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On log canonical rings

Osamu Fujino and Yoshinori Gongyo

Dedicated to Professor Yujiro Kawamata on the occasion of his sixtieth birthday

Abstract.

We discuss the relationship among various conjectures in the minimal model theory including the finite generation conjecture of the log canonical rings and the abundance conjecture. In particular, we show that the finite generation conjecture of the log canonical rings for log canonical pairs can be reduced to that of the log canonical rings for purely log terminal pairs of log general type.

Contents

§1. Introduction

In this article, we discuss the relationship among the following conjectures:

Conjecture A. Let (X, Δ) be a projective log canonical pair and Δ a $\mathbb{Q}$-divisor. Then the log canonical ring

$$R(X, \Delta) := \bigoplus_{m \geq 0} H^0(X, \mathcal{O}_X(\lfloor m(K_X + \Delta) \rfloor))$$

is finitely generated.

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Conjecture B. Let (X, Δ) be a projective purely log terminal pair such that $\lfloor \Delta \rfloor$ is irreducible and that Δ is a $\mathbb{Q}$-divisor. Suppose that $K_X + \Delta$ is big. Then the log canonical ring

$$R(X, \Delta) = \bigoplus_{m \geq 0} H^0(X, \mathcal{O}_X(\lfloor m(K_X + \Delta) \rfloor))$$

is finitely generated.

Conjecture C (Good minimal model conjecture)**.** Let (X, Δ) be a $\mathbb{Q}$-factorial projective divisorial log terminal pair and Δ an $\mathbb{R}$-divisor. If $K_X + \Delta$ is pseudo-effective, then (X, Δ) has a good minimal model.

From now on, Conjecture $\bullet_n$ (resp. Conjecture $\bullet_{\leq n}$) stands for Conjecture $\bullet$ with $\dim X = n$ (resp. $\dim X \leq n$). Remark that in Conjectures A, B, and C we may assume that (X, Δ) is log smooth, i.e., X is smooth and Δ has a simple normal crossing support by taking suitable resolutions.

The following result is the main theorem:

Theorem 1.1 (Main Theorem)**.** *Conjectures A_n, B_n, and $C_{\leq n-1}$ are all equivalent.*

We remark that Conjecture B_n implies Conjecture $\mathrm{A}_{\leq n}$ by Theorem 1.1 because Conjecture $\mathrm{A}_{\leq n-1}$ directly follows from Conjecture $\mathrm{C}_{\leq n-1}$. We also remark that the equivalence of Conjecture A_n and Conjecture $\mathrm{C}_{\leq n-1}$ seems to be a folklore statement, though we have never seen the explicit statement in the literature.

In [FM], the first author and Shigefumi Mori proved that the finite generation of the log canonical rings for projective klt pairs can be reduced to the case when the log canonical divisors are big by using the so-called Fujino–Mori canonical bundle formula (see [FM, Theorem 5.2]). This reduction seems to be indispensable for the finite generation of the log canonical rings for klt pairs (see, for example, [BCHM], and [L]). Unfortunately, the reduction arguments in [FM] can not be directly applied to log canonical pairs because the usual perturbation techniques do not work well for log canonical pairs (cf. Remark 3.7). The following statement is contained in our main theorem: Theorem 1.1.

Corollary 1.2. *Conjecture B_n implies Conjecture A_n.*

Corollary 1.2 is one of the motivations of this paper. The proof of Theorem 1.1 (and Corollary 1.2) heavily depends on the recent developments in the minimal model theory after [BCHM], for example, [B2], [DHP], [FG1], [G2], and [HMX]. It is completely different from the reduction techniques discussed in [FM].

In Conjecture B, we may assume that X is smooth, Δ has a simple normal crossing support, $\lfloor \Delta \rfloor$ is irreducible, and $K_X + \Delta$ is big. Hence Conjecture B looks more approachable than Conjecture A from the analytic viewpoint (cf. [DHP]).

As corollaries of Theorem 1.1 and its proof, we can also see the following:

Corollary 1.3. *Assume that Conjecture* B_n *holds. Let* (X, Δ) *be an* n*-dimensional* $\mathbb{Q}$*-factorial projective divisorial log terminal pair such that* Δ *is a* $\mathbb{Q}$*-divisor. If* $\kappa(X, K_X + \Delta) \geq 1$*, then* (X, Δ) *has a good minimal model.*

Corollary 1.4. *Assume that Conjecture* B_n *holds. Let* (X, Δ) *be an* n*-dimensional log canonical pair,* Δ *a* $\mathbb{Q}$*-divisor, and* $f : X \to S$ *a proper morphism onto an algebraic variety* S*. Then the relative log canonical ring*

$$R(X/S, \Delta) := \bigoplus_{m \geq 0} f_*\mathcal{O}_X(\lfloor m(K_X + \Delta) \rfloor)$$

is a finitely generated $\mathcal{O}_S$*-algebra.*

If Conjecture B_n implies Conjecture C_n, then Conjectures A, B, and C hold in any dimension by Theorem 1.1. Unfortunately, Corollary 1.3 is far from the complete solution of Conjecture C_n under Conjecture B_n. For the details of Conjecture C, we recommend the reader to see [FG1, Section 5] (see also Section 4: Appendix).

In [F6], the first author solved Conjecture A_4. Conjecture A_n with $n \geq 5$ is widely open. For surfaces, $R(X, \Delta)$ is known to be finitely generated under the assumption that Δ is a boundary $\mathbb{Q}$-divisor and X is $\mathbb{Q}$-factorial. When $\dim X = 2$, we do not have to assume that the pair (X, Δ) is log canonical for the minimal model theory. For the details, see [F6].

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We will work over $\mathbb{C}$, the field of complex numbers, throughout this paper. We will make use of the standard notation as in [KMM], [KM], [BCHM], [F3] and [F5].

§2. Preliminaries

In this section, we collect together some definitions and notation.

2.1 (Pairs)**.** A pair (X, Δ) consists of a normal variety X over $\mathbb{C}$ and an effective $\mathbb{R}$-divisor Δ on X such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. A pair (X, Δ) is called *klt* (resp. *lc*) if for any projective birational morphism $g : Z \to X$ from a normal variety Z, every coefficient of Δ_Z is < 1 (resp. ≤ 1) where $K_Z + \Delta_Z := g^*(K_X + \Delta)$. Moreover a pair (X, Δ) is called *canonical* (resp. *plt*) if for any projective birational morphism $g : Z \to X$ from a normal variety Z, every coefficient of g-exceptional components of Δ_Z is ≤ 0 (resp. < 1). Let (X, Δ) be an lc pair. If there is a projective birational morphism $g : Z \to X$ from a smooth projective variety Z such that every coefficient of g-exceptional components of Δ_Z is < 1, the exceptional locus $\mathrm{Exc}(g)$ of g is a divisor, and $\mathrm{Exc}(g) \cup \mathrm{Supp}\,\Delta_Z$ is a simple normal crossing divisor on Z, then (X, Δ) is called *dlt*.

We note that *klt*, *plt*, *dlt*, and *lc* stand for *kawamata log terminal*, *purely log terminal*, *divisorial log terminal*, and *log canonical*, respectively.

Let us recall the definition of *log minimal models*. In Definition 2.2, all the varieties are assumed to be projective.

Definition 2.2 (cf. [B2, Definition 2.1])**.** A pair (Y, Δ_Y) is a *log birational model* of (X, Δ) if we are given a birational map $\phi\colon X \dashrightarrow Y$ and $\Delta_Y = \Delta^{\sim} + E$ where $\Delta^{\sim}$ is the birational transform of Δ and E is the reduced exceptional divisor of ϕ^{-1}, that is, $E = \sum E_j$ where E_j is a prime divisor on Y which is exceptional over X for every j. A log birational model (Y, Δ_Y) is a *nef model* of (X, Δ) if in addition

(1) (Y, Δ_Y) is $\mathbb{Q}$-factorial dlt, and
(2) $K_Y + \Delta_Y$ is nef.

And we call a nef model (Y, Δ_Y) a *log minimal model of* (X, Δ) (*in the sense of Birkar–Shokurov*) if in addition

(3) for any prime divisor D on X which is exceptional over Y, we have
$$a(D, X, \Delta) < a(D, Y, \Delta_Y).$$

Let (Y, Δ_Y) be a log minimal model of (X, Δ). If $K_Y + \Delta_Y$ is semi-ample, then (Y, Δ_Y) is called a *good minimal model* of (X, Δ).

When (X, Δ) is plt, a log minimal model of (X, Δ) in the sense of Birkar–Shokurov is a log minimal model in the traditional sense (see [KM] and [BCHM]), that is, $\phi\colon X \dashrightarrow Y$ extracts no divisors. For the details, see [B1, Remark 2.6].

Remark 2.3. Assume that Conjecture $C_{\leq n}$ holds. Let (X, Δ) be a projective $\mathbb{Q}$-factorial dlt pair with $\dim X = n$ such that $K_X + \Delta$ is pseudo-effective. Then, by [B2, Corollary 1.6], there is a sequence of divisorial contractions and flips starting with (X, Δ) and ending up with a good minimal model (Y, Δ_Y). In particular, $X \dashrightarrow Y$ extracts no divisors. Therefore, (Y, Δ_Y) is a log minimal model of (X, Δ) in the traditional sense.

§3. Proof of Main Theorem

For the proof of the main theorem: Theorem 1.1, we discuss the relationship among the following conjectures:

Conjecture D (Abundance conjecture)**.** Let (X, Δ) be a projective log canonical pair. If $K_X + \Delta$ is nef, then $K_X + \Delta$ is semi-ample.

Conjecture E (Non-vanishing conjecture)**.** Let (X, Δ) be a projective log canonical pair. If $K_X + \Delta$ is pseudo-effective, then there exists some effective $\mathbb{R}$-divisor D such that $D \sim_{\mathbb{R}} K_X + \Delta$.

Conjecture F (Non-vanishing conjecture for smooth varieties)**.** Let X be a smooth projective variety. If K_X is pseudo-effective, then there exists some effective $\mathbb{Q}$-divisor D such that $D \sim_{\mathbb{Q}} K_X$.

For the above conjectures, we show the following lemmas:

Lemma 3.1. *Conjecture B_n and Conjecture $E_{\leq n-1}$ imply Conjecture $D_{\leq n-1}$.*

Lemma 3.2. *Conjecture B_n implies Conjecture $F_{\leq n-1}$.*

Lemma 3.3. *Conjecture $F_{\leq n}$ and Conjecture $D_{\leq n-1}$ imply Conjecture $E_{\leq n}$.*

Lemma 3.4. *Conjecture B_n implies Conjecture $C_{\leq n-1}$.*

Lemma 3.5. *Assume that Conjecture $C_{\leq n-1}$ holds. Let (X, Δ) be an n-dimensional $\mathbb{Q}$-factorial projective divisorial log terminal pair such that Δ is a $\mathbb{Q}$-divisor and $\kappa(X, K_X + \Delta) \geq 1$. Then (X, Δ) has a good minimal model. In particular, Conjecture $C_{\leq n-1}$ implies Conjecture $A_{\leq n}$.*

Let us start the proof of the lemmas.

Proof of Lemma 3.1. By taking a dlt blow-up and using Shokurov polytope (cf. [B2, Proposition 3.2. (3)] and [F5, Theorem 18.2]), we may assume that (X, Δ) is a $\mathbb{Q}$-factorial dlt pair and that Δ is a $\mathbb{Q}$-divisor. Moreover by taking a product with an Abelian variety we may further assume $\dim X = n-1$. The abundance conjecture follows from [L, Theorem A.6] and Conjecture B_n when (X, Δ) is klt. For a log canonical pair (X, Δ) with nef $K_X + \Delta$, its semi-ampleness follows from [FG1, Theorem 5.5] by Conjecture $\mathrm{E}_{\leq n-1}$ and the abundance theorem for klt pairs established above. Q.E.D.

Proof of Lemma 3.2. We may assume $\dim X = n-1$ by taking a product with an Abelian variety. Let $X \subset \mathbb{P}^N$ be a projectively normal embedding. We consider the $\mathbb{P}^1$-bundle

$$p : Y := \mathbb{P}_X(\mathcal{O}_X \oplus \mathcal{O}_X(-1)) \to X.$$

Let $f : Y \to Z$ be the birational contraction of the negative section E on Y and H a general sufficiently ample $\mathbb{Q}$-divisor on Z such that $\lfloor H \rfloor = 0$ and $K_Y + E + f^*H$ is big. Set $\Delta_Y = E + f^*H$. Without loss of generality, we may assume that (Y, Δ_Y) is a canonical pair with $\lfloor E + f^*H \rfloor = E$. By the assumption (Conjecture B_n), $R(Y, \Delta_Y)$ is finitely generated. Then $(Y^\dagger, \Delta_{Y^\dagger})$, where $Y^\dagger = \operatorname{Proj} R(Y, \Delta_Y)$ and $\Delta_{Y^\dagger}$ is the pushforward of Δ_Y on $Y^\dagger$, is the log canonical model of (Y, Δ_Y) (see, for example, [KMM, Theorem 0-3-12]). By taking a suitable dlt blow-up of $(Y^\dagger, \Delta_{Y^\dagger})$, we obtain a good minimal model $(Y', \Delta_{Y'})$ of (Y, Δ_Y) (cf. [B1]). See also [B3, Theorem 3.7]. Note that $\varphi : Y \dashrightarrow Y'$ extracts no divisors since (Y, Δ_Y) is plt. Moreover, we may assume that this birational map

$$\varphi : Y \dashrightarrow Y'$$

is a composition of $(K_Y + \Delta_Y)$-flips and $(K_Y + \Delta_Y)$-divisorial contractions by [HX, Corollary 2.9]. We note that E is not contracted by φ. Indeed, if E is contracted, then E is uniruled. However, by [BDPP, 0.3 Corollary], E is not uniruled since K_E is pseudo-effective. Note that $E \simeq X$. Now we see that $K_{Y'} + \Delta_{Y'}$ is semi-ample by the finite generation of $R(Y', \Delta_{Y'})$, where $\Delta_{Y'} = \varphi_*\Delta_Y$. Take a general member $D' \in |m(K_{Y'} + \Delta_{Y'})|$ such that $\varphi_*E \not\subset \operatorname{Supp} D'$ for some sufficiently divisible positive integer m. Then D' induces some effective $\mathbb{Q}$-divisor D such that $D \sim_{\mathbb{Q}} K_Y + \Delta_Y$ and $E \not\subset \operatorname{Supp} D$. Thus we can see $\kappa(X, K_X) = \kappa(E, K_E) \geq 0$ since

$$K_E = (K_Y + \Delta_Y)|_E \sim_{\mathbb{Q}} D|_E \geq 0.$$

Therefore, we obtain Conjecture $\mathrm{F}_{\leq n-1}$. Q.E.D.

The following proof is pointed out by the referee:

Alternative proof of Lemma 3.2. Let Y, Z and E be as in the above proof of Lemma 3.2 and let A be an ample Cartier divisor such that

$\mathcal{O}_X(1) \simeq \mathcal{O}_X(A)$. Since K_X is pseudo-effective, $K_X + A$ is big. Let H' be a hyperplane on $Z \subset \mathbb{P}^{N+1}$. Then we can easily check that

$$K_Y + E + 2f^*H' \sim E + p^*(K_X + A).$$

Let $\widetilde{H}$ be a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on Z such that $2\widetilde{H}$ is a general member of $|4H'|$. Then $(Y, E + f^*\widetilde{H})$ is canonical, $\lfloor E + f^*\widetilde{H} \rfloor = E$, and

$$K_Y + E + f^*\widetilde{H} \sim_{\mathbb{Q}} E + p^*(K_X + A).$$

It is easy to see that $K_Y + E + f^*\widetilde{H}$ is big. Therefore, $R(Y, E + f^*\widetilde{H})$ is finitely generated by the assumption (Conjecture B_n). Since $\mathcal{O}_Y(E + p^*(K_X + A))$ is the tautological line bundle associated to the $\mathbb{P}^1$-bundle $\mathbb{P}_X(\mathcal{O}_X(K_X) \oplus \mathcal{O}_X(K_X + A)) \to X$, the finite generation of $R(Y, E + f^*\widetilde{H})$ is equivalent to that of

$$R(X; K_X, K_X + A) := \bigoplus_{m_1, m_2 \geq 0} H^0(X, \mathcal{O}_X(m_1K_X + m_2(K_X + A))).$$

By [CL, Theorem 3], this implies that $\kappa(X, K_X) \geq 0$ since K_X is pseudo-effective and $K_X + A$ is big. Q.E.D.

Proof of Lemma 3.3. This follows from [DHP, Theorem 8.8] and [G2, Theorem 1.5]. Note that we can use the ACC theorems in [HMX]. Q.E.D.

Proof of Lemma 3.4. By [B2], it is enough to show Conjecture $\mathrm{D}_{\leq n-1}$ and Conjecture $\mathrm{E}_{\leq n-1}$. We show these conjectures by induction on the dimension. Now we assume that Conjecture $\mathrm{D}_{\leq d-1}$ and Conjecture $\mathrm{E}_{\leq d-1}$ hold for $d < n$. Note that Conjecture $\mathrm{F}_{\leq n-1}$ holds by Lemma 3.2. By Lemma 3.3, Conjecture $\mathrm{E}_{\leq d}$ holds. On the other hand, by Lemma 3.1 and its proof, Conjecture $\mathrm{D}_{\leq d}$ holds. Thus we see that Conjecture $\mathrm{D}_{\leq n-1}$ and Conjecture $\mathrm{E}_{\leq n-1}$ hold. Q.E.D.

Remark 3.6. By [B2], Conjecture $\mathrm{D}_{\leq n}$ and Conjecture $\mathrm{E}_{\leq n}$ imply Conjecture $\mathrm{C}_{\leq n}$. This fact was used in the proof of Lemma 3.4. On the other hand, it is easy to see that Conjecture $\mathrm{C}_{\leq n}$ implies Conjecture $\mathrm{D}_{\leq n}$ and Conjecture $\mathrm{E}_{\leq n}$ by using dlt blow-ups. See also Remark 2.3.

Proof of Lemma 3.5. By [B2], we may assume that $K_X + \Delta$ is nef. By [Fk, Proposition 3.1] (cf. [K, Theorem 7.3]), we obtain that $K_X + \Delta$ is abundant, i.e. $\kappa(X, K_X + \Delta) = \nu(X, K_X + \Delta)$, where $\nu(\bullet)$ is the numerical dimension (see, for example, [KMM, Lemma 6-1-1]). Thus we see that $K_X + \Delta$ is semi-ample by [FG1, Theorem 4.6]. Q.E.D.

Now we give the proof of Theorem 1.1.

Proof of Theorem 1.1. It is obvious that Conjecture A_n implies Conjecture B_n. By Lemma 3.4, Conjecture B_n implies Conjecture $\mathrm{C}_{\leq n-1}$. By Lemma 3.5, Conjecture $\mathrm{C}_{\leq n-1}$ implies Conjecture $\mathrm{A}_{\leq n}$. Thus we finish the proof of Theorem 1.1. Q.E.D.

Finally, we discuss the corollaries. Corollary 1.2 is contained in Theorem 1.1. Corollary 1.3 is a direct consequence of Lemma 3.4 and Lemma 3.5. The proof of [F4, Theorem 1.1] works for Corollary 1.4. Note that [F4] depends on [B1] and [F1]. Now we can use more powerful results in [B2] and [FG1].

We close this section with a remark on [FM, Theorem 5.2].

Remark 3.7. Let (X, Δ) be a projective log canonical pair such that Δ is a $\mathbb{Q}$-divisor. Let $\Phi : X \dashrightarrow Z$ be the Iitaka fibration with respect to $K_X + \Delta$. By taking a suitable resolution, we assume that Φ is a morphism, X is smooth, and $\operatorname{Supp} \Delta$ is a simple normal crossing divisor on X. Suppose that every log canonical center of (X, Δ) is dominant onto Z. Then $R(X, \Delta)$ is finitely generated.

By using a generalization of the semipositivity theorem (see [F2, Theorem 3.9] and [FG2, Theorem 3.6]), we can formulate a canonical bundle formula for log canonical pairs as in [FM, Section 4]. By using the canonical bundle formula for log canonical pairs, the proof of [FM, Theorem 5.2] works for the above setting. We leave the details as exercises for the reader. Note that the finite generation of the log canonical rings for projective klt pairs holds by [BCHM].

§4. Appendix

In this appendix, we discuss Conjecture C. The results in this appendix are essentially contained in [FG1, Section 5].

Let us recall the following conjecture (see [DHP, Conjecture 1.3] and [FG1, Conjecture 1.10]).

Conjecture G (DLT extension conjecture)**.** Let (X, Δ) be a projective divisorial log terminal pair such that Δ is a $\mathbb{Q}$-divisor, $\lfloor \Delta \rfloor = S$, $K_X + \Delta$ is nef, and $K_X + \Delta \sim_{\mathbb{Q}} D \geq 0$ where $S \subset \operatorname{Supp} D$. Then the restriction map

$$H^0(X, \mathcal{O}_X(m(K_X + \Delta))) \to H^0(S, \mathcal{O}_S(m(K_X + \Delta)))$$

is surjective for all sufficiently divisible integers $m \geq 2$.

Theorem 4.1. *Conjecture $F_{\leq n}$ and Conjecture $G_{\leq n}$ imply Conjecture $C_{\leq n}$.*

Proof. By induction on the dimension, we may assume that Conjecture $\mathrm{C}_{\leq n-1}$ holds true. Therefore, we obtain Conjecture $\mathrm{D}_{\leq n-1}$ (cf. Remark 3.6). Lemma 3.3, Conjecture $\mathrm{F}_{\leq n}$, and Conjecture $\mathrm{D}_{\leq n-1}$ imply Conjecture $\mathrm{E}_{\leq n}$. Finally, by [FG1, Theorem 5.9 and Corollary 5.10], Conjecture $\mathrm{E}_{\leq n}$ and Conjecture $\mathrm{G}_{\leq n}$ imply Conjecture $\mathrm{C}_{\leq n}$. Q.E.D.

We note that, for Theorem 4.1, it is sufficient to prove Conjecture G under the extra assumptions: X is $\mathbb{Q}$-factorial, $\kappa(X, K_X + \Delta) = 0$, and $\operatorname{Supp} D \subset \operatorname{Supp} \Delta$. For the details, see the proof of [FG1, Theorem 5.9].

Remark 4.2. Conjecture G holds true if $K_X + \Delta$ is semi-ample (see [FG1, Proposition 5.12]). Therefore, Conjecture G follows from Conjecture D.

Remark 4.3. In [DHP, Conjecture 1.3], it is assumed that

$$S \subset \operatorname{Supp} D \subset \operatorname{Supp} \Delta$$

in Conjecture G.

Conjecture H (Abundance conjecture for klt pairs with $\kappa = 0$)**.** Let (X, Δ) be a projective kawamata log terminal pair such that Δ is a $\mathbb{Q}$-divisor with $\kappa(X, K_X + \Delta) = 0$. Then $\kappa_\sigma(X, K_X + \Delta) = 0$, where κ_σ denotes Nakayama's numerical dimension.

Remark 4.4. It is known that the condition $\kappa_\sigma(X, K_X + \Delta) = 0$ is equivalent to the existence of good minimal models of (X, Δ) (see, for example, [D] and [G1]).

We can easily check the following statement (cf. the proof of [FG1, Theorem 5.9]).

Theorem 4.5. *Conjecture $F_{\leq n}$ and Conjecture $H_{\leq n}$ imply Conjecture $C_{\leq n}$.*

We leave the details as exercises for the reader. The proof of Theorem 4.5 is almost the same as that of Theorem 4.1.

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Department of Mathematics, Graduate School of Science, Osaka University, Toyonaka, Osaka 560-0043, Japan
E-mail address: fujino@math.osaka-u.ac.jp

Graduate School of Mathematical Sciences, The University of Tokyo, 3-8-1 Komaba, Meguro, Tokyo, 153-8914 Japan.
E-mail address: gongyo@ms.u-tokyo.ac.jp