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On isotropic divisors on irreducible symplectic manifolds

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Abstract.

Let X be an irreducible symplectic manifold and L a divisor on X. Assume that L is isotropic with respect to the Beauville-Bogomolov quadratic form. We define the rational Lagrangian locus and the movable locus on the universal deformation space of the pair (X, L). We prove that the rational Lagrangian locus is empty or coincides with the movable locus of the universal deformation space.

§1. Introduction

We start by recalling the definition of an irreducible symplectic manifold.

Definition 1.1 ([4, Théorèm 1]). *Let X be a compact Kähler manifold. The manifold X is said to be irreducible symplectic if X satisfies the following three properties.*

(1) *X carries a holomorphic symplectic form.*
(2) *X is simply connected.*
(3) $\dim H^0(X, \Omega_X^2) = 1$.

Together with Calabi-Yau manifolds and complex tori, irreducible symplectic manifolds form a building block of compact Kähler manifolds with $c_1 = 0$. It is shown in [21], [20] and [13] that fibre space structures of irreducible symplectic manifolds are very restricted. To state the result, we recall the definition of a Lagrangian fibration.

Definition 1.2. *Let X be an irreducible symplectic manifold and L a line bundle on X. A surjective morphism $g : X \to S$ is said to be Lagrangian if a general fibre is connected and Lagrangian. A dominant*

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map $g : X \dashrightarrow S$ is said to be rational Lagrangian if there exist another irreducible symplectic manifold X' and a birational map $\phi : X \dashrightarrow X'$ such that the composite map $g \circ \phi^{-1} : X' \to S$ is Lagrangian. We say that L defines a Lagrangian fibration if the linear system $|L|$ defines a Lagrangian fibration. We also say that L defines a rational Lagrangian fibration if $|L|$ defines a rational Lagrangian fibration.

Theorem 1.1 ([20], [21] and [13]). *Let X be a projective irreducible symplectic manifold. Assume that X admits a surjective morphism $g : X \to S$ over a smooth projective manifold S. Assume that $0 < \dim S < \dim X$ and g has connected fibres. Then g is Lagrangian and $S \cong \mathbb{P}^{1/2 \dim X}$.*

It is a natural to ask when a line bundle L defines a Lagrangian fibration. If L defines a rational Lagrangian fibration, then L is isotropic with respect to the Beauville-Bogomolov quadratic form. Moreover the first Chern class $c_1(L)$ of L belongs to the closure of the birational Kähler cone which is defined in [12, Definition 4.1].

Conjecture 1.1 (D. Huybrechts and J. Sawon). *Let X be an irreducible symplectic manifold and L a line bundle on X. Assume that L is isotropic with respect to the Beauville-Bogomolov quadratic form on $H^2(X, \mathbb{C})$. We also assume that $c_1(L)$ belongs to the closure of the birational Kähler cone of X. Then L will define a rational Lagrangian fibration.*

At this moment, partial results about Conjecture 1.1 are shown in [1], [6], [22] and [25]. In this note, we consider the concerning conjecture by a different approach. To state the result, we recall the basic facts of deformations of pairs which consists of a symplectic manifold and a line bundle.

Definition 1.3. *Let X be a Kähler manifold and L a line bundle on X. A deformation of the pair (X, L) consists of a proper smooth morphism $\mathfrak{X} \to S$ over a smooth manifold S with a reference point o and a line bundle $\mathfrak{L}$ on $\mathfrak{X}$ such that the fibre $\mathfrak{X}_o$ at o is isomorphic to X and the restriction $\mathfrak{L}|_{\mathfrak{X}_0}$ is isomorphic to L.*

If X is an irreducible symplectic manifold, it is known that there exists the universal deformation of deformations of a pair (X, L).

Proposition 1.1 ([10, (1.14)]). *Let X be an irreducible symplectic manifold and L a line bundle on X. We also let $\mathfrak{X} \to \mathrm{Def}(X)$ be the Kuranishi family of X. Then there exists a smooth hypersurface $\mathrm{Def}(X, L)$ of $\mathrm{Def}(X)$ such that the restriction family $\mathfrak{X}_L := \mathfrak{X} \times_{\mathrm{Def}(X)} \mathrm{Def}(X, L) \to \mathrm{Def}(X, L)$ forms the universal family of deformations of the pair (X, L).*

Namely, $\mathfrak{X}_L$ carries a line bundle $\mathfrak{L}$ and every deformation $\mathfrak{X}_S \to S$ of (X, L) is isomorphic to the pull back of $(\mathfrak{X}_L, \mathfrak{L})$ via a uniquely determined map $S \to \mathrm{Def}(X, L)$.

Now we can state the result.

Theorem 1.2. *Let X be an irreducible symplectic manifold and L a line bundle on X. We also let $\pi : \mathfrak{X}_L \to \mathrm{Def}(X, L)$ be the universal family of deformations of the pair (X, L) and $\mathfrak{L}$ the universal bundle. We denote by q the Beauville-Bogomolov form on $H^2(X, \mathbb{C})$. Assume that $q(L) = 0$. We define the locus of movable $\mathrm{Def}(X, L)_{\mathrm{mov}}$ by*

$$\{t \in \mathrm{Def}(X, L); c_1(\mathfrak{L}_t) \text{ belongs to the closure of the birational Kähler cone of } X.\}$$

We also define two more subsets of $\mathrm{Def}(X, L)$. The first is the locus of rational Lagrangian fibrations V, which is defined by

$$\{t \in \mathrm{Def}(X, L); \mathfrak{L}_t \text{ defines a rational Lagrangian fibration over projective space.}\}$$

The second is the locus of Lagrangian fibrations V_{reg}, which is defined by

$$\{t \in \mathrm{Def}(X, L); \mathfrak{L}_t \text{ defines a Lagrangian fibration over projective space.}\}$$

Then $V = \emptyset$ or $V = \mathrm{Def}(X, L)_{\mathrm{mov}}$. Moreover if $V \neq \emptyset$, V_{reg} is a dense open subset of $\mathrm{Def}(X, L)$ and $\mathrm{Def}(X, L) \setminus V_{\mathrm{reg}}$ is contained in a union of countably hypersurfaces of $\mathrm{Def}(X, L)$.

Remark 1.1. *Professors L. Kamenova and M. Verbitsky obtained V_{reg} is a dense open set of $\mathrm{Def}(X, L)$ under the assumption $V_{\mathrm{reg}} \neq \emptyset$ in* [14, Theorem 3.4].

To state an application of Theorem 1.2, we need the following two definitions.

Definition 1.4. *Two compact Kähler manifolds X and X' are said to be deformation equivalent if there exists a proper smooth morphism $\pi : \mathfrak{X} \to S$ over a smooth connected complex manifold S such that both X and X' form fibres of π.*

Definition 1.5. *An irreducible symplectic manifold X is said to be of $K3^{[n]}$-type if X is deformation equivalent to the n-pointed Hilbert scheme of a $K3$ surface. An irreducible symplectic manifold X is said to be generalized Kummer-type if X is deformation equivalent to a generalized Kummer variety which as defined in* [5, Théorèm 4].

It was shown in [3, Conjecture 1.4, Theorem 1.5 and Remark 11.4] and [18, Theorem 1.3 and Theorem 6.3] and [26, Proposition 3.36] that if X is isomorphic to the n-pointed Hilbert scheme of a $K3$ surface or a generalized Kummer variety, then Conjecture 1.1 holds. In fact, they proved that an isotropic line bundle L on X defines a rational Lagrangian fibration if $c_1(L)$ belongs to the closure of the birational Kähler cone and $c_1(L)$ is primitive in $H^2(X,\mathbb{Z})$. Combining these results and Theorem 1.2, we obtain the following result.

Corollary 1.1. *Let X be an irreducible symplectic manifold of type $K3^{[n]}$ or of generalized Kummer-type. We also let L be a line bundle L on X which is not trivial, isotropic with respect to the Beauville-Bogomolov quadratic form on $H^2(X,\mathbb{C})$ and such that $c_1(L)$ belongs to the closure of the birational Kähler cone of X. Then L define a rational Lagrangian fibration over the projective space.*

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§2. Birational correspondence of deformation families

In this section we study a relationship between deformation families. We start by introducing the following Lemma.

Lemma 2.1 ([10, Lemma 2.6])**.** *Let X and X' be irreducible symplectic manifolds. Assume that there exists a bimeromorphic map $\phi : X \dashrightarrow X'$. Then ϕ induces an isomorphism*

$$\phi_* : H^2(X,\mathbb{C}) \cong H^2(X',\mathbb{C})$$

which is compatible with the Hodge structures and the Beauville-Bogomolov quadratic forms.

We consider the relationship between the Kuranishi families of bimeromorphic irreducible symplectic manifolds.

Proposition 2.1. *Let X and X' are irreducible symplectic manifolds. We denote by $\pi : \mathfrak{X} \to \mathrm{Def}(X)$ the universal family of deformations of X. We also denote by $\pi' : \mathfrak{X}' \to \mathrm{Def}(X')$ the universal family of* *deformations of X'. Assume that X and X' are bimeromorphic. Then there exist dense open subset U of* $\mathrm{Def}(X)$ *and U' of* $\mathrm{Def}(X')$ *which satisfy the following three properties.*

(1) *The set* $\mathrm{Def}(X) \setminus U$ *is contained in a union of countably many hypersurfaces in* $\mathrm{Def}(X)$ *and* $\mathrm{Def}(X') \setminus U'$ *is also contained in a union of countably many hypersurfaces in* $\mathrm{Def}(X')$.

(2) *The pull backs of $\mathfrak{X} \to \mathrm{Def}(X)$ and $\mathfrak{X}' \to \mathrm{Def}(X')$ by the inclusions $U \hookrightarrow \mathrm{Def}(X)$ and $U' \hookrightarrow \mathrm{Def}(X')$ are isomorphic, that is, they satisfy the following diagram:*

$$\begin{tikzcd} \mathfrak{X} \times_{\mathrm{Def}(X)} U \arrow[r, "\cong", "\tilde{\phi}"'] \arrow[d] & \mathfrak{X}' \times_{\mathrm{Def}(X')} U' \arrow[d] \\ U \arrow[r, "\varphi", "\cong"'] & U', \end{tikzcd} \tag{1}$$

where $\tilde{\phi}$ and φ are isomorphic.

(3) *Let s be a point of U and s' the point $\varphi(s)$. We also let $\phi_s : \mathfrak{X}_s \cong \mathfrak{X}'_{s'}$ be the restriction of the isomorphism $\tilde{\phi} : \mathfrak{X} \times_{\mathrm{Def}(X)} U \cong \mathfrak{X}' \times_{\mathrm{Def}(X')} U'$ in the above diagram to the fibre $\mathfrak{X}_s$ at s and the fibre $\mathfrak{X}'_{s'}$ at s'. We denote by η a parallel transport in the local system $R^2\pi_*\mathbb{C}$ along a path from the reference point to s. We also denote by η' a parallel transport in the local system $R^2\pi'_*\mathbb{C}$ along a path from the reference point to s'. Then the composition of the isomorphisms*

$$H^2(X,\mathbb{C}) \overset{\eta}{\cong} H^2(\mathfrak{X}_s,\mathbb{C}) \overset{\phi_s}{\cong} H^2(\mathfrak{X}'_{s'},\mathbb{C}) \overset{\eta'^{-1}}{\cong} H^2(X',\mathbb{C})$$

coincides with ϕ_* which is the isomorphism induced by $\phi : X \dashrightarrow X'$.

Proof. The proof of this proposition is a mimic of the proof of [10, Theorem 5.9]. The proof consists of two steps. First, we show that there exist open sets U of $\mathrm{Def}(X)$ and U' of $\mathrm{Def}(X')$ which satisfy the the assertions (2) and (3) of Proposition 2.1. Since X and X' are bimeromorphic, we have a deformation $\mathfrak{X}_S \to S$ of X and a deformation $\mathfrak{X}'_S \to S$ of X' over a small disk S which are isomorphic to each other over the punctured disk $S \setminus 0$ by [12, Theorem 2.5]. By the universality, $\mathfrak{X}_S \to S$ is isomorphic to the base change $\mathfrak{X} \to \mathrm{Def}(X)$ by a uniquely determined morphism $S \to \mathrm{Def}(X)$. The family $\mathfrak{X}'_S \to S$ is also isomorphic to the base change of $\mathfrak{X}' \to \mathrm{Def}(X')$ by a uniquely determined morphism $S \to \mathrm{Def}(X')$. Thus there exist points $t \in \mathrm{Def}(X)$ and $t' \in \mathrm{Def}(X')$ such that the fibres $\mathfrak{X}_t$ and $\mathfrak{X}'_{t'}$ are isomorphic. Let

η be a parallel transportation of $R^2\pi_*\mathbb{C}$ along a path from the reference point to t and η' a parallel transportation of $R^2\pi'_*\mathbb{C}$ along a path from the reference point to t'. Let us consider the composition of the isomorphisms

$$(2) \qquad H^2(X,\mathbb{C}) \overset{\eta}{\cong} H^2(\mathfrak{X}_t,\mathbb{C}) \cong H^2(\mathfrak{X}'_{t'},\mathbb{C}) \overset{\eta'^{-1}}{\cong} H^2(X',\mathbb{C}).$$

By [12, Theorem 2.5], we have a birational map $\mathfrak{X}_S \dashrightarrow \mathfrak{X}'_S$ which commutes with the two projections. Moreover the restriction of this map to the special fibres coincides with ϕ. Thus the composition of the isomorphisms (2) coincides with ϕ_*. By [4, Théorèm 5 (b)], we can extend the isomorphism $\mathfrak{X}_t \cong \mathfrak{X}'_{t'}$ over open sets of $\mathrm{Def}(X)$ and $\mathrm{Def}(X')$, that is, there exist open sets U of $\mathrm{Def}(X)$ and U' of $\mathrm{Def}(X')$ such that the restriction families $\mathfrak{X} \times_{\mathrm{Def}(X)} U$ and $\mathfrak{X}' \times_{\mathrm{Def}(X')} U'$ are isomorphic and this isomorphism is compatible with the two projections $\mathfrak{X} \to \mathrm{Def}(X)$ and $\mathfrak{X}' \to \mathrm{Def}(X')$. By this construction, the restriction of the isomorphism $\tilde{\phi} : \mathfrak{X} \times_{\mathrm{Def}(X)} U \cong \mathfrak{X}' \times_{\mathrm{Def}(X')} U'$ to the fibres satisfies the assertion (3) of Proposition 2.1.

Next we show that U and U' satisfies the assertion (1) of Proposition 2.1. Let s be a point of $\bar{U}$. By [10, Theorem 4.3], the fibres $\mathfrak{X}_s$ and $\mathfrak{X}'_s$ are bimeromorphic. If $\dim H^{1,1}(\mathfrak{X}_s,\mathbb{Q}) = 0$, then $\mathfrak{X}_s$ and $\mathfrak{X}'_s$ carries neither curves nor effective divisors. Thus $\mathfrak{X}_s$ and $\mathfrak{X}'_s$ are isomorphic by [12, Proposition 2.1] and $s \in U$. Thus if $s \in \bar{U} \setminus U$ then $\dim H^{1,1}(\mathfrak{X}_s,\mathbb{C}) \geq 1$. Since $H^{1,1}(\mathfrak{X}_s,\mathbb{Q}) = H^2(\mathfrak{X}_s,\mathbb{Q}) \cap H^{2,0}(\mathfrak{X}_s,\mathbb{C})^{\perp}$, $\bar{U} \setminus U$ is contained in a union of countably many hypersurfaces. Q.E.D.

For the proof of Theorem 1.2, we also need a correspondence of deformation families of pairs. Before we state the assertion, we give a proof of the following Lemma.

Lemma 2.2. *Let X and X' be irreducible symplectic manifolds. Assume that there exists a bimeromophic map $\phi : X \dashrightarrow X'$. We also assume* $\dim H^{1,1}(X,\mathbb{Q}) = 1$ *and $q(\beta) \geq 0$ for every element β of $H^{1,1}(X,\mathbb{Q})$, where q_X is the Beauville-Bogomolov quadratic form on $H^2(X,\mathbb{C})$. Then X and X' are isomorphic.*

Proof. Since X and X' are bimeromorphic, we have an isomorphism

$$\phi_* : H^2(X,\mathbb{C}) \cong H^2(X',\mathbb{C})$$

by Lemma 2.1. Since ϕ_* respects the Beauville-Bogomolov quadratic forms and the Hodge structures, $\dim H^{1,1}(X',\mathbb{Q}) = 1$ and $H^{1,1}(X',\mathbb{Q})$ is generated by a class $\gamma \in H^{1,1}(X',\mathbb{Q})$ such that $q_{X'}(\gamma) \geq 0$, where $q_{X'}$ is the Beauville-Bogomolov quadratic form on $H^2(X',\mathbb{C})$. Let $\mathcal{C}_X$ and

$\mathcal{C}_{X'}$ be the positive cones in $H^{1,1}(X,\mathbb{R})$ and $H^{1,1}(X',\mathbb{R})$, respectively. By [10, Corollary 7.2], $\mathcal{C}_X$ and $\mathcal{C}_{X'}$ coincide with the Kähler cones of X and X', respectively. Since ϕ_* maps $\mathcal{C}_X$ to $\mathcal{C}_{X'}$, $\phi_*\alpha$ is Kähler for every Kähler class of $\alpha \in H^{1,1}(X,\mathbb{R})$. By [7, Corollary 3.3], ϕ can be extended to an isomorphism. Q.E.D.

Now we can state a correspondence of deformation families of pairs.

Proposition 2.2. *Let X be an irreducible symplectic manifold and L a line bundle on X. We also let X' be an irreducible symplectic manifold and L' a line bundle on X'. We denote the universal family of deformations of the pair (X,L) by $(\mathfrak{X}_L,\mathfrak{L})$ and the parametrizing space by $\mathrm{Def}(X,L)$. We also denote by the universal family of deformations of the pair (X',L') by $(\mathfrak{X}'_{L'},\mathfrak{L}')$ and the parameter space by $\mathrm{Def}(X',L')$. Assume that there exists a birational map $\phi : X \dashrightarrow X'$ such that $\phi_*L \cong L'$ and $q_X(L) \geq 0$, where q_X is the Beauville-Bogomolov quadratic form on $H^2(X,\mathbb{C})$. Then we have the following.*

(1) *There exist open subsets U_L of $\mathrm{Def}(X,L)$ and $U'_{L'}$ of $\mathrm{Def}(X',L')$ such that they satisfy the following diagram*

$$\begin{tikzcd} \mathfrak{X}_L \times_{\mathrm{Def}(X,L)} U_L \arrow[r, "\cong"] \arrow[d] & \mathfrak{X}'_{L'} \times_{\mathrm{Def}(X',L')} U'_{L'} \arrow[d] \\ U_L \arrow[r, "\varphi", "\cong"'] & U'_{L'} \end{tikzcd},$$

where φ is the isomorphism in the diagram of the assertion (2) of Proposition 2.1. Moreover $\mathrm{Def}(X,L)\setminus U_L$ is contained in a union of countably many hypersurfaces of $\mathrm{Def}(X,L)$ and $\mathrm{Def}(X,L)\setminus U'_{L'}$ is also contained in a union of countably many hypersurfaces of $\mathrm{Def}(X,L)$.

(2) *For every point $s \in U_L$, $(\phi_s)_*\mathfrak{L}_s \cong \mathfrak{L}'_{s'}$, where $s' = \varphi(s)$ and ϕ_s is the restriction of the isomorphism $\mathfrak{X}_L \times_{\mathrm{Def}(X,L)} U_L \to \mathfrak{X}'_{L'} \times_{\mathrm{Def}(X',L')} U'_{L'}$ to the fibres $\mathfrak{X}_{L,s}$ and $\mathfrak{X}'_{L',s'}$.*

Proof. We use the same notation in the statements and the proof of Proposition 2.1. If $U \cap \mathrm{Def}(X,L) \neq \emptyset$, then $U_L := U \cap \mathrm{Def}(X,L)$ and $U'_{L'} := U' \cap \mathrm{Def}(X',L')$ satisfies the assertion (1) and every point $s \in U_L$ satisfies the assertion of (2) because the restricted isomorphism satisfies the assertion (3) of Proposition 2.1. In order to show that $U \cap \mathrm{Def}(X,L) \neq \emptyset$, let s be a point of $\mathrm{Def}(X,L)$ such that $\dim H^{1,1}(\mathfrak{X}_s,\mathbb{Q}) = 1$, where $\mathfrak{X}_s$ is the fibre at s. We note that $\dim H^{1,1}(\mathfrak{X}_s,\mathbb{Q}) \geq 1$ for every point $s \in \mathrm{Def}(X,L)$. Since $H^{1,1}(\mathfrak{X}_s,\mathbb{Q}) = H^2(\mathfrak{X}_s,\mathbb{Q}) \cap H^{2,0}(\mathfrak{X}_s,\mathbb{C})^{\perp}$, $\dim H^{1,1}(\mathfrak{X}_s,\mathbb{Q}) = 1$ for a very general point s of $\mathrm{Def}(X,L)$. We will prove that $s \in U$. Since U is dense and open, there exists a small disk S of $\mathrm{Def}(X)$ such that $s \in S$ and $S \setminus \{s\} \subset U$. We denote $\varphi(s)$ by s' and $\varphi(S)$ by S'. If we consider the base changes $\mathfrak{X} \to \mathrm{Def}(X)$ by S and $\mathfrak{X}' \to \mathrm{Def}(X')$ by S', we obtain the following diagram:

$$\begin{tikzcd} \mathfrak{X}_{S\setminus\{s\}} \arrow[r, "\cong"] \arrow[d] & \mathfrak{X}'_{S'\setminus\{s'\}} \arrow[d] \\ S \setminus s \arrow[r, "\varphi", "\cong"'] & S' \setminus \{s'\}, \end{tikzcd}$$

By [10, Theorem 4.3], there exists a birational map $\mathfrak{X}_s \dashrightarrow \mathfrak{X}'_{s'}$. By the definition of the Beauville-Bogomolov quadratic form [4, Page 772], the function

$$\mathrm{Def}(X, L) \ni s \mapsto q_{\mathfrak{X}_s}(\mathfrak{L}_s) \in \mathbb{Z}$$

is constant, where $q_{\mathfrak{X}_s}$ stands for the Beauville-Bogomolov quadratic form on $H^2(\mathfrak{X}_s, \mathbb{C})$. Thus we have $q_X(L) = q_{\mathfrak{X}_s}(\mathfrak{L}_s) \geq 0$. By Lemma 2.2, ϕ_s is an isomorphism and we obtain the following diagram:

$$\begin{tikzcd} \mathfrak{X}_S \arrow[r, "\cong"] \arrow[d] & \mathfrak{X}'_{S'} \arrow[d] \\ S \arrow[r, "\varphi", "\cong"'] & S'. \end{tikzcd}$$

By the local Torelli theorem [4, Théorèm 5 (b)], there exist an open set U_S of S in $\mathrm{Def}(X)$ and an open set $U'_{S'}$ of S' in $\mathrm{Def}(X')$ such that the restriction of $\mathfrak{X}$ over U_S which satisfy the following diagram:

$$\begin{tikzcd} \mathfrak{X} \times_{\mathrm{Def}(X)} U_S \arrow[r, "\cong"] \arrow[d] & \mathfrak{X}' \times_{\mathrm{Def}(X')} U_{S'} \arrow[d] \\ U_S \arrow[r, "\cong"'] & U_{S'}. \end{tikzcd}$$

Since $\mathfrak{X} \to \mathrm{Def}(X)$ and $\mathfrak{X}' \to \mathrm{Def}(X')$ are locally universal, the isomorphism in the above diagram coincides with φ in the diagram (1). Thus $U_S \subset U$ and we have $s \in U$. Q.E.D.

§3. Proof of Theorem

We start with giving a numerical criterion of existence of Lagrangian fibrations.

Lemma 3.1. *Let X be an irreducible symplectic manifold and L a line bundle on X. The linear system $|L|$ defines a Lagrangian fibration over the projective space if and only if L is nef and L has the following property:*

$$\dim H^0(X, L^{\otimes k}) = \dim H^0(\mathbb{P}^{1/2 \dim X}, \mathcal{O}(k)) \tag{3}$$

for every positive integer k.

Proof. If $|L|$ defines a Lagrangian fibration over the projective space, it is trivial that L is nef and the dimension of global sections of $L^{\otimes k}$ satisfies the equation (3) by Definition 1.2. Thus we prove that $|L|$ defines a Lagrangian fibration under the assumption that L is nef and $\dim H^0(X, L^{\otimes k})$ satisfy the equation (3). By the assumption, the linear system $|L|$ defines a rational map $X \dashrightarrow \mathbb{P}^{1/2 \dim X}$. Let $\nu : Y \to X$ be a resolution of indeterminacy and $g : Y \to \mathbb{P}^{1/2 \dim X}$ is the induced morphism. Comparing $\nu^* L$ and $g^* \mathcal{O}(1)$, we have

$$\nu^* L \cong g^* \mathcal{O}(1) + F,$$

where F is an effective ν-exceptional divisor. By taking k-th multiples on both sides, we have

$$k\nu^* L \cong g^* \mathcal{O}(k) + kF.$$

If $F \neq 0$, then the above isomorphism and the equality (3) imply that L is not semiample. By the assumption, L is nef. If $L^{\dim X} \neq 0$, then L is also big and $\dim H^0(X, L^{\otimes k})$ does not satisfy the equation (3). Thus $L^{\dim X} = 0$. By [8, Theorem 4.7], we obtain

$$c_X q_X (kL + \alpha)^{1/2 \dim X} = (kL + \alpha)^{\dim X},$$

where q_X is the Beauville-Bogomolov quadratic form on $H^2(X, \mathbb{C})$, c_X is the positive constant of X and α is a Kähler class of $H^{1,1}(X, \mathbb{C})$. Comparing the degrees of both hand sides of the above equation, we obtain that the numerical Kodaira dimension $\nu(L)$ is $(1/2)\dim X$. By the equation (3), the Kodaira dimension $\kappa(L)$ is also equal to $(1/2)\dim X$. Since K_X is trivial, the equality $\nu(L) = \kappa(L)$ implies that L is semiample by [15, Theorem 6.1] and [9, Theorem 1.1]. Thus $F = 0$ and the linear system $|L|$ defines the morphism $f : X \to \mathbb{P}^{1/2 \dim X}$. The linear system $|lL|$ defines a morphism

$$f_l : X \to \operatorname{Proj} \oplus_{m \geq 0} H^0(X, L^{\otimes ml}) \cong \mathbb{P}^{\binom{n+l}{n} - 1}.$$

This morphism has connected fibres if l is sufficiently large. The morphism

$$\operatorname{Proj} \oplus_{m\geq 0} H^0(X, L^{\otimes m}) \to \operatorname{Proj} \oplus_{m\geq 0} H^0(X, L^{\otimes ml})$$

is the l-th Veronese embedding. Thus f_l is the composition of f and the Veronese embedding. This implies that f has connected fibres. Q.E.D.

We introduce a criterion which asserts local freeness of direct images of line bundles.

Lemma 3.2. *Let $\pi : \mathfrak{X}_S \to S$ be a smooth morphism over a small disk S with the reference point o. We also let $\mathfrak{L}_S$ be a line bundle on $\mathfrak{X}_S$. Assume that $\mathfrak{X}_S$ and $\mathfrak{L}_S$ satisfy the following conditions.*

(1) *The canonical bundle of every fibre is trivial.*

(2) *For every point t of $S \setminus \{o\}$, the restriction $\mathfrak{L}_{S,t}$ of $\mathfrak{L}_S$ to the fibre $\mathfrak{X}_{S,t}$ at t is semiample.*

(3) *The restriction $\mathfrak{L}_{S,o}$ of $\mathfrak{L}_S$ to the fibre $\mathfrak{X}_{S,o}$ at o is nef.*

Then the higher direct images $R^q\pi_*\mathfrak{L}_S^{\otimes k}$ are locally free for all $q \geq 0$ and $k \geq 1$. Moreover the morphisms

$$R^q\pi_*\mathfrak{L}_S^{\otimes k} \otimes k(o) \to H^q(\mathfrak{X}_{S,o}, \mathfrak{L}_{S,o}) \tag{4}$$

are isomorphic for all $q \geq 0$ and $k \geq 1$.

Proof. The first part is a special case of [24, Corollary 3.14]. By the criteria of cohomological flatness in [2, page 134], if $R^q\pi_*\mathfrak{L}_S^{\otimes k}$ is locally free and the morphism (4) is isomorphic, then the morphism

$$R^{q-1}\pi_*\mathfrak{L}_S^{\otimes k} \otimes k(o) \to H^{q-1}(\mathfrak{X}_{S,o}, \mathfrak{L}_{S,o})$$

is also isomorphic for every $k \geq 1$. If $q \geq \dim \mathfrak{X}_{S,s} + 1$, the both hand sides of the morphism (4) are zero. By a reverse induction, we obtain the last part of the assertions of Lemma 3.2. Q.E.D.

We need one more lemma to prove Theorem 1.2.

Lemma 3.3. *Let X be an irreducible symplectic manifold. We let also L be a line bundle such that $q_X(L) = 0$, where q_X is the Beauville-Bogomolov quadratic form on $H^2(X, \mathbb{C})$. Assume that X is not projective. Then $-L$ or L is nef.*

Proof. We may assume that L is belong to the closure of the positive cone in $H^2(X, \mathbb{C})$. If L is not nef, there exists a line bundle M on X such that $q_X(M, L) < 0$ and $q_X(M, \alpha) \geq 0$ for all Kähler class $\alpha \in H^2(X, \mathbb{C})$ by [10, Theorem 7.1]. If we choose a suitable rational number λ, we have $q_X(L+\lambda M) > 0$. This implies that X is projective by [11, Theorem 2]. That is a contradiction. Q.E.D.

Now we prove that if $V_{\mathrm{reg}} \neq \emptyset$ then V_{reg} is a dense open subset of $\mathrm{Def}(X,L)$.

Lemma 3.4. *We use the same notation as in Theorem 1.2. If $V_{\mathrm{reg}} \neq \emptyset$, V_{reg} is dense and open in* $\mathrm{Def}(X,L)$*. Moreover* $\mathrm{Def}(X,L)\setminus V_{\mathrm{reg}}$ *is contained in a countable union of hypersurfaces of* $\mathrm{Def}(X,L)$*.*

Proof. Let t be a point of V_{reg} and we denote by $\mathfrak{X}_t$ the fibre at t and by $\mathfrak{L}_t$ the restriction of $\mathfrak{L}$ to $\mathfrak{X}_t$. First we prove that V_{reg} is open. By the definition of V_{reg} in Theorem 1.2, the linear system $|\mathfrak{L}_t|$ defines a Lagrangian fibration $f_t : \mathfrak{X}_t \to \mathbb{P}^{1/2 \dim \mathfrak{X}_t}$. By [19, Corollary 1.3], there exists a neighbourhood U' of t such that $\mathfrak{L}$ is π-free over U', where π is the projection $\mathfrak{X} \to \mathrm{Def}(X,L)$. By [19, Theorem 1.2], there exists an open neighbourhood U_i of t such that $R^i\pi_*\mathfrak{L}$ is locally free on U_i. Let s be a point of $U' \cap (\cap_i U_i)$, $\mathfrak{X}_s$ the fibre of π at s and $\mathfrak{L}_s$ the restriction of $\mathfrak{L}$ to $\mathfrak{X}_s$. By the same argument of the proof of Lemma 3.2, a natural morphism

$$\pi_*\mathfrak{L} \otimes k(s) \to H^0(\mathfrak{X}_s, \mathfrak{L}_s)$$

is bijective. Hence $U' \cap (\cap_i U_i) \subset V_{\mathrm{reg}}$. Since $R^i\pi_*\mathfrak{L} = 0$ for $i > \dim \mathfrak{X}_t$, $U_i = \mathrm{Def}(X,L)$ for $i > \dim \mathfrak{X}_t$. This implies that V_{reg} is open. Next we prove that $\mathrm{Def}(X,L)\setminus V_{\mathrm{reg}}$ is contained in a union of countable hypersurfaces of $\mathrm{Def}(X,L)$. Since a union of real codimension two subsets cannot separate two non-empty open subsets, this implies that V_{reg} is dense. Let t' be a point of the closure of V_{reg} such that $\dim H^{1,1}(\mathfrak{X}_{t'}, \mathbb{Q}) = 1$, where $\mathfrak{X}_{t'}$ is the fibre at t'. We aim to prove that $t' \in V_{\mathrm{reg}}$. We denote by $\mathfrak{L}_{t'}$ the restriction of $\mathfrak{L}$ to $\mathfrak{X}_{t'}$. By the definition of the Beauville-Bogomolov quadratic form in [4, page 772], the function

$$\mathrm{Def}(X,L) \ni t \mapsto q_{\mathfrak{X}_t}(\mathfrak{L}_t) \in \mathbb{Z}$$

is a constant function, where $q_{\mathfrak{X}_t}$ is the Beauville-Bogomolov quadratic form on $H^2(\mathfrak{X}_t, \mathbb{C})$. Thus $q_{\mathfrak{X}_{t'}}(\mathfrak{L}_{t'}) = 0$. Since $H^{1,1}(\mathfrak{X}_{t'}, \mathbb{Q})$ is spanned by $\mathfrak{L}_{t'}$, $\mathfrak{X}_{t'}$ is not projective by [11, Theorem 2]. Thus $\mathfrak{L}_{t'}$ is nef by Lemma 3.3. We choose a small disk S in $\mathrm{Def}(X,L)$ such that $t' \in S$ and $S\setminus\{t'\} \subset V_{\mathrm{reg}}$. We also consider the restricted family $\pi_S : \mathfrak{X}_L \times_{\mathrm{Def}(X,L)} S \to S$. Then $\mathfrak{L}_{t''}^{\otimes k}$ is free for every point t'' of $S \setminus \{t'\}$ and $k \geq 1$, where $\mathfrak{L}_{t''}$ is the restriction of $\mathfrak{L}$ to the fibre $\mathfrak{X}_{t''}$ at t''. By Lemma 3.2, $(\pi_S)_*\mathfrak{L}^{\otimes k}$ is locally free and the morphism

$$(\pi_S)_*\mathfrak{L}^{\otimes k} \otimes k(t') \to H^0(\mathfrak{X}_{t'}, \mathfrak{L}_{t'}^{\otimes k})$$

is bijective for every $k \geq 1$. By Lemma 3.1, $t' \in V_{\mathrm{reg}}$. Let W be the subset of $\mathrm{Def}(X, L)$ defined by

$$W := \{t \in \mathrm{Def}(X, L); \dim H^{1,1}(\mathfrak{X}_t, \mathbb{Q}) \geq 2\}.$$

By the above argument, $\mathrm{Def}(X, L) \setminus V_{\mathrm{reg}} \subset W$. Let $\omega_{\mathfrak{X}_t}$ be a symplectic form on $\mathfrak{X}_t$. Since $H^{1,1}(\mathfrak{X}_t, \mathbb{Q}) = H^2(\mathfrak{X}_t, \mathbb{Q}) \cap H^{2,0}(\mathfrak{X}_t, \mathbb{C})^{\perp}$, W is contained in a union of countable hypersurfaces of $\mathrm{Def}(X, L)$ and we are done. Q.E.D.

We give a proof of Theorem 1.2.

Proof of Theorem 1.2. The proof consists of three parts. We start by proving the following Claim.

Claim 3.1. *If $V \neq \emptyset$, then $V_{\mathrm{reg}} \neq \emptyset$.*

Proof. We may assume that the reference point o of $\mathrm{Def}(X, L)$ is contained in V. By Definition 1.2, there exists a birational map $\phi : X \dashrightarrow X'$ such that the linear system $|\phi_* L|$ defines a Lagrangian fibration $X' \to \mathbb{P}^{1/2 \dim X}$. Let $L' := \phi_* L$ and $(\mathfrak{X}'_{L'}, \mathfrak{L}')$ be the universal family of deformations of the pair (X', L'). Let V'_{reg} be the locus of Lagrangian fibration of $\mathrm{Def}(X', L')$. Then the reference point o' of $\mathrm{Def}(X', L')$ is contained in V'_{reg}. By Lemma 3.4, V'_{reg} is a dense open set of $\mathrm{Def}(X', L')$. By Proposition 2.2, we also have dense open sets $U'_{L'}$ of $\mathrm{Def}(X', L')$ and U_L of $\mathrm{Def}(X, L)$ which satisfy the following diagram:

$$\begin{tikzcd}
\mathfrak{X}_L \times_{\mathrm{Def}(X,L)} U_L \arrow[r, "\cong"] \arrow[d] & \mathfrak{X}'_{L'} \times_{\mathrm{Def}(X',L')} U_{L'} \arrow[d] \\
U_L \arrow[r, "\varphi", "\cong"'] & U'_{L'}
\end{tikzcd}$$

By the assertion (2) of Proposition 2.2, $\varphi^{-1}(U'_{L'} \cap V'_{\mathrm{reg}}) \subset V_{\mathrm{reg}}$. Since $U'_{L'} \cap V'_{\mathrm{reg}} \neq \emptyset$, we obtain $V_{\mathrm{reg}} \neq \emptyset$. Q.E.D.

By Claim 3.1 and Lemma 3.4, $\mathrm{Def}(X, L)$ coincides with the closure of V_{reg} under the assumption that $V \neq \emptyset$.

Claim 3.2. *Assume that the reference point o of $\mathrm{Def}(X, L)$ is contained in the closure of V_{reg} and L is nef. Then $o \in V_{\mathrm{reg}}$.*

Proof. By the assumption that $o \in \overline{V}_{\mathrm{reg}}$, we choose a small disk S in $\mathrm{Def}(X, L)$ which has the following properties:

(1) $o \in S$.
(2) $S \setminus \{o\} \subset V_{\mathrm{reg}}$.

Let $\pi_S : \mathfrak{X}_L \times_{\mathrm{De}(X,L)} S \to S$ be the restriction family and $\mathfrak{L}_S$ the restriction of the universal bundle $\mathfrak{L}$ to $\mathfrak{X}_L \times_{\mathrm{Def}(X,L)} S$. Then $\pi_S : \mathfrak{X}_L \times_{\mathrm{Def}(X,L)} S \to S$ and $\mathfrak{L}_S$ satisfy the all assumptions of Lemma 3.2. Hense $\pi_* \mathfrak{L}_S^{\otimes k}$ are locally free and the morphisms

$$\pi_* \mathfrak{L}_S^{\otimes k} \otimes k(s) \to H^0(\mathfrak{X}_s, \mathfrak{L}_s^{\otimes k})$$

are isomorphic for all $k \geq 0$ and all points $s \in S$. Let s be a point of $S \setminus \{o\}$. Since s is contained in V_{reg},

$$\dim H^0(\mathfrak{X}_s, \mathfrak{L}_s^{\otimes k}) = \dim H^0(\mathbb{P}^{1/2 \dim X}, \mathcal{O}(k))$$

for all $k \geq 0$ by Lemma 3.1. This implies that

$$\dim H^0(X, L^{\otimes k}) = \dim H^0(\mathbb{P}^{1/2 \dim X}, \mathcal{O}(k)),$$

for all $k \geq 0$. Hence the pair (X, L) satisfies the all assumptions of Lemma 3.1 and we obtain $o \in V_{\mathrm{reg}}$. Q.E.D.

Claim 3.3. *Assume that the reference point o of* $\mathrm{Def}(X, L)$ *is contained in the closure of* V_{reg}*,* $c_1(L)$ *belongs to the closure of the birational Kähler cone and L is not nef. Then $o \in V$.*

Proof. We remark that X is projective by Lemma 3.3. We consider the same restriction family $\pi : \mathfrak{X}_L \times_{\mathrm{Def}(X,L)} S \to S$ in the proof of Claim 3.2. By the upper semicontinuity of the function

$$s \in S \mapsto \dim H^0(\mathfrak{X}_s, \mathfrak{L}_s),$$

$\mathfrak{L}_o = L$ is effective. By [23, Theorem 1.2], there exists another irreducible symplectic manifold X' and a birational map $\phi : X \dashrightarrow X'$ such that L' is nef, where $L' = \phi_* L$. By Proposition 2.2, we have dense open sets $U'_{L'}$ of $\mathrm{Def}(X', L')$ and U_L of $\mathrm{Def}(X, L)$ which satisfy the following diagram:

$$\begin{tikzcd} \mathfrak{X}_L \times_{\mathrm{Def}(X,L)} U_L \arrow[r, "\cong"] \arrow[d] & \mathfrak{X}'_{L'} \times_{\mathrm{Def}(X',L')} U_{L'} \arrow[d] \\ U_L \arrow[r, "\varphi", "\cong"'] & U'_{L'} \end{tikzcd}$$

Let V'_{reg} be the locus of Lagrangian fibrations of $\mathrm{Def}(X', L')$. Then $V'_{\mathrm{reg}} \neq \emptyset$ because the image $\varphi(V_{\mathrm{reg}} \cap U_L)$ is contained in V'_{reg} by Proposition 2.2 (2). By Lemma 3.4, V'_{reg} is dense. Hence the reference point o' of $\mathrm{Def}(X', L')$ is contained in the closure of V'_{reg}. By Claim 3.2, $o' \in V'_{\mathrm{reg}}$. This implies that $o \in V$. Q.E.D.

We finish the proof of Theorem 1.2. If $V \neq \emptyset$, V_{reg} is open and dense in $\mathrm{Def}(X, L)$ by Claim 3.1. Thus every point s of $\mathrm{Def}(X, L)_{\text{mov}}$ is contained in the closure of V_{reg}. Then $s \in V$ by Claim 3.3 and we are done. Q.E.D.

Proof of Corollary 1.1. We use the same notation of Theorem 1.2 and Corollary 1.1. We also define the subset Λ of $\mathrm{Def}(X)$ by

$$\Lambda := \{s \in \mathrm{Def}(X);\quad \mathfrak{X}_s \quad \begin{array}{l}\text{is isomorphic to the } n\text{-pointed Hilbert scheme}\\ \text{of } K3 \text{ or a generalized Kummer variety } \}\end{array}$$

First we will prove the following.

Lemma 3.5. *Let X be the n-pointed Hilbert scheme of a $K3$ surface or a generalized Kummer variety and L an isotropic line bundle on X. If $c_1(L)$ belongs to the closure of the birational Kähler cone, then L defines a Lagrangian fibration.*

Proof. We first assume that X is the n-pointed Hilbert scheme of a $K3$ surface S. We may assume that $c_1(L)$ is primitive in $H^2(X, \mathbb{Z})$. If S is projective, X is also projective and the assertion of Lemma is a part of [3, Conjecture 1.4, Theorem 1.5 and Remark 11.4] or [18, Theorem 1.3 and Theorem 6.3]. Thus we assume that S is not projective. By [4, Proposition 6 and Remarque], we have an injection $\imath : H^2(S, \mathbb{C}) \to H^2(X, \mathbb{C})$ such that

$$H^2(X, \mathbb{Z}) \cong \imath(H^2(S, \mathbb{Z})) \oplus \mathbb{Z}\delta,$$

where δ is the half of the cohomology class of the exceptional divisor of the Hilbert-Chow morphism $S^{[n]} \to S^{(n)}$. Thus $c_1(L)$ is represented by

$$c_1(L) = a\iota(c_1(L_S)) + b\delta,$$

where L_S is a line bundle on S. Since S is not projective, $L_S^2 \leq 0$. We have $q_X(\delta^2) = -2(n-1)$, $q_X(\iota(c_1(L_S)), \delta) = 0$ and $q_X(L) = 0$ where q_X is the Beauville-Bogomolov quadratic form. Thus we have $b = 0$ and $L_S^2 = 0$. Since $c_1(L)$ is primitive in $H^2(X, \mathbb{Z})$, we have $a = 1$ and $c_1(L_S)$ is primitive in $H^2(S, \mathbb{Z})$. By Lemma 3.3, L_S is nef. Hence S admits an elliptic fibration induced by the linear system of $|L_S|$. This implies that X admits a Lagrangian fibration which is induced by the linear system $|L|$. Next we assume that X is isomorphic to a generalizd Kummer variety associated with a complex two-dimensional torus A in [4, Théorème 4]. If A is projective, the assertion of Lemma is a part of [26, Proposition 3.36]. If A is not projective, we obtain the assertion of Lemma by the same argument in the case that X is isomorphic to the

n-pointed Hilbert scheme of a non-projective $K3$ surface if we replace [4, Proposition 6] by [4, Proposition 8]. Q.E.D.

Assume that $\Lambda \cap \mathrm{Def}(X, L)_{\mathrm{mov}} \neq \emptyset$. For a point $s \in \Lambda \cap \mathrm{Def}(X, L)_{\mathrm{mov}}$, we let $\mathfrak{L}_s$ the restriction of the universal bundle $\mathfrak{L}$ to the fibre $\mathfrak{X}_s$ at s. Then $\mathfrak{L}_s$ defines a rational Lagrangian fibration by Lemma 3.5, and consequently, Theorem 1.2 implies the claim.

Next we will prove the following Lemma.

Lemma 3.6. *The set* $\Lambda \cap \mathrm{Def}(X, L)$ *is dense in* $\mathrm{Def}(X, L)$.

Proof. Let $p_X : \mathrm{Def}(X) \to \mathbb{P}H^2(X, \mathbb{C})$ be the period map and Ω_X the period domain defined by

$$\Omega_X := \{\alpha \in \mathbb{P}H^2(X, \mathbb{C}); q_X(\alpha) = 0, q_X(\alpha + \bar{\alpha}) > 0\},$$

where q_X is the Beauville-Bogomolov quadratic form. We need the following claim.

Claim 3.4. (1) *The preimage of* $\Omega_X \cap c_1(L)^{\perp}$ *via* p_X *is contained in* $\mathrm{Def}(X, L)$.

(2) *For* $t \in \Lambda$*, there exists an element* $\delta_t \in H^2(\mathfrak{X}_t, \mathbb{Z}) \cap H^{1,1}(\mathfrak{X}_t, \mathbb{C})$ *and an isomorphism* $u_t^* : H^2(X, \mathbb{C}) \to H^2(\mathfrak{X}_t, \mathbb{C})$ *which respects the Beauville-Bogomolov quadratic forms such that the preimage of* $\Omega_X \cap (u_t^*)^{-1}(\delta_t^{\perp})$ *via* p_X *is contained in* Λ.

Proof. (1) This is a part of [4, Corollarie 1].
(2) The proof consists of four steps. From the first step to the third step, we assume that X is of $K3^{[n]}$-type.
Step 1. We define an element δ_t and an isomorphism u_t^*. By the definition of Λ, there exists a $K3$ surface S such that $\mathfrak{X}_t \cong S^{[n]}$. By [4, Proposition 6], we have an injection

$$H^2(S, \mathbb{C}) \to H^2(\mathfrak{X}_t, \mathbb{C}). \tag{5}$$

which respects the Hodge structures. Moreover the image of the above injection coincides with $(\delta_t)^{\perp}$ where δ_t is the half of the cohomology class of the exceptional divisor of the Hilbert-Chow morphism $S^{[n]} \to S^{(n)}$. Since $\mathfrak{X} \to \mathrm{Def}(X)$ is smooth, we have a diffeomorphism

$$u : X \times \mathrm{Def}(X) \to \mathfrak{X} \tag{6}$$

which is compatible with the projections. We denote by u_t the restriction of this diffeomorphism to the fibres at t. Then u_t induces the isomorphism

$$u_t^* : H^2(X, \mathbb{C}) \to H^2(\mathfrak{X}_t, \mathbb{C})$$

Step 2. We give a relationship between $\mathrm{Def}(X)$ and $\mathrm{Def}(\mathfrak{X}_t)$. Let us consider the diffeomorphism

$$\mathfrak{X}_t \times \mathrm{Def}(X) \overset{u_t^{-1}\times \mathrm{id}}{\to} X \times \mathrm{Def}(X) \to \mathfrak{X}.$$

For each point s of $\mathrm{Def}(X)$, we have a diffeomorphism $u'_s : \mathfrak{X}_t \to \mathfrak{X}_s$ which is the restriction of the above diffeomorphism to the fibres at s. The pull back of $(u'_s)^* H^{2,0}(\mathfrak{X}_s, \mathbb{C})$ gives the period map $\mathrm{Def}(X) \to \mathbb{P}H^2(\mathfrak{X}_t, \mathbb{C})$ which satisfies the following diagram:

$$\begin{tikzcd} & \mathbb{P}H^2(X,\mathbb{C}) \arrow[dd, "u_t^*"] \\ \mathrm{Def}(X) \arrow[ur, "p_X"] \arrow[dr, "p'"'] & \\ & \mathbb{P}H^2(\mathfrak{X}_t,\mathbb{C}) \end{tikzcd}$$

Let $\Omega_{\mathfrak{X}_t}$ be the period domain defined by

$$\Omega_{\mathfrak{X}_t} := \{\alpha \in \mathbb{P}H^2(\mathfrak{X}_t, \mathbb{C}); q_{\mathfrak{X}_t}(\alpha) = 0, q_{\mathfrak{X}_t}(\alpha + \bar{\alpha}) > 0\}.$$

Since u_t^* respects the Beauville-Bogomolov quadratic forms, u_t^* defines an isomorphism between Ω_X and $\Omega_{\mathfrak{X}_t}$. Then p' is also locally isomorphic, because p_X is locally isomorphic, This implies that $\mathfrak{X} \to \mathrm{Def}(X)$ is a universal family of local deformations of $\mathfrak{X}_t$ with the reference point t by [16, (5.7) Corollary]. Hence we have an isomorphism $v_t : \mathrm{Def}(X) \to \mathrm{Def}(\mathfrak{X}_t)$ such that the pull back of the Kuranishi family of $\mathfrak{X}_t$ via v_t is isomorphic to $\mathfrak{X}$. Let $p_{\mathfrak{X}_t}$ be the period map $\mathrm{Def}(\mathfrak{X}_t) \to \Omega_{\mathfrak{X}_t}$. Since $p_{\mathfrak{X}_t} \circ v_t = p'$, v_t satisfies the following diagram:

$$\begin{tikzcd} \mathrm{Def}(X) \arrow[r, "v_t"] \arrow[d, "p_X"'] & \mathrm{Def}(\mathfrak{X}_t) \arrow[d, "p_{\mathfrak{X}_t}"] \\ \mathbb{P}H^2(X,\mathbb{C}) \arrow[r, "u_t^*"'] & \mathbb{P}H^2(\mathfrak{X}_t,\mathbb{C}), \end{tikzcd}$$

Step 3. We prove that u_t^* and δ_t have the properties in the assertion of Claim. By [8, Theorem 4.7], the cup products and the $(1/2 \dim X)$-th power of the Beauville-Bogomolov quadratic form are proportional. Since u_t^* respects the cup products, u_t^* respects also the Beauville-Bogomolov quadratic forms. Let $\mathfrak{S}$ be the Kuranishi family of $\mathrm{Def}(S)$.

We also let $\mathfrak{S}^{[n]}$ be a crepant resolution of the relative symmetric n-th product $\mathfrak{S}^{(n)}$. By [4, Proposition 10], there exists an injection $\jmath$: $\mathrm{Def}(S) \to \mathrm{Def}(\mathfrak{X}_t)$ such that the pull back of the Kuranishi family of $\mathfrak{X}_t$ by $\jmath$ is isomorphic to $\mathfrak{S}^{[n]}$. Thus the image of $\mathrm{Def}(S)$ by $(v_t)^{-1} \circ \jmath$ is contained in Λ. By [4, Proposition 10], the image of $p_{\mathfrak{X}_t} \circ \jmath$ coincides with $\Omega_{\mathfrak{X}_t} \cap \delta_t^{\perp}$. This implies that the preimage of $\Omega \cap (u_t^*)^{-1}(\delta_t^{\perp})$ is contained in Λ.

Step 4. We consider the case that X is of generalized Kummer-type. By [4, Page 781, The second paragraph], [4, Lemma 1] and [4, Proposition 10] hold for a complex two-dimensional torus A and a generalized Kummer variety associated with A. Thus if we replace [4, Proposition 6] by [4, Proposition 8] and δ_t by F in [4, Proposition 8], the same argument works in the case that X is of generalized Kummer-type. Q.E.D.

We go back to the proof of Lemma 3.6. Let t_0 be a point of $\mathrm{Def}(X, L)$. The subset Λ is dense in $\mathrm{Def}(X)$ by [17, Theorem 1.1 and Theorem 4.1]. Thus there exists a sequence $t_m \in \Lambda$ such that $\lim_{m\to\infty} t_m = t_0$. By perturbing the sequence $\{p_X(t_m)\}$, we will construct a sequence of points $\{t'_m\}$ of Ω_X which has the following three properties:

(1) $\lim_{m\to\infty} t'_m = p_X(t_0)$.
(2) $t'_m \in \Omega_X \cap c_1(L)^{\perp}$ for $m \gg 0$.
(3) $t'_m \in \Omega_X \cap (u_m^*)^{-1}(\delta_m^{\perp})$ for $m \gg 0$, where u_m^* and δ_m are the same objects in Claim 3.4 for $\mathfrak{X}_{t_m}$.

Let G_2^+ be the set of oriented two-dimensional subspaces of $H^2(X, \mathbb{R})$ on which the restriction of q_X is positive definite. To construct the sequence, we need a bijection between Ω_X and G_2^+. For a point $t \in \Omega_X$, we denote by $[t]$ an element of $H^2(X, \mathbb{C})$ such that $\mathbb{C}[t]$ defines the line corresponding to the point $t \in \mathbb{P}H^2(X, \mathbb{C})$. The subspace of $H^2(X, \mathbb{R})$ spanned by $\mathrm{Re}[t]$ and $\mathrm{Im}[t]$ gives an element of G_2^+. On the contrary, for an element T of G_2^+, we have an oriented orthogonal basis $\{\omega_1, \omega_2\}$ and the line spanned by $\omega_1 + \sqrt{-1}\omega_2$ gives a point of Ω_X. Let T_m, $(m \ge 0)$ be the oriented two dimensional linear subspace $H^2(X, \mathbb{R})$ corresponding to $p_X(t_m)$, $(m \ge 0)$. We fix a base $\{p_0, q_0\}$ of T_0 and choose a sequence of bases$\{p_m, q_m\}$ of T_m such that $\lim_{m\to\infty} p_m = p_0$ and $\lim_{m\to\infty} q_m = q_0$. We fix an element $\alpha \in H^2(X, \mathbb{Z})$ such that $q_X(\alpha, c_1(L)) \neq 0$. For sequences $\{p_m, q_m\}$, $\{\delta_m\}$ and $\{u_m^*\}$ we define the following five sequences:

(1) $\lambda_m = q_X(p_m, c_1(L))/q_X(\alpha, c_1(L))$.
(2) $\mu_m = q_X(q_m, c_1(L))/q_X(\alpha, c_1(L))$.
(3) $\gamma_m = q_X(\alpha, \delta_m)$.
(4) $a_m = q_X(p_0, ((u_m^*)^{-1}(\delta_m)))$.

(5) $b_m = q_X(q_0, ((u_m^*)^{-1}(\delta_m)))$.

We may assume that the vectors (λ_m, μ_m) and (a_m, b_m) are linearly independent after perturbing $\{p_m, q_m\}$ if necessary. We define two sequences $\{r_m\}$ and $\{s_m\}$ as follows:

$$\gamma_m \frac{\lambda_m r_m + \mu_m s_m}{a_m r_m + b_m s_m} \geq 0 \tag{7}$$

Let T'_m be the linear subspace spanned by the following two elements:

$$\begin{cases} e_1 := p_m - \lambda_m \alpha + \frac{\lambda_m \gamma_m}{a_m r_m + b_m s_m}(r_m p_0 + s_m q_0), \\ e_2 := q_m - \mu_m \alpha + \frac{\mu_m \gamma_m}{a_m r_m + b_m s_m}(r_m p_0 + s_m q_0), \end{cases}$$

Since $t_0 \in \mathrm{Def}(X, L)$, $T_0 \subset c_1(L)^\perp$. Moreover $T_m \subset (u_m^*)^{-1}(\delta_m^\perp)$ because $\delta_m \in H^2(\mathfrak{X}_{t_m}, \mathbb{Z}) \cap H^{1,1}(\mathfrak{X}_{t_m}, \mathbb{C})$ and T_m corresponds to the period point $p_X(t_m)$. Hence $T'_m \subset c_1(L)^\perp \cap \delta_m^\perp$. Since $\lim_{m\to\infty} p_m = p_0$ and $\lim_{m\to\infty} q_m = q_0$, $\lim_{m\to\infty} \lambda_m = \lim_{m\to\infty} \mu_m = 0$. This implies that $\lim_{m\to\infty} T'_m \subset T_0$. We show that $\dim T'_m = 2$ for $m \gg 0$. Let us consider the following two elements:

$$\begin{cases} f_1 := p_0 + \frac{\lambda_m \gamma_m}{r_m a_m + s_m b_m}(r_m p_0 + s_m q_0), \\ f_2 := q_0 + \frac{\mu_m \gamma_m}{r_m a_m + s_m b_m}(r_m p_0 + s_m q_0), \end{cases} .$$

The above elements are linearly independent if and only if the determinant of

$$\begin{pmatrix} 1 + \frac{\lambda_m \gamma_m r_m}{r_m a_m + s_m b_m} & \frac{\lambda_m \gamma_m s_m}{r_m a_m + s_m b_m} \\ \frac{\mu_m \gamma_m r_m}{r_m a_m + s_m b_m} & 1 + \frac{\mu_m \gamma_m s_m}{r_m a_m + s_m b_m} \end{pmatrix}$$

is not zero. The determinant of the above matrix is equal to

$$1 + \gamma_m \frac{\lambda_m r_m + \mu_m s_m}{a_m r_m + b_m s_m}.$$

This is greater than 1 by the inequality (7). Thus f_1 and f_2 are linearly independent in $H^2(X, \mathbb{R})$. By the direct calculation, we have

$$\begin{cases} e_1 - f_1 = p_m - p_0 - \lambda_m \alpha \\ e_2 - f_2 = q_m - q_0 - \mu_m \alpha \end{cases} .$$

Since $\lim_{m\to\infty} p_m = p_0$, $\lim_{m\to\infty} q_m = q_0$ and $\lim_{m\to\infty} \lambda_m = \lim_{m\to\infty} \mu_m = 0$, these differences are arbitrary small for $m \gg 0$. This implies that $\dim T'_m = 2$ for $m \gg 0$. Therefore we have $\lim_{m\to\infty} T'_m = T_0$. Since the restriction of q_X on T_0 is positive definite, the restriction of q_X on T'_m is also positive definite for $m \gg 0$. Let t'_m be the period point corresponding to T'_m. Since $T'_m \subset c_1(L)^{\perp} \cap (u_m^*)^{-1}(\delta_m^{\perp})$ and T_0 corresponds to $p_X(t_0)$, the sequence $\{t'_m\}$ has the properties (1), (2) and (3).

Since $\lim_{m\to\infty} t'_m = t_0$, t'_m is contained in the image of p_X for $m \gg 0$. By Claim 3.4, the sequence of the preimage of $\{t'_m\}$ via p_X gives a sequence of points of $\Lambda \cap \mathrm{Def}(X, L)$ which converge at t_0. Q.E.D.

Finally we will prove the following Lemma.

Lemma 3.7. *Under the same assumptions and notation of Theorem 1.2, the closure of* $\mathrm{Def}(X, L) \setminus \mathrm{Def}(X, L)_{\mathrm{mov}}$ *is a proper closed subset of* $\mathrm{Def}(X, L)$.

Proof. We derive a contradiction assuming that the closure of $\mathrm{Def}(X, L) \setminus \mathrm{Def}(X, L)_{\mathrm{mov}}$ coincides with $\mathrm{Def}(X, L)$. For a point $s \in \mathrm{Def}(X, L) \setminus \mathrm{Def}(X, L)_{\mathrm{mov}}$, we denote by $\mathfrak{L}_s$ the restriction of the universal bundle $\mathfrak{L}$ to the fibre $\mathfrak{X}_s$ at s. We will prove that $\mathfrak{L}_s$ is big. By Corollary [10, Corollary 3.10], the interior of the positive cone of an irreducible symplectic manifold is contained in the effective cone. By the assumption, L belongs to the closure of the positive cone of X. Hence $\mathfrak{L}_s$ also belongs to the closure of the positive cone of $\mathfrak{X}_s$. Thus $\mathfrak{L}_s$ is pseudo-effective. By [23, Theorem 3.1], We obtain the q-Zariski decomposition

$$\mathfrak{L}_s = P_s + N_s,$$

which has the following three properties:

(1) The Chern class $c_1(P_s)$ belongs to the closure of the birational Kähler cone.
(2) The divisor N_s is effective or trivial. If N_s is not trivial, then the matrix $(q_{\mathfrak{X}_s}(N_i, N_j))_{1\le i,j\le n}$ is negative definite, where N_i, $(1 \le i \le n)$ is an irreducible component of N_s.
(3) $q_{\mathfrak{X}_s}(P_s, N_i) = 0$ for all i.

By the above properties, we have

$$0 = q_{\mathfrak{X}_s}(\mathfrak{L}_s) = q_{\mathfrak{X}_s}(P_s + N_s) = q_{\mathfrak{X}_s}(P_s) + q_{\mathfrak{X}_s}(N_s).$$

Since $\mathfrak{L}_s$ does not belong to the closure of the birational Kähler cone of $\mathfrak{X}_s$, $N_s \neq 0$. This implies that $q_{\mathfrak{X}_s}(P_s) > 0$. We deduce P_s is big by [10, Corollary 3.10]. Hence $\mathfrak{L}_s$ is also big.

Let us consider the following function

$$\mathrm{Def}(X, L) \ni s \mapsto h_n(s) := \dim H^0(\mathfrak{X}_s, \mathfrak{L}_s^n) \in \mathbb{Z}.$$

By the upper semicontinuity of $h_n(s)$, there exists an open set W of $\mathrm{Def}(X,L)$ such that for every point s of W,

$$h_n(t) \geq h_n(s)$$

for all points $t \in \mathrm{Def}(X,L)$. By the assumption that the closure of $\mathrm{Def}(X,L) \setminus \mathrm{Def}(X,L)_{\mathrm{mov}}$ coincides with $\mathrm{Def}(X,L)$, $W \cap (\mathrm{Def}(X,L) \setminus \mathrm{Def}(X,L)_{\mathrm{mov}}) \neq \emptyset$. In the first half of the proof of this Lemma, we have proved that $\mathfrak{L}_s$ is big for every point $s \in \mathrm{Def}(X,L) \setminus \mathrm{Def}(X,L)_{\mathrm{mov}}$. This implies that $\mathfrak{L}_t$ is big for every point $\mathrm{Def}(X,L)$. Let t be a point of $\mathrm{Def}(X,L)$ such that $\dim H^{1,1}(\mathfrak{X}_t, \mathbb{Q}) = 1$. Then $\mathfrak{L}_t$ is nef by Lemma 3.3. Since $\mathfrak{L}_t$ is nef and big, the higher cohomologies of $\mathfrak{L}_t$ vanishes. By the Riemann-Roch formula in [10, (1.11)], we obtain

$$\dim H^0(\mathfrak{X}_t, \mathfrak{L}_t^m) = \sum_{j=0}^{\dim \mathfrak{X}_t/2} \frac{a_j}{2j} m^{2j} q_{\mathfrak{X}_t}(\mathfrak{L}_t)^j = \chi(\mathcal{O}_{\mathfrak{X}_t}),$$

because $q_{\mathfrak{X}_t}(\mathfrak{L}_t) = q_X(L) = 0$. That is a contradiction. Q.E.D.

We finish the proof of Corollary 1.1. If $\Lambda \cap \mathrm{Def}(X,L)_{\mathrm{mov}} = \emptyset$, $\mathrm{Def}(X,L) \setminus \mathrm{Def}(X,L)_{\mathrm{mov}}$ contains dense subsets of $\mathrm{Def}(X,L)$. This contradicts Lemma 3.7. Q.E.D.

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