4 Existence results

In this section, we present some existence results for viscosity solutions of second-order (degenerate) elliptic PDEs.

We first present a convenient existence result via Perron's method, which was established by Ishii in 1987.

Next, for Bellman and Isaacs equations, we give representation formulas for viscosity solutions. From the dynamic programming principle below, we will realize how natural the definition of viscosity solutions is.

4.1 Perron's method

In order to introduce Perron's method, we need the notion of viscosity solutions for semi-continuous functions.

Definition. For any function $u : \overline{\Omega} \to \mathbf{R}$, we denote the upper and lower semi-continuous envelope of u by u^* and u_*, respectively, which are defined by

$$u^*(x) = \lim_{\varepsilon \to 0} \sup_{y \in B_\varepsilon(x) \cap \overline{\Omega}} u(y) \quad \text{and} \quad u_*(x) = \lim_{\varepsilon \to 0} \inf_{y \in B_\varepsilon(x) \cap \overline{\Omega}} u(y).$$

We give some elementary properties for u^* and u_* without proofs.

Proposition 4.1. *For $u : \overline{\Omega} \to \mathbf{R}$, we have*

(1) $u_*(x) \le u(x) \le u^*(x)$ *for* $x \in \overline{\Omega}$,
(2) $u^*(x) = -(-u)_*(x)$ *for* $x \in \overline{\Omega}$,
(3) u^**(resp.,* u_**) is upper (resp., lower) semi-continuous in* $\overline{\Omega}$*, i.e.*
$\limsup_{y \to x} u^*(y) \le u^*(x)$, *(resp.,* $\liminf_{y \to x} u_*(y) \ge u_*(x)$*) for* $x \in \overline{\Omega}$,
(4) *if u is upper (resp., lower) semi-continuous in* $\overline{\Omega}$,
then $u(x) = u^*(x)$ *(resp.,* $u(x) = u_*(x)$*) for* $x \in \overline{\Omega}$.

With these notations, we give our definition of viscosity solutions of

$$F(x, u, Du, D^2u) = 0 \quad \text{in } \Omega. \tag{4.1}$$

Definition. We call $u : \overline{\Omega} \to \mathbf{R}$ a viscosity subsolution (resp., supersolution) of (4.1) if u^* (resp., u_*) is a viscosity subsolution (resp., supersolution) of (4.1).

We call $u : \overline{\Omega} \to \mathbf{R}$ a viscosity solution of (4.1) if it is both a viscosity sub- and supersolution of (4.1).

Remark. We note that we supposed that viscosity sub- and supersolutions are, respectively, upper and lower semi-continuous in our comparison principle in section 3. Adapting the above new definition, we omit the semi-continuity for viscosity sub- and supersolutions in Propositions 3.1, 3.3 and Theorems 3.4, 3.5, 3.7, 3.9.

In what follows,

we use the above definition.

Remark. We remark that the comparison principle Theorem 3.7 implies the continuity of viscosity solutions.

"Continuity of viscosity solutions"

$$\left.\begin{array}{c}\textbf{viscosity solution } u\\ \textbf{satisfies } u^* = u_* \textbf{ on } \partial\Omega\end{array}\right\} \Longrightarrow \quad u \in C(\overline{\Omega})$$

Proof of the continuity of u. Since u^* and u_* are, respectively, a viscosity subsolution and a viscosity supersolution and $u^* \le u_*$ on $\partial\Omega$, Theorem 3.7 yields $u^* \le u_*$ in $\overline{\Omega}$. Because $u_* \le u \le u^*$ in $\overline{\Omega}$, we have $u = u^* = u_*$ in $\overline{\Omega}$; $u \in C(\overline{\Omega})$. □

We first show that the "point-wise" supremum (resp., infimum) of viscosity subsolutions (resp., supersolution) becomes a viscosity subsolution (resp., supersolution).

Theorem 4.2. *Let $\mathcal{S}$ be a non-empty set of upper (resp., lower) semi-continuous viscosity subsolutions (resp., supersolutions) of* (4.1).

Set $u(x) := \sup_{v\in\mathcal{S}} v(x)$ (resp., $u(x) := \inf_{v\in\mathcal{S}} v(x)$). If $\sup_{x\in K} |u(x)| < \infty$ for any compact sets $K \subset \Omega$, then u is a viscosity subsolution (resp., supersolution) of (4.1).

Proof. We only give a proof for subsolutions since the other can be proved in a symmetric way.

For $\hat{x} \in \Omega$, we suppose that $0 = (u^* - \phi)(\hat{x}) > (u^* - \phi)(x)$ for $x \in \Omega \setminus \{\hat{x}\}$ and $\phi \in C^2(\Omega)$. We shall show that

$$F(\hat{x}, \phi(\hat{x}), D\phi(\hat{x}), D^2\phi(\hat{x})) \le 0. \qquad (4.2)$$

Let $r > 0$ be such that $B_{2r}(\hat{x}) \subset \Omega$. We can find $s > 0$ such that

$$\max_{\partial B_r(\hat{x})} (u^* - \phi) \leq -s. \tag{4.3}$$

We choose $x_k \in B_r(\hat{x})$ such that $\lim_{k\to\infty} x_k = \hat{x}$, $u^*(\hat{x}) - k^{-1} \leq u(x_k)$ and $|\phi(x_k) - \phi(\hat{x})| < 1/k$. Moreover, we select upper semi-continuous $u_k \in \mathcal{S}$ such that $u_k(x_k) + k^{-1} \geq u(x_k)$.

By (4.3), for $3/k < s$, we have

$$\max_{\partial B_r(\hat{x})} (u_k - \phi) < (u_k - \phi)(x_k).$$

Thus, for large $k > 3/s$, there is $y_k \in B_r(\hat{x})$ such that $u_k - \phi$ attains its maximum over $\overline{B}_r(\hat{x})$ at y_k. Hence, we have

$$F(y_k, u_k(y_k), D\phi(y_k), D^2\phi(y_k)) \leq 0. \tag{4.4}$$

Taking a subsequence if necessary, we may suppose $z := \lim_{k\to\infty} y_k$. Since

$$(u^* - \phi)(\hat{x}) \leq (u_k - \phi)(x_k) + \frac{3}{k} \leq (u_k - \phi)(y_k) + \frac{3}{k} \leq (u^* - \phi)(y_k) + \frac{3}{k}$$

by the upper semi-continuity of u^*, we have

$$(u^* - \phi)(\hat{x}) \leq (u^* - \phi)(z),$$

which yields $z = \hat{x}$, and moreover, $\lim_{k\to\infty} u_k(y_k) = u^*(\hat{x}) = \phi(\hat{x})$. Therefore, sending $k \to \infty$ in (4.4), by the continuity of F, we obtain (4.2). $\square$

Our first existence result is as follows.

Theorem 4.3. *Assume that F is elliptic. Assume also that there are a viscosity subsolution $\xi \in USC(\overline{\Omega}) \cap L^\infty_{loc}(\Omega)$ and a viscosity supersolution $\eta \in LSC(\overline{\Omega}) \cap L^\infty_{loc}(\Omega)$ of* (4.1) *such that*

$$\xi \leq \eta \quad \text{in } \overline{\Omega}.$$

Then, $u(x) := \sup_{v\in\mathcal{S}} v(x)$ (resp., $\hat{u}(x) = \inf_{w\in\hat{\mathcal{S}}} w(x)$) is a viscosity solution of (4.1), *where*

$$\mathcal{S} := \left\{ v \in USC(\Omega) \;\middle|\; \begin{array}{c} v \text{ is a viscosity subsolution} \\ \text{of (4.1) such that } \xi \leq v \leq \eta \text{ in } \Omega \end{array} \right\}$$

$$\left(\textit{resp., } \hat{\mathcal{S}} := \left\{ w \in LSC(\Omega) \;\middle|\; \begin{array}{c} w \textit{ is a viscosity supersolution} \\ \textit{of } (4.1) \textit{ such that } \xi \le w \le \eta \textit{ in } \Omega \end{array} \right\}\right).$$

Sketch of proof. We only give a proof for u since the other can be shown in a symmetric way.

First of all, we notice that $\mathcal{S} \neq \emptyset$ since $\xi \in \mathcal{S}$.

Due to Theorem 4.2, we know that u is a viscosity subsolution of (4.1). Thus, we only need to show that it is a viscosity supersolution of (4.1).

Assume that $u \in LSC(\overline{\Omega})$. Assuming that $0 = (u - \phi)(\hat{x}) < (u - \phi)(x)$ for $x \in \Omega \setminus \{\hat{x}\}$ and $\phi \in C^2(\Omega)$, we shall show that

$$F(\hat{x}, \phi(\hat{x}), D\phi(\hat{x}), D^2\phi(\hat{x})) \ge 0.$$

Suppose that this conclusion fails; there is $\theta > 0$ such that

$$F(\hat{x}, \phi(\hat{x}), D\phi(\hat{x}), D^2\phi(\hat{x})) \le -2\theta.$$

Hence, there is $r > 0$ such that

$$F(x, \phi(x) + t, D\phi(x), D^2\phi(x)) \le -\theta \quad \text{for } x \in B_r(\hat{x}) \subset \Omega \text{ and } |t| \le r. \quad (4.5)$$

First, we claim that $\phi(\hat{x}) < \eta(\hat{x})$. Indeed, otherwise, since $\phi \le u \le \eta$ in Ω, $\eta - \phi$ attains its minimum at $\hat{x} \in \Omega$. See Fig 4.1.

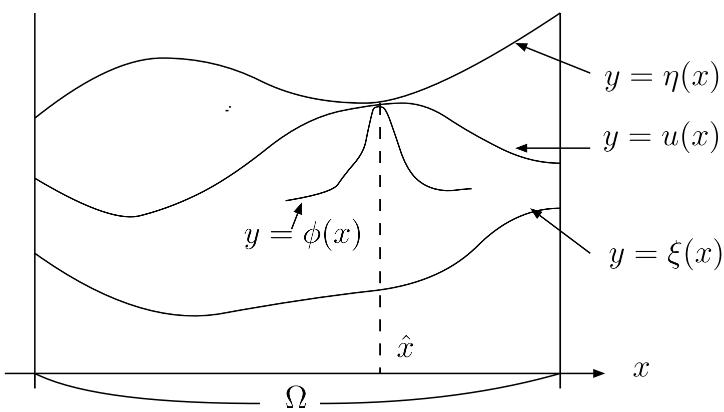

Fig 4.1

Hence, from the definition of supersolution η, we get a contradiction to (4.5) for $x = \hat{x}$ and $t = 0$.

We may suppose that $\xi(\hat{x}) < \eta(\hat{x})$ since, otherwise, $\xi = \phi = \eta$ at $\hat{x}$. Setting $3\hat{\tau} := \eta(\hat{x}) - u(\hat{x}) > 0$, from the lower and upper semi-continuity of η and ξ, respectively, we may choose $s \in (0, r]$ such that

$$\xi(x) + \hat{\tau} \le \phi(x) + 2\hat{\tau} \le \eta(x) \quad \text{for } x \in B_{2s}(\hat{x}).$$

Moreover, we can choose $\varepsilon \in (0, s)$ and $\tau_0 \in (0, \min\{\hat{\tau}, r\})$ such that $\phi(x) + 2\tau_0 \le u(x)$ for $x \in \overline{B}_{s+\varepsilon}(\hat{x}) \setminus B_{s-\varepsilon}(\hat{x})$.

If we can define a function $w \in \mathcal{S}$ such that $w(\hat{x}) > u(\hat{x})$, then we finish our proof because of the maximality of u at each point.

Now, we set

$$w(x) := \begin{cases} \max\{u(x), \phi(x) + \tau_0\} & \text{in } B_s(\hat{x}), \\ u(x) & \text{in } \Omega \setminus B_s(\hat{x}). \end{cases}$$

See Fig 4.2.

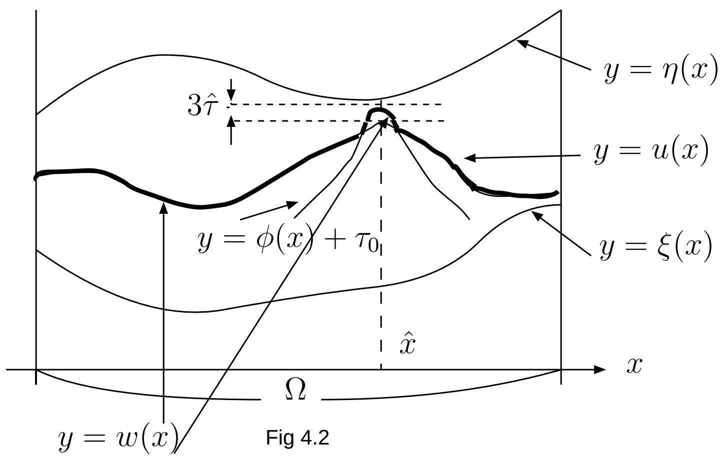

Fig 4.2

It suffices to show that $w \in \mathcal{S}$. Because of our choice of $\tau_0, s > 0$, it is easy to see $\xi \le w \le \eta$ in Ω. Thus, we only need to show that w is a viscosity subsolution of (4.1).

To this end, we suppose that $(w^* - \psi)(x) \le (w^* - \psi)(z) = 0$ for $x \in \Omega$, and then we will get

$$F(z, w^*(z), D\psi(z), D^2\psi(z)) \le 0. \tag{4.6}$$

If $z \in \Omega \setminus \overline{B}_s(\hat{x}) =: \Omega'$, by Proposition 2.4, then $u^* - \psi$ attains its maximum at $z \in \Omega'$, we get (4.6).

If $z \in \partial B_s(\hat{x})$, then (4.6) holds again since $w = u$ in $B_{s+\varepsilon}(\hat{x}) \setminus \overline{B}_{s-\varepsilon}(\hat{x})$.

It remains to show (4.6) when $z \in B_s(\hat{x})$. Since $\phi + \tau_0$ is a viscosity subsolution of (4.1) in $B_s(\hat{x})$, Theorem 4.2 with $\Omega := B_s(\hat{x})$ yields (4.6). □

Correct proof, which the reader may skip first. Since we do not suppose that $u \in LSC(\Omega)$ here, we have to work with u_*.

Suppose that $0 = (u_* - \phi)(\hat{x}) < (u_* - \phi)(x)$ for $x \in \Omega \setminus \{\hat{x}\}$ for some $\phi \in C^2(\Omega)$, $\hat{x} \in \Omega$, $\theta > 0$ and

$$F(\hat{x}, \phi(\hat{x}), D\phi(\hat{x}), D^2\phi(\hat{x})) \leq -2\theta.$$

Hence, we get (4.5) even in this case.

We also show that the w defined in the above is a viscosity subsolution of (4.1). It only remains to check that $\sup_\Omega(w - u) > 0$.

In fact, choosing $x_k \in B_{1/k}(\hat{x})$ such that

$$u_*(\hat{x}) + \frac{1}{k} \geq u(x_k),$$

we easily verify that if $1/k \leq \min\{\tau_0/2, s\}$ and $|\phi(\hat{x}) - \phi(x_k)| < \tau_0/2$, then we have

$$w(x_k) \geq \phi(x_k) + \tau_0 > \phi(\hat{x}) + \frac{\tau_0}{2} = u_*(\hat{x}) + \frac{\tau_0}{2} \geq u(x_k). \quad \square$$

4.2 Representation formula

In this subsection, for given Bellman and Isaacs equations, we present the expected solutions, which are called "value functions". In fact, via the dynamic programming principle for the value functions, we verify that they are viscosity solutions of the corresponding PDEs.

Although this subsection is very important to learn how the notion of viscosity solutions is the right one from a view point of applications in optimal control and games,

if the reader is more interested in the PDE theory than these applications, he/she may skip this subsection.

We shall restrict ourselves to

investigate the formulas only for first-order PDEs because in order to extend the results below to second-order ones, we need to introduce some terminologies from stochastic analysis. However, this is too much for this thin book.

As will be seen, we study the minimization of functionals associated with ordinary differential equations (ODEs for short), which is called a "deterministic" optimal control problem. When we adapt "stochastic" differential equations instead of ODEs, those are called "stochastic" optimal control problems. We refer to [10] for the later.

Moreover, to avoid mentioning the boundary condition, we will work on the whole domain $\mathbf{R}^n$.

Throughout this subsection, we also suppose (3.7); $\nu > 0$.

4.2.1 Bellman equation

We fix a control set $A \subset \mathbf{R}^m$ for some $m \in \mathbf{N}$. We define $\mathcal{A}$ by

$$\mathcal{A} := \{\alpha : [0,\infty) \to A \mid \alpha(\cdot) \text{ is measurable}\}.$$

For $x \in \mathbf{R}^n$ and $\alpha \in \mathcal{A}$, we denote by $X(\cdot; x, \alpha)$ the solution of

$$\begin{cases} X'(t) = g(X(t), \alpha(t)) & \text{for } t > 0, \\ \qquad\quad X(0) = x, \end{cases} \tag{4.7}$$

where we will impose a sufficient condition on continuous functions $g : \mathbf{R}^n \times A \to \mathbf{R}^n$ so that (4.7) is uniquely solvable.

For given $f : \mathbf{R}^n \times A \to \mathbf{R}$, under suitable assumptions (see (4.8) below), we define the cost functional for $X(\cdot; x, \alpha)$:

$$J(x, \alpha) := \int_0^\infty e^{-\nu t} f(X(t; x, \alpha), \alpha(t)) dt.$$

Here, $\nu > 0$ is called a discount factor, which indicates that the right hand side of the above is finite.

Now, we shall consider the optimal cost functional, which is called the value function in the optimal control problem;

$$u(x) := \inf_{\alpha \in \mathcal{A}} J(x, \alpha) \quad \text{for } x \in \mathbf{R}^n.$$

Theorem 4.4. (Dynamic Programming Principle) *Assume that*

$$\begin{cases} (1) & \sup_{a \in A} \left(\|f(\cdot, a)\|_{L^\infty(\mathbf{R}^n)} + \|g(\cdot, a)\|_{W^{1,\infty}(\mathbf{R}^n)} \right) < \infty, \\ (2) & \sup_{a \in A} |f(x, a) - f(y, a)| \le \omega_f(|x - y|) \quad \textit{for } x, y \in \mathbf{R}^n, \end{cases} \tag{4.8}$$

where $\omega_f \in \mathcal{M}$.

For any $T > 0$, we have

$$u(x) = \inf_{\alpha\in\mathcal{A}} \left(\int_0^T e^{-\nu t} f(X(t;x,\alpha),\alpha(t))dt + e^{-\nu T} u(X(T;x,\alpha)) \right).$$

Proof. For fixed $T > 0$, we denote by $v(x)$ the right hand side of the above.

Step 1: $u(x) \geq v(x)$. Fix any $\varepsilon > 0$, and choose $\alpha_\varepsilon \in \mathcal{A}$ such that

$$u(x) + \varepsilon \geq \int_0^\infty e^{-\nu t} f(X(t;x,\alpha_\varepsilon),\alpha_\varepsilon(t))dt.$$

Setting $\hat{x} = X(T;x,\alpha_\varepsilon)$ and $\hat{\alpha}_\varepsilon \in \mathcal{A}$ by $\hat{\alpha}_\varepsilon(t) = \alpha_\varepsilon(T+t)$ for $t \geq 0$, we have

$$\begin{aligned}\int_0^\infty e^{-\nu t} f(X(t;x,\alpha_\varepsilon),\alpha_\varepsilon(t))dt = & \int_0^T e^{-\nu t} f(X(t;x,\alpha_\varepsilon),\alpha_\varepsilon(t))dt \\ & + e^{-\nu T} \int_0^\infty e^{-\nu t} f(X(t;\hat{x},\hat{\alpha}_\varepsilon),\hat{\alpha}_\varepsilon(t))dt.\end{aligned}$$

Here and later, without mentioning, we use the fact that

$$X(t+T;x,\alpha) = X(t;\hat{x},\hat{\alpha}) \quad \text{for } T > 0, t \geq 0 \text{ and } \alpha \in \mathcal{A},$$

where

$$\hat{\alpha}(t) := \alpha(t+T) \ (t \geq 0) \quad \text{and} \quad \hat{x} := X(T;x,\alpha).$$

Indeed, the above relation holds true because of the uniqueness of solutions of (4.7) under assumptions (4.8). See Fig 4.3.

Thus, taking the infimum in the second term of the right hand side of the above among $\mathcal{A}$, we have

$$u(x) + \varepsilon \geq \int_0^T e^{-\nu t} f(X(t;x,\alpha),\alpha(t))dt + e^{-\nu T} u(\hat{x}),$$

which implies one-sided inequality by taking the infimum over $\mathcal{A}$ since $\varepsilon > 0$ is arbitrary.

Step 2: $u(x) \leq v(x)$. Fix $\varepsilon > 0$ again, and choose $\alpha_\varepsilon \in \mathcal{A}$ such that

$$v(x) + \varepsilon \geq \int_0^T e^{-\nu t} f(X(t;x,\alpha_\varepsilon),\alpha_\varepsilon(t))dt + e^{-\nu T} u(\hat{x}),$$

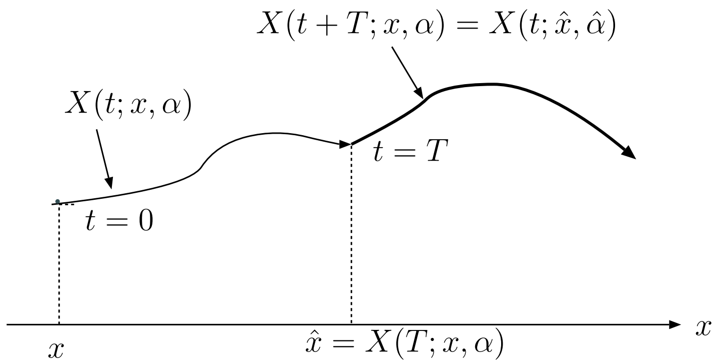

Fig 4.3

where $\hat{x} := X(T;x,\alpha_\varepsilon)$. We next choose $\alpha_1 \in \mathcal{A}$ such that

$$u(\hat{x}) + \varepsilon \geq \int_0^\infty e^{-\nu t} f(X(t;\hat{x},\alpha_1),\alpha_1(t))dt.$$

Now, setting

$$\alpha_0(t) := \begin{cases} \alpha_\varepsilon(t) & \text{for } t \in [0,T), \\ \alpha_1(t-T) & \text{for } t \geq T, \end{cases}$$

we see that

$$v(x) + 2\varepsilon \geq \int_0^\infty e^{-\nu t} f(X(t;x,\alpha_0),\alpha_0(t))dt,$$

which gives the opposite inequality by taking the infimum over $\alpha_0 \in \mathcal{A}$ since $\varepsilon > 0$ is arbitrary again. □

Now, we give an existence result for Bellman equations.

Theorem 4.5. *Assume that* (4.8) *holds. Then, u is a viscosity solution of*

$$\sup_{a\in A}\{\nu u - \langle g(x,a), Du\rangle - f(x,a)\} = 0 \quad \textit{in } \mathbf{R}^n. \tag{4.9}$$

Sketch of proof. In Steps 1 and 2, we give a proof when $u \in USC(\mathbf{R}^n)$ and $u \in LSC(\mathbf{R}^n)$, respectively.

Step 1: Subsolution property. Fix $\phi \in C^1(\mathbf{R}^n)$, and suppose that $0 = (u-\phi)(\hat{x}) \geq (u-\phi)(x)$ for some $\hat{x} \in \mathbf{R}^n$ and any $x \in \mathbf{R}^n$.

Fix any $a_0 \in A$, and set $\alpha_0(t) := a_0$ for $t \geq 0$ so that $\alpha_0 \in \mathcal{A}$.

For small $s > 0$, in view of Theorem 4.4, we have

$$\begin{aligned}\phi(\hat{x}) - e^{-\nu s}\phi(X(s;\hat{x},\alpha_0)) &\le u(\hat{x}) - e^{-\nu s}u(X(s;\hat{x},\alpha_0)) \\ &\le \int_0^s e^{-\nu t} f(X(t;\hat{x},\alpha_0), a_0)dt.\end{aligned}$$

Setting $X(t) := X(t;\hat{x},\alpha_0)$ for simplicity, by (4.7), we see that

$$e^{-\nu t}\{\nu\phi(X(t)) - \langle g(X(t),\alpha_0), D\phi(X(t))\rangle\} = -\frac{d}{dt}\left(e^{-\nu t}\phi(X(t))\right). \quad (4.10)$$

Hence, we have

$$0 \ge \int_0^s e^{-\nu t}\{\nu\phi(X(t)) - \langle g(X(t),a_0), D\phi(X(t))\rangle - f(X(t),a_0)\}dt.$$

Therefore, dividing the above by $s > 0$, and then sending $s \to 0$, we have

$$0 \ge \nu\phi(\hat{x}) - \langle g(\hat{x},a_0), D\phi(\hat{x})\rangle - f(\hat{x},a_0),$$

which implies the desired inequality of the definition by taking the supremum over A.

Step 2: Supersolution property. To show that u is a viscosity supersolution, we argue by contradiction.

Suppose that there are $\hat{x} \in \mathbf{R}^n$, $\theta > 0$ and $\phi \in C^1(\mathbf{R}^n)$ such that $0 = (u-\phi)(\hat{x}) \le (u-\phi)(x)$ for $x \in \mathbf{R}^n$, and that

$$\sup_{a\in A}\{\nu\phi(\hat{x}) - \langle g(\hat{x},a), D\phi(\hat{x})\rangle - f(\hat{x},a)\} \le -2\theta.$$

Thus, we can find $\varepsilon > 0$ such that

$$\sup_{a\in A}\{\nu\phi(x) - \langle g(x,a), D\phi(x)\rangle - f(x,a)\} \le -\theta \quad \text{for } x \in B_\varepsilon(\hat{x}). \quad (4.11)$$

By assumption (4.8) for g, setting $t_0 := \varepsilon/(\sup_{a\in A}\|g(\cdot,a)\|_{L^\infty(\mathbf{R}^n)}+1) > 0$, we easily see that

$$|X(t;\hat{x},\alpha) - \hat{x}| \le \int_0^t |X'(s;\hat{x},\alpha)|ds \le \varepsilon \quad \text{for } t \in [0,t_0] \text{ and } \alpha \in \mathcal{A}.$$

Hence, by setting $X(t) := X(t;\hat{x},\alpha)$ for any fixed $\alpha \in \mathcal{A}$, (4.11) yields

$$\nu\phi(X(t)) - \langle g(X(t),\alpha(t)), D\phi(X(t))\rangle - f(X(t),\alpha(t)) \le -\theta \quad (4.12)$$

for $t \in [0, t_0]$. Since (4.10) holds for α in place of α_0, multiplying $e^{-\nu t}$ in (4.12), and then integrating it over $[0, t_0]$, we obtain

$$\phi(\hat{x}) - e^{-\nu t_0}\phi(X(t_0)) - \int_0^{t_0} e^{-\nu t} f(X(t), \alpha(t))dt \le -\frac{\theta}{\nu}(1 - e^{-\nu t_0}).$$

Thus, setting $\theta_0 = \theta(1 - e^{-\nu t_0})/\nu > 0$, which is independent of $\alpha \in \mathcal{A}$, we have

$$u(\hat{x}) \le \int_0^{t_0} e^{-\nu t} f(X(t), \alpha(t))dt + e^{-\nu t_0} u(X(t_0)) - \theta_0.$$

Therefore, taking the infimum over $\mathcal{A}$, we get a contradiction to Theorem 4.4. □

Correct proof, which the reader may skip first.

Step 1: Subsolution property. Assume that there are $\hat{x} \in \mathbf{R}^n$, $\theta > 0$ and $\phi \in C^1(\mathbf{R}^n)$ such that $0 = (u^* - \phi)(\hat{x}) \ge (u^* - \phi)(x)$ for $x \in \mathbf{R}^n$ and that

$$\sup_{a \in A}\{\nu\phi(\hat{x}) - \langle g(\hat{x}, a), D\phi(\hat{x})\rangle - f(\hat{x}, a)\} \ge 2\theta.$$

In view of (4.8), there are $a_0 \in A$ and $r > 0$ such that

$$\nu\phi(x) - \langle g(x, a_0), D\phi(x)\rangle - f(x, a_0) \ge \theta \quad \text{for } x \in B_{2r}(\hat{x}). \tag{4.13}$$

For large $k \ge 1$, we can choose $x_k \in B_{1/k}(\hat{x})$ such that $u^*(\hat{x}) \le u(x_k) + k^{-1}$ and $|\phi(\hat{x}) - \phi(x_k)| < 1/k$. We will only use k such that $1/k \le r$.

Setting $\alpha_0(t) := a_0$, we note that $X_k(t) := X(t; x_k, \alpha_0) \in B_{2r}(\hat{x})$ for $t \in [0, t_0]$ with some $t_0 > 0$ and for large k.

On the other hand, by Theorem 4.4, we have

$$u(x_k) \le \int_0^{t_0} e^{-\nu t} f(X_k(t), a_0)dt + e^{-\nu t_0} u(X_k(t_0)).$$

Thus, we have

$$\phi(x_k) - \frac{2}{k} \le \phi(\hat{x}) - \frac{1}{k} \le u(x_k) \le \int_0^{t_0} e^{-\nu t} f(X_k(t), a_0)dt + e^{-\nu t_0}\phi(X_k(t_0)).$$

Hence, by (4.13) as in Step 1 of Sketch of proof, we see that

$$\begin{aligned} -\frac{2}{k} &\le \int_0^{t_0} e^{-\nu t}\{f(X_k(t), a_0) + \langle g(X_k(t), a_0), D\phi(X_k(t))\rangle - \nu\phi(X_k(t))\}dt \\ &\le -\frac{\theta}{\nu}(1 - e^{-\nu t_0}), \end{aligned}$$

which is a contradiction for large k.

Step 2: Supersolution property. Assume that there are $\hat{x} \in \mathbf{R}^n$, $\theta > 0$ and $\phi \in C^1(\mathbf{R}^n)$ such that $0 = (u_* - \phi)(\hat{x}) \le (u_* - \phi)(x)$ for $x \in \mathbf{R}^n$ and that

$$\sup_{a\in A}\{\nu\phi(\hat{x}) - \langle g(\hat{x},a), D\phi(\hat{x})\rangle - f(\hat{x},a)\} \le -2\theta.$$

In view of (4.8), there is $r > 0$ such that

$$\nu\phi(x) - \langle g(x,a), D\phi(x)\rangle - f(x,a) \le -\theta \quad \text{for } x \in B_{2r}(\hat{x}) \text{ and } a \in A. \tag{4.14}$$

For large $k \ge 1$, we can choose $x_k \in B_{1/k}(\hat{x})$ such that $u_*(\hat{x}) \ge u(x_k) - k^{-1}$ and $|\phi(\hat{x}) - \phi(x_k)| < 1/k$. In view of (4.8), there is $t_0 > 0$ such that

$$X_k(t; x_k, \alpha) \in B_{2r}(\hat{x}) \quad \text{for all } k \ge \frac{1}{r}, \alpha \in \mathcal{A} \text{ and } t \in [0, t_0].$$

Now, we select $\alpha_k \in \mathcal{A}$ such that

$$u(x_k) + \frac{1}{k} \ge \int_0^{t_0} e^{-\nu t} f(X(t; x_k, \alpha_k), \alpha_k(t))dt + e^{-\nu t_0} u(X(t_0; x_k, \alpha_k)).$$

Setting $X_k(t) := X(t; x_k, \alpha_k)$, we have

$$\phi(x_k) + \frac{3}{k} \ge \phi(\hat{x}) + \frac{2}{k} \ge u(x_k) + \frac{1}{k} \ge \int_0^{t_0} e^{-\nu t} f(X_k(t), \alpha_k(t))dt + e^{-\nu t_0}\phi(X_k(t)).$$

Hence, we have

$$\frac{3}{k} \ge \int_0^{t_0} e^{-\nu t}\{\langle g(X_k(t), \alpha_k(t)), D\phi(X_k(t))\rangle + f(X_k(t), \alpha_k(t)) - \nu\phi(X_k(t))\}dt.$$

Putting (4.14) with α_k in the above, we have

$$\frac{3}{k} \ge \theta \int_0^{t_0} e^{-\nu t} dt,$$

which is a contradiction for large $k \ge 1$. □

4.2.2 Isaacs equation

In this subsection, we study fully nonlinear PDEs (*i.e.* $p \in \mathbf{R}^n \to F(x,p)$ is neither convex nor concave) arising in differential games.

We are given continuous functions $f : \mathbf{R}^n \times A \times B \to \mathbf{R}$ and $g : \mathbf{R}^n \times A \times B \to \mathbf{R}^n$ such that

$$\begin{cases} (1) \quad \sup_{(a,b)\in A\times B} \left\{ \|f(\cdot,a,b)\|_{L^\infty(\mathbf{R}^n)} + \|g(\cdot,a,b)\|_{W^{1,\infty}(\mathbf{R}^n)} \right\} < \infty, \\ (2) \quad \sup_{(a,b)\in A\times B} |f(x,a,b) - f(y,a,b)| \le \omega_f(|x-y|) \text{ for } x, y \in \mathbf{R}^n, \end{cases} \tag{4.15}$$

where $\omega_f \in \mathcal{M}$.

Under (4.15), we shall consider Isaacs equations:

$$\sup_{a\in A} \inf_{b\in B} \{\nu u - \langle g(x,a,b), Du\rangle - f(x,a,b)\} = 0 \quad \text{in } \mathbf{R}^n, \tag{4.16}$$

and

$$\inf_{b\in B} \sup_{a\in A} \{\nu u - \langle g(x,a,b), Du\rangle - f(x,a,b)\} = 0 \quad \text{in } \mathbf{R}^n. \tag{4.16$'$}$$

As in the previous subsection, we shall derive the expected solution.

We first introduce some notations: While we will use the same notion $\mathcal{A}$ as before, we set

$$\mathcal{B} := \{\beta : [0,\infty) \to B \mid \beta(\cdot) \text{ is measurable}\}.$$

Next, we introduce the so-called sets of "non-anticipating strategies":

$$\Gamma := \left\{ \gamma : \mathcal{A} \to \mathcal{B} \;\middle|\; \begin{array}{c} \text{for any } T > 0, \text{ if } \alpha_1 \text{ and } \alpha_2 \in \mathcal{A} \text{ satisfy} \\ \text{that } \alpha_1(t) = \alpha_2(t) \text{ for } a.a.\ t \in (0,T), \\ \text{then } \gamma[\alpha_1](t) = \gamma[\alpha_2](t) \text{ for } a.a.\ t \in (0,T) \end{array} \right\}$$

and

$$\Delta := \left\{ \delta : \mathcal{B} \to \mathcal{A} \;\middle|\; \begin{array}{c} \text{for any } T > 0, \text{ if } \beta_1 \text{ and } \beta_2 \in \mathcal{B} \text{ satisfy} \\ \text{that } \beta_1(t) = \beta_2(t) \text{ for } a.a.\ t \in (0,T), \\ \text{then } \delta[\beta_1](t) = \delta[\beta_2](t) \text{ for } a.a.\ t \in (0,T) \end{array} \right\}.$$

Using these notations, we will consider maximizing-minimizing problems of the following cost functional: For $\alpha \in \mathcal{A}$, $\beta \in \mathcal{B}$, and $x \in \mathbf{R}^n$,

$$J(x,\alpha,\beta) := \int_0^\infty e^{-\nu t} f(X(t;x,\alpha,\beta), \alpha(t), \beta(t))dt,$$

where $X(\cdot;x,\alpha,\beta)$ is the (unique) solutions of

$$\begin{cases} X'(t) = g(X(t),\alpha(t),\beta(t)) \quad \text{for } t > 0, \\ \qquad\qquad\qquad\qquad X(0) = x. \end{cases} \tag{4.17}$$

The expected solutions for (4.16) and (4.16′), respectively, are given by

$$u(x) = \sup_{\gamma\in\Gamma}\inf_{\alpha\in\mathcal{A}}\int_0^\infty e^{-\nu t} f(X(t;x,\alpha,\gamma[\alpha]),\alpha(t),\gamma[\alpha](t))dt,$$

and

$$v(x) = \inf_{\delta\in\Delta}\sup_{\beta\in\mathcal{B}}\int_0^\infty e^{-\nu t} f(X(t;x,\delta[\beta],\beta),\delta[\beta](t),\beta(t))dt.$$

We call u and v upper and lower value functions of this differential game, respectively. In fact, under appropriate hypotheses, we expect that $v \le u$, which cannot be proved easily. To show $v \le u$, we first observe that u and v are, respectively, viscosity solutions of (4.16) and (4.16′). Noting that

$$\sup_{a\in A}\inf_{b\in B}\{\nu r - \langle g(x,a,b),p\rangle - f(x,a,b)\} \le \inf_{b\in B}\sup_{a\in A}\{\nu r - \langle g(x,a,b),p\rangle - f(x,a,b)\}$$

for $(x,r,p) \in \mathbf{R}^n\times\mathbf{R}\times\mathbf{R}^n$, we see that u (resp., v) is a viscosity supersolution (resp., subsolution) of (4.16′) (resp., (4.16)). Thus, the standard comparison principle implies $v \le u$ in $\mathbf{R}^n$ (under suitable growth condition at $|x| \to \infty$ for u and v).

We shall only deal with u since the corresponding results for v can be obtained in a symmetric way.

To show that u is a viscosity solution of the Isaacs equation (4.16), we first establish the dynamic programming principle as in the previous subsection:

Theorem 4.6. (Dynamic Programming Principle) *Assume that* (4.15) *hold. Then, for* $T > 0$, *we have*

$$u(x) = \sup_{\gamma\in\Gamma}\inf_{\alpha\in\mathcal{A}}\left(\begin{array}{c}\displaystyle\int_0^T e^{-\nu t} f(X(t;x,\alpha,\gamma[\alpha]),\alpha(t),\gamma[\alpha](t))dt \\ +e^{-\nu T}u(X(T;x,\alpha,\gamma[\alpha]))\end{array}\right).$$

Proof. For a fixed $T > 0$, we denote by $w(x)$ the right hand side of the above.

Step 1: $u(x) \le w(x)$. For any $\varepsilon > 0$, we choose $\gamma_\varepsilon \in \Gamma$ such that

$$u(x) - \varepsilon \le \inf_{\alpha \in \mathcal{A}} \int_0^\infty e^{-\nu t} f(X(t; x, \alpha, \gamma_\varepsilon[\alpha]), \alpha(t), \gamma_\varepsilon[\alpha](t))dt =: I_\varepsilon.$$

For any fixed $\alpha_0 \in \mathcal{A}$, we define the mapping $\mathcal{T}_0 : \mathcal{A} \to \mathcal{A}$ by

$$\mathcal{T}_0[\alpha] := \begin{cases} \alpha_0(t) & \text{for } t \in [0, T), \\ \alpha(t - T) & \text{for } t \in [T, \infty) \end{cases} \quad \text{for } \alpha \in \mathcal{A}.$$

Thus, for any $\alpha \in \mathcal{A}$, we have

$$\begin{aligned} I_\varepsilon &\le \int_0^T e^{-\nu t} f(X(t; x, \alpha_0, \gamma_\varepsilon[\alpha_0]), \alpha_0(t), \gamma_\varepsilon[\alpha_0](t))dt \\ &\quad + \int_T^\infty e^{-\nu t} f(X(t; x, \mathcal{T}_0[\alpha], \gamma_\varepsilon[\mathcal{T}_0[\alpha]]), \mathcal{T}_0[\alpha](t), \gamma_\varepsilon[\mathcal{T}_0[\alpha]](t))dt \\ &=: I_\varepsilon^1 + I_\varepsilon^2. \end{aligned}$$

We next define $\hat{\gamma} \in \Gamma$ by

$$\hat{\gamma}[\alpha](t) := \gamma_\varepsilon[\mathcal{T}_0[\alpha]](t + T) \quad \text{for } t \ge 0 \text{ and } \alpha \in \mathcal{A}.$$

Note that $\hat{\gamma}$ belongs to Γ.

Setting $\hat{x} := X(T; x, \alpha_0, \gamma_\varepsilon[\alpha_0])$, we have

$$I_\varepsilon^2 = e^{-\nu T} \int_0^\infty e^{-\nu t} f(X(t; \hat{x}, \alpha, \hat{\gamma}[\alpha]), \alpha(t), \hat{\gamma}[\alpha](t))dt.$$

Taking the infimum over $\alpha \in \mathcal{A}$, we have

$$\begin{aligned} u(x) - \varepsilon &\le I_\varepsilon^1 + e^{-\nu T} \inf_{\alpha \in \mathcal{A}} \int_0^\infty e^{-\nu t} f(X(t; \hat{x}, \alpha, \hat{\gamma}[\alpha]), \alpha(t), \hat{\gamma}[\alpha](t))dt \\ &=: I_\varepsilon^1 + \hat{I}_\varepsilon^2. \end{aligned}$$

Since $\hat{I}_\varepsilon^2 \le e^{-\nu T} u(\hat{x})$, we have

$$u(x) - \varepsilon \le I_\varepsilon^1 + e^{-\nu T} u(\hat{x}),$$

which implies $u(x) - \varepsilon \le w(x)$ by taking the infimum over $\alpha_0 \in \mathcal{A}$ and then, the supremum over Γ. Therefore, we get the one-sided inequality since $\varepsilon > 0$ is arbitrary.

Step 2: $u(x) \geq w(x)$. For $\varepsilon > 0$, we choose $\gamma_\varepsilon^1 \in \Gamma$ such that

$$w(x) - \varepsilon \leq \inf_{\alpha \in \mathcal{A}} \left(\begin{array}{c} \int_0^T e^{-\nu t} f(X(t; x, \alpha, \gamma_\varepsilon^1[\alpha]), \alpha(t), \gamma_\varepsilon^1[\alpha](t)) dt \\ + e^{-\nu T} u(X(T; x, \alpha, \gamma_\varepsilon^1[\alpha])) \end{array} \right).$$

For any fixed $\alpha_0 \in \mathcal{A}$, setting $\hat{x} = X(T; x, \alpha_0, \gamma_\varepsilon^1[\alpha_0])$, we have

$$w(x) - \varepsilon \leq \int_0^T e^{-\nu t} f(X(t; x, \alpha_0, \gamma_\varepsilon^1[\alpha_0]), \alpha_0(t), \gamma_\varepsilon^1[\alpha_0](t)) dt + e^{-\nu T} u(\hat{x}).$$

Next, we choose $\gamma_\varepsilon^2 \in \Gamma$ such that

$$u(\hat{x}) - \varepsilon \leq \inf_{\alpha \in \mathcal{A}} \int_0^\infty e^{-\nu t} f(X(t; \hat{x}, \alpha, \gamma_\varepsilon^2[\alpha]), \alpha(t), \gamma_\varepsilon^2[\alpha](t)) dt. =: I.$$

For $\alpha \in \mathcal{A}$, we define the mapping $\mathcal{T}_1 : \mathcal{A} \to \mathcal{A}$ by

$$\mathcal{T}_1[\alpha](t) := \alpha(t + T) \quad \text{for } t \geq 0.$$

Thus, we have

$$I \leq \int_0^\infty e^{-\nu t} f(X(t; \hat{x}, \mathcal{T}_1[\alpha_0], \gamma_\varepsilon^2[\mathcal{T}_1[\alpha_0]]), \mathcal{T}_1[\alpha_0](t), \gamma_\varepsilon^2[\mathcal{T}_1[\alpha_0]](t)) dt =: \hat{I}.$$

Now, for $\alpha \in \mathcal{A}$, setting

$$\hat{\gamma}[\alpha](t) := \begin{cases} \gamma_\varepsilon^1[\alpha](t) & \text{for } t \in [0, T), \\ \gamma_\varepsilon^2[\mathcal{T}_1[\alpha]](t - T) & \text{for } t \in [T, \infty), \end{cases}$$

and $\hat{X}(t) := X(t; \hat{x}, \mathcal{T}_1[\alpha_0], \gamma_\varepsilon^2[\mathcal{T}_1[\alpha_0]])$, we have

$$\begin{aligned} \hat{I} &= \int_T^\infty e^{-\nu(t-T)} f(\hat{X}(t-T), \mathcal{T}_1[\alpha_0](t-T), \gamma_\varepsilon^2[\mathcal{T}_1[\alpha_0]](t-T)) dt \\ &= e^{\nu T} \int_T^\infty e^{-\nu t} f(\hat{X}(t-T), \alpha_0(t), \hat{\gamma}[\alpha_0](t)) dt. \end{aligned}$$

Since

$$X(t; x, \alpha_0, \hat{\gamma}[\alpha_0]) = \begin{cases} X(t; x, \alpha_0, \gamma_\varepsilon^1[\alpha_0]) & \text{for } t \in [0, T), \\ \hat{X}(t - T) & \text{for } t \in [T, \infty), \end{cases}$$

we have

$$w(x) - 2\varepsilon \le \int_0^\infty e^{-\nu t} f(X(t;x,\alpha_0,\hat{\gamma}[\alpha_0]),\alpha_0(t),\hat{\gamma}[\alpha_0](t))dt.$$

Since α_0 is arbitrary, we have

$$w(x) - 2\varepsilon \le \inf_{\alpha\in\mathcal{A}} \int_0^\infty e^{-\nu t} f(X(t;x,\alpha,\hat{\gamma}[\alpha]),\alpha(t),\hat{\gamma}[\alpha](t))dt,$$

which yields the assertion by taking the supremum over Γ and then, by sending $\varepsilon \to 0$. □

Now, we shall verify that the value function u is a viscosity solution of (4.16).

Since we only give a sketch of proofs, one can skip the following theorem. For a correct proof, we refer to [1], originally by Evans-Souganidis (1984).

Theorem 4.7. *Assume that* (4.15) *holds.*
(1) *Then, u is a viscosity subsolution of* (4.16).
(2) *Assume also the following properties:*

$$\begin{cases} (i) & A \subset \mathbf{R}^m \text{ is compact for some integer } m \ge 1. \\ (ii) & \text{there is an } \omega_A \in \mathcal{M} \text{ such that} \\ & |f(x,a,b) - f(x,a',b)| + |g(x,a,b) - g(x,a',b)| \le \omega_A(|a-a'|) \\ & \text{for } x \in \mathbf{R}^n,\ a,a' \in A \text{ and } b \in B. \end{cases} \tag{4.18}$$

Then, u is a viscosity supersolution of (4.16).

<u>*Remark.*</u> To show that v is a viscosity subsolution of (4.16′), instead of (4.18), we need to suppose the following hypotheses:

$$\begin{cases} (i) & B \subset \mathbf{R}^m \text{ is compact for some integer } m \ge 1. \\ (ii) & \text{there is an } \omega_B \in \mathcal{M} \text{ such that} \\ & |f(x,a,b) - f(x,a,b')| + |g(x,a,b) - g(x,a,b')| \le \omega_B(|b-b'|) \\ & \text{for } x \in \mathbf{R}^n,\ b,b' \in B \text{ and } a \in A, \end{cases} \tag{4.18′}$$

while to verify that v is a viscosity supersolution of (4.16′), we only need (4.15).

<u>*Sketch of proof.*</u> We shall only prove the assertion assuming that $u \in USC(\mathbf{R}^n)$ and $u \in LSC(\mathbf{R}^n)$ in Step 1 and 2, respectively.

To give a correct proof without the semi-continuity assumption, we need a bit careful analysis similar to the proof for Bellman equations. We omit the correct proof here.

Step 1: Subsolution property. Suppose that the subsolution property fails; there are $x \in \mathbf{R}^n$, $\theta > 0$ and $\phi \in C^1(\mathbf{R}^n)$ such that $0 = (u-\phi)(x) \geq (u-\phi)(y)$ (for all $y \in \mathbf{R}^n$) and

$$\sup_{a \in A} \inf_{b \in B} \{\nu u(x) - \langle g(x,a,b), D\phi(x)\rangle - f(x,a,b)\} \geq 3\theta.$$

We note that $X(\cdot; x, \alpha, \gamma[\alpha])$ are uniformly continuous for any $(\alpha, \gamma) \in \mathcal{A} \times \Gamma$ in view of (4.15).

Thus, we can choose that $a_0 \in A$ such that

$$\inf_{b \in B} \{\nu\phi(x) - \langle g(x,a_0,b), D\phi(x)\rangle - f(x,a_0,b)\} \geq 2\theta.$$

For any $\gamma \in \Gamma$, setting $\alpha_0(t) = a_0$ for $t \geq 0$, we simply write $X(\cdot)$ for $X(\cdot; x, \alpha_0, \gamma[\alpha_0])$. Thus, we find small $t_0 > 0$ such that

$$\nu\phi(X(t)) - \langle g(X(t), a_0, \gamma[\alpha_0](t)), D\phi(X(t))\rangle - f(X(t), a_0, \gamma[\alpha_0](t)) \geq \theta$$

for $t \in [0, t_0]$. Multiplying $e^{-\nu t}$ in the above and then, integrating it over $[0, t_0]$, we have

$$\begin{aligned} \frac{\theta}{\nu}(1 - e^{-\nu t_0}) &\leq -\int_0^{t_0} \left\{ \frac{d}{dt}\left(e^{-\nu t}\phi(X(t))\right) + e^{-\nu t} f(X(t), a_0, \gamma[\alpha_0](t)) \right\} dt \\ &= \phi(x) - e^{-\nu t_0}\phi(X(t_0)) - \int_0^{t_0} e^{-\nu t} f(X(t), a_0, \gamma[\alpha_0](t)) dt. \end{aligned}$$

Hence, we have

$$u(x) - \frac{\theta}{\nu}(1 - e^{-\nu t_0}) \geq \int_0^{t_0} e^{-\nu t} f(X(t), a_0, \gamma[\alpha_0](t)) dt + e^{-\nu t_0} u(X(t_0)) =: \hat{I}.$$

Taking the infimum over $\mathcal{A}$, we have

$$\hat{I} \geq \inf_{\alpha \in \mathcal{A}} \left(\begin{array}{c} \displaystyle\int_0^{t_0} e^{-\nu t} f(X(t; x, \alpha, \gamma[\alpha]), \alpha(t), \gamma[\alpha](t)) dt \\ + e^{-\nu t_0} u(X(t_0; x, \alpha, \gamma[\alpha])) \end{array} \right).$$

Therefore, since $\gamma \in \Gamma$ is arbitrary, we have

$$u(x) - \frac{\theta}{\nu}(1 - e^{-\nu t_0}) \geq \sup_{\gamma \in \Gamma} \inf_{\alpha \in \mathcal{A}} \left(\begin{array}{c} \displaystyle\int_0^{t_0} e^{-\nu t} f(X(t; x, \alpha, \gamma[\alpha]), \alpha(t), \gamma[\alpha](t)) dt \\ + e^{-\nu t_0} u(X(t_0; x, \alpha, \gamma[\alpha])) \end{array} \right),$$

which contradicts Theorem 4.6.

Step 2: Supersolution property. Suppose that the supersolution property fails; there are $x \in \mathbf{R}^n$, $\theta > 0$ and $\phi \in C^1(\mathbf{R}^n)$ such that $0 = (u - \phi)(x) \le (u - \phi)(y)$ for $y \in \mathbf{R}^n$, and

$$\sup_{a\in A}\inf_{b\in B}\{\nu u(x) - \langle g(x,a,b), D\phi(x)\rangle - f(x,a,b)\} \le -3\theta.$$

For any $a \in A$, there is $b(a) \in B$ such that

$$\nu u(x) - \langle g(x,a,b(a)), D\phi(x)\rangle - f(x,a,b(a)) \le -2\theta.$$

In view of (4.18), there is $\varepsilon(a) > 0$ such that if $|a - a'| < \varepsilon(a)$ and $|x - y| < \varepsilon(a)$, then we have

$$\nu\phi(y) - \langle g(y,a',b(a)), D\phi(y)\rangle - f(y,a',b(a)) \le -\theta.$$

From the compactness of A, we may select $\{a_k\}_{k=1}^M$ such that

$$A = \bigcup_{k=1}^M A_k,$$

where

$$A_k := \{a \in A \mid |a - a_k| < \varepsilon(a_k)\}.$$

Furthermore, we set $\hat{A}_1 = A_1$, and inductively, $\hat{A}_k := A_k \setminus \cup_{j=1}^{k-1} A_j$; $\hat{A}_k \cap \hat{A}_j = \emptyset$ for $k \ne j$. We may also suppose that $\hat{A}_k \ne \emptyset$ for $k = 1, \dots, M$.

For $\alpha \in \mathcal{A}$, we define

$$\gamma_0[\alpha](t) := b(a_k) \quad \text{provided } \alpha(t) \in \hat{A}_k.$$

Now, setting $X(t) := X(t;x,\alpha,\gamma_0[\alpha])$, we find $t_0 > 0$ such that

$$\nu\phi(X(t)) - \langle g(X(t),\alpha(t),\gamma_0[\alpha](t)), D\phi(X(t))\rangle - f(X(t),\alpha(t),\gamma_0[\alpha](t)) \le -\theta$$

for $t \in [0, t_0]$. Multiplying $e^{-\nu t}$ in the above and then, integrating it, we obtain

$$\phi(x) - e^{-\nu t_0}\phi(X(t_0)) - \int_0^{t_0} e^{-\nu t} f(X(t),\alpha(t),\gamma_0[\alpha](t))dt \le -\frac{\theta}{\nu}(1 - e^{-\nu t_0}).$$

Since $\alpha \in \mathcal{A}$ is arbitrary, we have

$$u(x) + \frac{\theta}{\nu}(1 - e^{-\nu t_0}) \le \inf_{\alpha\in\mathcal{A}} \left(\begin{array}{c} \displaystyle\int_0^{t_0} e^{-\nu t} f(X(t;x,\alpha,\gamma_0[\alpha]),\alpha(t),\gamma_0[\alpha](t))dt \\ +e^{-\nu t_0} u(X(t_0;x,\alpha,\gamma_0[\alpha])) \end{array} \right),$$

which contradicts Theorem 4.6 by taking the supremum over Γ. $\square$

4.3 Stability

In this subsection, we present a stability result for viscosity solutions, which is one of the most important properties for "solutions" as noted in section 1. Thus, this result justifies our notion of viscosity solutions.

However, since we will only use Proposition 4.8 below in section 7.3, the reader may skip the proof.

First of all, for possibly discontinuous $F : \Omega \times \mathbf{R} \times \mathbf{R}^n \times S^n \to \mathbf{R}$, we are concerned with

$$F(x, u, Du, D^2u) = 0 \quad \text{in } \Omega. \tag{4.19}$$

We introduce the following notation:

$$F_*(x, r, p, X) := \lim_{\varepsilon \to 0} \inf \left\{ F(y, s, q, Y) \;\middle|\; \begin{array}{c} y \in \Omega \cap B_\varepsilon(x), |s - r| < \varepsilon, \\ |q - p| < \varepsilon, \|Y - X\| < \varepsilon \end{array} \right\},$$

$$F^*(x, r, p, X) := \lim_{\varepsilon \to 0} \sup \left\{ F(y, s, q, Y) \;\middle|\; \begin{array}{c} y \in \Omega \cap B_\varepsilon(x), |s - r| < \varepsilon, \\ |q - p| < \varepsilon, \|Y - X\| < \varepsilon \end{array} \right\}.$$

Definition. We call $u : \Omega \to \mathbf{R}$ a viscosity subsolution (resp., supersolution) of (4.19) if u^* (resp., u_*) is a viscosity subsolution (resp., supersoluion) of

$$F_*(x, u, Du, D^2u) \leq 0 \quad \big(\text{resp., } F^*(x, u, Du, D^2u) \geq 0\big) \quad \text{in } \Omega.$$

We call $u : \Omega \to \mathbf{R}$ a viscosity solution of (4.19) if it is both a viscosity sub- and supersolution of (4.19).

Now, for given continuous functions $F_k : \Omega \times \mathbf{R} \times \mathbf{R}^n \times S^n \to \mathbf{R}$, we set

$$\begin{aligned} &\underline{F}(x, r, p, X) \\ &:= \lim_{k \to \infty} \inf \left\{ F_j(y, s, q, Y) \;\middle|\; \begin{array}{c} |y - x| < 1/k, |s - r| < 1/k, \\ |q - p| < 1/k, \|Y - X\| < 1/k \\ \text{and } j \geq k \end{array} \right\}, \end{aligned}$$

$$\begin{aligned} &\overline{F}(x, r, p, X) \\ &:= \lim_{k \to \infty} \sup \left\{ F_j(y, s, q, Y) \;\middle|\; \begin{array}{c} |y - x| < 1/k, |s - r| < 1/k, \\ |q - p| < 1/k, \|Y - X\| < 1/k \\ \text{and } j \geq k \end{array} \right\}. \end{aligned}$$

Our stability result is as follows.

Proposition 4.8. *Let $F_k : \Omega \times \mathbf{R} \times \mathbf{R}^n \times S^n \to \mathbf{R}$ be continuous functions. Let $u_k : \overline{\Omega} \to \mathbf{R}$ be a viscosity subsolution (resp., supersolution) of*

$$F_k(x, u_k, Du_k, D^2u_k) = 0 \quad \text{in } \Omega.$$

Setting $\overline{u}$ (resp., $\underline{u}$) by

$$\overline{u}(x) := \lim_{k\to\infty} \sup\{(u_j)^*(y) \mid y \in B_{1/k}(x) \cap \Omega,\ j \geq k\}$$

$$\left(\text{resp., } \underline{u}(x) := \lim_{k\to\infty} \inf\{(u_j)_*(y) \mid y \in B_{1/k}(x) \cap \Omega,\ j \geq k\}\right)$$

for $x \in \overline{\Omega}$, then $\overline{u}$ (resp., $\underline{u}$) is a viscosity subsolution (resp., supersolution) of

$$\underline{F}(x, u, Du, D^2u) \leq 0 \quad (\text{resp., } \overline{F}(x, u, Du, D^2u) \geq 0) \quad \text{in } \Omega.$$

Remark. We note that $\overline{u} \in USC(\overline{\Omega})$, $\underline{u} \in LSC(\overline{\Omega})$, $\underline{F} \in LSC(\Omega \times \mathbf{R} \times \mathbf{R}^n \times S^n)$ and $\overline{F} \in USC(\Omega \times \mathbf{R} \times \mathbf{R}^n \times S^n)$.

Proof. We only give a proof for subsolutions since the other can be shown similarly.

Given $\phi \in C^2(\Omega)$, we let $x_0 \in \Omega$ be such that $0 = (\overline{u} - \phi)(x_0) > (\overline{u} - \phi)(x)$ for $x \in \Omega \setminus \{x_0\}$. We shall show that $\underline{F}(x_0, \overline{u}(x_0), D\phi(x_0), D^2\phi(x_0)) \leq 0$.

We may choose $x_k \in B_r(x_0)$ (for a subsequence if necessary), where $r \in (0, \text{dist}(x_0, \partial\Omega))$, such that

$$\lim_{k\to\infty} x_k = x_0 \quad \text{and} \quad \lim_{k\to\infty} (u_k)^*(x_k) = \overline{u}(x_0). \tag{4.20}$$

We select $y_k \in \overline{B}_r(x_0)$ such that $((u_k)^* - \phi)(y_k) = \sup_{B_r(x_0)}((u_k)^* - \phi)$.

We may also suppose that $\lim_{k\to\infty} y_k = z$ for some $z \in \overline{B}_r(x_0)$ (taking a subsequence if necessary). Since $((u_k)^* - \phi)(y_k) \geq ((u_k)^* - \phi)(x_k)$, (4.20) implies

$$\begin{aligned} 0 = \liminf_{k\to\infty}((u_k)^* - \phi)(x_k) &\leq \liminf_{k\to\infty}((u_k)^* - \phi)(y_k) \\ &\leq \liminf_{k\to\infty}(u_k)^*(y_k) - \phi(z) \\ &\leq \limsup_{k\to\infty}(u_k)^*(y_k) - \phi(z) \leq (\overline{u} - \phi)(z). \end{aligned}$$

Thus, this yields $z = x_0$ and $\lim_{k\to\infty}(u_k)^*(y_k) = \overline{u}(x_0)$. Hence, we see that $y_k \in B_r(x_0)$ for large $k \geq 1$. Since $(u_k)^* - \phi$ attains a maximum over $B_r(x_0)$

at $y_k \in B_r(x_0)$, by the definition of u_k (with Proposition 2.4 for $\Omega' = B_r(x_0)$), we have

$$F_k(y_k, (u_k)^*(y_k), D\phi(y_k), D^2\phi(y_k)) \leq 0,$$

which concludes the proof by taking the limit infimum with the definition of $\underline{F}$. $\square$