi, \eta \in D(L)^\times).$$

[28] So, "the solutions of $\sqrt{C^{-1}}\Theta$" are $\sqrt{C^{-1}}$-times "the solutions of Θ." Clearly we have $\Theta' = \sqrt{C^{-1}}\Theta \Leftrightarrow \Theta = \sqrt{C}\Theta'$, $\sqrt{C_1C_2}\Theta = \sqrt{C_1}(\sqrt{C_2}\Theta)$, and $\sqrt{C}\Theta = \Theta \Leftrightarrow C \in \mathbf{C}^\times$.

By (105), we have $[\eta; 0, B_\eta] = [\xi; -\frac{w_2}{w_1}, B_\eta w_1^2]$. Therefore, if we put $C = w_1 = \eta/\xi$, then $\sqrt{C^{-1}}[\eta; 0, B_\eta] = [\xi; A', B']$, with $A' = -\frac{w_2}{w_1} + \frac{\mathrm{D}_\xi(C)}{C} = 0$, and

$$
\begin{aligned}
B' &= B_\eta w_1^2 + \frac{w_2}{2w_1} \times \left(-\frac{w_2}{w_1}\right) + \frac{2w_1w_3 - w_2^2}{4w_1^2} \\
&= B_\eta w_1^2 + \frac{2w_1w_3 - 3w_2^2}{4w_1^2};
\end{aligned} \tag{108}
$$

hence $4B' \cdot \xi^2 = 4B_\eta \cdot \eta^2 + \langle \eta, \xi \rangle = -S\langle \eta \rangle + \langle \eta, \xi \rangle = -S\langle \xi \rangle$; hence $B' = B_\xi$. □

Definition . In the situation of Proposition 15, the class of $[\xi; 0, B_\xi]$ will be called the S-*class* (corresponding to the S-operator S), and will be denoted by $\mathfrak{R}^S$. If $S^{\{L,e\}}$ is the canonical S-operator attached to $\{L, e\}$, then $\mathfrak{R}^{S^{\{L,e\}}}$ will be called *the canonical class attached to* $\{L, e\}$ and denoted by $\mathfrak{R}\{L, e\}$.

By the following proposition, *a class $\mathfrak{R}$ is an S-class (for some S) if and only if it contains an equation of the form $[\xi; 0, B]$ ($\xi \in D(L)^\times$, $B \in L$).*

Proposition 16. *Let $\mathfrak{R}$ be any class containing an equation of the form $\Theta = [\xi; 0, B]$ ($B \in L$). Then there exists a unique S-operator S (on L) such that $\mathfrak{R} = \mathfrak{R}^S$. Moreover, if Θ and S are as above, we have $B = B_\xi^S$; and for each $\Theta' \in \mathfrak{R}^S$, there exists a differential $\eta \in D(L)^\times$, unique up to constant multiple, such that $\Theta' = [\eta; 0, B_\eta^S]$. Thus, there are two bijections:*

$$
S\text{-operators} \underset{1:1}{\longleftrightarrow} S\text{-classes}, \tag{109}
$$

by $S \leftrightarrow \mathfrak{R}^S$, and

$$
\underset{(S:\text{fixed})}{\text{equations in } \mathfrak{R}^S} \underset{1:1}{\longleftrightarrow} \text{the canonical divisors}^{29} \text{on } L, \tag{110}
$$

by $[\eta; 0, B_\eta] \leftrightarrow$ the divisor (η) of η.

Definition . We shall call $W' = (\eta)$ the divisor of the equation $\Theta' = [\eta; 0, B_\eta]$. It is clear by (107) that the divisor of $\sqrt{C}\Theta'$ is $(C) \cdot W'$.

Proof. Let $\Theta = [\xi; 0, B]$, and let S be the S-operator defined by $S\langle \eta \rangle = \langle \eta, \xi \rangle - 4B \cdot \xi^2$ (η: a variable in $D^2(L)$). Then $B = B_\xi^S$; hence $\mathfrak{R} = \mathfrak{R}^S$. Suppose that S' is another S-operator with $\mathfrak{R} = \mathfrak{R}^{S'}$. Then $[\xi; 0, B_\xi^{S'}] \in \mathfrak{R}$; hence $[\xi; 0, B_\xi^{S'}] = \sqrt{C}[\xi; 0, B_\xi^S]$ with some $C \in L^\times$. But by (106), this implies $\mathrm{D}_\xi(C) = 0$ (hence $C \in \mathbf{C}^\times$); hence $B_\xi^{S'} = B_\xi^S$; hence $S'\langle \xi \rangle = S\langle \xi \rangle$; hence $S' = S$. That $B' = B_\xi^S$ follows exactly in the same manner. Finally, let $\Theta' \in \mathfrak{R}^S$, and put $\Theta' = \sqrt{C}\Theta$ ($C \in L^\times$). Put $\eta = C \cdot \xi$. Then by (107), we obtain $\Theta' = [\eta; 0, B_\eta^S]$. □

Remark 1. Thus, if $[\xi; A, B]$ is the equation in $\mathfrak{R}^S$ whose divisor is (η), then $A = -\frac{w_2}{w_1}$, $B = B_\eta w_1^2$.

[29] I.e., the divisors of non-zero differentials (of degree one).

[3]. Let Δ be the fuchsian group corresponding to $\{L, e\}$, and identify $\{L, e\}$ with $\{L_{(\Delta\backslash\mathfrak{H})^*}, e_\Delta\}$ (see §40). Let $S = S^{\{L,e\}}$ be the canonical S-operator, and put $\xi = f(\tau)d\tau$. Then $B_\xi = -\frac{2f(\tau)f''(\tau)-3f'(\tau)^2}{4f(\tau)^4}$; hence the equation $[\xi; 0, B_\xi]$ takes the form:

$$\mathrm{D}_\xi^2 u = \frac{2f(\tau)f''(\tau) - 3f'(\tau)^2}{4f(\tau)^4}u. \tag{111}$$

As is well-known (and can be checked directly), the general solution of (111) is

$$u = (a\tau + b)\sqrt{f(\tau)} \qquad (a, b \in \mathbf{C}).\text{[30]} \tag{112}$$

[4]. Local properties of the equations in the canonical class. Now let $\mathfrak{K} = \mathfrak{K}\{L, e\}$ be the canonical class, and let Θ be the equation in $\mathfrak{K}$ having a given divisor $W = \prod_P P^{w(P)}$. Then Θ has the following properties:

(Θ-1) Θ is fuchsian; i.e., regular at each prime divisor P of L.

(Θ-2) At each P, the exponents of Θ are given by

$$\frac{1}{2}\left\{1 + w(P) + \frac{1}{e(P)}\right\}, \ \frac{1}{2}\left\{1 + w(P) - \frac{1}{e(P)}\right\}; \tag{113}$$

thus if $e(P) = 1$ or ∞, the difference of exponents is integral; but:

(Θ-3) Unless $e(P) = \infty$, the local solutions of Θ at P do not involve logarithms.

These follow immediately from the above [3] and from the following Lemma 15.

Lemma 15. *Let X be any Riemann surface, X' its finite covering, P' a point on X', P the point of X lying below P', and let e be the ramification index of P'/P. Let ω be a non-zero differential of degree h ($h \geq 1$) on X. Then the order* $\mathrm{ord}_{P'}\,\omega$ *of ω (considered as a differential on X') at P' is given by*

$$\mathrm{ord}_{P'}\,\omega = e(\mathrm{ord}_P\,\omega + h) - h. \tag{114}$$

Proof of Lemma 15. Immediate by using the local coordinates. □

[5]. Notes. Now a question arises "to what extent is the equation $\Theta \in \mathfrak{K}\{L, e\}$ characterized by (Θ-1) (Θ-2) (Θ-3)?" The following is to answer this question. Roughly, the result we obtain is parallel to the result in [4], [5] of §40. All statements given below can be proved directly; so their proofs are omitted.

Definition. Let $\Theta = [\xi; A, B]$ be any equation (in any class). Then Θ is called *admissible* with respect to $\{L, e\}$ if Θ satisfies (Θ-1) (Θ-2) (Θ-3) with some canonical divisor $W = \prod_P P^{w(P)}$.

If Θ is such, W is unique. So, we shall call *W the divisor of Θ*.

Proposition 17. *Let Θ be admissible w.r.t. $\{L, e\}$, and let W be its divisor. Let $C \in L^\times$. Then $\sqrt{C}\Theta$ is also admissible w.r.t. $\{L, e\}$, and its divisor is $(C) \cdot W$.*

Thus, we may speak of "*admissible classes.*"

---

[30] Thus, the ratios of the two independent solutions of $[\xi; 0, B_\xi]$ are $v = \frac{a\tau+b}{c\tau+d}$; $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL_2(\mathbf{C})$. The differential equation having v as the general solution is, of course, $\langle dv, \xi\rangle = -S^{\{L,e\}}\langle\xi\rangle$.

Proposition 18. *Let* $\Theta = [\xi; A, B]$. *Then* (Θ-1), (Θ-2), (Θ-3) *are equivalent to the following* (Θ-1)', (Θ-2)', (Θ-3)' *respectively.*

(Θ-1)': $\mathrm{ord}_P(A\cdot\xi) \geq -1, \mathrm{ord}_P(B\cdot\xi^2) \geq -2$ *at each* P.

(Θ-2)': *Let* t *be a prime element of* P, *put* $e = e(P)$, $w = w(P)$, $n = \mathrm{ord}_P\,\xi$, *and*

$$\begin{cases} \xi & = ct^n(1 + c_1t + \cdots)dt, \quad c \neq 0,\ c_1, c_2, \cdots \in \mathbf{C}, \\ A\cdot\xi & = \left(\frac{a_0}{t} + a_1 + a_2t + \cdots\right)dt, \quad a_0, a_1, \cdots \in \mathbf{C}, \\ B\cdot\xi^2 & = \left(\frac{b_0}{t^2} + \frac{b_1}{t} + b_2 + \cdots\right)(dt)^2; \quad b_0, b_1, \cdots \in \mathbf{C}. \end{cases} \tag{115}$$

Then,

$$\begin{cases} a_0 = n - w, \\ b_0 = \frac{1}{4}\left\{(w+1)^2 - \frac{1}{e^2}\right\}. \end{cases} \tag{116}$$

(Θ-3)': *We have*

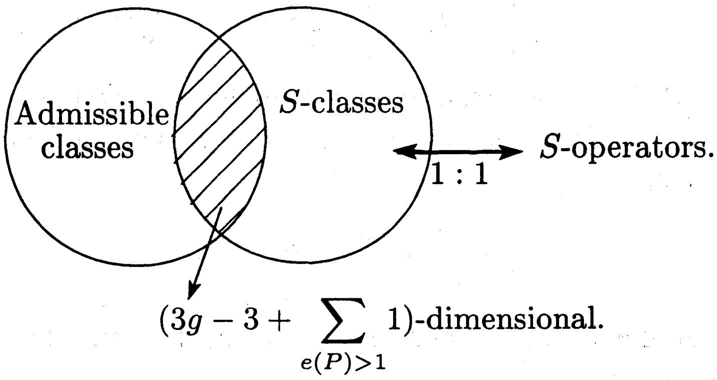
$$b_1 = \frac{1}{2}w(c_1 - a_1) \qquad \cdots \textit{ if } e = 1. \tag{117}$$

Corollary . *Let* $\mathfrak{R}^S$ *be an* S*-class and let* $\xi \in D(L)^\times$. *Put* $S\langle\xi\rangle = -4\beta = -4B_\xi^S\cdot\xi^2$, *so that* $[\xi; 0, B_\xi^S] \in \mathfrak{R}^S$. *Then* $\mathfrak{R}^S$ *is admissible with respect to* $\{L, e\}$ *if and only if* β *satisfies the conditions* (i) (ii) *of Proposition 9.*

Remark 2. Since $\mathfrak{R}\{L, e\}$ is admissible, this proves Proposition 9. Moreover, this shows that the conditions (i) (ii) of Proposition 9 are independent of ξ.

Remark 3. As can be seen easily from Proposition 18, admissible classes and S-classes are independent notions; i.e., there is no implication between them;

(118)



Thus, even if we restrict ourselves to S-classes, the conditions (Θ-1), (Θ-2), (Θ-3) do not characterize the canonical class. In fact, there is still $(3g - 3 + \sum_{e(P)>1} 1)$-dimensional freedom.

[6]. Now we shall give some results parallel to those of §40 [5]. Let Θ be an equation on L, and let L' be a finite extension of L. Consider Θ as an equation on L'. Then this Θ will be called the *extension of* Θ *to* L'. It is clear that the extension of Θ induces the extension of the class of Θ.

Proposition 19. *Let Θ be an admissible equation with respect to $\{L, e\}$, and let $\{L', e'\}$ be an admissible extension of $\{L, e\}$. Then, the extension Θ' of Θ to L' is also admissible with respect to $\{L', e'\}$. Moreover, if $(\xi)_L$ ($\xi \in D(L)^\times$) is the divisor of Θ, then the divisor of Θ' is $(\xi)_{L'}$. Finally, the induced extension map of classes:*

$$\text{(119)} \qquad \begin{array}{c}\textit{the admissible classes}\\ \textit{w.r.t. } \{L, e\}\end{array} \xrightarrow{\textit{extension}} \begin{array}{c}\textit{the admissible classes}\\ \textit{w.r.t. } \{L', e'\}\end{array}$$

is injective.

Use Proposition 18 to check this.

Proposition 20. *Let $\mathfrak{K}\{L, e\}$ be the canonical class attached to $\{L, e\}$, and let $\{L', e'\}$ be an admissible extension of $\{L, e\}$. Then the extension of $\mathfrak{K}\{L, e\}$ to $\{L', e'\}$ is the canonical class attached to $\{L', e'\}$.*

This is obvious by Proposition 10.
Now we shall prove:

Proposition 21. *Consider the situation* (93) *of Proposition 12. Suppose that there are admissible classes $\mathfrak{K}_1$, $\mathfrak{K}_2$ with respect to $\{L_1, e_1\}$, $\{L_2, e_2\}$ (respectively), such that their extensions to L are equal. Then such $\mathfrak{K}_1$, $\mathfrak{K}_2$ are unique and are the canonical classes attached to $\{L_1, e_1\}$, $\{L_2, e_2\}$ (respectively).*

Proof. Let $\mathfrak{K}$ be the extensions to L of $\mathfrak{K}_1$ and of $\mathfrak{K}_2$. Let $\xi \in D(L_1)^\times$, $\eta \in D(L_2)^\times$. Let $\Theta = [\xi; A, B]$ be the equation in $\mathfrak{K}$ whose divisor is $(\eta)_L$. Put $w_1 = \eta/\xi$, $w_{i+1} = dw_i/\xi$ ($i \geq 1$), and put $\Theta_1 = \sqrt{w_1^{-1}}\Theta = [\xi; A_1, B_1]$, so that

$$\text{(120)} \qquad A_1 = A + \frac{w_2}{w_1}, \quad B_1 = B + \frac{w_2}{2w_1}A + \frac{2w_1w_3 - w_2^2}{4w_1^2}.$$

Since the divisor of Θ_1 is $(\xi)_L$, Θ_1 must coincide with the extension to L of the equation in $\mathfrak{K}_1$ whose divisor is $(\xi)_{L_1}$. Therefore, $A_1, B_1 \in L_1$. On the other hand, Θ can be expressed as $\Theta = [\eta; A_2, B_2]$ with

$$\text{(121)} \qquad A_2 = \frac{A}{w_1} + \frac{w_2}{w_1^2}, \quad B_2 = \frac{B}{w_1^2},$$

and since the divisor of Θ is $(\eta)_L$, we have $A_2, B_2 \in L_2$ by the same reason as above. Now, by (120), (121), we obtain

$$\text{(122)} \qquad A_1 \cdot \xi = A_2 \cdot \eta;$$

hence $A_1 \cdot \xi = A_2 \cdot \eta \in D^1(L_1) \cap D^1(L_2)$. But by the Corollary of Lemma 14 (§42),

$$\text{(123)} \qquad D^h(L_1) \cap D^h(L_2) = \{0\} \quad (h \geq 1);$$

hence $A_1 \cdot \xi = A_2 \cdot \eta = 0$; hence

$$\text{(124)} \qquad A_1 = A_2 = 0.$$

Now by (120), (121) and (124), we obtain $B_1 \cdot \xi^2 = B_2 \cdot \eta^2 + \frac{1}{4}\langle\eta, \xi\rangle$; hence if we put $\alpha = -4B_1 \cdot \xi^2$, $\beta = -4B_2 \cdot \eta^2$, then we obtain

$$\langle\xi, \eta\rangle = \alpha - \beta; \quad \alpha \in D^2(L_1), \beta \in D^2(L_2). \tag{125}$$

Hence by the Corollary of Proposition 12, we obtain $\alpha = S^{\{L,e\}}\langle\xi\rangle = S^{\{L_1,e_1\}}\langle\xi\rangle, \beta = S^{\{L,e\}}\langle\eta\rangle = S^{\{L_2,e_2\}}\langle\eta\rangle$. But since $\Theta_1 = [\xi; 0, B_1]$ and $\Theta = [\eta; 0, B_2]$, this implies that $\mathfrak{K}_1$ and $\mathfrak{K}_2$ are the canonical classes attached to $\{L_1, e_1\}$ and $\{L_2, e_2\}$ respectively. □

§44. The second formulation.

[1]. Now let the notations be as in §41, so that L is any one-dimensional extension of $\mathbf{C}$ satisfying (L1), (L2) of §41. For such an L, we can define the equations $\Theta = [\xi; A, B]$ ($\xi \in D(L)^\times$; $A, B \in L$) and the classes $\left\{\sqrt{C}\Theta \mid C \in L^\times\right\}$ exactly in the same manner as in the previous section. Moreover, Propositions 15, 16 are also valid in this case (however, we must replace the right side of (110) by $D(L)^\times/\mathbf{C}^\times$, since we have not defined "the divisor of differential in L.") Thus, we have a one-to-one correspondence:

$$S\text{-operators on } L \underset{1:1}{\longleftrightarrow} S\text{-classes } \mathfrak{K}^S \text{ on } L. \tag{126}$$

Definition . Let $S = S^L$ be the canonical S-operator on L. Then the corresponding S-class $\mathfrak{K}^S$ will be called *the canonical class* on $\mathfrak{K}$, and denoted by $\mathfrak{K}\{L\}$.

Now $\mathrm{Aut}_{\mathbf{C}}\, L$ acts on the set of all equations, and hence also on the set of all classes, by

$$\mathrm{Aut}_{\mathbf{C}}\, L \ni \sigma : \ \Theta = [\xi; A, B] \to \Theta^\sigma = [\xi^\sigma; A^\sigma, B^\sigma]. \tag{127}$$

Then if S is any S-operator on L, it follows immediately from the definition of $\mathfrak{K}^S$ that $(\mathfrak{K}^S)^\sigma = \mathfrak{K}^{(S^\sigma)}$ ($\sigma \in \mathrm{Aut}_{\mathbf{C}}\, L$). Therefore, we obtain immediately from Theorem 9 the following:

Theorem 9′. (i) *The canonical class* $\mathfrak{K}\{L\}$ *is invariant by* $\mathrm{Aut}_{\mathbf{C}}\, L$. (ii) *If L is ample,* $\mathfrak{K}\{L\}$ *is the unique* $\mathrm{Aut}_{\mathbf{C}}\, L$*-invariant S-class on L. More strongly, if Φ is any closed non-compact subgroup of* $\mathrm{Aut}_{\mathbf{C}}\, L$, $\mathfrak{K}\{L\}$ *is already characterized by Φ-invariance.*

Remark . In the above (ii), the assumption "S-class" cannot be dropped. In fact, we can prove in the case of G-fields that the $\mathrm{Aut}_{\mathbf{C}}\, L$-invariant classes are finite in number, but may not be unique.[31] Also, we can prove in G-field cases that if we call $\left\{C^{1/n}\Theta \mid C \in L^\times,\ n \in \mathbf{Z}\right\}$ the weaker class, then the $\mathrm{Aut}_{\mathbf{C}}\, L$-invariant weaker class is unique.

[31]However, they can be obtained from the canonical class by a simple "twist," and are not important.