

## Cominuscule tableau combinatorics

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### Abstract.

We study “cominuscule tableau combinatorics” by generalizing constructions of M. Haiman, S. Fomin and M.-P. Schützenberger. In particular, we extend the *dual equivalence* ideas of [Haiman, 1992] to reformulate the generalized Littlewood-Richardson rule for cominuscule  $G/P$  Schubert calculus from [Thomas-Yong, 2006]. We apply dual equivalence to give an alternative and independent proof of the jeu de taquin results of [Proctor, 2004] needed in our earlier work. We also extend Fomin’s *growth diagram* description of jeu de taquin; the inherent symmetry of these diagrams leads to a generalization of Schützenberger’s *evacuation involution*. Finally, these results are applied to give an cominuscule extension of the *carton rule* of [Thomas-Yong, 2008].

### §1. Introduction

The cominuscule Schubert calculus rule of [ThYo09a] is based on results of R. Proctor [Pr04] on poset combinatorics, generalizing M.-P. Schützenberger’s [Sc77] *jeu de taquin* theory. In this paper, we begin with a cominuscule extension of M. Haiman’s *dual equivalence* [Ha92]. One consequence is an independent proof of those cases of R. Proctor’s theorem used in [ThYo09a]. It also permits us to reformulate our rule in a manner that avoids certain arbitrary choices demanded by the original version. In addition, we extend S. Fomin’s *growth diagrams* for jeu de taquin to the cominuscule setting and exploit their symmetry to give a simple formulation of this case of M.-P. Schützenberger’s *evacuation involution*. Finally, all of these results and constructions are then used to similarly extend the  $S_3$ -symmetric *carton rule* for Littlewood-Richardson coefficients [ThYo08].

This work contributes to the theory earlier developed in work of D. Peterson, R. Proctor and J. Stembridge, who show that many nice

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facts for maximal parabolic quotients of the symmetric group hold for  $d$ -complete posets, see, e.g., [St89, Pr04].

This paper is entirely combinatorial. Specifically, we do not discuss the geometry that connects this combinatorics to Schubert calculus. For more on that topic, we refer the reader to [ThYo09a] as well as its generalization due to P. E. Chaput-N. Perrin [ChPe12]. For additional context, we also mention that in [ThYo09b] we extended some of the combinatorics of this text to the context of  $K$ -theory. Further research in this direction may be found in, e.g., a paper of A. Buch–V. Ravikumar [BuRa12], a joint paper of the authors with E. Clifford [ClThYo14], as well as work of O. Pechenik [Pe12] and of A. Buch–M. Samuels [BuSa13].

### 1.1. Lie-theoretic data and jeu de taquin

We recall background used in [ThYo09a]. This paper centers around posets associated to seven families of generalized flag manifolds. These posets are explicitly described on the next page. Although we will present these posets in the Schubert calculus terminology of our previous work, these posets were earlier constructed starting from associated maximal parabolic subgroups, and called *minuscule posets* in [Pr84] (see in particular Section 12 of that paper for geometric remarks about cohomology of minuscule  $G/P$ 's).

Let  $G$  be a complex, connected, reductive Lie group with root system  $\Phi$ , positive roots  $\Phi^+$  and base of simple roots  $\Delta$ . Fix a choice of maximal parabolic subgroup  $P$  associated to a **cominuscule** simple root  $\beta(P)$ , i.e., if  $\beta(P)$  occurs in the simple root expansion of  $\gamma \in \Phi^+$ , it does so with coefficient one. Associated to  $G$  is the **poset of positive roots**  $\Omega_G = (\Phi^+, \prec)$  defined by the transitive closure of the covering relation  $\alpha \prec \gamma$  if  $\gamma - \alpha \in \Delta$ . Let

$$\Lambda_{G/P} = \{\alpha \in \Phi^+ : \alpha \text{ contains } \beta(P) \text{ in its simple root expansion}\} \subseteq \Omega_G,$$

the elements of which we refer to as **boxes**. Call the lower order ideals of  $\Lambda_{G/P}$  **straight shapes**, the set of which is denoted by  $\mathbb{Y}_{G/P}$ .

If  $\lambda \subseteq \nu$  are in  $\mathbb{Y}_{G/P}$ , their set-theoretic difference is the **skew shape**  $\nu/\lambda$ . A **standard filling** of  $\nu/\lambda$  is a bijection

$$\text{label} : \nu/\lambda \rightarrow \{1, 2, \dots, |\nu/\lambda|\} \text{ with } \text{label}(x) < \text{label}(y) \text{ if } x \prec y$$

(where  $|\nu/\lambda|$  denotes the number of boxes of  $\nu/\lambda$ ). This gives a **standard tableau**  $T$  of **shape**  $\nu/\lambda = \text{shape}(T)$ . Let  $\text{SYT}_{G/P}(\nu/\lambda)$  be the set of all such tableaux.



Given  $T \in \text{SYT}_{G/P}(\nu/\lambda)$  consider  $x \in \lambda$ , maximal in  $\prec$  subject to the condition that it is below *some* box of  $\nu/\lambda$ . Associate another standard tableau  $\text{jdt}_x(T)$ , called the **jeu de taquin slide** of  $T$  into  $x$ : Let  $y$  be the box of  $\nu/\lambda$  with the smallest label, among those covering  $x$ . Move  $\text{label}(y)$  to  $x$ , leaving  $y$  vacant. Look for boxes of  $\nu/\lambda$  covering  $y$  and repeat, moving into  $y$  the smallest label among those boxes covering it. Then  $\text{jdt}_x(T)$  results when no further slides are possible. A **rectification** of  $T$  is the result of iterating jeu de taquin slides until terminating at a straight shape standard tableau  $\text{rectification}(T)$ . Note that it is not obvious that there must be a unique rectification of a tableau; indeed, this is an important part of the theory which we present here (Corollary 2). As an example, one checks that there are two possible choices of orders of slides by which to rectify the  $\Lambda_{Gr(k, \mathbb{C}^n)}$

tableau above; using either order, the rectification is 

3			
1	2	4	

.

Given  $T \in \text{SYT}_{G/P}(\nu/\lambda)$ , consider  $x \in \Lambda_{G/P} \setminus \nu$  *minimal* in  $\prec$  subject to being *above* some element of  $\nu/\lambda$ . The **reverse jeu de taquin slide**  $\text{revjdt}_x(T)$  of  $T$  into  $x$  is defined similarly to a jeu de taquin slide, except we move into  $x$  the *largest* of the labels among boxes in  $\nu/\lambda$  covered by  $x$ .

We denote a sequence of slides by the sequence of boxes  $(x_1, \dots, x_k)$  utilized.

### 1.2. Dual equivalence

We now give a cominuscule extension of M. Haiman’s dual equivalence theory [Ha92]: Two tableaux  $T$  and  $U$  are **dual equivalent**, denoted  $T \equiv_D U$ , if any sequence of slides and reverse slides  $(x_1, \dots, x_k)$  for  $T$  and  $U$  results in tableaux of the same shape. Clearly,  $T \equiv_D U$  implies that  $\text{shape}(T) = \text{shape}(U)$  and moreover, it is easy to prove  $\equiv_D$  is an equivalence relation on tableaux.

One shape extends another if they can be written as  $\nu/\mu$  and  $\mu/\lambda$  respectively. If  $A$  and  $B$  are standard tableaux such that  $\text{shape}(B)$  extends  $\text{shape}(A)$ , let  $A \amalg B$  be the obvious standard tableau of shape  $\text{shape}(A) \cup \text{shape}(B)$  where the labels of  $B$  are increased by  $|\text{shape}(A)|$ .

Now suppose that  $\lambda \subseteq \mu \subseteq \nu \subseteq \rho$  are shapes, and let  $A, B, T$ , and  $U$  be tableaux such that

$$\text{shape}(A) = \mu/\lambda, \text{shape}(T) = \text{shape}(U) = \nu/\mu \text{ and } \text{shape}(B) = \rho/\nu.$$

Then it is straightforward [Ha92, Lemma 2.1] to show that

$$(1) \quad \text{if } T \equiv_D U \text{ then } A \amalg T \amalg B \equiv_D A \amalg U \amalg B.$$

Call the replacement of  $X := A \uparrow\uparrow T \uparrow\uparrow B$  by  $Y := A \uparrow\uparrow U \uparrow\uparrow B$  a **Haiman move**. Moreover, call a Haiman move **elementary** if the number of boxes  $m$  of  $T$  and  $U$  is:

$\Phi$	$A_{n-1}$	$B_n$	$C_n$	$D_n (\mathbb{Q}^{2n-2})$	$D_n (OG(n+1, 2n+2))$	$E_6$	$E_7$
$m$	3	—	4	$n$	4	5	6

(In  $B_n(\mathbb{Q}^{2n-1})$ , every shape has exactly one filling, and every dual equivalence class has exactly one member.)

**Theorem 1.** *For a cominuscule  $G/P$ :*

- (I) *Any two standard fillings of a straight shape  $\lambda$  are dual equivalent.*
- (II)  *$X, Y \in \text{SYT}_{G/P}(\nu/\lambda)$  are dual equivalent if and only if they are connected by a chain of **elementary Haiman moves**.*
- (III) *There is a unique straight shape of size  $m$  (as given in the table above) having two standard fillings  $T \equiv_D U$ . All other pairs of dual equivalent tableaux of this size are obtained by applying a sequence of jeu de taquin slides to this pair.*

In [Ha92] the main infinite cases ( $Gr(k, \mathbb{C}^n)$ ,  $LG(n, 2n)$  and  $OG(n+1, 2n+2)$ ) of the above theorem were proved. He moreover notes that many (but not all) aspects of his proof generalize to arbitrary posets. Indeed, our proof of Theorem 1 follows an approach similar to that used in his paper. In particular, part (II) is an extension of [Ha92, Proposition 2.4].

Besides the new exceptional cases of the above theorem, our proof differs in two ways from Haiman’s. First we avoid the need for “reading word orders” which were unavailable to us for the exceptional type cases of our theorem. Second, we introduce a simplification (Lemma 7) which reduces our proof of (III) in the exceptional case to a finite check that can be done (tediously) by hand, or, preferably, by computer, as is explained in our proof. This Lemma also simplifies the checks needed in the previously known cases. We will discuss these aspects in greater detail in Section 2.

R. Proctor [Pr04] has proved the following corollary in the greater generality of “ $d$ -complete posets” (not treated here), extending [Sc77, Sa87, Wo84]. We apply Theorem 1 to obtain an alternative proof for the cominuscule setting.

**Corollary 2.** *Given  $T \in \text{SYT}_{G/P}(\nu/\lambda)$ ,  $\text{rectification}(T)$  is independent of the order of jeu de taquin slides.*

In [Pr04, p. 5], R. Proctor credits D. Peterson for telling him that Corollary 2 is true; he writes that Peterson used a computer to prove this result. In view of our proof of Theorem 1, our proof of this Corollary (for the exceptional cases) is also ultimately computationally based. However, utilizing the technology of dual equivalence allows us to avoid checking rectifications of all standard tableaux in types  $E_6$  and  $E_7$ , and replaces this by a significantly smaller computer check (small enough to be carried out by hand in type  $E_6$ ).

We now apply dual equivalence to Schubert calculus. Each  $G/P$  is a union of  $B_-$ -orbits whose closures  $X_w := \overline{B_-wP/P}$  with  $wW_P \in W/W_P$  are the **Schubert varieties**. The cosets  $W/W_P$  correspond bijectively to straight shapes in  $\Lambda_{G/P}$ , so we can also refer to the Schubert varieties as  $X_\lambda$  for  $\lambda \in \mathbb{Y}_{G/P}$ . (The existence of such a natural correspondence between cosets and order ideals in a poset is a special feature of the cominuscule setting.) The Poincaré duals  $\{\sigma_\lambda\}$  of the Schubert varieties form the **Schubert basis** of the cohomology ring  $H^*(G/P; \mathbb{Z})$ . The **Schubert intersection numbers**  $\{c_{\lambda,\mu}^\nu(G/P)\}$  are defined by

$$(2) \quad \sigma_\lambda \cdot \sigma_\mu = \sum_{\nu \in \mathbb{Y}_{G/P}} c_{\lambda,\mu}^\nu(G/P) \sigma_\nu.$$

If the root system  $\Phi$  is not simply-laced, then its roots have two lengths, referred to as “long” and “short”. If  $\Phi$  is simply-laced, so all roots have the same length, we consider them all to be long. Let  $\mathbf{shortroots}(\cdot)$  be the number of boxes of a shape that are short roots. The following result relies on our earlier rule [ThYo09a, Main Theorem] to make the connection to Schubert calculus.

**Theorem 3.** *For cominuscule  $G/P$ , the number  $c_{\lambda,\mu}^\nu(G/P)$  equals  $2^{\mathbf{shortroots}(\nu/\lambda) - \mathbf{shortroots}(\mu)}$  times the number of dual equivalence classes of tableaux of shape  $\nu/\lambda$  rectifying to a tableau of shape  $\mu$ .*

Theorem 3 appears less explicit than our original rule [ThYo09a, Main Theorem] (reproduced below as Theorem 12), although both are computationally similar, see the remarks in Section 3. However, Theorem 3 has its advantages: it does not depend on a fixed choice of tableau of shape  $\mu$  to rectify to, and its statement is meaningful even in contexts where Corollary 2 is unavailable. In this sense, it is more transparent, and possibly useful, e.g., when finding rules for non-(co)minuscule  $G/P$ .

In [ThYo09a], a rule was also given for Schubert calculus for minuscule  $G/P$ . Every minuscule case has a corresponding cominuscule case  $(G/P)^\vee$  associated to the Langlands dual group  $G^\vee$  of  $G$ . The Schubert varieties and classes for the minuscule  $G/P$  can be indexed by

shapes in the corresponding cominuscule  $\Lambda_{(G/P)^\vee}$ . Thus, we also have the following reformulation of the minuscule rule of [ThYo09a]:

**Corollary 4.** *For minuscule  $G/P$ ,  $c_{\lambda^\vee}^\nu(G/P)$  is the number of dual equivalence classes of tableaux in  $\Lambda_{(G/P)^\vee}$  of shape  $\nu/\lambda$  which rectify to a tableau of shape  $\mu$ .*

### 1.3. Growth diagrams and their applications

S. Fomin’s *growth diagrams* provide a way to encode jeu de taquin. In section 2.1, we explain their straightforward generalization to cominuscule types. Growth diagrams make apparent a symmetry of jeu de taquin which we refer to as the “infusion involution” in [ThYo09a] (see also [Ha92, Lemma 2.7]).

M.-P. Schützenberger defined *evacuation* for an arbitrary finite poset. (See, e.g., the survey [St09] for background and references.) Growth diagrams allow us to give a new formulation of evacuation for cominuscule posets  $\Lambda_{G/P}$ . As for the classical setting of standard Young tableaux, the fact that evacuation is an involution is immediate from this perspective.

We refer to shapes of the form  $\Lambda_{G/P}/\rho$  as **reverse shapes**. There is a natural bijection between straight shapes and reverse shapes, as follows. Pick a tableau  $T$  of straight shape  $\lambda$ . If we apply as many **revjdt** slides as possible to  $T$ , the result is a tableau of reverse shape, say  $\Lambda_{G/P}/\rho$ . Since all standard fillings of  $\lambda$  are dual equivalent by Theorem 1(I), this shape only depends on  $\lambda$ , not on  $T$ , so we define  $\lambda^\vee = \rho$ . Since  $\Lambda_{G/P}$  is self-dual, the same procedure can be reversed to go from  $\lambda^\vee$  to  $\lambda$ . Thus the map  $\lambda \rightarrow \lambda^\vee$  is a bijection.

For  $\lambda, \mu, \nu \in \mathbb{Y}_{G/P}$  define  $c_{\lambda, \mu, \nu}(G/P) = c_{\lambda^\vee}^\nu(G/P)$ . As  $c_{\lambda, \mu, \nu}(G/P)$  is equal to the number of intersections of generic translates by elements of  $G$  of the Schubert varieties  $X_\lambda$ ,  $X_\mu$ , and  $X_\nu$ , one has the obvious  $S_3$ -symmetries:

$$(3) \quad \begin{aligned} c_{\lambda, \mu, \nu}(G/P) &= c_{\mu, \nu, \lambda}(G/P) = c_{\nu, \lambda, \mu}(G/P) \\ &= c_{\mu, \lambda, \nu}(G/P) = c_{\nu, \mu, \lambda}(G/P) = c_{\lambda, \nu, \mu}(G/P). \end{aligned}$$

In [ThYo08] we constructed a **carton rule** for  $c_{\lambda, \mu, \nu}$  in the Grassmannian case that transparently and uniformly explains all of the symmetries (3). As we explain in Section 5, dual equivalence, growth diagrams, and evacuation give us the tools we need to extend our construction to the cominuscule setting.

## §2. Growth diagrams and dual equivalence

As mentioned above, several steps in our development of cominuscule dual equivalence will be familiar to readers of [Ha92]. However, a



Note that the top row encodes the original tableau, while the left column  $\emptyset - (1) - (1^2) - (1^3)$  corresponds to the tableau  $R = \begin{bmatrix} 1 & 2 & 3 \\ & & \end{bmatrix}$ , which describes the order of the jeu de taquin slides in the rectification.

Growth diagrams can be characterized in the following way:

**Theorem 5.** *A rectangular array of straight shapes in  $\mathbb{Y}_{G/P}$  is a growth diagram if and only if for any  $2 \times 2$  subgrid  $\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$  the **Fomin growth conditions** hold:*

- (F0)  $\alpha/\gamma, \delta/\gamma, \beta/\alpha,$  and  $\beta/\delta$  each consist of a single box;
- (F1) if  $\alpha$  is the unique shape containing  $\gamma$  and contained in  $\beta$ , then  $\delta = \alpha$ ;
- (F2) otherwise there is a unique such shape other than  $\alpha$ , and this shape is  $\delta$ .

*Proof.* We first check that a growth diagram satisfies the growth conditions. The  $i$ -th row of the growth diagram defines a tableau  $T_i$ . To verify (F0), consider the  $2 \times 2$  subgrid located in rows  $i$  and  $i + 1$ , and columns  $j$  and  $j + 1$ . Then  $\beta/\alpha$  is the position of  $j$  in  $T_{i+1}$ , while  $\delta/\gamma$  is the position of  $j$  in  $T_i$ . In the course of the jdt slide which changes  $T_{i+1}$  to  $T_i$ , we have that  $\alpha/\gamma$  is the position of the empty box after the boxes numbered 1 to  $j - 1$  have moved from their positions in  $T_{i+1}$  to their positions in  $T_i$ , and  $\beta/\delta$  is the position of the empty box after box  $j$  has also moved. This establishes (F0). Next, we observe that condition (F1) is automatically satisfied given (F0). (It is included in the growth conditions for clarity.) Finally, if there is a unique shape between  $\beta$  and  $\gamma$  other than  $\alpha$ , then after we have moved 1 through  $j - 1$  from their positions in  $T_{i+1}$  to their positions in  $T_i$ , then the empty box and the box containing  $j$  are not adjacent, with the result that  $j$  occupies the same position in  $T_i$  as in  $T_{i+1}$ , which implies (F2). This establishes that the growth conditions hold for each  $2 \times 2$  subgrid.

Conversely, suppose that we have a rectangular array of shapes satisfying the growth conditions. Interpret the leftmost column as a straight shape  $A$ , and interpret the top row as a skew shape  $B$ . Now consider the growth diagram for the rectification of  $B$  in the rectification order given by  $A$ . This new diagram has the same left column and top row as our original diagram, and both satisfy the growth conditions. Since the growth conditions suffice to determine the whole array given the left column and top row, the two arrays must coincide, and the given array must be a growth diagram. Q.E.D.

Observe that the Fomin growth conditions are symmetric under a transposition about the bottom-left/top-right diagonal. This leads to an

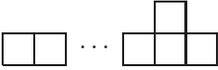
important tableau-theoretic involution, which we refer to as “infusion” (this is a much older concept, see [Ha92] as well as, e.g., [BeSoSt96]). Let  $A$  be a standard tableau of shape  $\lambda$ , and  $B$  a standard tableau of shape  $\nu/\lambda$ . We define  $\mathbf{infusion}(A, B) = (C, D)$  where  $C$  is the result of rectifying  $B$  according to the order given by  $A$ , and  $D$  is the tableau which records the order in which boxes of  $\nu$  were emptied in the rectification procedure. If we consider the growth diagram for the rectification of  $B$  in the order  $A$ , the bottom row gives the shape chain for  $C$ , and the rightmost column gives the shape chain for  $D$ . The fact that growth diagrams are transpose symmetric then implies that  $\mathbf{infusion}(C, D) = (A, B)$ ; that is to say, that infusion is an involution. For future use, we record the notation that if  $\mathbf{infusion}(A, B) = (C, D)$ , then  $\mathbf{infusion}_1(A, B) = C$  and  $\mathbf{infusion}_2(A, B) = D$ .

In fact, the same proof holds in a slightly more general setting. The following fact was also proved in [ThYo09a, Theorem 4.4].

**Lemma 6.** *For any standard tableaux  $T$  and  $U$  such that  $\mathbf{shape}(U)$  extends  $\mathbf{shape}(T)$  then  $\mathbf{infusion}(\mathbf{infusion}(T, U)) = (T, U)$ .*

### 2.2. Proof of Theorem 1

Consider the **basic shapes**, which are the minimal shapes in each  $\Lambda_{G/P}$  having two standard fillings, as displayed in Table 2.

$G/P$	$Gr(k, \mathbb{C}^n)$	$\mathbb{Q}^{2n-1}$	$LG(n, 2n)$	$\mathbb{Q}^{2n-2}$
$\lambda$	$(2, 1)$	—	$(1, 2, 1)$	$(1^{n-3}, 2, 1)$
				

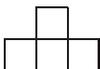
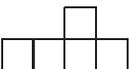
$G/P$	$OG(n+1, 2n+2)$	$\mathbb{O}\mathbb{P}^2$	$G_\omega(\mathbb{O}^3, \mathbb{O}^6)$
$\lambda$	$(1, 2, 1)$	$(1, 1, 2, 1)$	$(1, 1, 1, 2, 1)$
			

Table 2. The basic shapes for cominuscule posets.

We now establish a special case of Theorem 1(I), which turns out to be fundamental:

**Lemma 7.** *For each cominuscule  $G/P$ , the two tableaux of the basic shape are dual equivalent.*

*Proof.* Consider a sequence of slides

$$(4) \quad \text{revjdt}_{x_1}(\cdot), \dots, \text{jdt}_{x_i}(\cdot), \dots, \text{revjdt}_{x_j}(\cdot), \dots$$

associated to boxes  $\{x_i\} \subseteq \Lambda_{G/P}$ . Let  $T_1$  and  $T_2$  be the two standard tableaux of the basic shape. Call a **direction change** in (4) a **revjdt** slide followed by a **jdt** slide, or a **jdt** slide followed by a **revjdt** slide. We induct on the number of direction changes to show that the shapes of  $T_1$  and  $T_2$  under (4) are the same.

In the base case, there are none, and the conclusion is a straightforward (but tedious) verification; in the classical types analyzed in [Ha92], a similar approach was also suggested. However, here our task is actually simpler since we only need to check for size  $m = 3$  (in the  $Gr(k, \mathbb{C}^n)$  case) and  $m = 4$  (in the  $LG(n, 2n)$  and  $OG(n + 1, 2n + 2)$  cases) that reverse slides preserve the equality of shapes of the two tableaux of these sizes. Although this is an infinite check, the possibilities for how the relative positions of the  $m$  boxes can appear (in relation to  $\Lambda_{G/P}$ ) is small and can be indeed analyzed (although we omit the details).

The other classical types are easy to check.

Finally, in the exceptional types the check is finite. In type  $E_6$ , we carried this check out by hand. In type  $E_7$ , the number of cases is significantly larger. Though it is still within reach of human verification, we preferred to handle this case using a simple **Maple** program<sup>1</sup> (which we also used to reconfirm our hand-calculations for type  $E_6$ ). Our program constructs all possible (partial) reverse rectifications recursively (at each recursive step, each possible reverse jeu de taquin move is determined). This check takes no more than a few minutes on a computer.

This concludes the discussion of the base case of this proof.

Now we assume that there is at least one direction change. The first direction change is of the form “**revjdt** $_{x_c}(\cdot)$ , **jdt** $_{x_{c+1}}(\cdot)$ ”. Up until  $x_c$ , we have been solely applying **revjdt** slides, obtaining, by the base case,  $T'_1$  and  $T'_2$  of the same shape.

Recall that  $\Lambda_{G/P}$  is self-dual, and that we refer to a shape of the form  $\Lambda_{G/P}/\lambda$  as a reverse shape.

By the base case, there is sequence of slides

$$\text{revjdt}_{z_1}(\cdot), \dots, \text{revjdt}_{z_M}(\cdot)$$

that “reverse rectify”  $T'_1$  and  $T'_2$  to tableaux  $T''_1$  and  $T''_2$  of the same reverse shape. Suppose

$$\text{jdt}_{y_M}(\cdot), \dots, \text{jdt}_{y_1}(\cdot)$$

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<sup>1</sup>Software available at the authors’ websites.

are the slides that undo the  $\{z_i\}$  slide sequence, returning us to  $T'_1, T'_2$ . Observe that by the self-duality of  $\Lambda_{G/P}$  we can interpret

$$\mathbf{jdt}_{y_M}(\cdot), \dots, \mathbf{jdt}_{y_1}(\cdot), \mathbf{jdt}_{x_{c+1}}(\cdot)$$

as a sequence of **revjdt** slides for the dual poset to  $\Lambda_{G/P}$  which is, of course, isomorphic to  $\Lambda_{G/P}$ . Finally, concatenating the slides into  $x_{c+2}, \dots, x_N$  of (4) reinterpreted by  $\mathbf{jdt} \leftrightarrow \mathbf{revjdt}$ , we obtain a new sequence of slides with one fewer direction change that passes through  $T'_1, T'_2$ . Thus by induction, the resulting tableaux have the same shape, and therefore the same would be true of applying (4) to  $T_1, T_2$ . Q.E.D.

**Lemma 8.** *Let  $c$  and  $d$  be two distinct corners of  $\lambda \in \mathbb{Y}_{G/P}$ . There exists a sequence of jeu de taquin slides that when applied to one of the two standard tableaux of the basic shape  $\beta$ , the entry  $|\beta| - 1$  is sent to  $c$  and the entry  $|\beta|$  to  $d$ , while for the other standard tableau,  $|\beta|$  is sent to  $c$  and  $|\beta| - 1$  to  $d$ .*

*Proof.* Mark the two corners (i.e., those containing  $|\beta| - 1$  and  $|\beta|$ ) of  $\beta$  with a “ $\star$ ”. We wish to show that there is a sequence of jeu de taquin slides moving the two  $\star$ 's to  $c, d$ , without ever producing a situation where the two  $\star$ 's are trying to move into the same box. Clearly, such a sequence of jeu de taquin slides can be constructed by moving each of the  $\star$ 's along the boundary (as drawn in the plane) of the minimal straight shape containing  $c$  and  $d$ ; one takes the northwest boundary and the other the southeast boundary. Q.E.D.

**Lemma 9.** *Let  $c, d$  be two distinct corners of  $\lambda \in \mathbb{Y}_{G/P}$ . Then there exist two tableaux  $S_1$  and  $S_2$  of shape  $\lambda$ , related by a single elementary Haiman move, such that  $S_1$  has  $|\lambda|$  in  $c$  while  $S_2$  has  $|\lambda|$  in  $d$ .*

*Proof.* Start with the two fillings of the basic shape. By Lemma 7 these are dual equivalent. Apply the sequence of slides constructed in Lemma 8. The result is two dual equivalent fillings  $B_1, B_2$  of a shape  $\lambda/\gamma$  for some  $\gamma$ , one having its maximum entry in  $c$ , the other having its maximum entry in  $d$ . Let  $A$  be an arbitrary standard filling of  $\gamma$ . Then  $S_i := A \amalg B_i$  satisfy the statement of the lemma. Q.E.D.

For use below, we point out the following facts which follow immediately from the definition of dual equivalence:

**Lemma 10.** *If  $T \equiv_D U$  then  $\mathbf{jdt}_x(T) \equiv_D \mathbf{jdt}_x(U)$  and we also have  $\mathbf{revjdt}_x(T) \equiv_D \mathbf{revjdt}_x(U)$ .*

*If  $T \equiv_D U$  by an elementary Haiman move, then the same is true for the tableaux resulting from applying the same slide to  $T$  and  $U$ .*

*Conclusion of the proof of Theorem 1:* To prove (I), we induct on  $|\lambda|$ . The base case  $\lambda = \emptyset$  is obvious. Now suppose  $\lambda$  has at least one box. By induction, for any corner  $c$  of  $\lambda$ , there are (elementary) Haiman moves connecting those tableaux having  $|\lambda|$  in  $c$ . Thus we are done if there is only one corner of  $\lambda$ , so suppose there are at least two corners  $c$  and  $d$ , and  $T_1, T_2 \in \text{SYT}_{G/P}(\lambda)$  where  $T_1$  has  $|\lambda|$  in  $c$  and  $T_2$  has  $|\lambda|$  in  $d$ . By Lemma 9, there is

$$S_1 \equiv_D S_2 \text{ with } \text{shape}(S_1) = \text{shape}(S_2) = \lambda$$

such that  $S_1$  has  $|\lambda|$  in  $c$  and  $S_2$  has  $|\lambda|$  in  $d$ . Thus

$$T_1 \equiv_D S_1 \equiv_D S_2 \equiv_D T_2$$

as desired.

For (II), “ $\Leftarrow$ ” is trivial. Conversely, let  $T \in \text{SYT}_{G/P}(\lambda)$ . By (I), there is a chain of elementary Haiman moves

$$\text{infusion}_1(T, X) = C_0 \equiv_D C_1 \equiv_D \cdots \equiv_D C_N = \text{infusion}_1(T, Y).$$

Since  $X \equiv_D Y$ ,  $\text{infusion}_2(T, X) = \text{infusion}_2(T, Y)$ . Let

$$D_i = \text{infusion}_2(C_i, \text{infusion}_2(T, X)).$$

Then by Lemma 10, it follows that

$$X = D_0 \equiv_D D_1 \equiv_D \cdots \equiv_D D_N = Y$$

is a chain of elementary Haiman moves.

For (III), the assertion that there are only two fillings of size  $m$  is obvious. That these two fillings are dual equivalent is Lemma 7. The second claim follows by choosing a rectification sequence for the given dual equivalent tableaux. Since the two resulting tableaux must be different fillings of the same straight shape, the result follows by the first assertion. Q.E.D.

### 2.3. Proof of Corollary 2

Let  $T$  be a skew tableau in  $\text{SYT}_{G/P}(\nu/\lambda)$ , and write  $x_i$  for the box of  $T$  with entry  $i$ . Let  $A, B \in \text{SYT}_{G/P}(\lambda)$  encode two possible rectification orders for  $T$ . Since  $A \equiv_D B$ , we have that we have that  $\text{infusion}_1(A, T) = \text{infusion}_1(B, T)$ . Q.E.D.

**2.4. Proof of Theorem 3**

A pair of tableaux  $T, U$  are **jeu de taquin equivalent** if

$$\text{rectification}(T) = \text{rectification}(U).$$

They are merely **shape equivalent** if

$$\text{shape}(\text{rectification}(T)) = \text{shape}(\text{rectification}(U)).$$

**Proposition 11.** *Fix a shape  $\nu/\lambda \subseteq \Lambda_{G/P}$ . Within each shape equivalence class, each jeu de taquin equivalence class meets each dual equivalence class in a unique  $T \in \text{SYT}_{G/P}(\nu/\lambda)$ .*

*Proof.* Fix a choice of  $U \in \text{SYT}_{G/P}(\lambda)$ . We must show that for any  $A, B \in \text{SYT}_{G/P}(\nu/\lambda)$  that are shape equivalent, there exists a unique  $T \in \text{SYT}_{G/P}(\nu/\lambda)$  such that  $\text{infusion}_1(U, A) = \text{infusion}_1(U, T)$  (i.e.,  $T$  and  $A$  are in the same jeu de taquin class) and  $T \equiv_D B$ .

Notice that in fact, if we write  $R$  for  $\text{infusion}_2(U, B)$ , then

$$\text{infusion}_1(U, \cdot) \text{ and } \text{infusion}_2(\cdot, R)$$

are mutually inverse bijections between

the dual equivalence class of  $B$  and  $\text{SYT}_{G/P}(\text{shape}(\text{infusion}_1(U, B)))$ .

Therefore

$$T = \text{infusion}_2(\text{infusion}_1(U, A), R)$$

does the job.

Q.E.D.

Theorem 3 then follows immediately from Proposition 11 and

**Theorem 12.** ([ThYo09a, Main Theorem]) *For cominuscule  $G/P$ , let  $\lambda, \mu, \nu \in \mathbb{Y}_{G/P}$  and fix  $T_\mu \in \text{SYT}_{G/P}(\mu)$ . Then the number  $c'_{\lambda, \mu}(G/P)$  is  $2^{\text{shortroots}(\nu/\lambda) - \text{shortroots}(\mu)}$  times the number of standard tableaux of shape  $\nu/\lambda$  whose rectification is  $T_\mu$ .*

*For minuscule  $G/P$ , let  $\lambda, \mu, \nu \in \mathbb{Y}_{(G/P)^\vee}$  and fix  $T_\mu \in \text{SYT}_{G/P}(\mu)$ . Then  $c'_{\lambda, \mu}(G/P)$  is the number of standard tableaux of shape  $\nu/\lambda$  whose rectification is  $T_\mu$ .* Q.E.D.

**§3. Further discussion of dual equivalence**

**3.1. Computing  $c'_{\lambda, \mu}(G/P)$**

Consider  $G/P = \mathbb{O}\mathbb{P}^2$ , the Cayley plane associated to the root system  $E_6$ , and the skew shape  $\nu/\lambda = (1, 1, 2, 3, 1)/(1, 1, 1)$ . The seven

fillings are given in Table 3. In this **Haiman table**, the rows give the jeu de taquin equivalence classes, and the columns give the dual equivalence classes, in agreement with Proposition 11. The rightmost column computes the common **rectification** of the tableaux in a given row. Theorem 3 says, e.g., that  $c_{(1,1,1),(1,1,2,1)}^{(1,1,2,3,1)}(\mathbb{O}\mathbb{P}^2) = 3$  by counting the middle three columns. Meanwhile Theorem 12 says count the three tableaux in *either* the second or third row.

In practice, both rules are similar: in using Theorem 12, we do not know of any general way to avoid essentially checking all skew tableaux of shape  $\nu/\lambda$ . So, we basically produce much of the information needed to construct a Haiman table, which encodes all coefficients  $c_{\lambda,\gamma}^\nu(G/P)$  as  $\gamma$  varies. (To determine if two tableaux are dual equivalent, check if one tableau’s rectification sequence works for the other, and produces the same shape.)

### 3.2. The Haiman table and the generalized Robinson-Schensted correspondence

Organizing one’s thoughts about Schubert intersection numbers this way can be illuminating. For example, when

$$G/P = Gr(k, \mathbb{C}^n) \text{ and } \nu/\lambda = (k, k - 1, \dots, 3, 2)/(k - 1, k - 2, \dots, 2, 1)$$

the standard fillings are in obvious bijection with the symmetric group  $S_k$ . The last column is the “insertion tableau” of the Schensted insertion algorithm. The “recording tableau” of his algorithm labels the columns. Viewed this way, Proposition 11 generalizes Robinson-Schensted to arbitrary standard (cominuscule) tableaux, extending an observation of [Ha92].

### 3.3. Reading word order?

Further considering  $\Lambda_{\mathbb{O}\mathbb{P}^2}$ , we explain our difficulties in finding a reading word order for general cominuscule type. In [Ha92] the reading word of a shape is defined by reading its entries from left to right and from bottom to top, one row at a time. For shapes (respectively, shifted shapes), [Ha92] gives a short list of pairs of reading words such that if  $T$  and  $U$  are tableaux of size  $m = 3$  (respectively,  $m = 4$ ) then  $T \equiv_D U$  if and only if the reading words of  $T$  and  $U$  appear on this list. However, for  $\mathbb{O}\mathbb{P}^2$ , we have the following four tableaux:

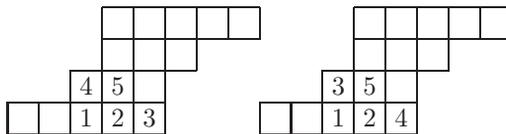
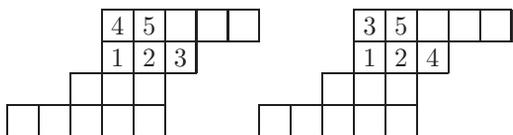



Table 3. A Haiman table: standard tableaux, their jeu de taquin and dual equivalence classes



The first two are dual equivalent while the second two are not. These pairs of tableaux, however, clearly have the same pairs of reading words, with respect to the obvious extension of the definition in [Ha92] or, indeed, with respect to any reading word order defined exclusively by planar geometry, since the corresponding entries are in the same relative positions in the two examples.

The question of a general cominuscule description of a reading word order is part of the broader question of finding a “semistandard” theory, together with a “lattice word” Schubert calculus rule; see, e.g., [St99, St89] and the references therein.

#### §4. Schützenberger’s evacuation involution

In this section we show how the cominuscule growth diagram approach leads to a simple proof that M. P. Schützenberger’s evacuation is an involution in the cominuscule setting. Again, our proofs parallel those in S. Fomin’s Appendix 1 to [St99, Chapter 7].

The classical evacuation involution appears prominently in combinatorial representation theory and algebraic geometry; see, e.g., [St96], and the references therein. For  $T \in \text{SYT}_{G/P}(\lambda)$ , let  $\tilde{T}$  be obtained by erasing the entry 1 of  $T$  in  $\beta(P)$  (the minimal element of  $\Lambda_{G/P}$ ) and subtracting 1 from the remaining entries. Let  $\Delta(T) = \mathbf{jdt}_{\beta(P)}(\tilde{T})$ . The **evacuation**  $\mathbf{evac}(T) \in \text{SYT}_{G/P}(\lambda)$  is defined by the shape chain

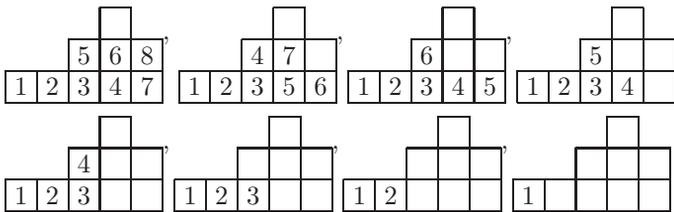
$$\emptyset = \mathbf{shape}(\Delta^{|\lambda|}(T)) - \mathbf{shape}(\Delta^{|\lambda|-1}(T)) - \dots - \mathbf{shape}(\Delta^1(T)) - \mathbf{shape}(T).$$

**Theorem 13.**  $\mathbf{evac} : \text{SYT}_{G/P}(\lambda) \rightarrow \text{SYT}_{G/P}(\lambda)$  is an involution, i.e.,  $\mathbf{evac}(\mathbf{evac}(T)) = T$ .

For example, if  $T =$ 

				7	
			4	6	9
1	2	3	5	8	

 $\in \text{SYT}_{G/P}((1, 1, 2, 3, 2))$ , iterating  $\Delta$  gives



and hence

$$\mathbf{evac}(T) =$$

				9	
			4	7	8
1	2	3	5	6	

The reader can check that  $\mathbf{evac}(\mathbf{evac}(T)) = T$ .

*Proof of Theorem 13:* Express each of the tableaux

$$T, \Delta^1(T), \dots, \Delta^{|\lambda|-1}(T), \Delta^{|\lambda|}(T) = \emptyset$$

as a shape chain and place them right justified in a triangular growth diagram. In the example above, we have Table 4. Noting that each “minor” of the table whose southwest corner contains a “ $\emptyset$ ” is in fact a growth diagram, it follows that the triangular growth diagram can be reconstructed using the top row and the growth conditions of Theorem 5. Observe that the right column encodes  $\mathbf{evac}(T)$ . By the symmetry of growth diagrams, it follows that applying the above procedure to  $\mathbf{evac}(T)$  would give the same triangular growth diagram, after a reflection across the antidiagonal. Thus the result follows. Q.E.D.

$\emptyset$	(1)	(1 <sup>2</sup> )	(1 <sup>3</sup> )	(1 <sup>2</sup> , 2)	(1 <sup>2</sup> , 2, 1)	(1 <sup>2</sup> , 2, 2)	(1 <sup>2</sup> , 2, 3)	(1 <sup>2</sup> , 2, 3, 1)	(1 <sup>2</sup> , 2, 3, 2)
	$\emptyset$	(1)	(1 <sup>2</sup> )	(1 <sup>3</sup> )	(1 <sup>4</sup> )	(1 <sup>2</sup> , 2, 1)	(1 <sup>2</sup> , 2 <sup>2</sup> )	(1 <sup>2</sup> , 2 <sup>2</sup> , 1)	(1 <sup>2</sup> , 2 <sup>3</sup> )
		$\emptyset$	(1)	(1 <sup>2</sup> )	(1 <sup>3</sup> )	(1 <sup>2</sup> , 2)	(1 <sup>2</sup> , 2, 1)	(1 <sup>2</sup> , 2, 1 <sup>2</sup> )	(1 <sup>2</sup> , 2 <sup>2</sup> , 1)
			$\emptyset$	(1)	(1 <sup>2</sup> )	(1 <sup>3</sup> )	(1 <sup>4</sup> )	(1 <sup>5</sup> )	(1 <sup>2</sup> , 2, 1 <sup>2</sup> )
				$\emptyset$	(1)	(1 <sup>2</sup> )	(1 <sup>3</sup> )	(1 <sup>4</sup> )	(1 <sup>2</sup> , 2, 1)
					$\emptyset$	(1)	(1 <sup>2</sup> )	(1 <sup>3</sup> )	(1 <sup>2</sup> , 2)
						$\emptyset$	(1)	(1 <sup>2</sup> )	(1 <sup>3</sup> )
							$\emptyset$	(1)	(1 <sup>2</sup> )
								$\emptyset$	(1)
									$\emptyset$

Table 4. A triangular growth diagram, for the proof of Theorem 13.

§5. Cartons

The goal of this section is to extend the main result of [ThYo08] to the cominuscule setting. Our description of the rule closely parallels the one for the original rule from our earlier paper.

5.1. Statement of the rule

Let  $\lambda$ ,  $\mu$ , and  $\nu$  be shapes in  $\mathbb{Y}_{G/P}$ , such that  $|\lambda| + |\mu| + |\nu| = |\Lambda_{G/P}|$ . (When this condition is not satisfied,  $c_{\lambda, \mu, \nu}(G/P)$  is necessarily 0.)

Figure 1 depicts a **carton**. This is a  $|\lambda| \times |\mu| \times |\nu|$  box, with a grid whose squares are  $1 \times 1$  drawn on each of the six faces. One vertex of the box is labelled  $\emptyset$ , and its opposite vertex is labelled  $\Lambda_{G/P}$ . A **carton filling** assigns a Young diagram to each vertex of the grid so that shapes increase one box at a time while moving away from the  $\emptyset$ , so that for any

$$\begin{array}{ccc}
 \alpha & - & \beta \\
 2 \times 2 \text{ subgrid } & \left| \begin{array}{c} \phantom{\alpha} \\ \phantom{\beta} \\ \phantom{\gamma} \\ \phantom{\delta} \end{array} \right| & \text{the Fomin growth conditions of Theorem 5} \\
 \gamma & - & \delta
 \end{array}$$

hold. Note that there are vertices of the grid which lie on edges of the box, and thus participate in  $2 \times 2$  subgrids on more than one face.

Fix a choice of standard tableaux  $T_\lambda, T_\mu$  and  $T_\nu$  of respective shapes  $\lambda, \mu$  and  $\nu$ . Initialize the edges  $\emptyset - T_\lambda, \emptyset - T_\mu$  and  $\emptyset - T_\nu$  with the shape chains for the corresponding tableaux. Let  $\text{CARTONS}_{\lambda, \mu, \nu}(G/P)$  be all carton fillings with the above initial data.

**Theorem 14.** *For cominuscule  $G/P$ , the number  $c_{\lambda, \mu, \nu}(G/P)$  equals  $2^{\text{shortroots}(\Lambda_{G/P}) - \text{shortroots}(\nu) - \text{shortroots}(\lambda) - \text{shortroots}(\mu)} \# \text{CARTONS}_{\lambda, \mu, \nu}(G/P)$ .*

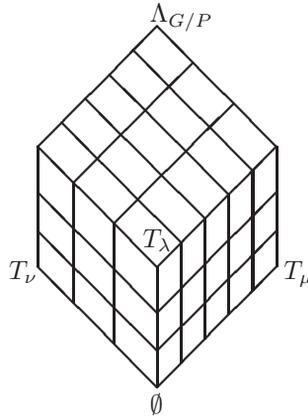


Fig. 1. Theorem 14 calculates  $c_{\lambda,\mu,\nu}(G/P)$  by assigning Young diagrams to the vertices of the six faces

This rule manifests bijections between the sets

$$\mathbf{CARTONS}_{\lambda,\mu,\nu}(G/P) \text{ and } \mathbf{CARTONS}_{\alpha,\beta,\gamma}(G/P)$$

for any permutation  $(\alpha, \beta, \gamma)$  of  $(\lambda, \mu, \nu)$ . In Figure 2 we give an example of Theorem 14.

### 5.2. The proof

The proof in the Grassmannian case is given in [ThYo08]. It carries over to the cominuscule setting using the tools developed for that setting in the previous sections (namely: dual equivalence, growth diagrams, and evacuation). Since the proof in the cominuscule case is the same as in the Grassmannian case, we do not give all the details, as the interested reader will have no trouble filling them in from [ThYo08].

Let  $\alpha \in \mathbb{Y}_{G/P}$ . Recall that we write  $\alpha^\vee$  for the shape obtained by taking any standard tableau of shape  $\alpha$  and reverse rectifying it as far as possible. It is easy to see that if  $\alpha \supset \beta$ , then  $\beta^\vee \supset \alpha^\vee$ . Thus  $\alpha \rightarrow \alpha^\vee$  is an anti-automorphism of  $Y_{G/P}$ . It therefore induces an isomorphism from the join-irreducibles of  $Y_{G/P}$  to the join-irreducibles of the poset of reverse shapes. This in turn induces an anti-automorphism of  $\Lambda_{G/P}$ . We call this anti-automorphism **rotate**, because in the Grassmannian case it amounts to rotation by 180 degrees. An explicit description of **rotate** in each of the cominuscule cases can be found in [ThYo09a, Section 2.2], whose equivalence with the definition we have given here is [ThYo09a, Proposition 4.6]. (We alluded earlier to the easily checked fact that  $\Lambda_{G/P}$  is self-dual, and thus admits *some* antiautomorphism.

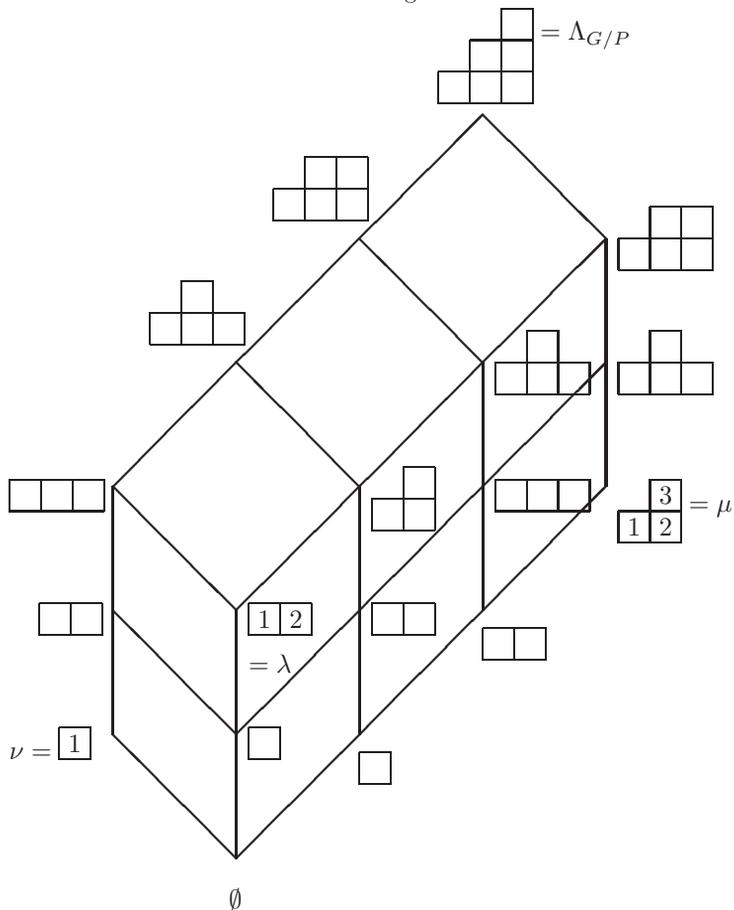


Fig. 2. The “front” three faces of the (unique) carton filling for  $c_{(1,1,0),(1,2,0),(1,0,0)}(LG(3,6)) = 2^{3-1-1} \cdot 1$ . The tableaux  $T_\lambda, T_\mu$  and  $T_\nu$  are as shown.

However, sometimes there is more than one anti-automorphism, and in those cases, it is important to use the correct one, defined as above.)

Given  $T$  a tableau of shape  $\alpha$ , we denote by  $\mathbf{rotate}(T)$  the tableau of shape  $\Lambda_{G/P}/\alpha^\vee$  obtained by putting the label from box  $x$  into the box  $\mathbf{rotate}(x)$ .

Given  $T \in \text{SYT}(\alpha)$  for a straight shape  $\alpha$ , define  $\tilde{T} \in \text{SYT}(\text{rotate}(\alpha))$  by computing  $\text{evac}(T) \in \text{SYT}(\alpha)$ , replacing entry  $i$  with  $|\alpha| - i + 1$  throughout and applying  $\text{rotate}$ .

The following fact extends [ThYo08, Lemma 2.1] with the same proof, given our definition of  $\text{rotate}$  above and Corollary 2.

**Lemma 15.** *Let  $\alpha, \beta, \gamma \in \mathbb{Y}_{G/P}$  and let  $T_\beta \in \text{SYT}(\beta), T_{\gamma^\vee/\alpha} \in \text{SYT}(\gamma^\vee/\alpha)$  be tableaux satisfying  $\text{rectification}(T_{\gamma^\vee/\alpha}) = T_\beta$ . Then*

$$\text{revrectification}(T_{\gamma^\vee/\alpha}) = \text{revrectification}(T_\beta) = \tilde{T}_\beta.$$

As in [ThYo08, Corollary 2.2], we have:

**Corollary 16.** *Fix a carton filling. The face joining the edges assigned the shape chains for  $T_\lambda$  and  $T_\mu$ , necessarily has assigned to its uninitialized corner the shape  $\nu^\vee$ . Similarly, the face joining the edges  $T_\lambda$  and  $T_\nu$ , has assigned to its uninitialized corner the shape  $\mu^\vee$ , and the face joining the edges  $T_\mu$  and  $T_\nu$ , has assigned to its uninitialized corner (the corner not visible in Figure 1) the shape  $\lambda^\vee$ . Thus, we can refer to the edges  $\lambda^\vee - \Lambda$ ,  $\mu^\vee - \Lambda$  and  $\nu^\vee - \Lambda$ . These edges are necessarily assigned the shape chains of  $\tilde{T}_\lambda, \tilde{T}_\mu$  and  $\tilde{T}_\nu$  respectively.*

Thus by Corollary 16, it makes sense to refer to a face by its corner vertices. Note any carton filling gives a growth diagram on the face  $\emptyset - \mu - \nu^\vee - \lambda$  for which the edge  $\lambda - \nu^\vee$  is a standard tableau of shape  $\nu^\vee/\lambda$  rectifying to  $T_\mu$ . By Theorem 12, fillings of this face count  $2^{\text{shortroots}(\mu) - \text{shortroots}(\nu^\vee/\lambda)} c_{\lambda, \mu, \nu}(G/P)$ .

Conversely, if we start with a filling of the  $\emptyset - \mu - \nu^\vee - \lambda$  face, then it is straightforward to use the Fomin growth conditions and Corollary 16 to show that there is at most one way to extend this filling to a filling of the entire carton. The proof that there is exactly one way to extend the filling follows exactly as in [ThYo08, Section 2.2], where references to the growth diagram encoding of evacuation from S. Fomin’s [St99, Appendix 1] are replaced by the cominuscule generalization given in Section 4 above. Finally, one observes that

$$2^{\text{shortroots}(\nu^\vee/\lambda) - \text{shortroots}(\mu)} = 2^{\text{shortroots}(\Lambda_{G/P}) - \text{shortroots}(\nu) - \text{shortroots}(\lambda) - \text{shortroots}(\mu)}.$$

Q.E.D.

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