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Existence of crepant resolutions

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Dedicated to Professor Yujiro Kawamata for his 60th birthday.

Abstract.

Let G be a finite subgroup of $\mathrm{SL}(n, \mathbb{C})$, then the quotient $\mathbb{C}^n/G$ has a Gorenstein canonical singularity. If $n = 2$ or 3, it is known that there exist crepant resolutions of the quotient singularity. In higher dimension, there are many results which assume existence of crepant resolutions. However, few examples of crepant resolutions are known. In this paper, we will show several trials to obtain crepant resolutions and give a conjecture on existence of crepant resolutions.

§1. Introduction

Let X be an irreducible algebraic variety over $\mathbb{C}$, and consider a crepant resolution:

Definition 1.1. A resolution $\pi : \widetilde{X} \to X$ is called a *crepant resolution* if the canonical isomorphism $\pi^* K_U \cong K_{\widetilde{U}}$ over $\widetilde{U} = \pi^{-1}(U) \subset \widetilde{X}$ extends to a bundle isomorphism $\pi^* K_X \cong K_{\widetilde{X}}$ over X, where $U \subset X$ is a non-singular open subset such that $\pi : \widetilde{U} = \pi^{-1}(U) \to U$ is an isomorphism.

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Let G be a finite subgroup of $\mathrm{SL}(n, \mathbb{C})$ and $X := \mathbb{C}^n/G$. The canonical bunle K_X is trivial, and if X has a crepant resolution, then $K_{\widetilde{X}} \cong \mathcal{O}_{\widetilde{X}}$ holds.

When $n = 2$, the quotient X has an isolated hypersurface singularity which is called a rational double point. The minimal resolution is a crepant resolution.

In the case $n = 3$, the quotient singularity is a canonical Gorenstein singularity (but not terminal), and a crepant resolution was constructed by Roan, Markushevich and Ito by using classification of the finite subgroups of $\mathrm{SL}(3, \mathbb{C})$ (cf. [12, 20, 23]). Moreover, the Euler number of $\widetilde{X}$ is just the number of conjugacy classes of G. In dimension two, it is known that the Euler number of the minimal resolution of the quotient singularity $\mathbb{C}^/G$ for $G \subset \mathrm{SL}(2, \mathbb{C})$ is the number of irreducible representation of G that is the same as the number of congulacy classes. Therefore, the formula of the Euler number in dimension three was thought as a generalization of the McKay correspondence.

If $n \geq 4$, then very few crepant resolutions are known. We can construct a resolution of an abelian quotient singularity via toric geometry, but it is not so easy to get a crepant resolution. Dais, Henk and Ziegler obtained some conditions on G to have a crepant resolution of the abelian quotient singularity ([6]). In addition, when the group G is a finite subgroup of $\mathrm{Sp}(n, \mathbb{C})$, the quotient $\mathbb{C}^{2n}/G$ has a symplectic singularity. It was proved that if the quotient admits a crepant resolution, then G is generated by symplectic reflections by Kaledin ([17]) and Verbitsky ([24]).

After the construction of crepant resolutions in dimension three, G-Hilbert scheme was introduced as a crepant resolution of $\mathbb{C}^n/G$ in dimension 2 by Ito and Nakamura ([16]) and generalized to 3-dimensional cases of abelian groups in $\mathrm{SL}(3, \mathbb{C})$ by Nakamura([22]). Moreover, Bridgeland, King and Reid proved that G-Hilbert schemes are crepant resolutions for any finite subgroup in $\mathrm{SL}(3, \mathbb{C})$ and constructed a generalized McKay correspondence as an equivalence of derived categories([2]).

The G-Hilbert scheme is the moduli space of G-clusters, where a *G-cluster* $\mathcal{Z}$ is a G-invariant subscheme $\mathcal{Z} \subset \mathbb{C}^n$ such that $H^0(\mathcal{O}_Z) \cong \mathbb{C}[G]$ the regular representation of G as $\mathbb{C}[G]$-modules. It was convenient tool for explaining the McKay correspondence. The generalized McKay correspondence was studied more after [2]. It is known that the Euler number of a crepant resolution is always the number of conjugacy classes of G ([1]). More generalized McKay correspondence via derived category was also known for abelian cases in arbitrary dimension by Kawamata ([18], Theorem 4.2), but we need to assume the existence of crepant resolutions to state these higher dimensional McKay correspondences.

Now, let us consider when crepant resolution exist in higher dimension. The G-Hilbert scheme is obtained uniquely from the construction and it may be related with a crepant resolution in higher dimension. When G is abelian, the second author showed that the G-Hilbert scheme in dimension 2 or 3 can be constructed as a toric variety determined by the Gröbner fan ([12], [14]). As the generalization of these results, the G-Hilbert scheme can be obtained in terms of Gröbner basis when G is a finite abelian subgroup of $\mathrm{GL}(n, \mathbb{C})$ as follows:

Theorem 1.2. (=Theorem 2.16) Let G be a finite abelian subgroup of $\mathrm{GL}(n, \mathbb{C})$, then the following hold.

(i) $\{W(\mathcal{G})\}$ is an affine covering of Hilb^G.

(ii) $Fan(G) = GF(I_G)$, therefore the normalization of Hilb^G is isomorphic to $T(GF(I_G))$.

In this theorem, Hilb^G is the component of G-Hilbert scheme which dominate $\mathbb{C}^n/G$ and $T(GF(I_G))$ is the toric variety given by the Gröbner fan of a G-invariant ideal I_G. By this theorem, we can see the structure of Hilb^G by convex geometry and check the smoothness and the crepantness by the simplicity and the structure of the cones in this Gröbner fan $GF(I_G)$.

When the group G is non-abelian, we cannot use toric geometry. However, if it is possible to divide the group into its abelian subgroups, we may obatin a crepant resolution as a combination of toric resolutions for the abelian subgroups. The second author constructed a crepant resolution by this method for 3-dimensional non-abelian (trihedral) groups and we will show you 4-dimensional examples.

Theorem 1.3. (=Theorem 3.1) Let $H \subset \mathrm{SL}(4, \mathbb{C})$ be one of the following groups:

(i) $H = \left\langle \frac{1}{n}(1,0,0,-1), \frac{1}{n}(0,1,0,-1), \frac{1}{n}(0,0,1,-1) \right\rangle$, where $n \in \mathbb{Z}$, $n \geq 2$.

(ii) $H = \left\langle \frac{1}{2n}(1, n-1, 1, n-1), \frac{1}{2}(0,1,0,1) \right\rangle$, where $n \in \mathbb{Z}$, $n \geq 1$.

(iii) $H = \left\langle \frac{1}{4n}(1, 2n-1, 1, 2n-1) \right\rangle$, where $n \in \mathbb{Z}$, $n \geq 2$.

(iv) $H = \left\langle \frac{1}{m}(1,-1,0,0), \frac{1}{n}(0,0,1,-1) \right\rangle$, where $m, n \in \mathbb{Z}$, $m \geq 3$, $n \geq 2$.

Let $K \subset \mathrm{SL}(4, \mathbb{C})$ be a group generated by

$$T = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \in \mathrm{SL}(4, \mathbb{C}),$$

and $G = \langle H, K \rangle$. Then G is a non-abelian finite subgroup of $\mathrm{SL}(4, \mathbb{C})$, and $\mathbb{C}^4/G$ has a crepant resolution.

Thus we tried to find examples of quotient singularities which admit a crepant resolution in higher dimension and propose a conjecture on existence of crepant resolutions here.

Conjecture 1.4. Let G be a finite subgroup of $\mathrm{SL}(n, \mathbb{C})$ and H_i be the abelian subgroups of G. The quotient singularity $\mathbb{C}^n/G$ admits a crepant resolution, if any of the quotients $\mathbb{C}^n/H_i$ admits a crepant resolution.

This conjecture is true for 2 and 3 dimensional cases. Moreover, in dimension two, we can construct the minimal resolution of non-cyclic singularities, that is, of type D_n, E_6, E_7 and E_8, as birational fiber product of the minimal resolution of quotient singularities by the maximal cyclic subgroups H_i, where *maximal cyclic subgroup* means that there are no larger cyclic subgroup containing H_i.

In dimension two or three, the opposite statement of this conjecture also holds: If $\mathbb{C}^n/G$ admits a crepant resolutions, then $\mathbb{C}^n/H_i$ for all abelian subgroups admit crepant resolutions. However, it is not true even in dimension four. In case a group $G \subset \mathrm{SL}(4, \mathbb{C})$ is generated by T of Theorem3.1 and $H = \frac{1}{4}(1, 3, 1, 3)$. As the quotient $\mathbb{C}^4/H$ has a terminal singularity, it does not admit a crepant resolution, but $\mathbb{C}^4/G$ has a crepant resolution (cf.[9]). Moreover, there are examples of non-abelian $G \subset \mathrm{Sp}(n, \mathbb{C})$ with abelian $H \subset G$ such that neither $\mathbb{C}^n/G$ nor $\mathbb{C}^n/H$ have crepant resolutions (cf. [17]).

Related with this conjecture, another description of crepant resolutions for non-abelian quotient singularity in dimension three has already obtained by Ishii, Nolla de Celis and the second author ([11]). They constructed a crepant resolution as iterated G-Hilbert scheme, that is, G/H-Hilbert scheme of H-Hilbert scheme, via toric geometry, and it is the moduli space of G-constellations, where a *G-constellation* $\mathcal{F}$ on X is a generalized notion of a G-cluster, and is a G-equivariant coherent sheaf on X such that $H^0(\mathcal{F}) \cong \mathbb{C}[G]$. This method works for any finite subgroups in $\mathrm{SL}(3, \mathbb{C})$ except two simple groups of order 60 and 168. Moreover, this may help to check the above conjecture in higher dimension.

This paper is organized as follows. We have already introduced known and new results on existence of crepant resolutions and proposed a conjecture in this section. In the next section, we discuss on the relation between G-Hilbert schemes and Gröbner bases for abelian finite subgroups in $\mathrm{GL}(n, \mathbb{C})$ and show Theorem 1.2. There are also examples of G-Hilbert schemes which were computed by Gröbner method, and you will see the singular one and reducible one among them. In section 3, we will show Theorem 1.3 which gives examples of crepant resolutions of 4-dimensional non-abelian quotient singularities. It is hard to draw picture even for 4-dimensional toric varieties, and we hope you feel the difficulties.

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§2. G-Hilbert schemes and Gröbner bases

In this section, we show that (the normalizations of) the G-Hilbert scheme for a finite abelian subgroup of $G \subset GL(n, \mathbb{C})$ is constructed by using Gröbner bases. Though such a result is already known in [4, 21], we prove it by a different and simpler way. It seems that G-graphs developed by Nakamura [22] is a basic tool to study the G-Hilbert scheme and it is easy to understand for non-experts. So here we consider a relation between G-graphs and Gröbner bases and as its consequence we show that the G-Hilbert scheme is constructed by Gröbner bases.

2.1. Notations

In this section, G denote a finite abelian subgroup of $\mathrm{GL}(n, \mathbb{C})$ of order r. Since G is abelian, we can assume that any $g \in G$ is of the form $g = \mathrm{diag}(\epsilon^{a_1}, \cdots, \epsilon^{a_n})$ with $0 \leq a_i \leq r-1$, where ϵ is a primitive rth root of unity. Let $\rho_0, \cdots, \rho_r$ be the all irreducible representations of G up to isomorphism and put $\mathrm{Irr}(G) = \{\rho_0, \cdots, \rho_r\}$. Let $S = \mathbb{C}[x_1, \ldots, x_n]$ be the coordinate ring of $\mathbb{C}^n$. A monomial in S are denoted by $\mathbf{x}^{\mathbf{u}} = x_1^{u_1} \cdots x_n^{u_n}$ and identified with a lattice point $\mathbf{u} = (u_1, \ldots, u_n)$ in $\mathbb{Z}^n_{\geq 0}$, where $\mathbb{Z}^n_{\geq 0}$ stands for the non-negative integers. We denote by $\mathcal{M}$ a set of all monomials in S.

For toric geometry, we prepare following notations. Let $N = \mathbb{Z}^n + \sum_{g \in G} \mathbb{Z}\bar{g}$ be a free $\mathbb{Z}$-module of rank n, where $\bar{g} = \frac{1}{r}(a_1, \cdots, a_n) \in \mathbb{R}^n$ for $g = \mathrm{diag}(\epsilon^{a_1}, \cdots, \epsilon^{a_n}) \in G$. Let M be the dual $\mathbb{Z}$-module of N, and $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$, $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}$. Let Δ be the region of $N_{\mathbb{R}}$ whose all entries are non-negative. Then the toric variety determined by Δ is isomorphic to $\mathbb{C}^n/G$.

2.2. G-graphs and G-Hilbert schemes

We recall the definition of G-graphs. We write $\deg(\mathbf{x}^{\mathbf{u}}) = \rho_i$ if $\mathbf{x}^{\mathbf{u}}(g \cdot p) = \rho_i(g)\mathbf{x}^{\mathbf{u}}(p)$ holds for any $g \in G$ and $p \in \mathbb{C}^n$. Since any monomial is contained in some ρ_i, we can define a map $\deg : \mathcal{M} \to \mathrm{Irr}(G)$.

Definition 2.1. A subset $\Gamma \subset \mathcal{M}$ is called a *G-graph* if the following conditions are satisfied.
(1) If $\mathbf{x}^{\mathbf{u}} \in \Gamma$ and $\mathbf{x}^{\mathbf{u}}$ is divided by $\mathbf{x}^{\mathbf{v}}$, then $\mathbf{x}^{\mathbf{v}} \in \Gamma$.
(2) The restriction map $\deg : \Gamma \to \mathrm{Irr}(G)$ is a bijection.

Remark 2.2. In the original definition, a G-graph is defined as a subset of $\mathbb{Z}^n_{\geqq 0}$. The above definition is equivalent to the original one via the bijection $\mathbb{Z}^n_{\geqq 0} \to \mathcal{M}; \mathbf{u} \mapsto \mathbf{x}^{\mathbf{u}}$.

We may regard G-graphs as n-dimensional Young diagrams with r boxes having irreducible representations different from each other. By the definition, for a G-graph Γ, $1 \in \Gamma$ and the cardinal of Γ is r.

Next we recall the definition of G-Hilbert scheme. The G-Hilbert scheme has two inequivalent definitions as follows. The first version, denoted by G-Hilb, is the moduli space of G-clusters. A G-cluster is a G-invariant subscheme $Z \subset \mathbb{C}^n$ such that $\mathcal{O}_Z$ is isomorphic to the regular representation $\mathbb{C}[G]$ as a $\mathbb{C}[G]$-module. For a G-cluster Z, we also call its defining ideal $I(Z)$ a G-cluster since we mainly consider defining ideals rather than subschemes. Note that an ideal $I \subset S$ is a G-cluster if it is G-invariant and the quotient ring S/I is isomorphic to $\mathbb{C}[G]$ as a $\mathbb{C}[G]$-module. We write $I \in G$-Hilb if I is a G-cluster. G-Hilb is a union of connected components of $(\mathrm{Hilb}^r(\mathbb{C}^n))^G$ (see [15]).

The second (and the original) version, denoted by Hilb^G, is called G-Hilbert scheme of Ito-Nakamura type. It is the irreducible component of $(\mathrm{Hilb}^r(\mathbb{C}^n))^G$ which dominates $\mathbb{C}^n/G$. Hilb^G is birational to $\mathbb{C}^n/G$ and projective over $\mathbb{C}^n/G$ via the Hilbert-Chow morphism. Of course, Hilb^G is a irreducible component of G-Hilb.

A relation between G-graphs and the G-Hilbert scheme is as follows. For a G-graph Γ we define

$$I(\Gamma) := \langle \mathbf{x}^{\mathbf{u}} \in S \mid \mathbf{x}^{\mathbf{u}} \not\in \Gamma \rangle,$$

and for a monomial ideal $I \in G$-Hilb we define

$$\Gamma(I) := \{\mathbf{x}^{\mathbf{u}} \in \mathcal{M} \mid \mathbf{x}^{\mathbf{u}} \not\in I\}.$$

Lemma 2.3. The following hold.

(i) $I(\Gamma) \in G$-Hilb.
(ii) $\Gamma(I)$ is a G-graph.
(iii) There is a one-to-one correspondence between a set of G-graphs and a set of monomial G-clusters.

Proof. (1) Since $I(\Gamma)$ is a monomial ideal, it is G-invariant. By the definition of G-graph, the basis of $S/I(\Gamma)$ as a $\mathbb{C}$-vector space consists of r monomials with degree $\rho_i \in \mathrm{Irr}(G)$ different from each other. So $S/I(\Gamma) \cong \mathbb{C}[G]$ holds. Hence $I(\Gamma) \in G$-Hilb.

(2) It is clear that $\Gamma(I)$ satisfies the condition (1) in Definition 2.1. Moreover since $S/I \cong \mathbb{C}[G]$, it also satisfies the conditions (2). Hence $\Gamma(I)$ is a G-graph.

(3) If Γ is a G-graph, we have $\Gamma = \Gamma(I(\Gamma))$. Conversely if I is a monomial G-cluster, we have $I = I(\Gamma(I))$. Therefore the assertion holds. Q.E.D.

Since G-graphs such that $I(\Gamma) \in \mathrm{Hilb}^G$ are important, we name them as follows.

Definition 2.4. A G-graph Γ is *distinguished* if $I(\Gamma) \in \mathrm{Hilb}^G$.

By Lemma 2.3 there are one-to-one correspondences.

$$\{G\text{-graphs}\} \overset{1-1}{\leftrightarrow} \{\text{monomial } G\text{-clusters in } G\text{-Hilb}\},$$
$$\{\text{distinguished } G\text{-graphs}\} \overset{1-1}{\leftrightarrow} \{\text{monomial } G\text{-clusters in } \mathrm{Hilb}^G\}$$

Remark 2.5. G-Hilb is a semi-projective toric variety, that is, for each component there exists a Zariski dense algebraic torus T whose action on itself can be extended to an action on the whole space (see [3, §3,4] for details). Monomial G-clusters are torus fixed points on G-Hilb.

Next we recall a way of construction of Hilb^G and its normalization by using G-graphs. When we fix a G-graph Γ, for any $\mathbf{u} \in \mathbb{Z}^n_{\geqq 0}$ there exists a unique element $\mathrm{wt}_\Gamma(\mathbf{u}) \in \mathbb{Z}^n_{\geqq 0}$ such that $\mathbf{x}^{\mathrm{wt}_\Gamma(\mathbf{u})} \in \Gamma$ and $\deg(\mathbf{x}^{\mathbf{u}}) = \deg(\mathbf{x}^{\mathrm{wt}_\Gamma(\mathbf{u})})$.

Definition 2.6. Let A_Γ be a set of minimal generators of $I(\Gamma)$. For a distinguished G-graph Γ, we define the cone

$$\sigma(\Gamma) := \{w \in N_\mathbb{R} \mid w \cdot (\mathbf{u} - \mathrm{wt}_\Gamma(\mathbf{u})) > 0 \text{ for all } \mathbf{x}^{\mathbf{u}} \in A_\Gamma\}$$

in $N_\mathbb{R}$. We define a fan $Fan(G)$ in $N_\mathbb{R}$ as a set of all closed cone $\overline{\sigma(\Gamma)}$ and all their faces where $\overline{\sigma(\Gamma)}$ is the closure of $\sigma(\Gamma)$.

It is known that for any distinguished G-graph, $\dim \sigma(\Gamma) = n$ (See [22, Definition 2.2]).

Definition 2.7. For any distinguished G-graph Γ, we define affine schemes

$$\begin{aligned} W(\Gamma) &:= \operatorname{Spec} \mathbb{C} \left[\frac{\mathbf{x}^{\mathbf{u}}}{\mathbf{x}^{\mathrm{wt}_\Gamma(\mathbf{u})}} \;\middle|\; \mathbf{x}^{\mathbf{u}} \in A_\Gamma \right], \\ U(\Gamma) &:= \operatorname{Spec} \mathbb{C}[\Delta^\vee \cap M]. \end{aligned}$$

Note that $U(\Gamma)$ is the normalization of $W(\Gamma)$. We denote the toric variety determined by $Fan(G)$ by $T(Fan(G))$, which is obtained by gluing of $U(\Gamma)$.

Theorem 2.8 ([22, Theorem 2.11])**.** The following hold.

(i) $Fan(G)$ is a finite fan with its support Δ.
(ii) Hilb^G is given by gluing of $W(\Gamma)$ for all distinguished G-graphs Γ.
(iii) The normalization of Hilb^G is isomorphic to $T(Fan(G))$.

Remark 2.9. It is stated that G-Hilb is irreducible in the original version [22]. However there are counterexamples (see subsection 2.5). A proof of the above theorem is almost same as [22] without considering distinguished G-graphs.

2.3. Gröbner fans

We assume that the readers know the definitions of term orders and (reduced) Gröbner bases.

We fix an arbitrary point $p \in (\mathbb{C}^*)^n$. Let $I_G := I(G \cdot p)$ be the defining ideal of the G-orbit $G \cdot p$. By the definition, I_G is a G-cluster. In this paper we only consider Gröbner bases for I_G. In general any reduced Gröbner basis for a binomial ideal consists of only binomials. So since I_G is a binomial ideal, any element of each reduced Gröbner basis $\mathcal{G}$ for I_G is of the form $f = \mathbf{x}^{\mathbf{u}} - c\mathbf{x}^{\mathbf{v}}$ where $c = \mathbf{x}^{\mathbf{u}}(p)/\mathbf{x}^{\mathbf{v}}(p)$ and $\mathbf{x}^{\mathbf{u}} > \mathbf{x}^{\mathbf{v}}$. For an ideal I, it is known that there are only finitely many reduced Gröbner bases for I.

In the following, we call an element of $\mathbb{R}^n$ a weight vector. For a weight vector w and a polynomial $f = \sum c_i \mathbf{x}^{\mathbf{u}_i}$, we define the initial form $\mathrm{in}_w(f)$ of f to be the sum of all terms $c_i \cdot \mathbf{x}^{\mathbf{u}_i}$ such that the inner product $w \cdot \mathbf{u}_i$ is maximal. Note that $\mathrm{in}_w(f)$ is not necessarily a monomial. For an ideal $I \subset S$, we call $\mathrm{in}_w(I) := \langle \mathrm{in}_w(f) \mid f \in I \rangle$ the initial ideal of I.

Next we define the Gröbner fan of I_G.

Definition 2.10. For any reduced Gröbner basis $\mathcal{G}$ for an ideal I with respect to $<$, we define a Gröbner cone $\sigma(\mathcal{G})$ by

$$\sigma(\mathcal{G}) := \{ w \in \mathbb{R}^n \mid \mathrm{in}_w(f) = \mathrm{in}_<(f) \text{ for all } f \in \mathcal{G} \}.$$

The Gröbner fan $GF(I_G)$ of an ideal I is a set of all faces of $\overline{\sigma(\mathcal{G})}$ for all reduced Gröbner bases $\mathcal{G}$ for I.

Since $f \in \mathcal{G}$ is of the form $\mathbf{x}^{\mathbf{u}} - c\mathbf{x}^{\mathbf{v}}$ with $\mathbf{x}^{\mathbf{u}} > \mathbf{x}^{\mathbf{v}}$ we have

$$\sigma(\mathcal{G}) = \{w \in \mathbb{R}^n \mid w \cdot (\mathbf{u} - \mathbf{v}) > 0 \text{ for all } f = \mathbf{x}^{\mathbf{u}} - c\mathbf{x}^{\mathbf{v}} \in \mathcal{G} \text{ s.t. } \mathbf{x}^{\mathbf{u}} > \mathbf{x}^{\mathbf{v}}\}.$$

In the rest, we consider Gröbner fan in $N_{\mathbb{R}} \cong \mathbb{R}^n$. Actually it is possible because all $\mathbf{x}^{\mathbf{u}} - c\mathbf{x}^{\mathbf{v}} \in \mathcal{G}$ are G-semi-invariant, so $\mathbf{x}^{\mathbf{u}-\mathbf{v}}$ is G-invariant, hence $\sigma(\mathcal{G})$ is a cone in $N_{\mathbb{R}}$.

Proposition 2.11. The support of $GF(I_G)$ is equals to Δ.

Proof. Since $x_i^r - p_i^r \in I_G$ for each $i = 1, 2, \dots, n$, we have $GF(I_G) \subset \Delta$. Conversely for any $w \in \Delta$, if w is zero, then it is trivial that $w \in GF(I_G)$. If w is not zero, for an arbitrary term order $<$, we get a new term order $<_w$ as follows. $\mathbf{x}^{\mathbf{u}} <_w \mathbf{x}^{\mathbf{v}}$ if and only if $w \cdot \mathbf{u} < w \cdot \mathbf{v}$ or $(w \cdot \mathbf{u} = w \cdot \mathbf{v}$ and $\mathbf{x}^{\mathbf{u}} < \mathbf{x}^{\mathbf{v}})$. Then we have $w \in \overline{\sigma(\mathcal{G})}$ for the reduced Gröbner basis $\mathcal{G}$ for I_G with respect to $<_w$. Q.E.D.

Definition 2.12. For each reduced Gröbner basis $\mathcal{G}$ for I_G with respect to $>$, we define an affine scheme

$$W(\mathcal{G}) := \operatorname{Spec} \mathbb{C} \left[\frac{\mathbf{x}^{\mathbf{u}}}{\mathbf{x}^{\mathbf{v}}} \;\middle|\; \mathbf{x}^{\mathbf{u}} - c\mathbf{x}^{\mathbf{v}} \in \mathcal{G}, \mathbf{x}^{\mathbf{u}} > \mathbf{x}^{\mathbf{v}} \right].$$

We denote by $T(GF(I_G))$ the toric variety determined by the Gröbner fan $GF(I_G)$.

In the next subsection, we observe a relation between G-graphs and reduced Gröbner bases and show that there is a natural correspondence between them.

2.4. G-graphs and reduced Gröbner bases

The purpose of this subsection is to prove that Hilb^G and its normalization are completely described in terms of Gröbner bases.

Lemma 2.13. For any term order $<$, we have $\mathrm{in}_<(I_G) \in \mathrm{Hilb}^G$.

Proof. We consider a one parameter group in G-Hilb as follows. If we take a weight vector $w = (w_1, \dots, w_n) \in \sigma(\mathcal{G}) \cap \mathbb{Z}^n_{>0}$, then by the definition we have $\mathrm{in}_<(I_G) = \mathrm{in}_w(I_G)$. For any $t \in \mathbb{C}^*$, we define $p_t = (t^{w_1}p_1, \dots, t^{w_n}p_n)$ and $I_{G,t} = I(G \cdot p_t)$. Since p is a point on $(\mathbb{C}^*)^n$ and $t \neq 0$, $G \cdot p_t$ is a free G-orbit, so we have $I_{G,t} \in G$-Hilb. Moreover since $\pi(I_{G,t})$ is a non-singular point on $\mathbb{C}^n/G$, we have $I_{G,t} \in \mathrm{Hilb}^G$ where π is the Hilbert-Chow morphism.

Next we show that $\lim_{t\to 0} I_{G,t} = \langle \mathrm{in}_<(I_G)\rangle$. Let $\mathcal{G}$ be the reduced Gröbner basis for I_G with respect to $<$. Each element $f \in \mathcal{G}$ is of the form $f = \mathbf{x}^{\mathbf{u}} - c\mathbf{x}^{\mathbf{v}}$ with $\mathbf{x}^{\mathbf{u}} > \mathbf{x}^{\mathbf{v}}$ and $c = \mathbf{x}^{\mathbf{u}}(p)/\mathbf{x}^{\mathbf{v}}(p)$. Thus if we put

$$f_t = \mathbf{x}^{\mathbf{u}} - c_t\mathbf{x}^{\mathbf{v}}, \quad c_t = \mathbf{x}^{\mathbf{u}}(p_t)/\mathbf{x}^{\mathbf{v}}(p_t),$$

then a set $\mathcal{G}_t = \{f_t \mid f \in \mathcal{G}\}$ is the reduced Gröbner basis for $I_{G,t}$ with respect to $<$. Since $c_t = \mathbf{x}^{\mathbf{u}}(p_t)/\mathbf{x}^{\mathbf{v}}(p_t) = t^{\langle w,\mathbf{u}\rangle}\mathbf{x}^{\mathbf{u}}(p)/t^{\langle w,\mathbf{v}\rangle}\mathbf{x}^{\mathbf{v}}(p) = t^{\langle w,\mathbf{u}-\mathbf{v}\rangle}c$ and $\langle w, \mathbf{u}-\mathbf{v}\rangle > 0$ hold, we have $\lim_{t\to 0} c_t = 0$, hence $\lim_{t\to 0} I_{G,t} = \langle \mathrm{in}_<(I_G)\rangle$.

It follows that $\langle \mathrm{in}_<(I_G)\rangle \in G$-Hilb since we have $S/\langle \mathrm{in}_<(I_G)\rangle \cong S/I_G \cong \mathbb{C}[G]$ as a $\mathbb{C}[G]$-module. Therefore we have $\mathrm{in}_<(I_G) \in \mathrm{Hilb}^G$ since Hilb^G is an irreducible component of G-Hilb. Q.E.D.

For a reduced Gröbner basis $\mathcal{G}$ for I_G with respect to $<$, we define

$$\varphi(\mathcal{G}) := \Gamma(\langle \mathrm{in}_<(\mathcal{G})\rangle) = \{\mathbf{x}^{\mathbf{u}} \in \mathcal{M} \mid \mathbf{x}^{\mathbf{u}} \notin \langle \mathrm{in}_<(\mathcal{G})\rangle\},$$

and for a distinguished G-graph Γ, we define

$$\psi(\Gamma) = \{\mathbf{x}^{\mathbf{u}} - c\mathbf{x}^{\mathrm{wt}_\Gamma(\mathbf{u})} \mid \mathbf{x}^{\mathbf{u}} \in A_\Gamma, c = \mathbf{x}^{\mathbf{u}}(p)/\mathbf{x}^{\mathrm{wt}_\Gamma(\mathbf{u})}(p)\}.$$

Note that we have $\mathrm{in}_w(\mathcal{G}) = A_\Gamma$ for any $w \in \sigma(\Gamma)$ by the definition.

Proposition 2.14. The following hold.

(i) $\varphi(\mathcal{G})$ is a distinguished G-graph.

(ii) $\psi(\Gamma)$ is a reduced Gröbner basis for I_G with respect to $w \in \sigma(\Gamma)$.

(iii) φ and ψ are mutually inverse maps. Therefore we have a one-to-one correspondence:

$$\{\text{distinguished } G\text{-graphs}\} \overset{1-1}{\longleftrightarrow} \{\text{reduced Gröbner bases for } I_G\}$$

Proof. (1) By Lemma 2.13 $\langle \mathrm{in}_<(\mathcal{G})\rangle$ is a monomial G-cluster in Hilb^G. So by Lemma 2.3 (2), $\varphi(\mathcal{G})$ is a distinguished G-graph.

(2) We put $\mathcal{G} = \psi(\Gamma)$. It is clear that $\mathcal{G} \subset I_G$. We have $\dim_{\mathbb{C}} S/\langle \mathrm{in}_w(\mathcal{G})\rangle = \dim_{\mathbb{C}} S/\langle A_\Gamma\rangle = r = \dim_{\mathbb{C}} S/I_G = \dim_{\mathbb{C}} S/\mathrm{in}_w(I_G)$ and $\langle \mathrm{in}_w(\mathcal{G})\rangle \subset \mathrm{in}_w(I_G)$, hence we have $\langle \mathrm{in}_w(\mathcal{G})\rangle = \mathrm{in}_w(I_G)$. Therefore $\mathcal{G}$ is a Gröbner basis for I_G. Moreover, since A_Γ is the minimal generators of $I(\Gamma)$ and $I(\Gamma) = \langle \mathrm{in}_w(\mathcal{G})\rangle$, any term contained in $f \in \mathcal{G}$ is not divisible by any $g \in \mathrm{in}_w(\mathcal{G} \setminus \{f\})$, so $\mathcal{G}$ is reduced.

(3) For any distinguished G-graph Γ and reduced Gröbner basis $\mathcal{G}$, we have $\varphi(\psi(\Gamma)) = \Gamma$ and $\psi(\varphi(\mathcal{G})) = \mathcal{G}$, so the assertion follows. Q.E.D.

Proposition 2.15. Let Γ be a distinguished G-graph and $\mathcal{G} = \psi(\Gamma)$ the corresponding reduced Gröbner basis. Then we have

(i) $W(\Gamma) = W(\mathcal{G})$

(ii) $\sigma(\Gamma) = \sigma(\mathcal{G})$.

Proof. Let $<$ be a term order such that $\mathcal{G}$ is the reduced Gröbner basis with respect to $<$, then any $f \in \mathcal{G}$ is of the form $\mathbf{x}^{\mathbf{u}} - c\mathbf{x}^{\mathrm{wt}_\Gamma(\mathbf{u})}$ such that $\mathbf{x}^{\mathbf{u}} \in A_\Gamma$ and $\mathbf{x}^{\mathbf{u}} > \mathbf{x}^{\mathrm{wt}_\Gamma(\mathbf{u})}$. So we have

$$W(\Gamma) = \operatorname{Spec} \mathbb{C}\left[\frac{\mathbf{x}^{\mathbf{u}}}{\mathbf{x}^{\mathrm{wt}_\Gamma(\mathbf{u})}} \;\middle|\; \mathbf{x}^{\mathbf{u}} \in A_\Gamma\right] = W(\mathcal{G})$$

$$\sigma(\Gamma) = \{w \in N_{\mathbb{R}} \mid w \cdot (\mathbf{u} - \mathrm{wt}_\Gamma(\mathbf{u})) > 0 \text{ for all } \mathbf{x}^{\mathbf{u}} \in A_\Gamma\} = \sigma(\mathcal{G}).$$

Q.E.D.

As a consequence of the above results we have the following. This is the main result in this section.

Theorem 2.16. Let G be a finite abelian subgroup of $\mathrm{GL}(n, \mathbb{C})$, then the following hold.

(i) $\{W(\mathcal{G})\}$ is an affine covering of Hilb^G.

(ii) $Fan(G) = GF(I_G)$, therefore the normalization of Hilb^G is isomorphic to $T(GF(I_G))$.

Proof. It follows from Theorem 2.8 and Proposition 2.15. Q.E.D.

It is known that G-Hilb is the minimal resolution of $\mathbb{C}^2/G$ for $G \subset \mathrm{GL}(2, \mathbb{C})$ ([10, 16, 19]) and a crepant resolution of $\mathbb{C}^3/G$ for $G \subset \mathrm{SL}(3, \mathbb{C})$ ([2, 22]), and in this case G-Hilb $= \mathrm{Hilb}^G$. In particular we have next corollaries.

Corollary 2.17 (Ito [13, Theorem 1.1])**.** For a finite abelian subgroup G of $\mathrm{GL}(2, \mathbb{C})$, $T(GF(I_G))$ is the minimal resolution of $\mathbb{C}^2/G$.

Corollary 2.18 (cf. Ito [14])**.** For a finite abelian subgroup G of $\mathrm{SL}(3, \mathbb{C})$, $T(GF(I_G))$ is a crepant resolution of $\mathbb{C}^3/G$.

In general, G-Hilb is not necessarily smooth even if $G \subset \mathrm{GL}(3, \mathbb{C})$. Moreover, Craw-Maclagan-Thomas found a non-normal Hilb^G for a finite abelian subgroup $G \subset \mathrm{GL}(6, \mathbb{C})$ ([4, Example 5.7]).

2.5. Examples

We show several examples of the Gröbner fans of I_G in dimension three. In this subsection we assume $G \subset \mathrm{GL}(3, \mathbb{C})$ and $p = (1, 1, 1)$. We say a cyclic subgroup $G \subset \mathrm{GL}(3, \mathbb{C})$ is of type $\frac{1}{r}(a_1, a_2, a_3)$ if it is generated by an element $\mathrm{diag}(\epsilon^{a_1}, \epsilon^{a_2}, \epsilon^{a_3})$ where $0 \leq a_i < r$ for each $i = 1, 2, 3$. We write $x = x_1, y = x_2, z = x_3$.

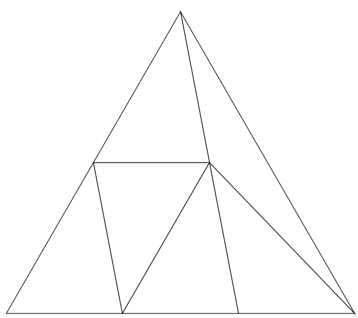
Lemma 2.19. If G is of type $\frac{1}{r}(1, a, b)$, a set

$$\{x^r - 1, y - x^a, z - x^b\}$$

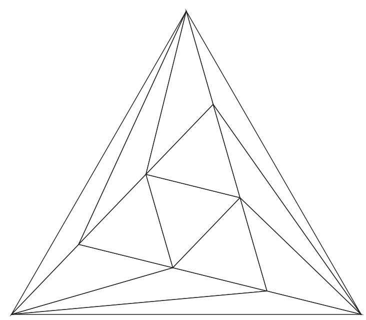
is the reduced Gröbner basis for I_G with respect to $w = (w_1, w_2, w_3)$ with $w_1 > 0, w_2 > aw_1, w_3 > bw_1$.

Proof. For a G-graph $\Gamma = \{1, x, \ldots, x^{r-1}\}$, we have the reduced Gröbner basis $\psi(\Gamma)$ for I_G with respect to $w \in \sigma(\Gamma)$ by Proposition 2.14. We have $\psi(\Gamma) = \{x^r - 1, y - x^a, z - x^b\}$ and $\sigma(\Gamma) = \{w = (w_1, w_2, w_3) \in \Delta \mid w_1 > 0, w_2 > aw_1, w_3 > bw_1\}$ since $A_\Gamma = \{x^r, y, z\}$ and $\deg(x^r) = \deg(1), \deg(y) = \deg(x^a), \deg(z) = \deg(x^b)$. Q.E.D.

Example 2.20. Next we show examples in $\mathrm{SL}(3, \mathbb{C})$. The following picture is the Gröbner fan of G of type $\frac{1}{6}(1, 2, 3)$ and $\frac{1}{13}(1, 3, 9)$ respectively. By Corollary 2.18 this fan gives a crepant resolution of $\mathbb{C}^3/G$. We remark that the number of chambers is equal to the order of G. In general this number equals to the Euler number of Hilb^G. This shows a generalized McKay correspondence, that is, the Euler number of a crepant resolution equals to the number of conjugacy classes of G.



$\frac{1}{6}(1, 2, 3)$



$\frac{1}{13}(1, 3, 9)$

Example 2.21. Let G be of type $\frac{1}{4}(1, 2, 3)$. Note that $G \not\subset \mathrm{SL}(3, \mathbb{C})$. This is a singular example where the order of the group G is minimum. By Lemma 2.19 $\{x^4 - 1, y - x^2, z - x^3\}$ is a reduced Gröbner basis for I_G. Then all reduced Gröbner bases, the corresponding G-graphs and the Gröbner fan for I_G are as follows. In G-graphs, for example Γ_2 stands for $\{1, y, z, yz\}$. These can be calculated by software "Gfan" [8]. An affine cover $W(\mathcal{G})$ of Hilb^G is smooth if $\sigma(\mathcal{G})$ is a triangle and has a singularity if a quadrangle. In particular this singularity is a terminal singularity described by $xy - zw = 0$.

The reduced Gröbner bases

$\mathcal{G}_1 = \{z^4 - 1, y - z^2, x - z^3\}$

$\mathcal{G}_2 = \{z^2 - y, y^2 - 1, x - yz\}$

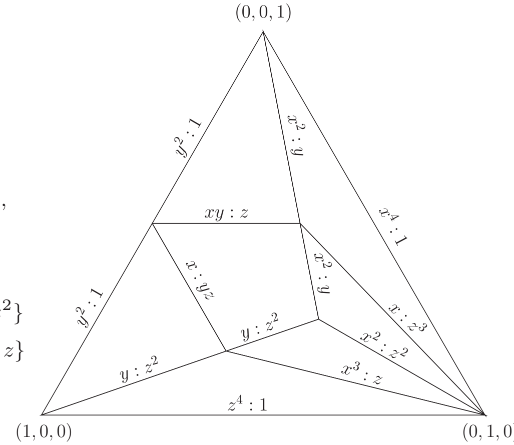
$\mathcal{G}_3 = \{z^2 - y, yz - x, y^2 - 1, xz - 1,$
$\quad xy - z, x^2 - y\}$

$\mathcal{G}_4 = \{z - xy, y^2 - 1, x^2 - y\}$

$\mathcal{G}_5 = \{z^3 - x, y - z^2, xz - 1, x^2 - z^2\}$

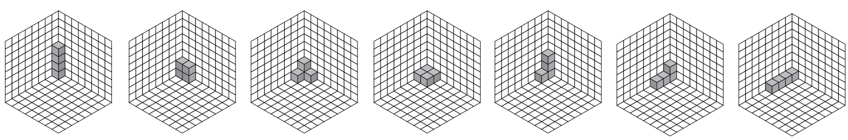
$\mathcal{G}_6 = \{z^2 - x^2, y - x^2, xz - 1, x^3 - z\}$

$\mathcal{G}_7 = \{z - x^3, y - x^2, x^4 - 1\}$



The Gröbner fan for I_G

The G-graphs



Γ_1 Γ_2 Γ_3 Γ_4 Γ_5 Γ_6 Γ_7

Example 2.22. Let G be of type $\frac{1}{101}(1, 7, 93)$ which is a exercise appearing in [5]. Though they provide a method to calculate $Fan(G)$, we can also obtain it by calculating $GF(I_G)$. We can check directly that there are 101 reduced Gröbner bases for I_G and so $GF(I_G)$ consists of 101 small triangles.

Finally we consider examples such that G-Hilb is reducible. Craw-Maclagan-Thomas [4] show that if G is of type $\frac{1}{14}(1, 9, 11)$ then Hilb^G is reducible. However, there are more reducible examples. The next is a criterion to determine whether G-Hilb is reducible or not.

Proposition 2.23. If the number of G-graphs is bigger than the number of reduced Gröbner bases for I_G, then G-Hilb is reducible.

Proof. It is clear by Proposition 2.14. Q.E.D.

Example 2.24. Table 1 shows all reducible G-Hilb of type $\frac{1}{r}(1, a, b)$ for $r \leq 26$. A. Ishii tell the author that G-Hilb of type $\frac{1}{8}(1, 3, 5, 7)$ is also reducible.

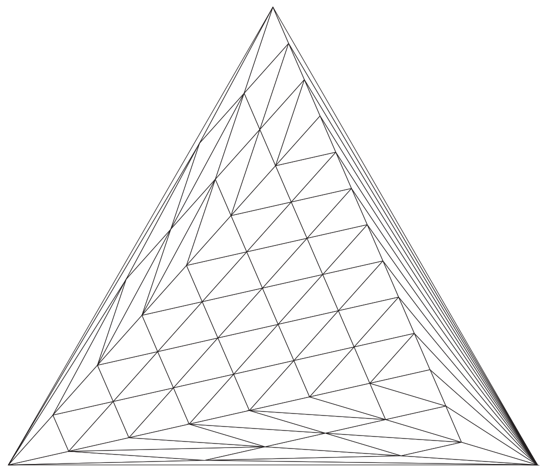


Fig. 1. Craw-Reid's exercise : $\frac{1}{101}(1,7,93)$

type	$\#\{G\text{-graphs}\}$	$\#\{\text{reduced G-bases}\}$
$\frac{1}{14}(1,9,11)$	35	33
$\frac{1}{19}(1,6,8)$	44	42
$\frac{1}{20}(1,12,17)$	37	35
$\frac{1}{23}(1,9,20)$	58	56
$\frac{1}{24}(1,15,19)$	59	57
$\frac{1}{25}(1,11,20)$	39	37
$\frac{1}{26}(1,5,7)$	46	44
$\frac{1}{26}(1,8,11)$	52	49

Table 1. reducible G-Hilb of type $\frac{1}{r}(1,a,b)$ with $r \leq 26$

§3. Examples of crepant resolutions of 4-dimensional non-abelian quotient singularities

In this section, we introduce some examples of four-dimensional quotient singularities by non-abelian finite groups, for which we can construct crepant resolutions. For the rest of this section, we denote

$\frac{1}{n}(a_1, \ldots, a_4)$ a diagonal matrix $\mathrm{diag}(\epsilon^{a_1}, \epsilon^{a_2}, \epsilon^{a_3}, \epsilon^{a_4})$ where $\epsilon = \exp(2\pi i/n)$ and $0 \leq a_i < n$ for each $i = 1, 2, 3, 4$, and call $(a_1 + \cdots + a_4)/n$ its *age*.

Theorem 3.1. Let $H \subset \mathrm{SL}(4, \mathbb{C})$ be one of the following groups:

(i) $H = \left\langle \frac{1}{n}(1, 0, 0, -1), \frac{1}{n}(0, 1, 0, -1), \frac{1}{n}(0, 0, 1, -1) \right\rangle$, where $n \in \mathbb{Z}$, $n \geq 2$.

(ii) $H = \left\langle \frac{1}{2n}(1, n-1, 1, n-1), \frac{1}{2}(0, 1, 0, 1) \right\rangle$, where $n \in \mathbb{Z}$, $n \geq 1$.

(iii) $H = \left\langle \frac{1}{4n}(1, 2n-1, 1, 2n-1) \right\rangle$, where $n \in \mathbb{Z}$, $n \geq 2$.

(iv) $H = \left\langle \frac{1}{m}(1, -1, 0, 0), \frac{1}{n}(0, 0, 1, -1) \right\rangle$, where $m, n \in \mathbb{Z}$, $m \geq 3$, $n \geq 2$.

Let $K \subset \mathrm{SL}(4, \mathbb{C})$ be a group generated by

$$T = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \in \mathrm{SL}(4, \mathbb{C}),$$

and $G = \langle H, K \rangle$. Then G is a non-abelian finite subgroup of $\mathrm{SL}(4, \mathbb{C})$, and $\mathbb{C}^4/G$ has a crepant resolution.

Proof. For each group G of Theorem 3.1, H coincides with the set of all diagonal matrices in G, and G is decomposed into a semi-direct product $H \rtimes K$. Now, we construct a crepant resolution in two steps, as is done in the paper by Ito [12].

$$\begin{tikzcd}
 & & \widetilde{X} \arrow[d, "\text{(2) crepant resolution}"] \\
 & \widetilde{Y} \arrow[loop left, "K"] \arrow[r, "/K"'] \arrow[d, "\text{(1) K-equivariant crepant resolution}"'] & \widetilde{Y}/K \arrow[d, "\text{crepant morphism}"] \\
\mathbb{C}^4 \arrow[r] & Y = \mathbb{C}^4/H \arrow[r, "/K"'] & X = \mathbb{C}^4/G.
\end{tikzcd}$$

(1) Construct a K-equivariant crepant resolution $\widetilde{Y} \to Y = \mathbb{C}^4/H$. By taking quotient of $\widetilde{Y} \to Y$ by K, we can obtain a crepant morphism $\widetilde{Y}/K \to X = \mathbb{C}^4/G$.

(2) Construct a crepant resolution $\widetilde{X} \to \widetilde{Y}/K$. Then the composite $\widetilde{X} \to X$ is also a crepant resolution.

Since H is abelian, crepant resolutions of Y can be constructed by toric methods. For that purpose, we only need to subdivide a tetrahedron P, which is an intersection of a cone $\sigma = \sum_{i=1}^{4} \mathbb{R}_{\geq 0} e_i$ and a hyperplane $\sum_{i=1}^4 x_i = 1$, into $\#H$ pieces of small tetrahedrons, using points of age one. In order the corresponding crepant resolution to be K-equivariant, we should take a K-invariant subdivision. (K acts on P by the natural action on the standard basis of $\mathbb{C}^4$.)

For the groups of Theorem 3.1, points in P of age one are drawn in Figures 2–5, and subdivisions that give K-equivariant crepant resolutions are shown in Figures 6–9. Note that each octahedron in Figure 6 is divided into four tetrahedrons as in Figure 10.

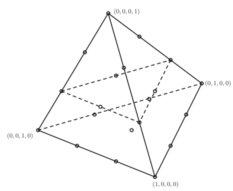


Fig. 2. (i) $n = 3$

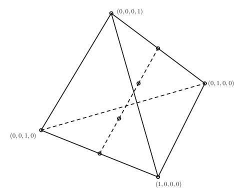


Fig. 3. (ii) $n = 3$

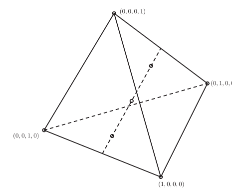


Fig. 4. (iii) $n = 3$

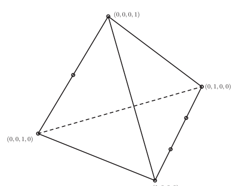


Fig. 5. (iv) $m = 3$, $n = 2$

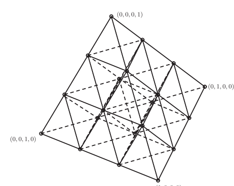


Fig. 6. (i) $n = 3$

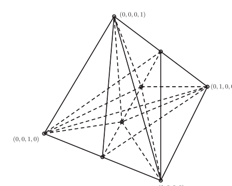


Fig. 7. (ii) $n = 3$

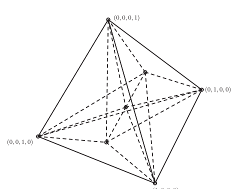


Fig. 8. (iii) $n = 3$

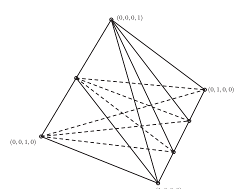


Fig. 9. (iv) $m = 3$, $n = 2$

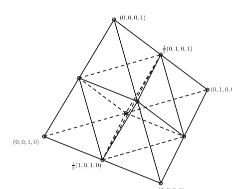


Fig. 10. subdivision of an octahedron

Now, by changing coordinates analytically around the fixed locus of K, we can find all singularities in $\widetilde{Y}/K$ are written as a product of A_1-singularity and $\mathbb{C}^2$. Therefore, $\widetilde{Y}/K$ has a crepant resolution. Q.E.D.

Remark 3.2. In dimension four, we have to choose H and K carefully. For $\mathbb{C}^4/H$ does not necessarily have a K-equivariant crepant resolution, and terminal singularities could appear in $\widetilde{Y}/K$. Although this method is applicable to only a few groups, there exist examples for other H and K than we presented here.

Remark 3.3. The crepant resolutions $\widetilde{Y}$ are H-Hilb except 6(i) in the figures. For these H-Hilb, $\widetilde{X}$ could be the iterated G-Hilb, that is, G/H-Hilb of H-Hilb because the crepant resolution of $\widetilde{Y}/K$ is K-Hilb.

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