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Singularities of mappings and the vanishing homology of images and discriminants

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Abstract.

These notes provide an introduction to the theory of singularities of mappings and right-left equivalence. They cover a part of Mather theory, concerned with stability, classification and deformations, and go on to study the vanishing homology of images and discriminants of families of mappings. A number of open questions are discussed in the last three sections.

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§1. Preface

These lecture notes accompanied the series of four lectures on singularities of mappings given at the July 2012 summer school in the ICMC São Carlos. They inevitably reflect my own interests and knowledge. I am aware of omitting vast areas of wonderful mathematics, and of lamentable incompleteness even in what I have attempted to cover. A

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good reference for Mather's foundational work on stability, versality and finite determinacy, and its subsequent refinements, is still needed, as is an account of the geometrical aspects of the theory of singularities of mappings following the line of development initiated by Milnor's book. These notes give an brief introduction to both aspects; I hope they can provide a way into the subject.

I am very grateful to the organisers for the opportunity to speak on this subject, for the stimulus of preparing these lecture notes and for their generous and delightful hospitality over the two weeks of the summer school and workshop. São Carlos has become a world-class centre for singularity theory, and is a wonderful place to do mathematics.

I am also grateful to the anonymous referee for a very careful reading of these notes, and many helpful suggestions.

§2. Introduction

The crucial notion is of course the *derivative* of a smooth or analytic mapping: if $f : X \to Y$ is a map of manifolds and $x \in X$ then $d_x f : T_x X \to T_{f(x)} Y$ is the derivative, defined by

$$d_x f(\hat{x}) = \lim_{h \to 0} \frac{f(x + h\hat{x}) - f(x)}{h}$$

if X and Y are open sets in linear spaces. If X and Y are contained, but not open, in linear spaces, $d_x f$ can be defined by restricting to $T_x X$ the derivative of a suitable extension of f to an open set in the linear ambient space; otherwise one uses charts. It is also worth recalling that every tangent vector $\hat{x} \in T_x X$ is the tangent vector $\gamma'(0)$ to a parameterised curve $\gamma : (\mathbb{R}, 0) \to (X, x)$ (or $\gamma : (\mathbb{C}, 0) \to (X, x)$ in the complex analytic category), and that $d_x f$ satisfies

$$d_x f(\gamma'(0)) = (f \circ \gamma)'(0). \tag{2.1}$$

This may be taken as the definition. It is particularly useful in infinite dimensional cases, such as where X is a group of diffeomorphisms.

A point $x \in X$ is a *regular point* of f if $d_x f$ is surjective, and a *critical point* if it is not. The image of a critical point is a *critical value* of f; any point in Y which is not a critical value is a *regular value* (even if it has no preimages). The set of all critical values is often called the *discriminant* of the map f. If x_0 is a regular point then f is said to be a *submersion* at x_0. If x_0 is a regular point, then a simple argument based on the inverse function theorem establishes

Theorem 2.1. *(Normal form for submersions) Let dim $X = n \geq k =$ dim Y, and suppose that x_0 is a regular point of $f : X \to Y$. Then one can choose coordinates $x_1, \ldots, x_n$ on X around x_0, and $y_1, \ldots, y_k$ on Y around $f(x_0)$, such that f takes the form $f(x_1, \ldots, x_n) = (x_1, \ldots, x_k)$.*

These notions are only of interest when $\dim X \geq \dim Y$; when $\dim X < \dim Y$, all points of X are critical points, and the set of critical values of f is the whole image of f. In this case one is interested in whether or not $d_x f$ is *injective.* If it is, f is an *immersion* at x_0, and one has

Theorem 2.2. *(Normal form for immersions) Let dim $X = n \leq k =$ dim Y and suppose that $f : X \to Y$ is an immersion at x_0. Then one can choose coordinates around x_0 and $f(x_0)$ such that f takes the form $f(x_1, \ldots, x_n) = (x_1, \ldots, x_n, 0, \ldots, 0)$.*

Exercise 2.3. *(1) Find proofs of 2.1 and 2.2. Both follow from the inverse function theorem, by incorporating f into a suitable auxiliary mapping whose derivative is invertible.*

(2) Prove that if $f : (k^n, 0) \to (k^p, 0)$ has rank k at 0 then in suitable coordinates f takes the form

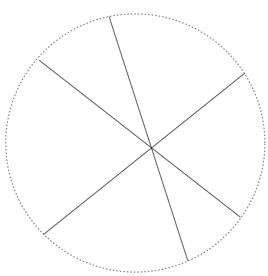
$$(x_1, \ldots, x_n) \to (x_1, \ldots, x_k, f_{k+1}(x), \ldots, f_p(x)).$$

Singularity theory begins where these two theorems end: it is concerned with what happens at points where f is neither a submersion nor an immersion. It concentrates on the local behaviour of mappings, and for this reason uses the notion of *germ of mapping*, which we study briefly in Subection 2.2. Geometrical singularity theory for the two cases $\dim X \geq \dim Y$ and $\dim X < \dim Y$ is rather different. In the first case, classical singularity theory is interested in *preimages* $f^{-1}(y_0)$, and there is also a theory of the discriminant, initiated by Teissier in [51]. In the second case, to which much less attention has been devoted, one studies the *images* of maps. In fact very little is known about the geometry of maps in case $\dim X < \dim Y - 1$, and the theory for the case $\dim X = \dim Y - 1$ has an embarassing gap, in the form of an unproved (and unrefuted) conjecture which I made twenty five years ago.

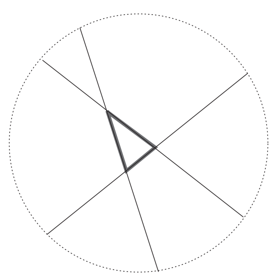
This minicourse will concentrate on two key invariants for singularities of mappings, and the relation between them. The first comes from deformation theory: it is the *deformation-theoretic codimension*, and is the subject of Section 4. Until then, one can use the following relatively non-technical working definition: it is the minimal number of parameters for a family of mappings in which a singularity equivalent to the one in question occurs 'stably' or 'irremovably'. Clearly the codimension depends on which equivalence relation one is using. These notes focus on right-left equivalence, or $\mathcal{A}$-equivalence. The second, studied in Section 3, comes from topology: it is the "rank of the vanishing homology (of a nearby stable object)". This vague phrase will be made more precise; for now, we make do with two examples. The first is the non-degenerate critical point of a polynomial or analytic function, equivalent, by the Morse Lemma, to the germ defined by

$$f(x_1, \ldots, x_n) = x_1^2 + \cdots + x_n^2.$$

Here $f^{-1}(0)$ is contractible, but for $t \neq 0$, $f^{-1}(t)$ has the homotopy-type of an n-sphere [1]. When t returns to 0, the rank of the homology of $f^{-1}(t)$ diminishes by 1; this is the 'rank of the vanishing homology' for this example. The second is the three pieces of plane curve which meet at a point in the Reidemeister move of type III. This configuration is evidently unstable: one can move any one of the three to form a triangle. Since now all intersections are transverse, this configuration is stable. It is the 'nearby stable object' for this example, and its vanishing homology, generated by the 1-cycle highlighted in the drawing on the right, once again has rank 1.



Unstable



Stable

The deformation-theoretic codimension in the second example is also equal to 1; therein lies its importance in knot theory. Given two plane projections of the same knot, one can be deformed to the other in such a way that during the deformation, only three types of qualitative change occur. These are the three 'Reidemeister moves', and our example shows the third of these. They cannot be avoided in a 1-parameter family of projections; other more complicated singularities can be.

[1]This is true for any $t \neq 0$ when $k = \mathbb{C}$; when $k = \mathbb{R}$ it holds for $t > 0$. Indeed in this case the inclusion of real in complex is a homotopy equivalence. It is an example of a "good real picture".

Notation and Terminology 2.4. Let X and Y be manifolds, and $f : X \to Y$ a differentiable map.

(1) A *singular point*, or *singularity* of f is a point where f is not a submersion, in case $\dim X \geq \dim Y$, and not an *immersion*, in case $\dim X \leq \dim Y$.

(2) A map $X \to Y$ has *corank* r at x_0 if the rank of $d_{x_0} f$ is r less than the greatest possible value, $\min\{\dim X, \dim Y\}$. Thus if $\dim X \leq \dim Y$ then f has corank r at x_0 if r is the dimension of the kernel of $d_{x_0} f$, and if $\dim X \geq \dim Y$ then the corank is the dimension of the cokernel of $d_{x_0} f$.

(3) If $Z \subset X$ then a *singular point* of Z is a point at which Z is not a submanifold of X.

2.1. Real or complex?

Real singularities in dimension ≤ 3 can be drawn; for complex objects the drawing stops in dimension 1. Over the complex numbers, the relation between geometry and algebra is simpler, beginning with the fact that every complex degree n polynomial has n roots in $\mathbb{C}$. So both fields have their advantages. Here we state and prove theorems mostly in the complex context, but try to draw their real versions.

2.2. Germs, cones and local rings

Definition 2.5. Let $f, g : X \to Y$ be maps of topological spaces, and let $S \subset X$.

(1) We say that f and g *have the same germ at* S (or *along* S if S is not a finite point set), if there is a neighbourhood U of S in X such that f and g coincide on U. This is evidently an equivalence relation, and a *germ of mapping at* S is an equivalence class under this relation.

(2) Two subsets X_1 and X_2 of X *have the same germ at* (or *along*) S if there is a neighbourhood U of S in X such that $X_1 \cap U = X_2 \cap U$. A *germ at* S *of subset of* X is an equivalence class of subset under this relation.

We denote a germ at S of mapping $X \to Y$ by $f : (X, S) \to Y$, or $f : (X, S) \to (Y, T)$ if $f(S) \subset T \subset Y$. To determine a germ of mapping at S, it is enough to specify the behaviour of f on some neighbourhood of S in X. Usually X is $\mathbb{C}^n$ or an analytic variety embedded in $\mathbb{C}^n$, S is a single point or a finite set, and we specify f by means of power series which converge in some neighbouhood of the points of S. Not every power series can be extended to a globally defined map $X \to Y$, so really our subject is not 'germs at S of maps $X \to Y$', but 'germs at

S of maps to Y from some neighbourhood of S'. In practice this will not cause any difficulty.

Germs of maps to $\mathbb{C}$ can be added and multiplied, and the set of germs at x_0 of analytic functions on X is a $\mathbb{C}$-algebra. It is denoted $\mathcal{O}_{X,x_0}$.

The notion of germ is particularly natural in the complex analytic category, because of uniqueness of analytic continuation: if U_1 and U_2 are connected open sets in $\mathbb{C}^n$ and $f_i : U_i \to \mathbb{C}^p$ are complex analytic maps, then if f_1 and f_2 coincide on some open $V \subset U_1 \cap U_2$, they coincide on all of $U_1 \cap U_2$.

Exercise 2.6. *Show that the same is not true of real C^∞ maps.*

If X and Y are spaces, and we select some class of germs of maps $X \to Y$ – e.g. germs of continuous maps, or germs of complex analytic maps in case X and Y are complex analytic varieties – then we can put together all of the germs into a global object, a *sheaf*. This notion is crucial in algebraic and analytic geometry, but I do not want to make it a prerequisite for this course. Instead, we will develop the notion as it is needed. We begin with a working definition sufficient to make some of the necessary theorems at least vaguely comprehensible.

The definition of sheaf requires an algebraic structure, so we take, as our target space Y, the field $\mathbb{C}$. It is natural to associate to each open $U \subset X$ the set

$$\mathcal{O}_X(U) := \{f : U \to \mathbb{C} : f \text{ is complex analytic}\}$$

and make it into a $\mathbb{C}$-algebra by defining the operations pointwise:
$(f+g)(x) = f(x) + g(x),\ (fg)(x) = f(x)g(x),$
$(\lambda f)(x) = \lambda f(x)$ for $\lambda \in \mathbb{C}$.

If $U \subset V \subset X$, there is a restriction map

$$\rho_{U,V} : \mathcal{O}_X(V) \to \mathcal{O}_X(U)$$

which is a homomorphism of $\mathbb{C}$-algebras, and if $U \subset V \subset W$ then evidently

$$\rho_{U,V} \circ \rho_{V,W} = \rho_{U,W}. \tag{2.2}$$

Let $\mathcal{U}_x$ be the collection of all neighbourhoods of a point x. The equivalence relation by which we arrived at the notion of germ of function or mapping becomes a relation on the disjoint union $\coprod_{U \in \mathcal{U}_x} \mathcal{O}_X(U)$:

$$\begin{aligned} &f \in \mathcal{O}_X(U) \text{ and } g \in \mathcal{O}_X(V) \ \text{ are equivalent if there exists} \\ &\qquad\qquad W \in \mathcal{U}_x \ \text{ such that } \rho_{W,U}(f) = \rho_{W,V}(g). \end{aligned} \tag{2.3}$$

The set of equivalence classes, $\mathcal{O}_{X,x_0}$, is in a natural way a $\mathbb{C}$-algebra: if $f, g \in \mathcal{O}_{X,x_0}$ then they can be represented by some $f_1 \in \mathcal{O}_X(U)$ and $g_1 \in \mathcal{O}_X(V)$, for some open neighbourhoods U, V of x_0, and then the restrictions $\rho_{U\cap V,U}(f)$ and $\rho_{U\cap V,V}(g)$ in $U \cap V$ can be added or multiplied in the usual way. The sum and product of these restrictions then determine germs at x_0, which, as one can easily check, are independent of the choices of representative f_1, g_1.

Exercise 2.7. *Show this.*

The map $\rho_{x_0,U} : \mathcal{O}_X(U) \to \mathcal{O}_{X,x_0}$ defined by sending $f \in \mathcal{O}_X(U)$ to its germ at x_0 is a $\mathbb{C}$-algebra homomorphism. Evidently

$$\rho_{x_0,V} = \rho_{x_0,U} \circ \rho_{U,V}.$$

Exercise 2.8. *Is $\rho_{x_0,U}$ surjective? Injective?*

We often write "$\mathcal{O}(U)$" instead of "$\mathcal{O}_X(U)$" when X is clear from the context.

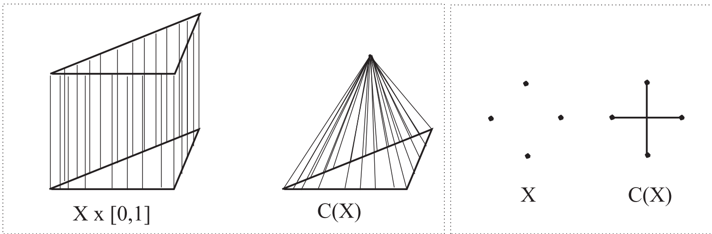
The procedure we have outlined can be applied equally well to functions of other types: continuous, or C^∞, or real analytic, etc. It also makes sense in a wider context:

Exercise 2.9. Let $f : X \to Y$ be a map of topological spaces. For $U \subset Y$ define $\mathcal{H}^p(U) := H^p(f^{-1}(U))$ (the p-th topological cohomology of $f^{-1}(U)$).

(1) Given $U \subset V \subset Y$, show how to define $\rho_{U,V} : \mathcal{H}^p(V) \to \mathcal{H}^p(U)$ so that (2.2) holds.

(2) Show that if f is a locally trivial fibre bundle then for U a sufficiently small and contractible neighbourhood of a point $y \in Y$, $\mathcal{H}^p(U) \simeq \mathcal{H}^p(\{y\})$.

A further justification for the use of the notion of germ in singularity theory comes from the fact that closed analytic spaces are 'locally conical'. This is particularly important in the definition of the vanishing homology, so we go into some detail here. If X is any topological space, the *cone on* X, which we denote by $C(X)$, is obtained by forming the Cartesian product $X \times [0, 1]$ and then identifying all of the points of $X \times \{1\}$ with one another. One writes $C(X) = (X \times [0, 1])/(X \times \{1\})$, where the notation B/A, for A a subset of B, means the quotient of B by the equivalence relation which identifies all the points of A to one another. If X is embedded in some $\mathbb{R}^n$ then the cone $C(X)$ can be described more concretely as follows: if v is an (arbitrary) point in $\mathbb{R}^n \times \{1\}$ then $C(X)$ is homeomorphic to the union of all of the line segments in $\mathbb{R}^n \times [0, 1]$ joining v to a point $(x, 0)$, for $x \in X$.



Exercise 2.10. For any space X, $C(X)$ can be contracted to its vertex.

Because cones are contractible, their homology is equal to that of a point.

For $x_0 \in \mathbb{C}^n$, let $S_\varepsilon(x_0)$ be the sphere of radius ε centred at x_0, and let $B_\varepsilon(x_0)$ be the ball of radius ε centred at x_0.

Theorem 2.11. ([5]) *Let $U \subset \mathbb{C}^n$ be open and let $X \subset U$ be the set of common zeros of k functions $f_1, \ldots, f_k \in \mathcal{O}(U)$. If $x_0 \in X$, there exists $\varepsilon > 0$ such that $X \cap B_\varepsilon(x_0)$ is homeomorphic to the cone on its boundary $X \cap S_\varepsilon(x_0)$.*

Exercise 2.12. *Show that this is true in the trivial case that $X = \mathbb{C}^n$, and therefore if X is a smooth manifold at x_0.*

Write $X_\varepsilon := S_\varepsilon(x_0) \cap X$ and $X_{\leq\varepsilon} := X \cap B_\varepsilon(x_0)$. If X is a k-dimensional manifold except at x_0 (i.e. X has isolated singularity at x_0) then the theorem can be proved by

(1) constructing a 'radial' vector field v, pointing in towards x_0, on a neighbourhood of x_0 in X, and adjusting the length of the vectors so that for each point $x \in X_\varepsilon$, the trajectory $\varphi_t(x)$ starting at x arrives at x_0 at time $t = 1$, and

(2) defining a homeomorphism $H : X_\varepsilon \times [0, 1) \to X_{\leq\varepsilon} \smallsetminus \{x_0\}$ by

$$H(x, t) = \varphi_t(x),$$

which (automatically) extends to a homeomorphism $(X_\varepsilon \times [0, 1])/(X_\varepsilon \times \{1\}) \to X_{\leq\varepsilon}$.

The theorem holds also for locally closed real analytic subsets of $\mathbb{R}^n$ with isolated singularities, but not in general for the zero loci of C^∞ functions. A more involved argument, using Whitney regular stratifications, proves the theorem for the case where X is a (real or complex) analytic set with arbitrary singularity at x_0 – see [5].

Exercises 2.13. (1) Give an example to show that the zero-loci of C^∞ functions need not be locally conical.

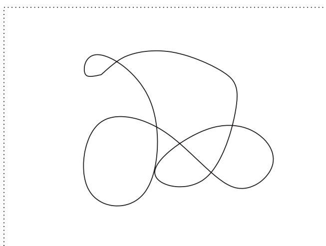
(2) Suppose that X has isolated singularity at 0, and that there is a function $\rho : X \to \mathbb{R}_{\geq 0}$ such that

(a) ρ has no critical point in $X_{\leq \varepsilon} \smallsetminus \{x_0\}$, and
(b) $\rho^{-1}(0) = \{x_0\}$.

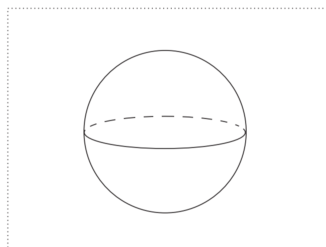
Use the gradient vector of ρ to construct the vector field of the sketched proof of 2.11. [2]

(3) Show that ρ_E satisfies condition 1. of the previous exercise iff for all ε' with $0 < \varepsilon' \leq \varepsilon$, $X \pitchfork S_{\varepsilon'}(x_0)$.

(4) Divide up the objects pictured below into subsets which are cones on their boundary.

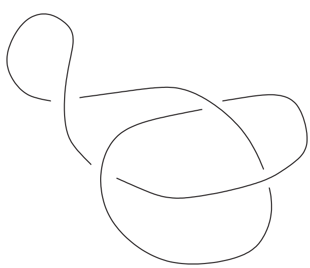


Planar projection of a knot



Sphere

(5) Take a thin copper wire (less than 1mm in diameter, but thick enough to form a self-supporting structure) and join the two ends after bending it to form a knot – which (making allowances for the fact that the wire is not infinitely thin) should be a C^∞ embedding of the circle in $\mathbb{R}^3$. You should obtain something looking like



[2]The hardest part of the proof of 2.11 comes in showing that such a function exists. In fact *any* real analytic function $\rho : X \to \mathbb{R}_{\geq 0}$ satisfying 2.13(2)(b) will do; one uses the curve selection lemma (*cf* [40]) to show that it also satisfies 2.13(2)(a) for some $\varepsilon > 0$. In particular, one can use the Euclidean distance-squared function $\rho_E(x) := \|x - x_0\|^2$.

The view shown here is "a generic projection' – the only singular points on the image are transverse crossings of two branches. Looking at the knot from different points of view, you should see different types of singular points. There are not many different types; it is instructive to see how many you can find, and to make sketches of conic neighbourhoods of them. See [53] and [43] for lists, drawings and properties.

(6) What is the appropriate version of locally conical structure for a *mapping*? It's worth trying to make up your own definition. For a good answer, see [16].

The local conical structure is crucially important in singularity theory. It gives a clear meaning to the term "local", and it makes possible the idea of local changes in a deformation. The simplest example along these lines is the Milnor fibre of an isolated hypersurface singularity. We have already seen that if f is an analytic function on some open set in $\mathbb{C}^n$ and has isolated singularity at x_0, then there exists $\varepsilon > 0$ such that $B_\varepsilon(x_0) \subset U$ and $f^{-1}(y_0) \cap B_\varepsilon(x_0)$ is homeomorphic to the cone on $f^{-1}(y_0) \cap S_\varepsilon(x_0)$ – indeed, that $f^{-1}(y_0) \pitchfork S_{\varepsilon'}(x_0)$ for all ε' with $0 < \varepsilon' \leq \varepsilon$. An argument involving properness shows also that

Proposition 2.14. *In this case, there exists $\eta > 0$ (depending on the choice of ε) such that provided $|y_0 - y| \leq \eta$ then $f^{-1}(y) \pitchfork S_\varepsilon(x_0)$. For such ε and η, the map*

$$f| : B_\varepsilon(x_0) \cap f^{-1}(B^*_\eta(y_0)) \to B^*_\eta(y_0)$$

is a locally trivial fibre bundle.

The same principle gives us the notion of the "nearby stable object" (near to a singularity with isolated instability) in other situations. The details may be more complicated but the basic idea is the same.

2.3. Background in commutative algebra

If X is any analytic space and $p \in X$, then the evaluation map

$$\mathcal{O}_{X,p} \to \mathbb{C}, \quad f \mapsto f(p)$$

is surjective, so that its image is the field $\mathbb{C}$. Its kernel is therefore a maximal ideal in $\mathcal{O}_{X,p}$,which is denoted by $\mathfrak{m}_{X,p}$. Indeed it is the only maximal ideal, since if $f \in \mathcal{O}_{X,p}$ is not in $\mathfrak{m}_{X,p}$ then $1/f \in \mathcal{O}_{X,p}$, so that any ideal containing f also contains 1 and therefore all of $\mathcal{O}_{X,p}$. This shows that every proper ideal of $\mathcal{O}_{X,p}$ is contained in $\mathfrak{m}_{X,p}$. Rings with a single maximal ideal are called *local rings*. Their properties play a very large rôle in singularity theory.

We will frequently abbreviate $\mathfrak{m}_{X,p}$ simply to $\mathfrak{m}$. If $x_1,\ldots x_n$ are coordinates on X around p, and $p = (p_1,\ldots,p_n)$ in these coordinates, then every germ $f \in \mathcal{O}_{X,p}$ can be written as a convergent power series in $x_1 - p_1,\ldots,x_n - p_n$. It follows that

$$\mathfrak{m}_{X,p} = (x_1 - p_1,\ldots,x_n - p_n) \tag{2.4}$$

(the ideal generated by $x_1 - p_1,\ldots,x_n - p_n$).

In any ring R, the sum and product of ideals I and J are defined simply by

$$\begin{aligned} I + J &= \{r + s : r \in I, s \in J\} \\ IJ &= \{\textstyle\sum_{i=0}^m r_i s_i : m \in \mathbb{N}, r_i \in I, s_i \in J \text{ for all } i\}. \end{aligned}$$

Exercise 2.15. *(1) Show that in any ring R, if I and J are ideals then so are $I + J$ and IJ.*

(2) Let $X = \mathbb{C}^n$ and $p = 0$.

(a) *Show that* $\mathfrak{m}^2 = \{f \in \mathcal{O}_{\mathbb{C}^n,0} : f(0) = \partial f/\partial x_i(0) = 0$ *for* $i = 1,\ldots,n\}$.

(b) *Show more generally that*

$$\mathfrak{m}^k = \{f \in \mathcal{O}_{\mathbb{C}^n,0} : \partial^\alpha f/\partial x^\alpha(0) = 0 \ \ \textit{for } 0 \le |\alpha| \le k - 1\}$$

where α is a multi-index $\alpha = (\alpha_1,\ldots,\alpha_n)$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, and by $\partial^0 f/\partial x^0$ we mean simply f.

In the C^∞ category, (2.4) and 2.15(2)(a) and (b) also hold. However (2.4) is no longer completely obvious, and is known as *Hadamard's Lemma* – see Martinet's book [32], Chapter 1.

We will make much use of the following statement.

Lemma 2.16. (Nakayama's Lemma) *Let M be a finitely generated module over a Noetherian local ring R with maximal ideal $\mathfrak{m}$. If $\mathfrak{m}\,M = M$ then $M = 0$.*

Corollary 2.17. *Let M and N be submodules of an R-module P, with M finitely generated, and suppose that*

$$M \subset N + \mathfrak{m}\,M. \tag{2.5}$$

Then $M \subset N$.

Proof Let $m_1,\ldots,m_r$ generate M over R. Since $M = \mathfrak{m}\,M$, for each i there exist $\alpha_{ij} \in \mathfrak{m}$ such that for $i = 1,\ldots,r$,

$$m_i = \alpha_{11}m_1 + \cdots + \alpha_{1r}m_r.$$

Rewriting these r equations as a single matrix equation we get

$$\begin{pmatrix} m_1 \\ \vdots \\ m_n \end{pmatrix} = \begin{pmatrix} \alpha_{11} & \cdots & \alpha_{n1} \\ \vdots & \cdots & \vdots \\ \alpha_{1n} & \cdots & \alpha_{nn} \end{pmatrix} \begin{pmatrix} m_1 \\ \vdots \\ m_n \end{pmatrix}$$

and therefore

$$(I_n - A) \begin{pmatrix} m_1 \\ \vdots \\ m_n \end{pmatrix} = 0,$$

where I_n is the $n \times n$ identity matrix and A is the matrix $[\alpha_{ij}]$. Multiplying both sides by the matrix of cofactors of $I_n - A$ we deduce that

$$\det[I_n - A] m_i = 0$$

for all i. But $\det[I_n - A]$ is a unit in the ring R, since it is equal to $1 + \alpha$ for some $\alpha \in \mathfrak{m}$. Hence $m_i = 0$ for $i = 1, \ldots, r$, and so $M = 0$. □

Proof of Corollary Let $M_0 = (M + N)/N$. The hypothesis $M \subset N + \mathfrak{m}M$ implies that $M_0 = \mathfrak{m}M_0$. It follows by the Lemma that $M_0 = 0$, so that $M \subset N$. □

2.4. Conservation of multiplicity

Suppose that U is open in $\mathbb{C}^n$, that$f : U \to \mathbb{C}^n$ is analytic, that $f(a) = b$, and that a is isolated in $f^{-1}(b)$ – that is, there exists $\varepsilon > 0$ such that $f^{-1}(b) \cap B_\varepsilon(a) = \{a\}$. Then the $\mathcal{O}_{\mathbb{C}^n,a}$-ideal $f^* \mathfrak{m}_{\mathbb{C}^n,b} := (f_1 - b_1, \ldots, f_n - b_n)$ must contain a power of the maximal ideal $\mathfrak{m}_{\mathbb{C}^n,a}$, since $\sqrt{f^* \mathfrak{m}_{\mathbb{C}^n,b}} = \mathfrak{m}_{\mathbb{C}^n,a}$. In fact

Proposition 2.18. *The following three statements are equivalent:*

(1) *a is isolated in $f^{-1}(b)$;*
(2) *$\dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^n,a} / f^* \mathfrak{m}_{\mathbb{C}^n,b} < \infty$;*
(3) *$f^* \mathfrak{m}_{\mathbb{C}^n,b} \supset \mathfrak{m}^k_{\mathbb{C}^n,a}$ for some $k < \infty$.*

Proof. That (3) implies (2) implies (1) is obvious. The converse follows from Ruckert's *Nullstellensatz*: that for any ideal $I \subset \mathcal{O}_{\mathbb{C}^n,a}$, the ideal of all functions vanishing on $V(I)$ is the radical $\sqrt{I} := \{f \in \mathcal{O}_{\mathbb{C}^n,a} : f^k \in I \text{ for some } k\}$. Since each coordinate function $x_i - a_i$ vanishes on $V(f^* \mathfrak{m}_{\mathbb{C}^n,b})$ it follows that $(x_i - a_i)^{k_i} \in f^* \mathfrak{m}_{\mathbb{C}^n,b}$ for some k_i. Then 3 holds with $k = n \max_i\{k_i\} - 1$. Q.E.D.

Exercise 2.19. Show that if I is any ideal in $\mathcal{O}_{\mathbb{C}^n,x_0}$ such that $\dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^n,x_0} / I = k < \infty$ then $I \supset \mathfrak{m}^k$.

The dimension of $\mathcal{O}_{\mathbb{C}^n,a}/f^*\mathfrak{m}_{\mathbb{C}^n,b}$ is the *multiplicity* of f at a; we will denote it by $\mathrm{mult}_a(f)$.

Theorem 2.20. *Let U be open in $\mathbb{C}^n$, let $f : U \to \mathbb{C}^n$ be analytic, and let x_0 be isolated in $f^{-1}(y_0)$. Then there exists $\varepsilon > 0$ and $\eta > 0$ such that for all $y \in B_\eta(y_0)$,*

$$\sum_{x \in f^{-1}(y) \cap B_\varepsilon(x_0)} mult_x(f) = mult_{x_0} f. \tag{2.6}$$

The equality (2.6) is the basis for a number of statements about conservation of multiplicity. Here are some examples.

Conservation of Milnor number: If U is open in $\mathbb{C}^n$ and $f : U \to \mathbb{C}$ has isolated singularity at x_0 then the *Milnor number* of f at x_0 is defined to be $\mathrm{mult}_{x_0}(j^1 f)$ where $j^1 f : (\mathbb{C}^n, x_0) \to (\mathbb{C}^n, 0)$ is the map with component functions $\partial f/\partial x_1, \ldots, \partial f/\partial x_n$. That is,

$$\mu_{x_0}(f) = \dim\ \mathcal{O}_{\mathbb{C}^n,x_0}/J_f,$$

where J_f is the jacobian ideal $(\partial f/\partial x_1, \ldots, \partial f/\partial x_n)$.

Corollary 2.21. *Let U be open in $\mathbb{C}^n$ and let $f : U \to \mathbb{C}$ have isolated singularity at x_0 with Milnor number $\mu < \infty$. Then in any deformation $F : U \times \mathbb{C}^d \to \mathbb{C}$ of f, there exists $\varepsilon > 0$ and $\eta > 0$ such that for $|u| < \eta$,*

$$\sum_{x \in B_\varepsilon(x_0)} \mu_x(f_u) = \mu_{x_0}(f).$$

Proof. Suppose first that the set

$$S_F^{\mathrm{rel}} := \{(x,u) : \partial F/\partial x_1 = \cdots = \partial F/\partial x_n = 0 \text{ at } (x,u)\}$$

is smooth. Its dimension is necessarily equal to d, since $j^1 f$ must be a submersion outside x_0.

Let $\pi : S_F^{\mathrm{rel}} \to U$ be projection. Since S_F^{rel} is locally isomorphic to $\mathbb{C}^{\dim U}$, we can apply 2.20 to the map π. If $(u,x) \in S_F^{\mathrm{rel}}$ then

$$\mathcal{O}_{S_F^{\mathrm{rel}},(u,x)}/\pi^*\mathfrak{m}_{U,(v,u)} \simeq \mathcal{O}_{\mathbb{C}^n,x}/J_{f_u} \tag{2.7}$$

and thus

$$\mathrm{mult}_{(u,x)}(\pi) = \mu_x f_u.$$

It follows from 2.20 that there exists $\varepsilon > 0$ and $\eta > 0$ such that for $|u| < \eta$,

$$\sum_{x \in B_\varepsilon(x_0)} \mu_x(f_u) = \mu_{x_0}(f).$$

If $S_F^{\rm rel}$ is not smooth, one can further deform F by a deformation $G : U \times \mathbb{C}^d \times \mathbb{C}^e$ such that $S_G^{\rm rel}$ is smooth of the requisite dimension – for example $G(x,u,v) = F(u,x) + \sum_i v_i x_i$. The first part of the argument applies to G, and the conclusion is obtained by restricting to $\{v = 0\}$. Q.E.D.

Exercise 2.22. (1) Prove the equality (2.7).

(2) Show that if $S_F^{\rm rel}$ is smooth then u is a regular value of π if and only if f_u has only non-degenerate critical points.

Conservation of intersection number of plane curves: If $C = \{f = 0\}$ and $D = \{g = 0\}$ are plane analytic curves meeting at x_0, their *intersection number* at x_0, $I_{x_0}(C, D)$, is defined to be the multiplicity at x_0 of the map (f, g).

Corollary 2.23. *Suppose the two curves C and D meet at x_0 with $I_{x_0}(C, D) < \infty$, and let C_t and D_t be parameterised families of plane curves with $C_0 = C, D_0 = D$. Then there exist $\varepsilon > 0$ and $\eta > 0$ such that for $|t| < \eta$,*

$$\sum_{x \in C_t \cap D_t \cap B_\varepsilon(x_0)} I_x(C_t, D_t) = I_{x_0}(C, D).$$

Proof. **Exercise** Q.E.D.

Conservation of cross-cap number: Suppose $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ is given by $f(x, y) = (x, f_2(x, y), f_3(x, y))$. Its non-immersive locus S_f is determined by the equations

$$\partial f_2/\partial y = \partial f_3/\partial y = 0.$$

Suppose this set consists just of 0. We define the *cross-cap number of f*, $C_0(f)$, as $\mathrm{mult}_0(\partial f_2/\partial y, \partial f_3/\partial y)$.

Exercise 2.24. (a) Find $C_0(f)$ in each of the following cases:

(1) $f(x, y) = (x, y^2, xy)$ (this is the parameterisation of the Whitney umbrella, and is known as the *cross-cap*);

(2) $f(x, y) = (x, y^2, y^3 + x^{k+1}y)$

(3) $f(x, y) = (x, y^3, xy + y^{3k-1})$.

(b) Suppose that $F(x, y, u) = (x, F_2(x, y, u), F_3(x, y, u), u)$ is an unfolding of f with $u \in \mathbb{C}^d$, and for fixed u let

$$f_u(x, y) = (x, F_2(x, y, u), F_3(x, y, u)).$$

Let S_F be the non-immersive locus of F, and consider the projection $\pi : S_F \to \mathbb{C}^d$. Show that

(1) It is possible to choose F so that S_F is smooth of codimension 2 in $\mathbb{C}^2 \times \mathbb{C}^d$.
(2) In this case $\text{mult}_{(x,y,u)}(\pi) = C_{(x,y)}(f_u)$.
(3) There exist $\varepsilon > 0$ and $\eta > 0$ such that for $|u| < \eta$,

$$\sum_{(x,y)\in S_{f_u}\cap B_\varepsilon(0)} C_{(x,y)}(f_u) = C_0(f).$$

(4) One can show that if $C_0(f) = 1$ then f is $\mathcal{A}$-equivalent to the cross-cap, the germ of (a)(1). Conclude that there exist deformations f_u of f with $C_0(f)$ cross-caps.

(c) Suppose that $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$ has corank 1. Show that the ideal of $(n-1) \times (n-1)$ minor determinants of the matrix of df (the *ramification ideal* of f, $\mathcal{R}_f$) is generated by some two of these minors. Hint: do this first when $n = 2$, where it's easier to see what is going on. How many generators does $\mathcal{R}_f$ need when f has corank 2? corank 3?

We will see other applications of 2.20 to prove conservation of multiplicity of one kind or another. However 2.20 is not sufficient in all cases. In the examples we have just seen, we applied 2.20 to the projection π from the singular or relative critical space S_F of a deformation F, to the parameter space $\mathbb{C}^d$. This relied upon being able to choose F such that S_F is smooth. However there are situations where this is not possible. For example, the non-immersive locus of an unfolding $F(x, y, u) = (F_1(x, y, u), F_2(x, y, u), F_3(x, y, u), u)$ has equations

$$(2.8)\quad \det\begin{vmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} \end{vmatrix} = \det\begin{vmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} \\ \frac{\partial F_3}{\partial x} & \frac{\partial F_3}{\partial y} \end{vmatrix} = \det\begin{vmatrix} \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} \\ \frac{\partial F_3}{\partial x} & \frac{\partial F_3}{\partial y} \end{vmatrix} = 0$$

and if F is an unfolding of a map-germ of corank 2, then all three determinants lie in the square of the maximal ideal, so that their locus of common zeroes is unavoidably singular.

Nevertheless, it is still true that, just as shown in Exercise 2.24(b) above, for a finitely determined map-germ $(\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$, the number of cross-caps appearing in a stable perturbation is equal to

$$\dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^2,0} / \mathcal{R}_f,$$

where $\mathcal{R}_f$ is the *ramification ideal* of f, generated by the three 2×2 minors of the matrix of df (as for F in (2.8) above). The proof of this makes use of the notion of *Cohen-Macaulay* rings and spaces, and involves some quite serious, though by now rather standardised, commutative algebra arguments. Instead of 2.20 we use

Theorem 2.25. *Let U be open in an n-dimensional Cohen Macaulay variety $X \subset \mathbb{C}^N$, let $f : U \to \mathbb{C}^n$ be analytic, and let x_0 be isolated in $f^{-1}(y_0)$. Then there exists $\varepsilon > 0$ and $\eta > 0$ such that for all $y \in B_\eta(y_0)$,*

$$\sum_{x \in f^{-1}(y) \cap B_\varepsilon(x_0) \cap X} mult_x(f) = mult_{x_0} f. \tag{2.9}$$

In the example described above, $V(\mathcal{R}_f)$ is Cohen Macaulay provided its codimension in the domain of the unfolding F is equal to 2. This is a consequence of Theorem 2.35 below.

The proofs of Theorems 2.20 and 2.25 run along the same lines. The first step is to show that $\mathcal{O}_{X,x_0}$ is a finitely generated module over $\mathcal{O}_{\mathbb{C}^n,0}$. For this one uses the Preparation Theorem, 2.26 below. The second step is to use the Cohen-Macaulayness of $\mathcal{O}_{X,x_0}$ to show that it is not only finitely generated but free over $\mathcal{O}_{\mathbb{C}^n,0}$.

Proof that $\mathcal{O}_{X,x_0}$ is Cohen Macaulay generally uses the technique of "pulling back algebraic structures" discussed in Subsection 2.7 below.

2.5. The preparation theorem

The following theorem has rather an algebraic appearance, but is in fact a theorem of analysis. The classical Weierstrass Preparation Theorem on which it is based concerns division of analytic functions, and is more evidently "analytic".

Theorem 2.26. *Let X and Y be complex manifolds (or, more generally, analytic spaces) and $f : (X, x_0) \to (Y, y_0)$ an analytic map germ. Let M be a finitely generated module over $\mathcal{O}_{X,0}$. The following statements are equivalent.*

(1) *M is also finitely generated over $\mathcal{O}_{Y,y_0}$ via f.*
(2) *$dim_{\mathbb{C}} M/f^*\mathfrak{m}_{Y,y_0} M < \infty$.*

It is extensively used in analytic geometry and singularity theory. The statement also holds, verbatim, for C^∞ mappings and modules over the ring $\mathcal{E}_n$ of C^∞ germs. This much harder theorem was proved by Bernard Malgrange, at the urging of René Thom, in the 1960's, and made possible Thom's Catastrophe Theory, and Mather's celebrated series of papers on the stability of C^∞ mappings, [33], [35], [34], [36], [37], [38]. Alternative proofs were published by Łojasiewicz and by Mather himself.

2.6. Jet spaces and jet bundles

We denote by $J^k(n, p)$ the space of p-tuples of polynomials of degree $\leq k$ in n variables with no constant term. A germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$

determines a germ of map $j^k f : (\mathbb{C}^n, 0) \to J^k(n,p)$, the k-jet extension of f, defined by

$$j^k f(x) = \text{degree } k \text{ Taylor polynomial of } f \text{ at } x, \text{ without its constant term.}$$

The Taylor polynomial of f is determined by partial derivatives of order $\leq k$ of the component functions of f at x, so the k-jet can be thought of as simply recording these partial derivatives. There is a also a *jet bundle* $J^k(X,Y)$ over any pair of manifolds X and Y, whose fibre over $(x_0, y_0) \in X \times Y$, which we denote by $J^k(X,Y)_{(x,y)}$, is the set of k-jets of germs of maps $(X, x_0) \to (Y, y_0)$. Two such map-germs determine the same k-jet at x if they have the same partials of order $\leq k$ at x, with respect to some, and therefore to any, local coordinate systems on X and Y. So in coordinate free terms, a k-jet is an equivalence class of map-germs $(X, x) \to (Y, y)$.

Although $J^k(n,p)$ is a vector space, the fibre of $J^k(X,Y)$ over (x_0, y_0) is not; for the identifications between the two spaces depends on a choice of coordinate system, and when we change coordinates the higher derivatives of f change in a non-linear way. Thus there is no natural way of providing $J^k(X,Y)_{(x_0,y_0)}$ with the operations of a vector space, and $J^k(X,Y)$ is not a vector bundle over $X \times Y$.

Nevertheless, $J^k(X,Y)$ is a locally trivial fibre bundle over $X \times Y$.

Its importance for us is because of its role as a kind of Platonic Heaven which houses ideal versions of all of the singularities which appear in mappings. I will spend the rest of this section justifying this metaphysical remark.

Consider first the 1-jet-bundle $J^1(X,Y)$. By a choice of local coordinates on $U_X \subset X$ and $U_Y \subset Y$ we can identify $\pi^{-1}(U_X \times U_Y)$ with a product $U_X \times U_Y \times J^1(n,p)$ where $U_X \subset \mathbb{C}^n, U_Y \subset \mathbb{C}^p$ are open sets. The information contained in the 1-jet $j^1 f(x)$ is just the values of the first order partials of f, so we can think of $j^1 f$ as the map

$$x \mapsto (x, f(x), [d_x f]) \in \mathbb{C}^n \times \mathbb{C}^p \times \mathrm{Mat}_{p \times n}(\mathbb{C})$$

where $[d_x f]$ is the $n \times p$ jacobian matrix of f at x. Let us suppose, to fix ideas, that $n \leq p$, and define $\Sigma^k(n,p)$ (or Σ^k when the dimensions are clear from the context) to be the set of $p \times n$ complex matrices of kernel rank k.

Exercise 2.27. $\Sigma^k(n,p)$ *is a submanifold of* $Mat_{p\times n}(\mathbb{C})$ *of codimension* $k(p-n+k)$. *The formula for the codimension can be recalled as follows: a* $p \times n$ *matrix of the form*

$$\begin{pmatrix} I_{n-k} & B \\ 0 & D \end{pmatrix}$$

has kernel rank k if and only if $D = 0$. The same is true if we have an invertible $(n-k) \times (n-k)$ matrix A in place of I_{n-k}. A more general matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

in which A is of size $(n-k) \times (n-k)$ and invertible can be brought to this form by left-multiplying by

$$\begin{pmatrix} I_{n-k} & 0 \\ -CA^{-1} & I_{p-n+k} \end{pmatrix}$$

The matrix is in Σ^k if all entries in the transformed D are equal to zero. This gives $(p-n+k)k$ independent equations.

Let $f : X \to Y$ be a mapping, and denote now by $\Sigma^k(f)$ the set of points in X where $d_x f$ has kernel rank k. Then $\Sigma^k(f) = (j^1 f)^{-1}(\Sigma^k)$. Note, incidentally, that if we change coordinates on X then of course $j^1 f$ also changes, but $(j^1 f)^{-1}(\Sigma^k)$ is, evidently, unchanged. This is because Σ^k has the important property that it is preserved by the action of coordinate changes on X (or on Y).

Observation: suppose $x_0 \in \Sigma^k(f)$ and $j^1 f \pitchfork \Sigma^k$ at x_0. Then

- $\Sigma^k(f)$ is a smooth submanifold of X of codimension $k(p-n+k)$.
- Slightly less obvious: for $\ell < k$, $j^1 f \pitchfork \Sigma^\ell$ also.
- Indeed, writing $m_0 := j^1 f(x_0)$, up to product with smooth factors, there is a local diffeomorphism of germs of filtered spaces between

$$(\mathrm{Mat}_{p\times n}, m_0) \supset (\overline{\Sigma^1}, m_0) \supset \cdots \supset (\overline{\Sigma^{k-1}}, m_0) \supset (\Sigma^k, m_0)$$

and

$$(X, x_0) \supset (\overline{\Sigma^1(f)}, x_0) \supset \cdots \supset (\overline{\Sigma^{k-1}(f)}, x_0) \supset (\Sigma^k(f), x_0).$$

The second statement is a consequence of the fact that the corresponding stratification

$$\mathrm{Mat}_{p\times n}(\mathbb{C}) \supset (\Sigma^1 \smallsetminus \overline{\Sigma^2}) \supset \cdots \supset (\Sigma^\ell \smallsetminus \overline{\Sigma^{\ell+1}}) \cdots$$

is Whitney regular. We do not dwell on this now. The aim is simply to make clear that the transversality of $j^1 f$ to certain submanifolds of the jet bundle $J^k(X, Y)$ gives us a lot of information about submanifolds (subsets) of X determined by the geometry of f. The subsets that we are interested in are those which are preserved by the action of the group of diffeomorphisms of X and Y – the so-called left-right invariant subsets of $J^k(X, Y)$. The hypothesis on the transversality of $j^1 f$ to Σ^k that we invoked in our observation is motivated by the following statement.

Proposition 2.28. *Let $W \subset J^k(X,Y)$ be a left-right invariant submanifold. Then*

(1) *If $f : X \to Y$ is a stable* [3] *map, then $j^k f \pitchfork W$.*
(2) *If $f : (X, x_0) \to (Y, y_0)$ is a germ of finite $\mathcal{A}_e$-codimension, then $j^k f \pitchfork W$ on $X \smallsetminus \{x_0\}$.*

Proof. Suppose f is stable.

Step 1: Suppose that $j^k f(x_0) \in W$. There exists a germ of unfolding $F : (X \times S, (x_0, 0)) \to (Y \times S, (f(x_0), 0))$ of f such that the "relative" jet extension map $j^k_x F : X \times S \to J^k(X,Y)$ is transverse to W at $(x_0, 0)$. This can be arranged by choosing coordinates on X and Y around x_0 and y_0, and then taking as parameter space $S = J^k(n,p)$, and regarding its members as polynomial maps, which can be added to f. The resulting family is defined by $F(x,u) = f(x) + u(x)$, and $j^k_x F|_{\{x_0\} \times S} \to J^k(X,Y)_{(x_0,y_0)}$ is the identity map. It is thus transverse to W.

Step 2: f is stable, so F is a trivial unfolding. Thus, there exist germs of diffeomorphisms Φ of $(X \times S, (x_0, 0))$ with $\Phi(x,u) = (\varphi_u(x), u)$ and Ψ of $(Y \times S, (y_0, 0))$ with $\Psi(y,u) = (\psi_u(y), u)$ such that $\Psi \circ (f \times \mathrm{id}_S) \circ \Phi = F$. As $j^k_x F \pitchfork W$, we have $j^k_x \Psi \circ F \circ \Phi \pitchfork W$. As W is left-right invariant, it follows that $j^k f \pitchfork W$ (Exercise).

The second statement follows by the geometric criterion for finite codimension, Theorem 5.19. Since f is stable outside x_0, $j^k f$ is transverse to W outside x_0. Q.E.D.

Using an auxiliary map such as $j^k f$ to pull back a universal object from jet space can give useful information. Provided the codimension of the pulled back object is the same as the codimension of the universal object, much of the associated algebraic structure pulls back also. We will see this in Subsection 2.7.

A second important application of jet-space is through the Thom Transversality Theorem, which concerns the behaviour of smooth maps between smooth manifolds. A *residual* subset of a topological space is the intersection of a countable number of dense open sets, and a property is *generic* if it is held by all members of a residual subset. If M and N are smooth manifolds, the *Whitney C^k Topology* on the space $C^\infty(M,N)$ of

[3]The notions of stability and $\mathcal{A}_e$-codimension are defined and discussed in Section 4 below. See also Theorem 5.19 for the geometrical import of finite codimension - essentially it means "isolated instabiity".

smooth maps from M to N has as base the collection of subsets modelled on open sets $U \subset J^k(M,N)$:

$$C_U = \{f \in C^\infty(M,N) : j^k f(M) \subset U\}$$

and the Whitney C^∞ topology allows such sets for all values of k. We will always consider $C^\infty(M,N)$ with this topology. It is a Baire Space – residual sets are dense. A property of mappings $M \to N$ is said to be *generic* if it is held by the members of a residual subset of $C^\infty(M,N)$.

Exercise 2.29. If A is a residual subset of a Baire space S, can $S \smallsetminus A$ contain a residual subset of S?

Theorem 2.30. (Thom Transversality Theorem) *Let M and N be C^∞ manifolds, let $W \subset J^k(M,N)$ be a smooth submanifold, and let $T(W)$ be the set of smooth maps $f : M \to N$ such that $j^k f \pitchfork W$. Then*

(1) *$T(W)$ is residual in $C^\infty(M,N)$.*
(2) *If W is closed in $J^k(M,N)$ then $T(W)$ is open in $C^\infty(M,N)$.*

Note that if $\operatorname{codim} W > \dim M$, then $j^k f : M \to J^k(M,N)$ can only be transverse to W if $(j^k f)^{-1}(W) = \emptyset$. This is often the way that one proves that sets of mappings with certain properties are residual.

An immersion is an *embedding* if it is a diffeomorphism onto its image.

It is just a short step to prove Whitney's 'easy' embedding theorem from 2.30:

Theorem 2.31. *Let M be an n-dimensional smooth manifold. If $p \geq 2n+1$ then the set of embeddings $M \to \mathbb{R}^p$ is residual in $C^\infty(M,\mathbb{R}^p)$.*

If the domain M is compact, one has only to prove that immersions are residual, and that injective maps are residual. Properness (that the preimage of every compact set is compact) is a global property with some subtlety, and we will not discuss it except to say that it is automatic if the domain is compact. Injectivity, on the other hand, is a property of jets, and can be arranged, if the dimensions are right, by requiring transversality to a suitable submanifold of the *multi-jet space* ${}_rJ^k(M,N)$, which is defined as follows: there is a natural map $p : J^k(M,N) \to M$ giving the source of each jet; ${}_rJ^k(M,N)$ is the preimage in $\left(J^k(M,N)\right)^r$ of the set

$$M^{(r)} = \{(x_1,\ldots,x_r) \in M^r : x_i \neq x_j \text{ if } i \neq j\},$$

under the r-fold product map $p^r : \left(J^k(M,N)\right)^r \to M^r$. Each map $f : M \to N$ gives rise to a natural map ${}_rj^k f : M^{(r)} \to {}_rJ^k(M,N)$.

Theorem 2.32. *Let M and N be C^∞ manifolds, and let $W \subset {}_rJ^k(M,N)$ be a smooth submanifold. Then the set of smooth maps $f : M \to N$ such that ${}_rj^k f \pitchfork W$ is residual in $C^\infty(M,N)$ with the Whitney topology.*

Exercises 2.33. (1) The "Elementary Transversality Theorem" says that if W is a smooth submanifold of N then the set

$$\{f \in C^\infty(M,N) : f \pitchfork W\}$$

is residual. Show how to deduce this from the Thom Transversality Theorem 2.30.

(2) Show that an immersion which is a homeomorphism onto its image is a diffeomorphism.

(3) Show that if M is compact then an injective immersion is an embedding.

(4) Give an example of an injective immersion of $\mathbb{R}$ in $\mathbb{R}^2$ which is not an embedding.

(5) Prove Whitney's easy embedding theorem 2.31 for compact manifolds M. The theorem does not *require* the hypothesis of compactness, but explaining this would lead us too far away from the main thrust of the lectures.

(6) Let $W = \{(x,0,0) \in \mathbb{R}^3 : -1 < x < 1\}$. Show that the set $\{f \in C^\infty(\mathbb{R},\mathbb{R}^3) : f \pitchfork W\}$ is *not* open. Hint: consider $f(t) = (-1,t,0)$.

(7) Let $W = \{(x,0) \in \mathbb{R}^2 : -1 < x < 1\}$. Show that the set $\{f \in C^\infty(S^1,\mathbb{R}^3) : f \pitchfork W\}$ is not open.

(8) If $n < 6$, the set of mappings $M^n \to N^{n+1}$ for which all singularities have corank 1 is residual (see 2.4 for the definition of corank). Is it open?

(9) What is the smallest value of n for which a stable map from an n-dimensional manifold to an $n+1$-dimensional manifold can have a corank 2 singularity? A corank 3 singularity?

(10) A critical point x_0 of a smooth real-valued function is *non-degenerate* if the Hessian matrix

$$\left[\frac{\partial^2 f}{\partial x_i x_j}(x_0)_{1\le i,j\le m}\right]$$

(with respect to some, and hence any, set of local coordinates) is invertible. A function $M \to \mathbb{R}$ is a *Morse function* if all of its critical points are non-degenerate and no two critical points share the same critical value. Show that for any smooth manifold M, Morse functions form a residual set in $C^\infty(M,\mathbb{R})$.

(11) A fixed point x_0 of a smooth map $f : M \to M$ is *non-degenerate* if $d_{x_0}f$ does not have 1 as an eigenvalue. Show that this condition can be expressed in terms of the transversality of some jet extension map to a suitable submanifold of jet space, and deduce that the set of maps $f : M \to M$ with only non-degenerate fixed points is residual in $C^\infty(M, M)$.

Further reading: Chapter II of the book [18] of Guillemin and Golubitsky.

2.7. Pulling back algebraic structures

The following result fits well with the idea that in singularity theory we study ideal objects, in the sense of Plato, and then attempt to wrestle their properties back to the reality of our concrete examples by some kind of pull-back procedure. The ideal objects are usually contained in spaces of $p \times q$ matrices, or jet spaces $J^k(N, P)$. The condition for the success of this strategy is usually that the codimension of the concrete object in its ambient space is the same as the codimension of the ideal object in its ambient space.

Theorem 2.34. *Let $f : X \to Y$ be a map of complex manifolds and let $W \subset Y$ be an analytic subspace.*
(1) If $f^{-1}(W) \neq \emptyset$ then

$$\mathit{codim}_X f^{-1}(W) \leq \mathit{codim}_Y(W). \tag{2.10}$$

(2) If W is Cohen-Macaulay, and the inequality in (2.10) *is an equality, then*

- (a) *$f^{-1}(W)$ is Cohen-Macaulay, and*
- (b) *If $\mathbf{L}_\bullet$ is a free resolution of the germ of $\mathcal{O}_{W,w_0}$ as $\mathcal{O}_{Y,w_0}$-module, then for each $x \in f^{-1}(W)$ with $f(x) = w_0$, $\mathbf{L}_\bullet \otimes_{\mathcal{O}_{Y,w_0}} \mathcal{O}_{X,x}$ is a free resolution of $\mathcal{O}_{f^{-1}(W),x}$ as $\mathcal{O}_{X,x}$ module.*

Later we will need a version of 2(b) of Theorem 2.34 with $M \otimes_{\mathcal{O}_{Y,y_0}} \mathcal{O}_{X,x_0}$, where M is an $\mathcal{O}_{Y,y}$-module, in place of $\mathcal{O}_{f^{-1}(W),x}$. Its proof is very similar to the proof of 2.34, and is left to the reader.

Before proving 2.34, let us look at an example of its application.

Corollary 2.35. *Let M be a $p \times n$ matrix of functions in $\mathcal{O}_{\mathbb{C}^n,x_0}$, with $p \geq n$. If the codimension in $\mathbb{C}^n$ of $V(\min_k(M))$ is equal to $(p - k + 1)(n - k + 1)$ then $V(\min_k(M))$ is Cohen-Macaulay.*

Proof. Denote the entries of M by m_{ij}. Let ψ_M denote the map

$$\mathbb{C}^n \to \mathrm{Mat}_{p\times q}(\mathbb{C}), \quad x \mapsto (m_{ij}(x) : 1 \leq i \leq p,\ 1 \leq j \leq q).$$

Then $V(\min_k(M)) = \psi_M^{-1}(W_k)$. A well known theorem of Eagon and Hochster in [13] tells us that the space W defined by the $k \times k$ minors of the generic matrix in matrix space $\mathrm{Mat}_{p\times n}(\mathbb{C})$ is Cohen-Macaulay of codimension $(p-k+1)(n-k+1)$. Now apply Theorem 2.34(a). Q.E.D.

Corollary 2.36. *Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ be an analytic germ, with $n < p$, and denote by Σ_f the non-immersive locus of f. Then*

$$\mathit{codim}\,(\Sigma_f) \leq p - n + 1,$$

and in case of equality, Σ_f is Cohen-Macaulay.

Proof. Σ_f is defined by the maximal ($= n \times n$) minors of the Jacobian matrix

$$\left(\frac{\partial f_i}{\partial x_j}\ 1 \leq i \leq p,\ 1 \leq j \leq n\right)$$

of f. So the corollary is just an application of 2.35. Q.E.D.

To prove 2.34 we need some (well known) preparatory lemmas.

Lemma 2.37. *Let M be a Cohen-Macaulay module over the ring R and let $a_1, \ldots, a_n \in R$. If*

$$\mathit{dim}\ M/(a_1, \ldots, a_n)M = \mathit{dim}\ M - n$$

then
(i) $a_1, \ldots, a_n$ is an M-sequence and
(ii) $M/(a_1, \ldots, a_n)M$ is Cohen-Macaulay.

Proof. We prove both statements simultaneously by induction on n. Let $M_j = M/(a_1, \ldots, a_j)M = M_{j-1}/a_j M_{j-1}$. The hypothesis implies that $\dim\ M_j/a_{j+1}M_j < \dim\ M_j$. We claim that a_{j+1} is not a zero divisor on M_j. This is equivalent to saying that a_{j+1} does not lie in any associated prime of M_j. Now $\mathrm{Ass}(M_j)$ is the set of minimal members (with respect to inclusion) of $\mathrm{supp}(M_j)$. The fact that M_j is Cohen-Macaulay means in particular that all of these have the same height, equal to $\dim\ R - \dim\ M_j$. Because $\dim\ M_j/a_{j+1}M_j < \dim\ M_j$, the minimal members of $\mathrm{supp}(M_j/a_{j+1}M_j) = \mathrm{supp}(M_j) \cap V(a_{j+1})$ are all of greater height than the minimal members of $\mathrm{supp}(M_j)$. Thus

$$\text{miminal members of } \big(\mathrm{supp}(M_j) \bigcap V(a_{j+1})\big)$$

contains none of the minimal members of $\mathrm{supp}(M_j)$. In other words, a_{j+1} lies in none of the minimal members of $\mathrm{supp}(M_j)$, i.e. in none of the associated primes of M_j. This means that a_{j+1} is regular on M_j.

The sequence

$$0 \longrightarrow M_j \xrightarrow{a_{j+1}} M_j \longrightarrow M_{j+1} \longrightarrow 0$$

is now exact. From this it follows by the depth lemma that

$$\text{depth } M_{j+1} = \text{depth } M_j - 1 = \dim M_j - 1$$

and hence

$$\dim M_j > \dim M_{j+1} \geq \text{depth } M_{j+1} = \dim M_j - 1.$$

Cohen-Macaulayness of M_{j+1} follows. Q.E.D.

Lemma 2.38. *Suppose that M is a Cohen-Macaulay module over R and that the elements $a_1, \ldots, a_n$ in R form an M-sequence and an R-sequence. Let I be the ideal in R generated by $a_1, \ldots, a_n$. If $\mathbf{L}_\bullet$ is a free resolution of M over R, then $\mathbf{L}_\bullet \otimes R/I$ is a free resolution of M/IM over R/I.*

Proof. Again we use induction on n, and the sequence M_j, $j = 0, \ldots, n$ of modules defined in the previous proof. Let $R_0 = R$ and $R_j = R/(a_1, \ldots, a_j)$ for $j = 1, \ldots, n$. Suppose that $\mathbf{L}_\bullet \otimes_R R_j$ is exact. Then it is a resolution of M_j. We have

$$H_i(\mathbf{L}_\bullet \otimes R_{j+1}) = \mathrm{Tor}_i^{R_j}\big(M_j, R_{j+1})$$

so to prove exactness we have to show that these Tor modules vanish. We calculate $\mathrm{Tor}^{R_j}\big(M_j, R_{j+1})\big)$ by tensoring the short exact sequence

$$0 \longrightarrow R_j \xrightarrow{a_{j+1}} R_j \longrightarrow R_{j+1} \longrightarrow 0$$

with M_j. This gives the long exact sequence

$$\to \mathrm{Tor}_i(M_j, R_j) \to \mathrm{Tor}_i(M_j, R_j) \to \mathrm{Tor}_i(M_j, R_{j+1}) \to$$

$$\cdots \to \mathrm{Tor}_1(M_j, R_{j+1}) \longrightarrow M_j \xrightarrow{a_{j+1}} M_j \longrightarrow M_{j+1} \longrightarrow 0\ .$$

From this it immediately follows that $\mathrm{Tor}_i^{R_j}(M_j, R_{j+1}) = 0$ for $i > 1$, since this module appears in the sequence flanked by Tor modules which are trivially zero. Vanishing of $\mathrm{Tor}_1^{R_j}(M_j, R_{j+1})$ follows from vanishing of $\mathrm{Tor}_1^{R_j}(M_j, R_j)$ and the injectivity of $M_j \xrightarrow{a_{j+1}} M_j$. Q.E.D.

Proof. of Theorem 2.34 The map

$$f^{-1}(W) \longrightarrow \widetilde{f^{-1}(W)} := \{(x,w) \in X \times W : w = f(x)\}$$

sending x to $(x, f(x))$ has inverse given by the restriction to $\widetilde{f^{-1}(W)}$ of the usual projection $X \times W \to X$. Thus $f^{-1}(W)$ and $\widetilde{f^{-1}(W)}$ are isomorphic, and it is enough to prove that $\widetilde{f^{-1}(W)}$ is Cohen Macaulay. As the product of a smooth space with a Cohen Macaulay space, $X \times W$ is Cohen Macaulay of dimension $\dim W + \dim X$. Taking local coordinates $y_1, \ldots, y_p$ on Y around w_0, we can then view $\widetilde{f^{-1}(W)}$ as the fibre over $0 \in \mathbb{C}^p$ of the map $\pi : X \times W \to Y$ given by $\pi(x) = (y_1 - f_1(x), \ldots, y_p - f_p(x))$. By the *hauptidealsatz*, $\dim X \times W - \dim \widetilde{f^{-1}(W)} \le p = \dim Y$, from which (2.10) follows.

Now suppose that (2.10) is an equality. Then by Lemma 2.37 the components of π form a regular sequence in $\mathcal{O}_{X \times W}$. Since $\mathcal{O}_{X \times W}$ is Cohen-Macaulay, so also is its quotient by the ideal generated by the components of π. This quotient is $\mathcal{O}_{\widetilde{f^{-1}(W)}} \simeq \mathcal{O}_{f^{-1}(W)}$, so $f^{-1}(W)$ is Cohen-Macaulay. The remaining statement is just Lemma 2.38 applied to the $\mathcal{O}_{X \times W}$-module $\mathcal{O}_{\widetilde{f^{-1}(W)}}$. Q.E.D.

§3. Equivalence of germs of mappings

Let $f, g : (X, S) \to (Y, y_0)$ be germs of analytic maps. They are

(1) *right-equivalent* if there exists a germ of analytic automorphism φ of (X, S) such that $f_2 = f_1 \circ \varphi$;

(2) *left-equivalent*, if there exists a germ of analytic automorphism ψ of (Y, y_0) such that $f_2 = \psi \circ f_1$;

(3) *left-right-equivalent*, if there exist germs of analytic automorphisms φ of (X, S) and ψ of (Y, y_0) such that $\psi \circ f \circ \varphi^{-1} = g$. This is the most natural equivalence relation if one is interested in the maps themselves.

(4) *contact equivalent*, if there exists a germ of automorphism Φ of $(X \times Y, S \times \{y_0\})$, of the form $\Phi(x,y) = (\varphi_1(x), \varphi_2(x,y))$, such that $\Phi(\text{graph } (f_1)) = \text{graph } (f_2)$.

We usually replace the term "analytic automorphism" by "diffeomorphism", because most of the theory works unchanged for C^∞ maps.

In each case there is a group of germs of diffeomorphisms acting on the set of mappings. The groups (or, more precisely, their actions) are denoted by $\mathcal{R}, L, A$ and $\mathcal{K}$ respectively. We will be most interested in

$\mathcal{A}$: it is the most natural if one is interested in the geometry of maps between complex spaces.

Exercise 3.1. (a) *Show that $\mathcal{A} \subset \mathcal{K}$, in the sense that $\mathcal{A}$-equivalence implies $\mathcal{K}$-equivalence.*

(b) *Show that if $f \sim_{\mathcal{K}} g$ then $f^{-1}(y_0)$ and $g^{-1}(y_0)$ are diffeomorphic.*

For a very good survey of these groups and their actions, see [54].

A big part of singularity theory has always been concerned with the problem of classification. Generally one classifies germs of analytic maps $(\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ up to $\mathcal{A}$-equivalence, and up to $\mathcal{R}$-equivalence if $p = 1$. Contact equivalence is a technical device which is of interest primarily if one is concerned with preimages of y_0, but also plays an important role in the theory of left-right equivalence, as we will see.

A key ingredient in classification is the notion of *finite determinacy.* Let us assume that $X = \mathbb{C}^n$, $Y = \mathbb{C}^p$ and $S = \{x_0\}$.

Definition 3.2. Let $f : (X, x_0) \to (Y, y_0)$ be a complex analytic or C^∞ map, and let $\mathcal{G}$ be one of the groups listed above. We say f is *k-determined for $\mathcal{G}$-equivalence* if whenever the Taylor series of another germ g coincides with that of f up to degree k, then $f \sim_{\mathcal{G}} g$, and *finitely determined* if it is k-determined for some $k \in \mathbb{N}$.

The notion has an obvious generalisation to the case where S consists of more than a single point, but has only been used in practice in case S is a finite point set. Here we will look only at the case where S is a single point.

In [34], John Mather showed that for all of the groups listed above, finite determinacy is equivalent to *isolated instability.* We will not prove this, but will explain the main ideas of the proof. The key is to understand how to construct diffeomorphisms. In all of singularity theory this is done by integrating vector fields. With very few exceptions, there is no other method!

3.1. Integration of vector fields

Proposition 3.3. *Let χ be an analytic vector field on the open set $U \subset \mathbb{C}^n$. Then for each $x_0 \in U$ there is an open neighbourhood $U(x_0) \subset U$, a disc $B_\eta(0)$ of radius $\eta > 0$ centred at $0 \in \mathbb{C}$, and an analytic map $\Phi : U(x_0) \times B_\eta(0) \to U$ such that for all (x, t)*

(1) $\Phi(x, 0) = x$;

(2) $\frac{d}{dt}\Phi(x, t) = \chi(\Phi(x, t))$. □

The curve described by $\Phi(x, t)$, for fixed x, as t varies, is called a *trajectory* of the vector field χ, and (2) above says that the tangent vector to this trajectory at the point $\Phi(x,t)$ is the vector $\chi(\Phi(x,t))$. Writing $\gamma_x(t)$ in place of $\Phi(x,t)$, and keeping x fixed, this becomes

$$\gamma_x'(t) = \chi(\gamma_x(t)).$$

If instead we fix t, we get a map $\varphi_t : U(x_0) \to U$. Notice that (1) above says that φ_0 is the identity map. From the theorem of existence and uniqueness of solutions of ordinary differential equations, one easily deduces

Corollary 3.4. (a) *Wherever the composite is defined, one has*

$$\varphi_s \circ \varphi_t = \varphi_{s+t}.$$

(b) *For each $x_0 \in U$ and each fixed value of $t \in B_\eta(0)$, the map $\varphi_t : U(x_0) \to \varphi_t(U(x_0))$ is a diffeomorphism (bianalytic isomorphism), with inverse φ_{-t}.*

The family of diffeomorphisms φ_t is called the *integral flow* of the vector field χ. All arguments involving the integration of vector fields to construct diffeomorphisms go via the following *Thom-Levine* theorem:

Corollary 3.5. *Suppose that $F : X \to Y$ is an analytic map of complex manifolds, and that χ and $\tilde{\chi}$ are vector fields on Y and X such that for each $x \in X$ one has*

$$d_xF(\tilde{\chi}(x)) = \chi(F(x)). \tag{3.1}$$

Then the integral flows Φ and $\tilde{\Phi}$ of χ and $\tilde{\chi}$ satisfy

$$F \circ \tilde{\varphi}_t = \varphi_t \circ F \tag{3.2}$$

wherever the composites are defined.

The two equations (3.1) and (3.2) can be expressed in terms of commutative diagrams. The vector fields χ and $\tilde{\chi}$ are sections of the tangent bundles TY and TX respectively, and (3.1) and (3.2) say that the diagrams

(3.3)
$$\begin{tikzcd} TX \arrow[r, "dF"] \arrow[d, dashed] & TY \arrow[d, dashed] \\ X \arrow[r, "F"] \arrow[u, bend left, "\tilde{\chi}"] & Y \arrow[u, bend right, "\chi"'] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} X \arrow[r, "F"] \arrow[d, "\tilde{\varphi}_t"'] & Y \arrow[d, "\varphi_t"] \\ X \arrow[r, "F"'] & Y \end{tikzcd}$$

commute.

The Thom-Levine theorem shows how an "infinitesimal condition" gives rise to a family of diffeomorphisms. Equalities like (3.1) are linear in χ and $\tilde{\chi}$, and these vector fields can often be constructed by the methods of commutative algebra. This is the entry-point of commutative algebra, which, through it, has a huge input into Singularity Theory.

As an example of what is involved, let us prove the simplest of the determinacy theorems of John Mather. If $f : (\mathbb{C}^n, 0) \to \mathbb{C}$ is an analytic germ of function, then the first order partials $\partial f/\partial x_1, \ldots, \partial f/\partial x_n$ generate the *jacobian ideal* J_f in the ring $\mathcal{O}_{\mathbb{C}^n,0}$.

Example 3.6. (a) *If* $f(x_1, \ldots, x_n) = x_1^2 + \cdots + x_n^2$ *then* J_f *is the maximal ideal* $\mathfrak{m}_n := \mathfrak{m}_{\mathbb{C}^n,0}$.

(b) *If* $f(x_1, x_2) = x_1^2 + x_2^{k+1}$ *then* $J_f = (x_1, x_2^k)$.

(c) *If* $f(x_1, x_2) = x_1^2 x_2 + x_2^{k-1}$ *then* $J_f = (x_1 x_2, x_1^2 + (k-1)x_2^{k-2})$.

Theorem 3.7. *(i) Suppose that* $f \in \mathcal{O}_{\mathbb{C}^n,0}$ *is* k*-determined for right equivalence. Then* $\mathfrak{m}_n\, J_f \supset \mathfrak{m}_n^{k+1}$.
(ii) Conversely, suppose that $f \in \mathcal{O}_{\mathbb{C}^n,0}$ *and*

$$\mathfrak{m}_n\, J_f \supset \mathfrak{m}_n^k\,. \tag{3.4}$$

Then f *is* k*-determined for* $\mathcal{R}$*-equivalence.*

Exercise 3.8. *Find the lowest value of* k *for which* (3.4) *holds for each of the functions in Example 3.6.*

Proof of 3.7.(i) Let $h \in \mathfrak{m}_n^{k+1}$. Then for all t there exists $\varphi_t \in \mathrm{Diff}(\mathbb{C}^n, 0)$ such that $f + th = f \circ \varphi_t$. If we could assume the existence of a smoothly parametrised family of diffeomorphisms φ_t with $\varphi_0 = \mathrm{id}$ such that $f \circ \varphi_t = f + th$ then we could reason as follows:

$$h = \frac{\partial(f + th)}{dt} = \frac{\partial(f \circ \varphi_t)}{dt} = \sum_i \left(\frac{\partial f}{\partial x_i} \circ \varphi_t\right) \frac{\partial \varphi_{t,i}}{\partial t}. \tag{3.5}$$

Note that since $\varphi_t(0) = 0$ for all t it follows that $\partial \varphi_{t,i}/\partial t \in \mathfrak{m}_n$. When $t = 0$, since $\varphi_0 = \mathrm{id}$, this gives

$$h = \sum_i \frac{\partial f}{\partial x_i} \frac{\partial \varphi_{t,i}}{\partial t} \in \mathfrak{m}_n\, J_f \tag{3.6}$$

so that $\mathfrak{m}_n^{k+1} \subset \mathfrak{m}_n\, J_f$ as required.

However, our hypothesis does not allow us immediately to assert that the diffeomorphisms φ_t fit together to give a smooth family. So instead we look in jet space $J^{k+1}(n, 1) = \mathfrak{m}_n \,/\, \mathfrak{m}_n^{k+2}$. As f is k-determined, the set

$$L := \{j^{k+1}(f + h) : h \in \mathfrak{m}_n^{k+1}\} \subset J^{k+1}(n, 1)$$

lies entirely in the $\mathcal{R}^{(k+1)}$-orbit of f, where $\mathcal{R}^{(k+1)}$ is the finite dimensional quotient of $\mathrm{Diff}(\mathbb{C}^n, 0)$ acting on jet space. Now $\mathcal{R}^{(k+1)}$ can be identified with the set

$$\{j^{k+1}\varphi(0) : \varphi \in \mathrm{Diff}(\mathbb{C}^n, 0)\}$$

and has a natural structure of algebraic group: the composite of two polynomial mappings depends polynomially on their coefficients, and in $\mathcal{R}^{(k+1)}$ one composes and then truncates at degree $k+1$. This group acts algebraically on $J^{k+1}(n, 1)$. Thus, as the set L lies in the orbit of $j^{k+1}f(0)$, writing $z = j^{k+1}f(0)$, and $\mathcal{R}^{(k+1)}z$ for the $\mathcal{R}^{(k+1)}$-orbit of z, one has

$$\frac{\mathfrak{m}_n^{k+1}}{\mathfrak{m}_n^{k+2}} = T_z L \subset T_z(\mathcal{R}^{(k+1)}z) = \frac{\mathfrak{m}_n\, J_f + \mathfrak{m}_n^{k+2}}{\mathfrak{m}_n^{k+2}}, \tag{3.7}$$

and thus

$$\mathfrak{m}_n^{k+1} \subset \mathfrak{m}_n\, J_f + \mathfrak{m}_n^{k+2}\,.$$

The conclusion we want follows by Nakayama's Lemma, 2.16.

The second equality in (3.7) is important and not completely obvious. It can be obtained along the lines of the argument leading up to (3.6), but using the crucial fact that if the Lie group G acts on the manifold M and for $x \in M$ we denote by α_x the orbit map $g \in G \mapsto gx$, then for each $x \in M$ with smooth orbit Gx,

$$T_x Gx = d_e\alpha_x(T_e G).$$

Now $d_e\alpha_x(T_eG)$ is equal to

$$\{\frac{d}{dt}(\gamma(t)\cdot x)|_{t=0} :\ \gamma \text{ is a curve germ } (\mathbb{C}, 0) \to (G, e)\};$$

every curve in $(\mathcal{R}^{(k+1)}, \mathrm{id})$ is of the form $j^{k+1}\varphi_t$ for a 1-parameter family of diffeomorphisms φ_t, so now it really is true that $T_z\mathcal{R}^{(k+1)}z$ is equal to the set of all $\frac{d}{dt}j^{k+1}(f\circ\varphi_t)_{t=0}$ where φ_t is a 1-parameter family in $\mathrm{Diff}(\mathbb{C}^n, 0)$ with $\varphi_0 = \mathrm{id}$. Reversing the order of differentiation, this derivative becomes

$$j^{k+1}\left(\frac{d}{dt}(f\circ\varphi_t)|_{t=0}\right),$$

and this proves the second equality in (3.7).

(ii) Suppose that g has the same degree k Taylor polynomial as f. Then $g - f \in \mathfrak{m}_n^{k+1}$. Let $F(x,t) = f(x) + t(g(x) - f(x))$, and write $f_t(x) = F(t, x)$. The idea of the proof is to show that for each value t_0 of t, there is a neighbourhood $U(t_0)$ of t_0 in $\mathbb{C}$ such that f_t and f_{t_0} are $\mathcal{R}$-equivalent for all $t \in U(t_0)$. A finite number of these neighbourhoods cover the compact interval $[0, 1] \subset \mathbb{C}$, so by transitivity $f = f_0 \sim_{\mathcal{R}} f_1 = g$.

Step 1: We do this first for $t_0 = 0$. As F is a function of the $n + 1$ variables $x_1, \ldots, x_n, t$, we consider the germ $F \in \mathcal{O}_{\mathbb{C}^{n+1},0}$. We will need to refer to the ideal in $\mathcal{O}_{\mathbb{C}^{n+1},0}$ generated by $x_1, \ldots, x_n$; rather than the cumbersome "$\mathfrak{m}_n \, \mathcal{O}_{\mathbb{C}^{n+1},0}$" we will use "$\tilde{\mathfrak{m}}_n$". Notice that $\partial F/\partial t = g - f \in \tilde{\mathfrak{m}}_n^{k+1}$. It follows from our hypothesis on f that

$$\frac{\partial F}{\partial t} \in \tilde{\mathfrak{m}}_n \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right). \tag{3.8}$$

We would like to show

$$\frac{\partial F}{\partial t} \in \tilde{\mathfrak{m}}_n \left(\frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_n} \right). \tag{3.9}$$

For if we have

$$\frac{\partial F}{\partial t} = \tilde{\chi}_1 \frac{\partial F}{\partial x_1} + \cdots + \tilde{\chi}_n \frac{\partial F}{\partial x_n} \tag{3.10}$$

for some functions $\tilde{\chi}_i \in \tilde{\mathfrak{m}}_n$, then defining a vector field $\tilde{\chi}$ on $\mathbb{C}^{n+1}$ by

$$\tilde{\chi} = \frac{\partial}{\partial t} - \sum_i \tilde{\chi}_i \frac{\partial}{\partial x_i},$$

(3.10) becomes

$$dF(\tilde{\chi}) = 0.$$

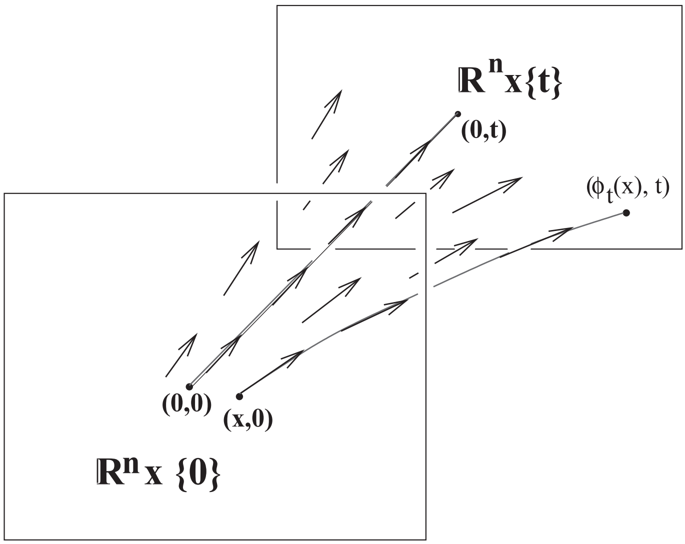
This is exactly (3.1) with $\chi = 0$. It implies that F is constant along the trajectories of the vector field $\tilde{\chi}$. Let $\tilde{\Phi}(x, t) = (\tilde{\phi}_t(x), t)$ be the integral flow of $\tilde{\chi}$. The integral flow of the zero vector field is the identity map, and therefore by the Thom-Levine lemma we have

$$F \circ \tilde{\Phi} = F. \tag{3.11}$$

Since the component of $\tilde{\chi}$ in the t-direction has constant length 1, it follows that $\tilde{\varphi}_t$ maps $\mathbb{C}^n \times \{0\}$ to $\mathbb{C}^n \times \{t\}$. Restricting both sides of (3.11) to $\mathbb{C}^n \times \{0\}$ we therefore get

$$f_t \circ \tilde{\varphi}_t = f.$$

This is not quite enough to show that the germs at 0 of f and of f_t are right-equivalent: we need to show also that $\varphi_t(0) = 0$. But this holds, because $\tilde{\chi}_i \in \tilde{\mathfrak{m}}_n$ for all i, and thus $\chi_i(0,t) = 0$, $\tilde{\chi}$ is tangent to the t-axis $\{0\} \times \mathbb{R}$, and $\varphi_t(0) = 0$ for all t. Thus $\tilde{\varphi}_t \in \mathcal{R}$ and $f_t \sim_{\mathcal{R}} f$ as required.



The arrows show a real version of the vector field $\tilde{\chi}$ of the proof. At all points of the t-axis, the vector field is tangent to the axis, so any trajectory beginning at a point on the axis remains on the axis. Thus $\varphi_t(0) = 0$.

Now we set about deducing (3.9) from (3.8). Since $\partial F/\partial t = g - f \in \tilde{\mathfrak{m}}_n^k$, to show (3.9), it will be enough to show

$$\tilde{\mathfrak{m}}_n^k \subset \tilde{\mathfrak{m}}_n \left(\frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_n} \right). \tag{3.12}$$

We know that

$$\tilde{\mathfrak{m}}_n^k \subset \tilde{\mathfrak{m}}_n \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right). \tag{3.13}$$

Because

$$\frac{\partial f}{\partial x_i} = \frac{\partial F}{\partial x_i} - t \frac{\partial (g-f)}{\partial x_i}, \tag{3.14}$$

it follows that

$$\frac{\partial f}{\partial x_i} \in \tilde{\mathfrak{m}}_n \left(\frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_n} \right) + \mathfrak{m}_{n+1} \, \tilde{\mathfrak{m}}_n^k$$

and therefore

$$\tilde{\mathfrak{m}}_n^k \subset \tilde{\mathfrak{m}}_n \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right) \subset \tilde{\mathfrak{m}}_n \left(\frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_n}\right) + \mathfrak{m}_{n+1}\, \tilde{\mathfrak{m}}_n^k. \tag{3.15}$$

Now some commutative algebra comes to our aid. By *Nakayama's Lemma*, 2.17, proved in Subsection 2.3, (3.15) implies at once that (3.12) holds: we apply it taking as R the local ring $\mathcal{O}_{\mathbb{C}^{n+1}}$ with maximal ideal $\mathfrak{m}_{n+1}$, and taking $M = \tilde{\mathfrak{m}}_n^k$ and $N = \tilde{\mathfrak{m}}_n J_f$ (where, as before $\tilde{\mathfrak{m}}_n$ means the ideal in $\mathcal{O}_{\mathbb{C}^{n+1}}$ generated by $x_1, \ldots, x_n$).

This completes the proof that the deformation $f + t(g - f)$ is trivial for t in some neighbourhood of 0.

Step 2: The remainder of the proof involves showing that the same procedure can be employed for every value of t: we want to show that for any t_0 the deformation $f+t(g-f)$ is trivial in a neighbourhood of t_0. This deformation can be written in the form $(f+t_0(g-f))+(t-t_0)(g-f)$, and taking as new parameter $s = t - t_0$, the problem reduces to what we have already discussed, except that instead of our original f we now have a new function, $f_{t_0} := f + t_0(g - f)$. In order that our earlier argument should apply, we have to show that f_{t_0} also satisfies the hypothesis of this argument: that

$$\mathfrak{m}\, J_{f_{t_0}} \supset \mathfrak{m}^k \tag{3.16}$$

Once again this is done by a simple argument involving Nakayama's Lemma, which I leave as an exercise.

Exercise 3.9. *Show that if* $\mathfrak{m}\, J_f \supset \mathfrak{m}^k$ *and* $g - f \in \mathfrak{m}^{k+1}$ *then* $\mathfrak{m}\, J_{f_{t_0}} \supset \mathfrak{m}^k$.

□

The first part of the proof of Theorem 3.7 justifies part (i) of the following definition.

Definition 3.10.

$$\begin{array}{llcl} (i) & T\mathcal{R}f & = & \mathfrak{m}_n\, J_f \\ (ii) & T\mathcal{R}_e f & = & J_f \end{array}$$

The second tangent space is the *extended* right tangent space. Its heuristic justification is less clear than that of $T\mathcal{R}f$; it can be obtained by the argument of the proof of Theorem 3.7(i) *if we remove the requirement that* $\varphi_t(0) = 0$ *for all* t.

§4. Left-right equivalence

In these lectures we are interested in left-right equivalence more than right equivalence. But Theorem 3.7 is a good indication of what is true and how, in principle, one goes about proving it. For left-right equivalence, the proof is necessarily more complicated, since one has simultaneously to produce families of diffeomorphisms of source and target. However the overall strategy is the same. First we need to define a suitable tangent space for $\mathcal{A}$-equivalence.

Mather and Thom, in their work in the 60's on smooth maps, thought in global terms: a C^∞ map $f : N \to P$ is *stable* if its orbit under the natural action of $\mathrm{Diff}(N) \times \mathrm{Diff}(P)$ is open in $C^\infty(N, P)$, with respect to a suitable topology. Here we are interested in local geometry, and so we give a local version of this definition: a map-germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ is stable if every deformation is trivial: roughly speaking, if f_t is a deformation of f then there should exist deformations of the identity maps of $(\mathbb{C}^n, 0)$ and $(\mathbb{C}^p, 0)$, φ_t and ψ_t, such that

$$f_t = \psi_t \circ f \circ \varphi_t. \tag{4.1}$$

A substantial part of Mather's six papers on the stability of C^∞ mappings [33]-[38] is devoted to showing that if all the germs of a mapping f are stable in this local sense then f is stable in the global sense. We will not discuss global stability any further.

Definition 4.1. (1) An *unfolding* of f is a map-germ

$$F : (\mathbb{C}^n \times \mathbb{C}^d, 0) \to (\mathbb{C}^p \times \mathbb{C}^d, 0)$$

of the form

$$F(x, u) = (\tilde{f}(x, u), u)$$

such that $\tilde{f}(x, 0) = f(x)$.

Retaining the parameters u in the second component of the map makes the following definition easier to write down:
(2) The unfolding F is *trivial* if there exist germs of diffeomorphisms

$$\Phi : (\mathbb{C}^n \times \mathbb{C}^d, 0) \to (\mathbb{C}^n \times \mathbb{C}^d, 0)$$

and

$$\Psi : (\mathbb{C}^p \times \mathbb{C}^d, 0) \to (\mathbb{C}^p \times \mathbb{C}^d, 0)$$

such that

(a) $\Phi(x, u) = (\varphi(x, u), u)$ and $\varphi(x, 0) = x$

(b) $\Psi(y,h) = (\psi(y,u),u)$ and $\psi(y,0) = y$
(c)

$$F = \Psi \circ (f \times \mathrm{id}) \circ \Phi \tag{4.2}$$

(where $f \times \mathrm{id}$ is the 'constant' unfolding $(x,u) \mapsto (f(x),u)$).
(3) The map-germ $f : (\mathbb{C}^n,0) \to (\mathbb{C}^p,0)$ is *stable* if every unfolding of f is trivial.

By writing $\varphi(x,u) = \varphi_u(x)$ and $\psi(y,u) = \psi_u(y)$, from (4.2) we recover the heuristic definition (4.1). We do not insist that the mappings φ_u and ψ_u preserve the origin of $\mathbb{C}^n$ and $\mathbb{C}^p$ respectively. After all, if the interesting behaviour merely changes its location, we should not regard the unfolding as non-trivial.

Example 4.2. Consider the map-germ $f(x) = x^2$, and its unfolding $F(x,u) = (x^2 + ux, u)$. This is trivialised by the families of diffeomorphisms $\Phi(x,u) = (x + u/2, u)$, $\Psi(y,u) = (y - u^2/4, u)$. Both Φ and Ψ are just families of translations.

Exercise 4.3. Check that in the previous example $F = \Psi \circ (f \times \mathrm{id}) \circ \Phi$.

Fortunately, there exists a simple and computable criterion for stability. If f is stable, then the quotient

$$T^1(f) := \frac{\{\frac{d}{dt}f_t|_{t=0} : f_0 = f\}}{\{\frac{d}{dt}(\psi_t \circ f \circ \varphi_t)|_{t=0} : \varphi_0 = \mathrm{id}\}}, \tag{4.3}$$

is equal to 0. In general this quotient is a vector space whose dimension, the$\mathcal{A}_e$-*codimension of* f, measures the failure of stability. Mather ([35]) proved

Theorem 4.4. Infinitesimal stability is equivalent to stability: *f is stable if and only if $T^1(f) = 0$.*

One of the aims of this lecture is to develop techniques for calculating $T^1(f)$, and apply them in some examples.

Exercise 4.5. Germs of submersions and immersions are infinitesimally stable and therefore stable. This is an easy calculation using the normal forms of Theorems 2.1 and 2.2

Before continuing, we note that the denominator in (4.3) is very close to being the tangent space to the orbit of f under the group

$\mathcal{A} = \mathrm{Diff}(\mathbb{C}^n, 0) \times \mathrm{Diff}(\mathbb{C}^p, 0)$. It is not quite equal to it, because we are allowing ϕ_t and ψ_t to move the origin (so they are not "paths in $\mathrm{Diff}(\mathbb{C}^n, 0)$ and $\mathrm{Diff}(\mathbb{C}^p, 0)$"). For this reason we denote the denominator call the denominator in (4.3) the 'extended' tangent space and denote it by $T\mathcal{A}_e f$. The tangent space to the $\mathcal{A}$-orbit of f is denoted $T\mathcal{A}f$. It is the subspace of $T\mathcal{A}_e f$ corresponding to families φ_t and ψ_t for which $\varphi_t(0) = 0$ and $\psi_t(0) = 0$ for all t. By the chain rule,

$$\frac{d}{dt}\Big(\psi_t \circ f \circ \varphi_t\Big)|_{t=0} = df\big(\frac{d\phi_t}{dt}|_{t=0}\big) + \big(\frac{d\psi_t}{dt}|_{t=0}\big) \circ f.$$

Both $(d\varphi_t/dt)|_{t=0}$ and $(d\psi_t/dt)|_{t=0}$ are germs of vector fields, on $(\mathbb{C}^n, 0)$ and $(\mathbb{C}^p, 0)$ respectively: $(d\varphi_t(x)/dt)|_{t=0}$ is the tangent vector at x to the trajectory $\varphi_t(x)$. In the same way, the elements of the numerator of 4.3 should be thought of as 'vector fields along f'; $(df_t/dt)|_{t=0}$ is the tangent vector at $f(x)$ to the trajectory $x \mapsto f_t(x)$. By associating to $(df_t/dt)|_{t=0}$ the map

$$\hat{f} : x \mapsto (x, (d/dt)f_t|_{t=0}) \in T\mathbb{C}^p,$$

we obtain a commutative diagram:

(4.4)
$$\begin{tikzcd} T\mathbb{C}^n \arrow[r, "df"] \arrow[d] & T\mathbb{C}^p \arrow[d] \\ \mathbb{C}^n \arrow[r, "f"'] \arrow[ur, dashed, "\hat{f}"] & \mathbb{C}^p \end{tikzcd}$$

in which the vertical maps are the bundle projections. Elements of $\theta_{\mathbb{C}^n,0}$ can be written in various ways: as n-tuples,

$$\xi(x) = (\xi_1(x), \ldots, \xi_n(x))$$

(sometimes as columns rather than rows), or as sums:

$$\xi(x) = \sum_{j=1}^{n} \xi_j(x)\partial/\partial x_j.$$

The second notation emphasizes the role of the coordinate system on $\mathbb{C}^n, 0$. Similarly, elements of $\theta(f)$ can be written as row vectors or column vectors, or as sums:

$$\hat{f}(x) = \sum_{j=1}^{p} \hat{f}_j(x)\partial/\partial y_j.$$

We denote by

$\theta(f)$	the numerator of (4.3)
$\theta_{\mathbb{C}^n,0}$	$\{$germs at 0 of vector fields on $\mathbb{C}^n\}$
$\theta_{\mathbb{C}^p,0}$	$\{$germs at 0 of vector fields on $\mathbb{C}^p\}$
$tf : \theta_{\mathbb{C}^n,0} \to \theta(f)$	the map $\xi \mapsto df \circ \xi$
$\omega f : \theta_{\mathbb{C}^p,0} \to \theta(f)$	the map $\eta \mapsto \eta \circ f$

The notation "tf" is slightly fussy. We use it instead of df here because we think of df as the bundle map between tangent bundles induced by f, as in the diagram (4.4), whereas tf is the map "left composition with df" from $\theta_{\mathbb{C}^n,0}$ to $\theta(f)$. Some authors use "df" for both. In any case,

$$(4.5)\qquad \begin{array}{rcl} T\mathcal{A}_e f & = & tf(\theta_{\mathbb{C}^n,0}) + \omega f(\theta_{\mathbb{C}^p,0}) \\ T\mathcal{A} f & = & tf(\mathfrak{m}_n\, \theta_{\mathbb{C}^n,0}) + \omega f(\mathfrak{m}_p\, \theta_{\mathbb{C}^p,0}) \end{array}.$$

These spaces are not just vector spaces:

$\theta_{\mathbb{C}^n,0}$	is an $\mathcal{O}_{\mathbb{C}^n,0}$-module
$\theta(f)$	is an $\mathcal{O}_{\mathbb{C}^n,0}$-module
$tf : \theta_{\mathbb{C}^n,0} \to \theta(f)$	is $\mathcal{O}_{\mathbb{C}^n,0}$-linear, so
$\theta(f)/tf(\theta_{\mathbb{C}^n,0})$	is an $\mathcal{O}_{\mathbb{C}^n,0}$-module

But $T^1(f)$ is not an $\mathcal{O}_{\mathbb{C}^n,0}$-module, because $\mathcal{O}_{\mathbb{C}^p,0}$ is not. It is, however, an $\mathcal{O}_{\mathbb{C}^p,0}$-module; for via composition with f, $\mathcal{O}_{\mathbb{C}^n,0}$ becomes an $\mathcal{O}_{\mathbb{C}^p,0}$- module: we can 'multiply' $g \in \mathcal{O}_{\mathbb{C}^n,0}$ by $h \in \mathcal{O}_{\mathbb{C}^p,0}$ using composition with f to transport $h \in \mathcal{O}_{\mathbb{C}^p,0}$ to $h \circ f \in \mathcal{O}_{\mathbb{C}^n,0}$:

$$h \cdot g := (h \circ f)g.$$

By this 'extension of scalars', every $\mathcal{O}_{\mathbb{C}^n,0}$-module becomes an $\mathcal{O}_{\mathbb{C}^p,0}$-module. This is where commutative algebra enters the picture. But we will not open the door to it in any serious way just yet. We simply note that

$\theta_{\mathbb{C}^p,0}$	is an $\mathcal{O}_{\mathbb{C}^p,0}$-module
$\omega f : \theta_{\mathbb{C}^p,0} \to \theta(f)$	is $\mathcal{O}_{\mathbb{C}^p,0}$-linear, so
$T^1(f)$	is an $\mathcal{O}_{\mathbb{C}^p,0}$-module

4.1. First calculations

Example 4.6. (1) The map-germ

$$f(x, y) = (x, y^2, xy)$$

parametrising the cross-cap (Whitney umbrella, pinch point) is stable. We use coordinates (x, y) on the source and (X, Y, Z) on the target. We now calculate that $T^1(f) = 0$. For this purpose we divide $\mathcal{O}_{\mathbb{C}^2,0}$ into *even* and *odd* parts with respect to the y variable, and denote them by $\mathcal{O}^e$ and $\mathcal{O}^o$. Every element of $\mathcal{O}^e$ can be written in the form $a(x, y^2)$, and every element of $\mathcal{O}^o$ in the form $ya(x, y^2)$. Then (we hope the notation is self-explanatory)

$$\theta(f) = \begin{pmatrix} \mathcal{O}^e \oplus \mathcal{O}^o \\ \mathcal{O}^e \oplus \mathcal{O}^o \\ \mathcal{O}^e \oplus \mathcal{O}^o \end{pmatrix}$$

and since

$$\omega f \begin{pmatrix} a(X,Y) \\ b(X,Y) \\ c(X,Y) \end{pmatrix} = \begin{pmatrix} a(x,y^2) \\ b(x,y^2) \\ c(x,y^2) \end{pmatrix} \tag{4.6}$$

we see that the even part of $\theta(f)$ is indeed contained in $T\mathcal{A}_e f$, and we need worry only about the odd part. Since

$$tf\big(a(x,y^2)\frac{\partial}{\partial x}\big) \tag{4.7}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 2y \\ y & x \end{pmatrix} \begin{pmatrix} a(x,y^2) \\ 0 \end{pmatrix} = \begin{pmatrix} a(x,y^2) \\ 0 \\ ya(x,y^2) \end{pmatrix}$$

we get all of the odd part of the third row. Since

$$tf\big(a(x,y^2)\frac{\partial}{\partial y}\big) \tag{4.8}$$

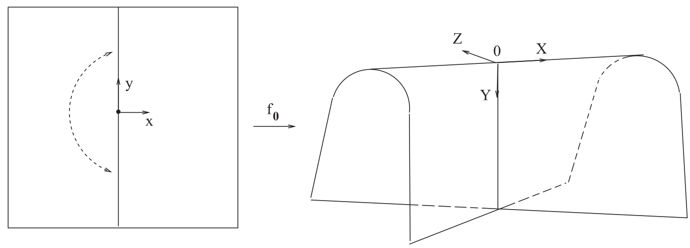
$$= \begin{pmatrix} 1 & 0 \\ 0 & 2y \\ y & x \end{pmatrix} \begin{pmatrix} 0 \\ a(x,y^2) \end{pmatrix} = \begin{pmatrix} 0 \\ 2ya(x,y^2) \\ xa(x,y^2) \end{pmatrix}$$

we get all of the odd part of the second row. Since

$$tf\big(ya(x,y^2)\frac{\partial}{\partial x}\big) \tag{4.9}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 2y \\ y & x \end{pmatrix} \begin{pmatrix} ya(x,y^2) \\ 0 \end{pmatrix} = \begin{pmatrix} ya(x,y^2) \\ 0 \\ y^2a(x,y^2) \end{pmatrix}$$

we get all of the odd part of the first row. So $T\mathcal{A}_e f = \theta(f)$, and f is stable.



(2) The map-germ $f(x,y) = (x, y^2, y^3 + x^2y)$ is not stable. The calculation of (4.6), (4.8) and (4.9) still apply, with insignificant modifications. The only change from (1) is that (4.7) now shows that

$$T\mathcal{A}_e f \supset \big(x\,\mathcal{O}^o\big)\partial/\partial Z \tag{4.10}$$

and we need an extra calculation

$$tf\big(ya(x,y^2)\frac{\partial}{\partial y}\big) = \begin{pmatrix} 1 & 0 \\ 0 & 2y \\ 2xy & x^2+3y^2 \end{pmatrix}\begin{pmatrix} 0 \\ ya(x,y^2) \end{pmatrix} = \tag{4.11}$$

$$\begin{pmatrix} 0 \\ 2y^2a(x,y^2) \\ x^2ya(x,y^2) + 3y^3a(x,y^2) \end{pmatrix}$$

In view of (4.10) and what we know about the even terms, this completes the proof that

$$T^1(f) = \begin{pmatrix} \mathcal{O}^e + \mathcal{O}^o \\ \mathcal{O}^e + \mathcal{O}^o \\ \mathcal{O}^e + x\,\mathcal{O}^o + y^2\,\mathcal{O}^o \end{pmatrix} \tag{4.12}$$

It follows that $T^1(f)$ is generated, as a vector space over $\mathbb{C}$, by $y\partial/\partial Z$.

Definition 4.7. The $\mathcal{A}_e$-codimension of the map-germ f is the dimension, as a $\mathbb{C}$-vector space, of $T^1(f)$.

Exercise 4.8. Calculate the $\mathcal{A}_e$-codimension, and a $\mathbb{C}$-basis for $T^1(f)$, when

(a) $f(x,y) = (x, y^2, y^3 + x^{k+1}y)$
(b) $f(x,y) = (x, y^2, x^2y + y^5)$
(c) $f(x,y) = (x, y^2, x^2y + y^{2k+1})$.

Remark 4.9. If $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ is not an immersion then the ideal $f^*m_{\mathbb{C}^3,0}$ generated in $\mathcal{O}_{\mathbb{C}^2,0}$ by the three component functions of f is strictly contained in $m_{\mathbb{C}^2,0} = (x,y)$. It follows that

$$\dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^2,0}/f^*m_{\mathbb{C}^3,0} \geq 2.$$

It can be shown (cf [41]) that every germ for which this dimension is exactly 2 (as in all the examples above) is $\mathcal{A}$-equivalent to one of the form $f(x,y) = (x, y^2, yp(x,y^2))$. Alternative characterisation: these are the map-germs $(\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ of Boardman type $\sum^{1,0}$.

Question to ponder for later: what is the significance here of the involution $(x,y) \mapsto (x,-y)$?

Since we are usually concerned with germs at 0, we write

$$\begin{array}{lll} \mathcal{O}_n & \text{in place of} & \mathcal{O}_{\mathbb{C}^n,0} \\ \theta_n & \text{in place of} & \theta_{\mathbb{C}^n,0} \\ \mathfrak{m}_n & \text{in place of} & \mathfrak{m}_{\mathbb{C}^n,0} \end{array}$$

The examples considered above are somewhat atypical. Calculating $T\mathcal{A}_e f$ is generally rather complicated. Checking that a given map-germ is *stable*, however, is made much easier by a theorem of John Mather, which makes use of the extended tangent space for contact equivalence (see the start of Section 3),

$$T\mathcal{K}_e f = tf(\theta_{\mathbb{C}^n,0}) + f^* \, \mathfrak{m}_p \, \theta(f).$$

Here $f^* m_{\mathbb{C}^p,0}$ is the ideal in $\mathcal{O}_{\mathbb{C}^n,0}$ generated by the component functions of f. When $p = 1$, $T\mathcal{K}_e f$ is just the ideal $(f, \partial f/\partial x_1, \ldots, \partial f/\partial x_n)$ of $\mathcal{O}_{\mathbb{C}^n,0}$. In any case it is always an $\mathcal{O}_{\mathbb{C}^n,0}$-module, which makes calculating with it very much easier than calculating $T\mathcal{A}_e f$. The role of $T\mathcal{K}_e f$ here does not involve its geometrical interpretation as extended tangent space. We will discuss the contact group $\mathcal{K}$ further in Section 5.

Let $v_1, \ldots, v_p$ be members of a vector space V over a field k. We denote the subspace spanned over k by $v_1, \ldots, v_p$ by $\mathrm{Sp}_k\{v_1, \ldots, v_p\}$.

Mather's theorem is

Theorem 4.10. *Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ be an analytic map-germ. The following are equivalent:*

(1) $T\mathcal{A}_e f = \theta(f)$ *(so f is stable).*
(2) $T\mathcal{K}_e f + Sp_{\mathbb{C}}\{\partial/\partial y_1, \ldots, \partial/\partial y_p\} = \theta(f)$
(3) $T\mathcal{K}_e f + Sp_{\mathbb{C}}\{\partial/\partial y_1, \ldots, \partial/\partial y_p\} + \mathfrak{m}_n^{p+1}\, \theta(f) = \theta(f)$.

Proof. (1) $\Longrightarrow$ (2) and (2) $\Longrightarrow$ (3) are trivial, since the left hand sides of the equalities increase from each statement to the next.

To see that (3) $\Longrightarrow$ (2), suppose that (3) holds and let $\alpha_1, \ldots, \alpha_p \in \mathfrak{m}_n$. We will show that $\alpha_1 \cdots \alpha_p \partial/\partial y_i \in T\mathcal{K}_e f + \mathfrak{m}_n^{p+1}\, \theta(f)$. Because every member of $\mathfrak{m}_n^p\, \theta(f)$ is a sum of such elements, it will follow that

$$\mathfrak{m}_n^p\, \theta(f) \subset T\mathcal{K}_e f + \mathfrak{m}_n^{p+1}\, \theta(f),$$

and therefore, by Nakayama's Lemma, that

$$\mathfrak{m}_n^p\,\theta(f) \subset T\mathcal{K}_e f.$$

To see that $\alpha_1\cdots\alpha_p\partial/\partial y_i \in T\mathcal{K}_e f + \mathfrak{m}_n^{p+1}\,\theta(f)$, observe that because, by (3),

$$\dim_{\mathbb{C}}\theta(f)/T\mathcal{K}_e f + \mathfrak{m}_n^{p+1}\,\theta(f) \leq p,$$

the $p+1$ elements

$$\partial/\partial y_i, \alpha_1\partial/\partial y_i, \ldots, \alpha_1\cdots\alpha_p\partial/\partial y_i$$

cannot be linearly independent. Thus there exist $c_0, \ldots, c_p \in \mathbb{C}$, not all zero, such that

$$c_0\partial/\partial y_i + c_1\alpha_1\partial/\partial y_i + \cdots + c_p\alpha_1\cdots\alpha_p\partial/\partial y_i = 0 \tag{4.13}$$

in $\theta(f)/T\mathcal{K}_e f + \mathfrak{m}_n^{p+1}\,\theta(f)$. Let c_j be the first of the c_i to be non-zero. Then (4.13) can be rewritten as

$$(c_j\alpha_1\cdots\alpha_j + \cdots + c_p\alpha_1\cdots\alpha_p)\partial/\partial y_i \in T\mathcal{K}_e f + \mathfrak{m}_n^{p+1}\,\theta(f).$$

The left hand side here is an $\mathcal{O}_{\mathbb{C}^n,0}$-unit times $\alpha_1\cdots\alpha_j\partial/\partial y_i$, and thus $\alpha_1\cdots\alpha_j\partial/\partial y_i$, and so $\alpha_1\cdots\alpha_p\partial/\partial Y_i$, are members of $T\mathcal{K}_e f + \mathfrak{m}_n^{p+1}\,\theta(f)$.

To see that (2) $\Longrightarrow$ (1), consider $M := \theta(f)/tf(\theta_{\mathbb{C}^n,0})$ as $\mathcal{O}_{\mathbb{C}^p,0}$ module via f. Then $\mathfrak{m}_p\,M$ is what up to now we have been denoting by $f^*\,\mathfrak{m}_p\,M$. We have

$$M/\,\mathfrak{m}_p\,M = M/f^*\,\mathfrak{m}_p\,M = \frac{\theta(f)}{tf(\theta_{\mathbb{C}^n,0}) + f^*\,\mathfrak{m}_p\,\theta(f)},$$

and by hypothesis this is generated as a $\mathbb{C}$-vector space by the classes of $\partial/\partial y_1, \ldots, \partial/\partial y_p$ in $M/\,\mathfrak{m}_p\cdot M$. It follows by the Preparation Theorem that M is generated as $\mathcal{O}_{\mathbb{C}^p,0}$-module by the classes of $\partial/\partial y_1, \ldots, \partial/\partial y_p$ in M. The $\mathcal{O}_{\mathbb{C}^p,0}$ submodule of $\theta(f)$ generated by $\partial/\partial y_1, \ldots, \partial/\partial y_p$ is just $\omega f(\theta_{\mathbb{C}^p,0})$; so from the fact that M is generated over $\mathcal{O}_{\mathbb{C}^p,0}$ by the classes of $\partial/\partial y_1, \ldots, \partial/\partial y_p$, we deduce simply that $\theta(f) = tf(\theta_{\mathbb{C}^n,0}) + \omega f(\theta_{\mathbb{C}^p,0})$.

Q.E.D.

Corollary 4.11. *Whether or not $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ is stable is determined by its $p+1$-jet.*

Proof. If $j^{p+1}f = j^{p+1}g$ then

$$T\mathcal{K}_e f + \mathfrak{m}_n^{p+1}\,\theta(f) = T\mathcal{K}_e g + \mathfrak{m}_n^{p+1}\,\theta(g).$$

So (3) holds for f if and only if it holds for g. Q.E.D.

Example 4.12. (1) We apply this theorem to the map-germ f of Example 4.6(1). We have

$$\begin{aligned} T\mathcal{K}_e f &= tf(\theta_2) + f^*\,\mathfrak{m}_3\,\theta(f) \\ &= \mathcal{O}_{\mathbb{C}^2,0}\cdot\{\partial f/\partial x, \partial f/\partial y\} + (x,y^2)\theta(f) \\ &= \mathcal{O}_{\mathbb{C}^2,0}\cdot\left\{\begin{pmatrix}1\\0\\y\end{pmatrix}, \begin{pmatrix}0\\2y\\x\end{pmatrix}\right\} + \begin{pmatrix}(x,y^2)\\(x,y^2)\\(x,y^2)\end{pmatrix}\end{aligned}$$

You can easily show that the condition of the theorem holds; in particular, since (x, y^2) contains the square of the maximal ideal of $\mathcal{O}_{\mathbb{C}^2,0}$, it's necessary only to check for terms of degree 0 and 1.
(2) The same theorem can be used to show that the map-germs

(a) $f : (\mathbb{C}^3, 0) \to (\mathbb{C}^3, 0)$ defined by

$$f(x_1, x_2, x_3) = (x_1, x_2, x_3^4 + x_1x_3^2 + x_2x_3)$$

(b) $f : (\mathbb{C}^4, 0) \to (\mathbb{C}^5, 0)$ defined by

$$f(x_1, x_2, x_3, x_4) = (x_1, x_2, x_3, x_4^3 + x_1x_4, x_2x_4^2 + x_3x_4)$$

(c) $f : (\mathbb{C}^5, 0) \to (\mathbb{C}^6, 0)$ defined by

$$f(x, y, a, b, c, d) = (x^2 + ay, xy + bx + cy, y^2 + dx, a, b, c, d)$$

are stable. These are left as **Exercises**.

Remark 4.13. The reader will note that each of the germs listed in Example 4.12(2) is itself an unfolding of a germ of rank 0 (i.e. whose derivative at 0 vanishes). Of course, by means of the inverse function theorem *any* germ can be put in this form, in suitable coordinates. But in fact there is a general procedure for finding *all* stable map-germs as unfoldings of lower-dimensional germs of rank zero, based on Mather's theorem quoted here. The procedure is the following:

if $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ of rank 0, then $T\mathcal{K}_e f$ is contained in $\mathfrak{m}\,\theta(f)$. Let $g_1, \ldots, g_d \in \theta(f)$ project to a basis for the quotient $\mathfrak{m}\,\theta(f)/T\mathcal{K}_e f$. Then the unfolding $F : (\mathbb{C}^n \times \mathbb{C}^d, (0,0)) \to (\mathbb{C}^p \times \mathbb{C}^d, (0,0))$ defined by

$$F(x, u_1, \ldots, u_d) = (f(x) + \sum_j u_j g_j(x), u_1, \ldots, u_d) \tag{4.14}$$

is a stable map-germ.

Exercise 4.14. *Apply this procedure starting with* $f(x, y) = (x^2, y^2)$.

An ingenious result, due to Terry Gaffney, and extending Mather's, allows one to transform a guess for $T\mathcal{A}_e f$, (based perhaps on a calculation modulo some power of the maximal ideal (i.e. ignoring all terms of degree higher than some fixed k)) into a rigorous calculation.

Theorem 4.15. *Suppose that* $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ *is a map-germ such that*

$$T\mathcal{K}_e f \supset \mathfrak{m}_n^\ell\, \theta(f)$$

and $C \subset \theta(f)$ *is an* $\mathcal{O}_{\mathbb{C}^p,0}$*-submodule such that*

$$C \supset \mathfrak{m}_n^k\, \theta(f)$$

(where $k > 0$*). Then*

$$C = T\mathcal{A}_e f \qquad \Leftrightarrow \qquad C = T\mathcal{A}_e f + f^*\, \mathfrak{m}_p\, C + \mathfrak{m}_n^{k+\ell}\, \theta(f).$$

A proof, due to Terry Gaffney, can be found in [41, 3:2]

Exercise 4.16. Find the smallest integer ℓ such that $T\mathcal{K}_e f \supset \mathfrak{m}_2^\ell\, \theta(f)$ when f is the map germ of Example 4.6(2).

§5. The contact group $\mathcal{K}$

The contact group $\mathcal{K}$ acting on the set of germs $(\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ is defined as follows. As a group, $\mathcal{K}$ is the set of diffeomorphisms of $(\mathbb{C}^n \times \mathbb{C}^p, (0, 0))$ of the form

$$\Phi(x, y) = (\varphi(x), \psi(x, y))$$

where $\psi(x, 0) = 0$ for all x. It is obvious that $\mathcal{K}$ is a subgroup of $\mathrm{Diff}(\mathbb{C}^n \times \mathbb{C}^p, (0, 0))$. It acts on the set of germs of maps $(\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ via its action on their graphs: if $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ and $\Phi \in \mathcal{K}$ then $\Phi \cdot f$ is the map-germ $(\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ whose graph is $\Phi(\mathrm{graph}(f))$. Since

$$\mathrm{graph}(f) = \{(x, f(x)) : x \in (\mathbb{C}^n, 0)\},$$

this means that

$$\mathrm{graph}\big((\Phi \cdot f)\big) = \{(\phi(x), \psi(x, f(x)) : x \in (\mathbb{C}^n, 0)\}$$

and thus

$$(\Phi \cdot f)(\varphi(x)) = \psi(x, f(x)),$$

so that

$$(\Phi \cdot f)(x) = \psi\big(\varphi^{-1}(x), f(\varphi^{-1}(x))\big). \tag{5.1}$$

We will see shortly that germs are contact-equivalent if and only if their fibres over 0 are isomorphic, and so contact equivalence has a clear geometric significance. Nevertheless, its significance for the theory of singularities of mappings goes much further than this. Theorem 4.10 has already given a glimpse of this.

Observe that $\mathcal{R}$ and $\mathcal{L}$ (and therefore $\mathcal{R} \times \mathcal{L} = \mathcal{A}$) are naturally embedded in $\mathcal{K}$: given $\varphi \in \mathcal{R}$ and $\eta \in \mathcal{L}$, define Φ_φ and Φ_η by $\Phi_\varphi(x, y) = (\varphi(x), y), \quad \Phi_\eta(x, y) = (x, \eta(y))$; then by (5.1)

$$(\Phi_\varphi \cdot f)(x) = f(\varphi^{-1}(x)), \quad (\Phi_\eta \cdot f)(x) = \eta \circ f(x).$$

We define another subgroup $\mathcal{C}$ of $\mathcal{K}$ to be the set of all those $\Phi = (\varphi, \psi) \in \mathcal{K}$ such that φ is the identity. Thus by (5.1), $\Phi = (\mathrm{id}, \psi) \in \mathcal{C}$ acts by

$$(\Phi \cdot f)(x) = \psi(x, f(x)).$$

Proposition 5.1. *$\mathcal{K}$ is the semi-direct product of $\mathcal{R}$ and $\mathcal{C}$.*

Proof. First we show that $\mathcal{K} = \mathcal{C}\mathcal{R}$. Given $\Phi = (\varphi, \psi) \in \mathcal{K}$, define $\Phi_\varphi \in \mathcal{R} \subset \mathcal{K}$ by $\Phi_\varphi(x, y) = (\varphi(x), y)$, and $\Phi_1 \in \mathcal{C} \subset \mathcal{K}$ by $\Phi_1(x, y) = (x, \psi(\varphi^{-1}(x), y))$. Then $\Phi = \Phi_1 \circ \Phi_\varphi$.

In view of this, to show that $\mathcal{C}$ is normal, we need only show that if $\Gamma \in \mathcal{C}$ and $\Phi_\varphi \in \mathcal{R} \subset \mathcal{K}$ then

$$\Phi_{\varphi^{-1}} \Gamma \Phi_\varphi \in \mathcal{C}.$$

This is straightforward:

$$\begin{aligned} \big(\Phi_{\varphi^{-1}} \Gamma \Phi_\varphi\big)(x, y) &= \big(\Phi_{\varphi^{-1}} \Gamma\big)(\phi(x), y) \\ &= \Phi_{\varphi^{-1}}(\phi(x), \psi(\varphi(x), y)) \\ &= (x, \psi(\varphi(x), y)). \end{aligned}$$

Q.E.D.

Let $\mathrm{Gl}_p(\mathcal{O})$ be the group of invertible $p \times p$ matrices over $\mathcal{O} = \mathcal{O}_{\mathbb{C}^n, 0}$. If $A \in \mathrm{Gl}_p(\mathcal{O})$ and $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$, define $A \cdot f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ by $(A \cdot f)(x) = A(x) f(x)$. The map $(x, y) \mapsto (x, A(x)y)$ is a diffeomorphism of $(\mathbb{C}^n \times \mathbb{C}^p, (0, 0))$ and maps $\mathbb{C}^n \times \{0\}$ to itself, and as such lies in the group $\mathcal{C}$. We will denote by $\mathcal{C}_L$ the subgroup of $\mathcal{C}$ consisting of all such maps.

Proposition 5.2. *Map-germs $f, g : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ are $\mathcal{C}$-equivalent only if they are $\mathcal{C}_L$-equivalent.*

Proof. Let $\Gamma \in \mathcal{C}$ with $\Gamma(x,y) = (x, \psi(x,y))$, and let ψ have components $\psi_1, \ldots, \psi_p$. Because $\psi(x,0) = 0$, for each $i = 1, \ldots, p$ we have

$$\psi_i(x,y) = \sum_{j=1}^{p} y_j \psi_{ij}(x,y)$$

for some functions ψ_{ij}. So for any $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ with components $f_1, \ldots, f_t$,

$$(5.2) \qquad (\Gamma \cdot f)(x) = \psi(x, f(x)) = \big(\psi_1(x, f(x)), \ldots, \psi_t(x, f(x))\big)$$

$$= \left(\sum_{j=1}^{p} \psi_{1j}(x, f(x)) f_j(x), \ldots, \sum_{j=1}^{p} \psi_{pj}(x, f(x)) f_j(x) \right).$$

Let $a_{ij}(x) = \psi_{ij}(x, f(x))$, define $A \in \mathrm{Gl}_p(\mathcal{O})$ by $A = (a_{ij})$, and let Γ_A be the corresponding element of $\mathcal{C}$. Then by (5.2), we have $\Gamma_A \cdot f = \Gamma \cdot f$. Note that $A \in \mathrm{Gl}_p(\mathcal{O})$, i.e. that the matrix $A(0)$ is invertible; this holds because the matrix of the linear isomorphism $d_0\Gamma$ is equal to

$$\begin{pmatrix} I_s & 0 \\ 0 & A(0) \end{pmatrix}.$$

Q.E.D.

It is an odd feature of this proof that the element $\Gamma_A \in \mathcal{C}_L$ that we construct depends on the map-germ f; we have not defined a retraction $\mathcal{C} \to \mathcal{C}_L$.

Proposition 5.3. *Let $f, g : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ be map germs and suppose that the ideals $(f_1, \ldots, f_p)$ and $(g_1, \ldots, g_p)$ of $\mathcal{O}$ are equal. Then f and g are $\mathcal{C}$-equivalent.*

Proof. Because the two ideals are equal, there exist $a_{ij} \in \mathcal{O}$ and $b_{ij} \in \mathcal{O}$, for $1 \le i, j \le p$, such that

$$(5.3) \qquad f_i = \sum_j a_{ij} g_j \quad \text{and } g_i = \sum_j b_{ij} f_j \quad \text{for } 1 \le i \le t.$$

Defining matrices $A = (a_{ij})$ and $B = (b_{ij})$ and writing $\mathbf{f}$ and $\mathbf{g}$ for the column vectors $(f_1, \ldots, f_p)^t$ and $(g_1, \ldots, g_p)^t$, (5.3) becomes

$$A\mathbf{f} = \mathbf{g} \text{ and } B\mathbf{g} = \mathbf{f},$$

so that $BA\mathbf{f} = \mathbf{f}$ and $AB\mathbf{g} = \mathbf{g}$. Unfortunately, despite this, A and B need not be invertible (consider for example the case where $f_1 = f_2$ and

$g_1 = g_2$; it's easy to find non-invertible A and B such that (5.3) holds); to find a suitable element of $\mathcal{C}$ transforming $\mathbf{f}$ to $\mathbf{g}$ we modify A to ensure its invertibility.

Lemma 5.4. *Let $A_0, B_0 : \mathbb{C}^p \to \mathbb{C}^p$ be linear maps. There exists a linear map $C_0 : \mathbb{C}^p \to \mathbb{C}^p$ such that $A_0 + C_0(I_p - B_0A_0)$ is invertible.*

Proof of Lemma. Let W be a complement to im A_0 in $\mathbb{C}^p$, and choose $Q_0 : \mathbb{C}^p \to \mathbb{C}^p$ such that $Q_0| : \ker A_0 \to W$ is an isomorphism. Define $C_0 = A_0 + Q_0(I_p - B_0A_0)$, where I_p is the $p \times p$ identity matrix. Then C_0 is injective and therefore an isomorphism. $\square$

We apply the lemma by taking A_0 and B_0 to be $A(0)$ and $B(0)$ respectively. Define the $p \times p$ matrix C by $C = A + Q_0(I_p - BA)$. Then $C(0)$ is the matrix C_0 of the lemma, so C is invertible. Clearly $(I_p - BA)$ annihilates $\mathbf{f}$, so $C \cdot \mathbf{f} = \mathbf{g}$, and f and g are $\mathcal{C}$-equivalent (indeed, $\mathcal{C}_L$-equivalent), as required. Q.E.D.

Theorem 5.5. *For map-germs $f, g : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$, the following are equivalent:*

(1) *the germs $(f^{-1}(0), 0)$ and $(g^{-1}(0), 0)$, with their possibly non-reduced stucture, are isomorphic.*
(2) *the map-germs f and g are $\mathcal{K}$-equivalent.*

Proof. (1) $\implies$ (2): Let $\varphi : (\mathbb{C}^n) \to (\mathbb{C}^n, 0)$ induce an isomorphism $(f^{-1}(0), 0) \simeq (g^{-1}(0), 0)$. Then the ideals $\big(f_1, \ldots, f_p\big)$ and $\big((g \circ \varphi)_1, \ldots, (g \circ \varphi)_p\big)$ of $\mathcal{O}_{\mathbb{C}^n,0}$ are equal, and therefore by 5.3 the germs f and $g \circ \varphi$ are $\mathcal{C}$-equivalent. It follows that f and g are $\mathcal{K}$-equivalent.

(2) $\implies$ (1): Suppose that $\Phi = (\varphi, \psi) \in \mathcal{K}$ transforms the graph of f to that of g. Then $g \circ \varphi$ and f are $\mathcal{C}$-equivalent and hence $\mathcal{C}_L$-equivalent. It follows immediately that the ideals $\big((g \circ \varphi)_1, \ldots, (g \circ \varphi)_p\big)$ and $\big(f_1, \ldots, f_p\big)$ are equal, and thus the (possibly non-reduced) germs $(f^{-1}, 0)$ and $((g \circ \varphi)^{-1}, 0)$ are the same. Thus $(f^{-1}(0), 0)$ and $(g^{-1}(0), 0)$ are isomorphic. Q.E.D.

The quotient $\mathcal{O}_{\mathbb{C}^n,0} / f^* \mathfrak{m}_{\mathbb{C}^p,0}$ is the *local algebra* of the germ f, and denoted by $Q(f)$. It is the algebra of germs on the fibre $f^{-1}(0)$. Theorem 5.5 says in effect that germs are $\mathcal{K}$-equivalent if and only if their local algebras are isomorphic.

Theorem 5.6. ([36]) *Stable germs $(\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ are $\mathcal{A}$-equivalent if and only if they are $\mathcal{K}$-equivalent, and thus stable germs are classified by the isomorphism classes of their local algebras.*

If $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ and $n > p$ then $Q(f)$ cannot be finite-dimensional, by the Hauptidealsatz. It is therefore of interest that 5.6 can be strengthened as follows. Let $\mathfrak{m}$ be the maximal ideal in $Q(f)$, and write $\mathcal{O}_n$ and $\mathfrak{m}_n$ in place of $\mathcal{O}_{\mathbb{C}^n,0}$ and $\mathfrak{m}_{\mathbb{C}^n,0}$. For each $k \in \mathbb{N}$, let

$$Q_k(f) = Q(f)/\,\mathfrak{m}^{k+1} = \mathcal{O}_n\,/(f^*\,\mathfrak{m}_p + \mathfrak{m}_n^{k+1})\,\mathcal{O}_n\,.$$

Corollary 5.7. ([36, Theorem A]) *Stable germs f and g are $\mathcal{A}$-equivalent if and only $Q_{p+1}(f) \simeq Q_{p+1}(g)$.*

The proof of 5.7 from 5.6 is very different from the proof (3) $\Longrightarrow$ (2) in 4.10. The stability of f does *not* imply that $f^*\,\mathfrak{m}_p\,\mathcal{O}_n \supset \mathfrak{m}_n^{p+1}$. What is obvious is that $Q_{p+1}(f)$ depends only on the $p+1$-jet of f; for if f and g agree up to degree $p+1$ then

$$(f^*\,\mathfrak{m}_p + \mathfrak{m}_n^{p+2})\,\mathcal{O}_n = (g^*\,\mathfrak{m}_p + \mathfrak{m}_n^{p+2})\,\mathcal{O}_n\,.$$

One can deduce 5.7 from 5.6 as follows:

(1) If $Q_{p+1}(f) \simeq Q_{p+1}(g)$ then there exists a diffeomorphism $\varphi : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ such that $\varphi^*(f^*\,\mathfrak{m}_p\,\mathcal{O}_n)) + \mathfrak{m}_n^{p+1} = g^*\,\mathfrak{m}_p\,\mathcal{O}_n + \mathfrak{m}_n^{p+1}$. By the argument of Proposition 5.3, there exists a matrix $C \in \mathrm{Gl}_p(\mathcal{O}_n)$ such that $C \cdot f \circ \varphi = g \mod \mathfrak{m}_n^{p+1}$. Thus f is $\mathcal{K}$ equivalent to a germ g_1 which agrees with g up to degree $p+1$. Because $j^{p+1}g_1 = j^{p+1}g$, g_1 is stable, by 4.11.

(2) Stable germs $(\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ are $p+1$-determined for $\mathcal{A}$-equivalence. We will shortly prove this as Theorem 5.9. From this, it follows that g and g_1 are $\mathcal{A}$-equivalent. Now by Theorem 5.6, g_1 and f are $\mathcal{A}$-equivalent, and the $\mathcal{A}$-equivalence of f and g follows.

In fact Theorem 5.9 is used in the proof of Theorem 5.6.

In preparation for the proof of Theorem 5.9, we need the following result.

Proposition 5.8. *If $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ is stable then $T\mathcal{A}f = T\mathcal{K}f$ and consequently $T\mathcal{A}f \supset \mathfrak{m}_n^{p+1}\,\theta(f)$.*

Proof. To show that

$$tf(\mathfrak{m}_n\,\theta_n) + \omega f(\mathfrak{m}_p\,\theta_p) = tf(\mathfrak{m}_n\,\theta_n) + f^*\,\mathfrak{m}_p\,\theta(f),$$

it is necessary only to show that $f^*\,\mathfrak{m}_p\,\theta(f)$ is contained in the left hand side of this equality. This is easy: because f is stable,

$$f^*\,\mathfrak{m}_p\,\theta(f) = f^*\,\mathfrak{m}_p(tf(\theta_n) + \omega f(\theta_p))$$

$$= tf(\mathfrak{m}_p\,\theta_n) + \omega f(\mathfrak{m}_p\,\theta_p) \subset tf(\mathfrak{m}_n\,\theta_n) + \omega f(\mathfrak{m}_p\,\theta_p).$$

Q.E.D.

Theorem 5.9. *Suppose $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ is infinitesimally stable and let $g : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ be any germ such that $j^{p+1}f = j^{p+1}g$. Then f and g are $\mathcal{A}$-equivalent.*

Proof. Write $f_u(x) = f(x) + u(g - f)(x)$. Since $j^{p+1}f_u = j^{p+1}f$ for all u, we know from 4.11 that the germ at 0 of f_u is infinitesimally stable for all u. Using this, we show that for any fixed representatives of f and g, for each value u_0 of u, there is a neighbourhood U of u_0 in the parameter space $\mathbb{C}$ such that the germs of f_u and f_{u_0} are $\mathcal{A}$-equivalent for all $u \in U$. We refer to this property as the *local $\mathcal{A}$-triviality* of the deformation f_u.

A finite number of such neighbourhoods cover the compact interval $[0, 1]$, and it follows by transitivity that $f = f_0 \simeq_{\mathcal{A}} f_1 = g$.

For simplicity of notation, we assume in the following proof that $u_0 = 0$. This does not sacrifice any generality; indeed, by re-baptising f_{u_0} as f, we are able to deduce the general statement from this apparently special case.

Proof of local $\mathcal{A}$-triviality for $u_0 = 0$:

Define $F : (\mathbb{C} \times \mathbb{C}^n, (0, 0)) \to (\mathbb{C} \times \mathbb{C}^p, (0, 0))$ by $F(u, x) = (u, f(x) + u(g(x) - f(x)))$.

To lighten the notation we write S ("source"), T ("target") and P ("parameter space') for $(\mathbb{C}^n, 0)$, $(\mathbb{C}^p, 0)$ and $(\mathbb{C}, 0)$ respectively. We denote the space of vector fields on $P \times S$ which are tangent to the fibres of the projection $P \times S \to P$ by $\theta_{P\times S/P}$; $\theta_{P\times T/T}$ is defined analogously, and, similarly, $\theta(F/P)$ is the space of vector fields along F which are tangent to the fibres of the projection $P \times T \to P$. Denote by $\bar{t}F$ and $\bar{\omega}F$ the obvious homomorphisms $\theta_{P\times S/P} \to \theta(F/P)$ and $\theta_{P\times T/P} \to \theta(F/P)$ obtained from tF and ωF by suppressing mention of the (null) $\partial/\partial u$ component. We extend the elements of θ_S, θ_T and $\theta(f)$ to elements, of the same name, of $\theta_{P\times S/P}$, $\theta_{P\times T/P}$ and $\theta(F/P)$ respectively, whose values at (u, x) and (u, y) are independent of u.

Since $g - f \in \mathfrak{m}_S^{p+2}\,\theta(f)$, we have $\partial F/\partial u \in \mathfrak{m}_S^{p+2}\,\theta(F/P)$. It follows, by the Thom-Levine Theorem, 3.5, that we need only show that

$$(5.4) \qquad \mathfrak{m}_S^{p+2}\,\theta(F/P) \;\subseteq\; \bar{t}F(\mathfrak{m}_S\,\theta_{P\times S/P}) + \bar{\omega}F(\mathfrak{m}_T\,\theta_{P\times T/P})$$

For suppose that

$$\frac{\partial F}{\partial u} = \bar{t}F(\xi) + \bar{\omega}F(\chi).$$

Then writing

$$\chi = \eta + \partial/\partial u \;\text{ and }\; \tilde{\chi} = \partial/\partial u - \xi,$$

we obtain

$$tF(\tilde{\chi}) = \omega F(\chi).$$

The integral flow Φ_t of χ has the form

$$\Phi_t(u, y) = (u, \phi_t(u, y))$$

and moreover $\phi_t(u, 0) = 0$ for all t, u, since $\eta \in \mathfrak{m}_T\, \theta_{P\times T/P}$. Similarly, the integral flow $\tilde{\Phi}$ of $\tilde{\chi}$ has the form

$$\tilde{\Phi}_t(u, x) = (u, \tilde{\varphi}_t(u, x))$$

with $\tilde{\varphi}_t(u, 0) = 0$ for all t, u. By Thom-Levine, we have

$$F \circ \tilde{\Phi}_t(u, x) = \Phi_t \circ F$$

and in particular

$$F \circ \tilde{\Phi}_u(0, x) = \Phi_u(F(0, x)). \tag{5.5}$$

Write

$$\Phi_u(0, y) = (u, \varphi_u(y)), \quad \text{and} \quad \tilde{\Phi}_u(0, x) = (u, \tilde{\varphi}_u(x)).$$

Then from (5.5) we get

$$(u, f_u(\tilde{\varphi}_u(x)))) = (u, \varphi_u(f(x)));$$

in other words,

$$f_u \circ \tilde{\varphi}_u = \varphi_u \circ f. \tag{5.6}$$

Since $\phi_u(0) = 0$ in T and $\tilde{\varphi}_u(0) = 0$ in S, this means that f_u and f are $\mathcal{A}$-equivalent.

Note that the diffeomorphisms we have constructed are merely germs at $(0, 0)$ in $P \times S$ and $P \times T$. By choosing representatives, we obtain a neighbouhood U of 0 in P such that the equation (5.6) holds for all $u \in U$.

Now we go on to prove (5.4). Let $\alpha \in \mathfrak{m}_S^{p+2}\, \theta(F/P)$, and let α_0 be the restriction of α to $\{u = 0\}$. Thus $\alpha_0 \in \mathfrak{m}_S^{p+2}\, \theta(f)$ and so there exist $\xi \in \mathfrak{m}_S\, \theta_S$ and $\eta \in \mathfrak{m}_T\, \theta_T$ such that $\alpha_0 = tf(\xi) + \omega f(\eta)$. Note that $\alpha - \alpha_0 = u\alpha_1$ for some $\alpha_1 \in \mathfrak{m}_S^{p+2}\, \theta(F/P)$, so

$$\alpha - \alpha_0 \in \mathfrak{m}_{P\times T}\, \mathfrak{m}_S^{p+2}\, \theta(F/P).$$

Now

$$\bar{t}F(\xi) - tf(\xi) \in \mathfrak{m}_{P\times T}\, \mathfrak{m}_S^{p+2}\, \theta(F/P),$$

for

$$\partial F/\partial x_i - \partial f/\partial x_i = u\partial(g-f)/\partial x_i \in \mathfrak{m}_{P\times T}\,\mathfrak{m}^{t+1}\,\theta(F/P),$$

and the components of ξ lie in $\mathfrak{m}_S$. It is easy to see that $\bar{\omega}F(\eta) - \omega f(\eta) \in \mathfrak{m}_{P\times T}\,\mathfrak{m}_S^{p+2}\,\theta(F/P)$.

It follows that

$$\alpha = \alpha_0 + u\alpha_1 = \bar{t}F(\xi) + \bar{\omega}F(\eta) + u\alpha_1$$

$$\in \bar{t}F(\mathfrak{m}_S\,\theta_{P\times S/P}) + \bar{\omega}F(\theta_{P\times T/P}) + \mathfrak{m}_{P\times T}\,\mathfrak{m}_S^{p+2}\,\theta(F/P);$$

thus

(5.7) $$\mathfrak{m}_S^{p+2}\,\theta(F/P) \subseteq$$

$$\bar{t}F(\mathfrak{m}_S\,\theta_{P\times S/P}) + \omega F(\mathfrak{m}_T\,\theta_{P\times T/P}) + \mathfrak{m}_{P\times T}\,\mathfrak{m}_S^{p+2}\,\theta(F/P).$$

The last line invites the application of Nakayama's Lemma in the form 2.17, except that if $n > p$ then $\mathfrak{m}_S^{p+2}\,\theta(F/P)$ is not a finitely generated $\mathcal{O}_{P\times T}$-module.

To circumvent this difficulty we project the inclusion (5.7) into $M := \theta(F/P)/\bar{t}F(\mathfrak{m}_S\,\theta_{P\times S/P})$. To spare the notation, write $Q := \bar{t}F(\mathfrak{m}_S\,\theta_{P\times S/P})$. From (5.7) we obtain

(5.8) $$\frac{\mathfrak{m}_S^{p+2}\,\theta(F/P) + Q}{Q}$$

$$\subseteq \frac{Q + \omega F(\mathfrak{m}_T\,\theta_{P\times T/P})}{Q} + \mathfrak{m}_{P\times T}\,\frac{\mathfrak{m}_S^{p+2}\,\theta(F/P) + Q}{Q}.$$

Now M is a finitely generated $\mathcal{O}_{P\times T}$ module. For it is a finitely generated $\mathcal{O}_{P\times S}$-module, and moreover

$$\begin{aligned}\dim_{\mathbb{C}} \frac{M}{F^*(\mathfrak{m}_{P\times T})M} &= \frac{\theta(F/P)}{\bar{t}F(\mathfrak{m}_S\,\theta_{P\times S/P}) + F^*\,\mathfrak{m}_{P\times T}\,\theta(F/P)} \\ &\simeq \frac{\theta(f)}{tf(\mathfrak{m}_S\,\theta_S) + \omega f(\mathfrak{m}_T\,\theta_T)},\end{aligned}$$

and by Proposition 5.8, the dimension of the last quotient is less than or equal to the dimension of $\theta(f)/\,\mathfrak{m}_S^{p+1}\,\theta(F)$, and is therefore finite. By the Preparation Theorem 2.26, this implies that M is finitely generated over $\mathcal{O}_{P\times T}$.

It follows that the left hand side of the inclusion (5.8), is finitely generated over $\mathcal{O}_{P\times T}$; for if $m_1, \ldots, m_N$ generate M, then the left hand side of (5.8) is generated by elements $x^c m_i$, where $i = 1, \ldots, N$ and x^c runs over all monomials of degree $p+2$ in $x_1, \ldots, x_n$. We can now conclude, by Nakayama's Lemma, that

$$\frac{\mathfrak{m}_S^{p+2}\,\theta(F/P) + Q}{Q} \subseteq \frac{Q + \omega F(\mathfrak{m}_T\,\theta_{P\times T/P})}{Q}$$

and therefore that (5.4) holds. Q.E.D.

Remarks on the proof Theorem 5.9 uses the "small increment" method introduced in the proof of Theorem 3.7. One starts with a statement concerning the tangent space $T\mathcal{G}f$ (where $\mathcal{G} = \mathcal{R}, \mathcal{A}$ or $\mathcal{K}$), of the form

$$T\mathcal{G} \supset \mathfrak{m}_S^k\,\theta(f) \tag{5.9}$$

for some k, and then shows using Nakayama's Lemma that if F is an unfolding or deformation of f on a single parameter, u, for which $\partial F/\partial u|_{\{u=0\}} \in T\mathcal{G}f$, then $\partial F/\partial u$ is contained in the parametrised version of $T\mathcal{G}f$, namely $\mathfrak{m}_S(\partial F/\partial x_1, \ldots, \partial F/\partial x_s)$ in the case of Theorem 3.7, and $\bar{t}F(\theta_{P\times S/P}) + \bar{\omega}F(\mathfrak{m}_T\,\theta_{P\times T/P})$ in the case of Theorem 5.9.

From this it follows by the Thom-Levine Theorem that in any representative, f_u is $\mathcal{G}$-equivalent to f for sufficiently small u. This step works in many different circumstances. To prove the stronger result, that f is not merely equivalent to f_u for u sufficiently close to zero, but to f_1, we have to show that the first step can be applied for each fixed value of $u_0 \in [0, 1]$ – that f_u is $\mathcal{G}$-equivalent to f_{u_0} for all u sufficiently close to u_0. This requires showing that for any u, the original estimate (5.9) holds with f_u in place of f. In the case of Theorem 3.7, this had to be done by an additional argument, which we left to the reader, as Exercise 3.9. In the proof we have just finished, the extra step was not needed, or, rather, had been taken care of before the proof began. The estimate (5.9) in this case was that $T\mathcal{A}f \supset \mathfrak{m}_S^{p+1}\,\theta(f)$, which follows from the stability of f (Proposition 5.8). If $j^{p+2}g = j^{p+2}f$ and $f_u = f + t(g-f)$ then f_u is infinitesimally stable for all u, by Corollary 4.11, so that (5.9) holds for f_u for all u.

Exercises 5.10. Since the isomorphism type of the local algebra Q_f determines f up to contact equivalence, algebraic properties of $Q(f)$ must reflect contact-invariant properties of f, and one should be able to determine invariants of f from $Q(f)$ alone.

(1) Let $F : (\mathbb{C}^n \times \mathbb{C}^d, 0) \to (\mathbb{C}^p \times \mathbb{C}^d, 0)$ be an unfolding of $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$. Show that $Q(F) \simeq Q(f)$.

(2) Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$. Show that the rank of $d_0 f$ is contact-invariant.

(3) Characterise the local algebra $Q(f)$ when f is an immersion, and when f is a submersion.

(4) Given $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$, how can one determine the rank of $d_0 f$ from $Q(f)$? There are several correct answers here; find one involving the dimension of a certain quotient.

5.1. Consequences of Finite Codimension

Let $f : \mathbb{C}^n \to \mathbb{C}^p$ (or $\mathbb{R}^n \to \mathbb{R}^p$) be an analytic (or C^∞) map. Its k-jet at a point x is the p-tuple consisting of the Taylor polynomials of degree k of its component functions. The k-jet of f at x is denoted by $j^k f(x)$. We say that a map-germ $f : (\mathbb{C}^n, x) \to (\mathbb{C}^p, y)$ is *k-determined for $\mathcal{A}$-equivalence* if any other map-germ having the same k-jet at x is $\mathcal{A}$-equivalent to f, and *finitely determined for $\mathcal{A}$-equivalence* if this holds for some finite value of k.

Theorem 5.11. (J.Mather [34]) *f is finitely determined if and only if $dim_{\mathbb{C}} T^1(f) < \infty$.*

The smallest value of k for which this holds is the *determinacy degree* of f. Finding good estimates for the determinacy degree of f in terms of easily calculable data was once a major endeavour. Mather's original estimates (in [34]) were impractically large. They were greatly improved by Terry Gaffney and Andrew du Plessis ([17], [12]). In particular the following estimate due to Gaffney is useful:

Theorem 5.12. ([17]) *If*

$$T\mathcal{A}_e f \supset \mathfrak{m}^k_{\mathbb{C}^n,0}\,\theta(f) \quad \textit{and} \quad T\mathcal{K}_e f \supset \mathfrak{m}^\ell_{\mathbb{C}^n,0}\,\theta(f)$$

then f is $k+\ell$-determined.

Since we are reaching conclusions about the $\mathcal{A}$-orbit of f, it is slightly curious that our hypotheses are framed in terms of $T\mathcal{A}_e f$ and not $T\mathcal{A}f$. Indeed it is (almost) obvious that if f is k-determined then

$$T\mathcal{A}f \supset \mathfrak{m}_n^{k+1}\,\theta(f) \tag{5.10}$$

To make it obvious, recall from Subsection 2.6 the jet spaces $J^k(n,p)$.

Definition/Reminder 5.13. (1) $\mathfrak{m}(n,p)$ is the vector space of all germs $(\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$. It can be identified with $\mathfrak{m}_n\,\theta(f)$ for any $f \in \mathcal{O}(n,p)$.

(2) $J^k(n,p)$ is the set of k-jets of germs $(\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$.

(3) $j^k : \mathfrak{m}(n,p) \to J^k(n,p)$ is the operation "take the k-jet". The map $j^k : \mathfrak{m}(n,p) \to J^k(n,p)$ is surjective. Its kernel is $\mathfrak{m}_n^k\,\mathfrak{m}(n,p)$, so

$$J^k(n,p) \simeq \mathfrak{m}(n,p)/\,\mathfrak{m}_n^k\,\mathfrak{m}(n,p).$$

(4) For $k \leq \ell$, $\pi^\ell_k : J^\ell(n,p) \to J^k(n,p)$ is the projection ("truncate at degree k")

(5) $\mathcal{A}^k = j^k(\mathcal{A}) \subset J^k(n,n) \times J^k(p,p)$ is the quotient of $\mathcal{A}$, which acts naturally on $J^k(n,p)$.

The diagram (in which the rows are group actions)

$$\begin{tikzcd} \mathcal{A} \times \mathfrak{m}_n\, \mathfrak{m}(n,p) \arrow[r] \arrow[d, "j^k \times j^k"'] & \mathfrak{m}_n\, \mathcal{O}(n,p) \arrow[d, "j^k"] \\ \mathcal{A}^{(k)} \times J^k(n,p) \arrow[r] & J^k(n,p) \end{tikzcd} \tag{5.11}$$

is commutative. The lower row is a finite-dimensional model of the upper row. In the lower row we really do have an algebraic group acting algebraically on an algebraic variety - indeed, on a finite dimensional complex vector space. This model provides motivation for many assertions, such as the statement that if f is k-determined then $T\mathcal{A}f \supset \mathfrak{m}_n^{k+1}\,\theta(f)$. What is clear is that if f is k-determined then

$$\mathcal{A}^{(\ell)} j^\ell f(0) = (\pi^\ell_k)^{-1}\big(\mathcal{A}^{(k)} j^k f(0)\big).$$

Now π^ℓ_k is linear, and its kernel is $j^\ell\big(\mathfrak{m}^{k+1}\,\theta(f)\big)$. So if f is k-determined,

$$T\mathcal{A}^{(\ell)} j^\ell f(0) \supset j^\ell\big(m^{k+1}\theta(f)\big)$$

Since

$$J^\ell(n,p) = \mathfrak{m}_n\,\theta(f)/\,\mathfrak{m}_n^{\ell+1}\,\theta(f),$$

this can be rewritten

$$T\mathcal{A}f + \mathfrak{m}_n^{\ell+1}\,\theta(f) \supset \mathfrak{m}_n^{k+1}\,\theta(f), \tag{5.12}$$

almost the statement (5.10) described as obvious above. If we knew that $\mathfrak{m}_n^{k+1}\,\theta(f)$ were a finitely generated module over $\mathcal{O}_{\mathbb{C}^p,0}$ then an application of Nakayama's Lemma would prove (5.10). But we don't know it, and in fact if $n > p$ it can't be true. Neverthless, it is possible to deduce (5.10) from (5.12) using some algebraic/analytic geometry:

(1) $T\mathcal{K}_e f \supset T\mathcal{A}f$, so (5.12) implies

$$T\mathcal{K}_e f + \mathfrak{m}_n^{\ell+1}\,\theta(f) \supset \mathfrak{m}_n^{k+1}\,\theta(f). \tag{5.13}$$

(2) Because (5.13) involves only $\mathcal{O}_{\mathbb{C}^n,0}$-modules, by Nakayama's Lemma we deduce that $T\mathcal{K}_e f \supset \mathfrak{m}_n^{k+1}\,\theta(f)$. This implies that

$$\dim_{\mathbb{C}}\big(\theta(f)/T\mathcal{K}_e f\big) < \infty$$

(f is "$\mathcal{K}$-finite", or has "finite singularity type".)

(3) Let J_f be the ideal in $\mathcal{O}_{\mathbb{C}^n,0}$ generated by the $p \times p$ minors of the matrix of df. Its locus of zeros is the critical set $\sum_f$, the set of points where f is not a submersion. By taking the determinants of p-tuples of elements of $\theta(f)$, from the fact that f is $\mathcal{K}$ finite we deduce that $\dim_{\mathbb{C}}(\mathcal{O}_{\mathbb{C}^n,0}/J_f + f^*\mathfrak{m}_p\,\mathcal{O}_{\mathbb{C}^n,0}) < \infty$. This condition has a clear geometrical significance (over the complex numbers!):

$$V(J_f + f^*\mathfrak{m}_p\,\mathcal{O}_{\mathbb{C}^n,0}) = \sum\nolimits_f \cap f^{-1}(0),$$

so f *is finite-to-one on its critical locus.*

(4) From this it follows that every coherent sheaf of $\mathcal{O}_{\mathbb{C}^n,0}$ modules supported on $\sum_f$ is finite over $\mathcal{O}_{\mathbb{C}^p,0}$. In particular

$$\big(\mathfrak{m}^{\ell+1}\,\theta(f) + tf(\theta_n)\big)/tf(\theta_n)$$

is a finite $\mathcal{O}_{\mathbb{C}^p,0}$-module! So now we can apply Nakayama's Lemma to deduce (5.10) from (5.12): simply take the quotient on both sides by $tf(\theta_n)$.

It took some quite non-elementary steps to get to the "obvious" statement (5.10) from the truly obvious statement (5.12)!

Exercise 5.14. Use the techniques just introduced to prove Theorem 4.10. Note that the hypothesis of 4.10 is equivalent to

$$\theta(f) = T\mathcal{A}_e f + T\mathcal{K}_e f = T\mathcal{A}_e f + f^*\,\mathfrak{m}_p\,\theta(f).$$

In view of the fact that (5.10) *is* true, one might hope that its converse, which also seems reasonable, should also be true. But things are not so simple. They become simpler if we replace the group $\mathcal{A}$ by its subgroup $\mathcal{A}_1$ consisting of pairs of germs of diffeomorphisms whose derivative at 0 is the identity. This observation by Bill Bruce led to what was probably the final major step forward on finite determinacy, [1], in which *unipotent groups* $\mathcal{G}$ are identified as those for which the determinacy degree is equal to one less than the smallest power k such that $m_n^k\theta(f) \subseteq T\mathcal{G}_e f$. The group $\mathcal{A}$ itself is not unipotent.

To prove a statement of the kind

$$T\mathcal{A}f \supset \mathfrak{m}_n^k\,\theta(f) \quad \Longrightarrow \quad f \text{ is } d(k)\text{-determined}$$

one has to show that if g and f differ by terms in $\mathfrak{m}_n^{d(k)+1}$ then two things happen:

(a) first, the germ of deformation $f + t(g - f)$ is trivial – so that for all t is some neighbourhood of 0, $f + t(g - f)$ is equivalent to f.

(b) Second, that for *any* value t_0 of t, we also have $T\mathcal{A}(f + t_0(g - f)) \supset \mathfrak{m}_n^k\,\theta(f)$ – so that by the first assertion, the deformation $f + tg$ is trivial also in the neighbourhood of any parameter value t_0.

In practice, one should not expect to obtain the precise determinacy degree of a map-germ from a general theorem like 5.12. Instead, one can often significantly improve an estimate by using another result due to Mather (in [34]) and known as "Mather's Lemma".

Proposition 5.15. *Let the Lie group G act smoothly on the manifold M, and let $W \subset M$ be a smooth connected submanifold. Then a necessary and sufficient condition for W to be contained in a single orbit is that*

(a) *for all $x \in W$, $T_xW \subset T_xGx$, and*
(b) *the dimension of T_xGx is the same for all $x \in W$.*

One uses the lemma as follows: suppose that it is possible to show, e.g. by applying a general theorem, that f is ℓ-determined, and wants to show that it is k-determined for some $k < \ell$. Let $M = J^\ell(n,p)$, $G = \mathcal{A}^{(\ell)}$ and

$$W = \{j^\ell g : j^k g = j^k f\}.$$

Exercise 5.16. *If W lies in a single $\mathcal{A}^{(k)}$-orbit then f is k-determined.*

Because we are working modulo $\mathfrak{m}^{\ell+1}$, terms of degree $\ell + 1$ and higher can be ignored in calculating $T\mathcal{A}^{(k)}g$, and this may make it relatively straightforward to show that the conditions of Mather's Lemma hold.

5.2. Multi-germs

We have spoken only of 'mono'-germs $(\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$. But many of the interesting phenomena associated with deformations of mono-germs require description in terms of multi-germs, so they cannot sensibly be avoided. For example, a parametrised plane curve singularity splits into a certain number of nodes on deformation; each of these is stable, and their number is an invariant of the singularity.

$t \mapsto (t^2, t^7)$ $\qquad$ $t \mapsto (t^2, t(t^2 - 4u)(t^2 - 9u)(t^2 - 16u))$

Example 5.17. (a) A *node* is a bi-germ consisting of two immersed branches meeting transversely: any node is right-left equivalent to the bi-germ

$$\left\{ \begin{array}{l} f^{(1)} : s \mapsto (s,0) \\ f^{(2)} : t \mapsto (0,t) \end{array} \right. \tag{5.14}$$

(**Exercise**) and is right-left stable. This is easy to prove once you have mastered the notation, which is explored in the next example.

(b) The bi-germ consisting of two germs of immersion from $\mathbb{C}$ to $\mathbb{C}^2$ which meet tangentially is not stable. In suitable coordinates such a germ can be written

$$\left\{ \begin{array}{l} f^{(1)} : s \mapsto (s,0) \\ f^{(2)} : t \mapsto (t,h(t)) \end{array} \right. \tag{5.15}$$

We will calculate $T\mathcal{A}_e f$. We use independent coordinate systems s, t on $\mathbb{C}$, centred on each of the base-points, which we label $0^{(1)}$ and $0^{(2)}$. The two branches meet tangentially if $h \in (t^2)$. We have $\theta(f) = \theta\big(f^{(1)}\big) \oplus \theta\big(f^{(2)}\big)$, $tf : \theta_{\mathbb{C},\{0^{(1)},0^{(2)}\}} \to \theta(f)$ is equal to $tf^{(1)} \oplus tf^{(2)}$, and $\omega f : \theta_{\mathbb{C}^2,0} \to \theta(f)$ is given by $\eta \mapsto (\eta \circ f^{(1)}, \eta \circ f^{(2)})$. We represent elements of $\theta(f)$ as 2×2-matrices, in which the first column is in $\theta\big(f^{(1)}\big)$ and the second in $\theta\big(f^{(2)}\big)$. Elements of $\theta_{\mathbb{C},\{0^{(1)},0^{(2)}\}}$ are written as pairs $\big(a(s)\frac{\partial}{\partial s}, b(t)\frac{\partial}{\partial t}\big)$. Then

$$tf\big(a(s)\partial/\partial s, 0\big) = \begin{bmatrix} a(s) & 0 \\ 0 & 0 \end{bmatrix} \tag{5.16}$$

so in $T\mathcal{A}_e f$ we have everything in the top left corner; also

$$tf\big(0, b(t)\partial/\partial t\big) = \begin{bmatrix} 0 & b(t) \\ 0 & h'(t)b(t) \end{bmatrix} \tag{5.17}$$

$$\omega f\left(\begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix}\right) = \begin{bmatrix} \eta_1(s,0) & \eta_1(t,h(t)) \\ \eta_2(s,0) & \eta_2(t,h(t)) \end{bmatrix}. \tag{5.18}$$

Using (5.18) with $\eta_2 = 0$, in view of (5.16) we get everything in the top right corner. Now using (5.17), in the bottom right hand corner we get everything in the Jacobian ideal J_h, and using (5.18) with $\eta_1 = 0$ and $\eta_2(X,Y) = p(X)$ we get everything of the form

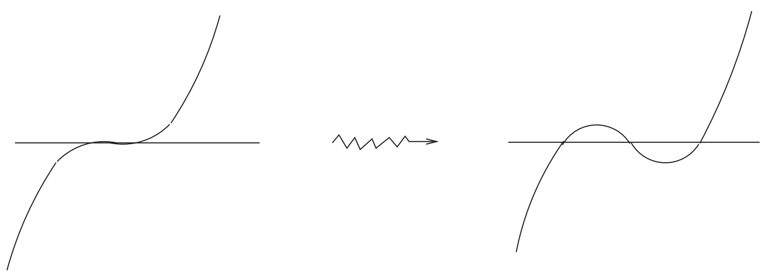
$$\begin{bmatrix} 0 & 0 \\ p(s) & p(t) \end{bmatrix}.$$

We have essentially shown

Proposition 5.18.

$$\theta(f)/T\mathcal{A}_e f \simeq \mathcal{O}_{\mathbb{C},0^{(2)}}/J_h$$

Notice that f can be perturbed to a bi-germ with ν nodes, where ν is the order of h.



So the number of nodes is one more than the codimension.

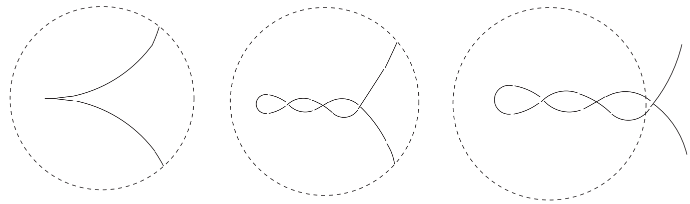
The relation between the $\mathcal{A}_e$-codimension of a map-germ and the geometry and topology of a stable perturbation is one of the most interesting aspects of the subject, and we will explore it further below.

5.3. Finite codimension = isolated instability

The next theorem is stated in two parts; the first is a special case of the second, but is easier to make sense of.

Theorem 5.19. (Terry Gaffney)*(1) The germ* $f : (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ *(*$n < p$*) has finite* $\mathcal{A}_e$*-codimension if and only if for every representative* $f : U \to V$ *of* f *there is a neighbourhood* V_0 *of* $0 \in V$ *such that for every* $y \in V_0 \smallsetminus \{0\}$ *the multi-germ* $f : (\mathbb{C}^n, f^{-1}(y)) \to (\mathbb{C}^p, y)$ *is stable.*
(2) $f : (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ *has finite* $\mathcal{A}_e$*-codimension if and only if for every representative* $f : U \to V$ *of* f *there is a neighbourhood* V_0 *of* $0 \in V$ *such that for every* $y \in V_0 \smallsetminus \{0\}$ *the multi-germ* $f : (\mathbb{C}^n, f^{-1}(y) \cap \Sigma_f) \to (\mathbb{C}^p, y)$ *is stable.*

This theorem is an easy application of the theory of coherent analytic sheaves; there is a a proof in [54]. As a consequence of 5.19, when a germ of finite codimension is deformed, the only qualitative changes occur in the vicinity of the unique unstable point. Near the boundary of the domain of any representative of the germ, nothing changes, in a sufficiently small deformation.



5.4. Versal Unfoldings

An unfolding of a map-germ f_0 is $\mathcal{A}_e$-*versal* if it contains, up to parametrised $\mathcal{A}$-equivalence, every possible unfolding of the germ. In this section we make precise sense of this idea, and study some examples.

Definition 5.20. (1) Let $F, G : (\mathbb{C}^n \times \mathbb{C}^d, 0) \to (\mathbb{C}^p \times \mathbb{C}^d, 0)$ be unfoldings of the same map germ f_0. They are *equivalent* if there exist germs of diffeomorphisms

$$\Phi : (\mathbb{C}^n \times \mathbb{C}^d, 0) \to (\mathbb{C}^n \times \mathbb{C}^d, 0)$$

and

$$\Psi : (\mathbb{C}^p \times \mathbb{C}^d, 0) \to (\mathbb{C}^p \times \mathbb{C}^d, 0)$$

such that

(a) $\Phi(x, u) = (\varphi(x, u), u)$ and $\varphi(x, 0) = x$
(b) $\Psi(y, h) = (\psi(y, u), u)$ and $\psi(y, 0) = y$
(c) $F = \Psi \circ G \circ \Phi$

Note that an unfolding is trivial (Definition 4.1) if it is equivalent to the constant unfolding.

(2) Let $h : (\mathbb{C}^e, 0) \to (\mathbb{C}^d, 0)$ be a map germ. With $F(x, u) = (f(x, u), u)$ as in (1), the unfolding $(\mathbb{C}^n \times \mathbb{C}^e, 0) \to (\mathbb{C}^p \times \mathbb{C}^e, 0)$ defined by

$$(x, v) \mapsto (f(x, h(v)), v)$$

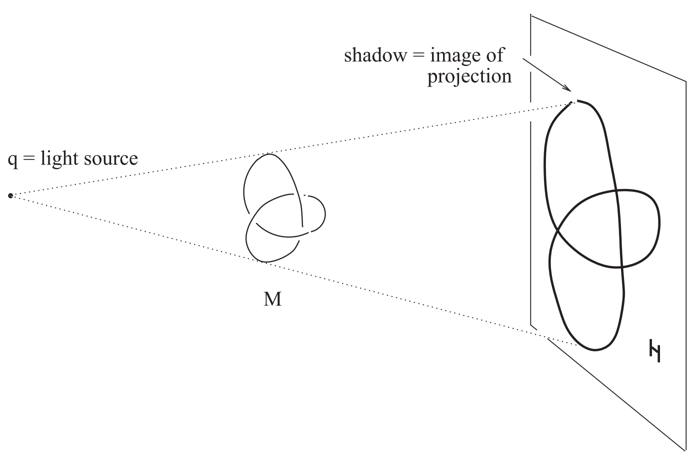
is called the *pull-back* of F by h, and denoted by h^*F. The map-germ h in this context is often called the 'base-change' map, and we say that h^*F is the unfolding *induced from* F by h.

(3) The unfolding F of f_0 is $\mathcal{A}_e$-*versal* if for every other unfolding $G : (\mathbb{C}^n \times \mathbb{C}^e, 0) \to (\mathbb{C}^p \times \mathbb{C}^e, 0)$ of f_0, there is a base-change map $h : (\mathbb{C}^e, 0) \to (\mathbb{C}^d, 0)$ such that G is equivalent (in the sense of (1)) to the unfolding h^*F (as defined in (2)).

The term 'versal' if the intersection of the words 'universal' and 'transversal'. Versal unfoldings were once upon a time called universal, but later it was decided that they did not deserve this term, because the base-change map h of part (3) of the definition is not in general unique. Uniqueness is an important ingredient in the "universal properties" which characterise many mathematical objects, and so universal unfoldings were stripped of their title. However the intersection with the word 'transversal' is serendipitous, as we will see.

Example 5.21. Some light relief Consider a manifold $M \subset \mathbb{C}^N$. Radial projection from a point q into a hyperplane H defines a map $P_q : M \to H$. If the hyperplane H is replaced by another hyperplane H', then the corresponding projection $P'_q : M \to H'$ is left-equivalent to P_q; composing P'_q with the restriction of P_q to H', we get P_q. On the other hand, if we vary the point q then we may well deform the projection P_q non-trivially. So we consider the unfolding

$$P : M \times \mathbb{C}^N \to H \times \mathbb{C}^N.$$



It is instructive to look at this over $\mathbb{R}$ with the help of a piece of bent wire and a point source of light situated at $q \in \mathbb{R}^3$. Are the unstable map-germs one sees versally unfolded in the family of all projections? This is discussed in [53] and again in [43].

Like stability, versality can be checked by means of an infinitesimal criterion. Let $F(x, u) = (f(x, u), u)$ be an unfolding of f_0. Write $\partial f/\partial u_j|_{u=0}$ as $\dot{F}_j$.

Theorem 5.22. (Infinitesimal versality is equivalent to versality) *The unfolding F of f_0 is versal if and only if*

$$T\mathcal{A}_e f_0 + Sp_{\mathbb{C}}\{\dot{F}_1, \ldots, \dot{F}_d\} = \theta(f_0)$$

– in other words, if the images of $\dot{F}_1, \ldots, \dot{F}_d$ in $T^1(f_0)$ generate it as (complex) vector space. □

For a proof, see Chapter X of Martinet's book [32]. Martinet proves the theorem for C^∞ map-germs; the proof in the analytic category is the same. Both use the Preparation Theorem, 2.26. □

Exercise 5.23. Prove 'only if' in Theorem 5.22. It follows in a straightforward way from the definitions: let g be an arbitrary element of $\theta(f_0)$ and take, as G, the 1-parameter unfolding $G(x,t) = (f(x) + tg(x), t)$. Show that if G is equivalent to an unfolding induced from F then $g \in T\mathcal{A}_e f_0 + \mathrm{Sp}_{\mathbb{C}}\{\dot{F}_1, \ldots, \dot{F}_d\}$

Example 5.24. Consider the map-germ of Example 4.6, $f_0(x,y) = (x, y^2, y^3 + x^2 y)$. We saw that $y\partial/\partial Z$ projects to a basis for $T^1(f_0)$. So

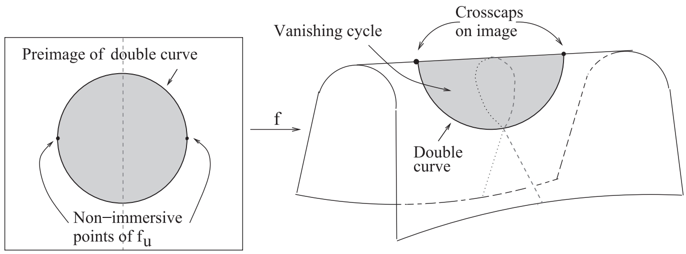
$$F(x,y,u) = (x, y^2, y^3 + x^2 y + uy, u)$$

is a versal deformation. What is the geometry here? Think of F as a family of mappings,

$$f_u(x,y) = (x, y^2, y^3 + x^2 y + uy).$$

The *ramification ideal* $\mathcal{R}_{f_u} \subset \mathcal{O}_{\mathbb{C}^2}$ generated by the 2×2 minors of the matrix $[df_u]$ defines the set of points where f_u fails to be an immersion. Here $\mathcal{R}_{f_u} = (y, x^2 + u)$. So for $u \neq 0$, f_u has two non-immersive points. They are only visible over $\mathbb{R}$ when $u < 0$.

How does f_u behave in the neighbourhood of each of these points? At each, $\mathcal{R}_{f_u}$ is equal to the maximal ideal; it follows that df_u is transverse to the submanifold $\Sigma^1 \subset L(\mathbb{C}^2, \mathbb{C}^3)$ consisting of linear maps of rank 1. In fact this transversality *characterises* the map-germ f of 4.6(1) up to $\mathcal{A}$-equivalence, though here we are not yet able to show that. Using this characterisation, we see that in a neighbourhood of the image of each of the two points $(\pm\sqrt{-u}, 0)$, the image of f_u looks like the drawing in Example 4.6. The key to assembling the image of f_u from its constituent parts is the curve of self-intersection. The only points mapped 2-1 by f_u are the points of the curve $\{x^2 + y^2 + u = 0\}$; for $u < 0$ this is a circle when viewed over $\mathbb{R}$. Here points $(x, \pm y)$ share the same image. The two non-immersive points of f_u are the fixed points of the involution $(x, y) \mapsto (x, -y)$ which interchanges pairs of points sharing the same image.



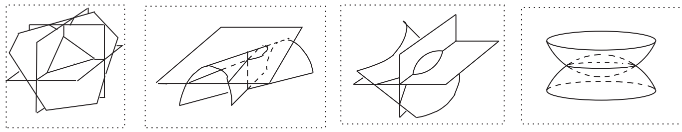
The image contains a chamber; indeed it is homotopy-equivalent to a 2-sphere. This is no coincidence. The next figure shows images of stable perturbations of each of the remaining codimension 1 singularities of maps from surfaces into 3-space. Each is homotopy-equivalent to a 2-sphere. Some choices have been made regarding the real form: sometimes a change of sign which makes no difference over $\mathbb{C}$ does make a difference over $\mathbb{R}$. Nevertheless in all of these cases it is possible to choose a suitable real form whose perturbation is a homotopy 2-sphere. We return to this example in Example 7.10

Exercise 5.25. Find versal unfoldings of the following germs:

(a) $f : (\mathbb{C}, 0) \to (\mathbb{C}^2, 0)$, $f(t) = (t^3, t^4)$.
(b) $f : (\mathbb{C}, 0) \to (\mathbb{C}^2, 0)$, $f(t) = (t^2, t^5)$.
(c) $f : (\mathbb{C}, 0) \to (\mathbb{C}^2, 0)$, $f(t) = (t^2, t^{2k+1})$.
(d) $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$, $f(x, y) = (x, y^2, y^3 + x^{k+1}y)$.
(e) $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$, $f(x, y) = (x, y^3 + x^2y)$.

5.5. Stable perturbations

We have looked at examples of mappings from $\mathbb{C}^n$ to $\mathbb{C}^{n+1}$ for $n = 1, 2$. By inspection, we can see that the perturbations of the unstable maps we considered were at least locally stable: every (mono- and multi-) germ they contain is stable. In the dimension range we have looked at, every germ of finite codimension can be perturbed so that it becomes stable. These are "nice dimensions", to use a term due to John Mather. Our definition is equivalent to the following property of the pair (n, p): in the base of a versal deformation of any germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ finite $\mathcal{A}_e$-codimension, the set of parameter-values u such that f_u has an unstable multi-germ is a proper analytic subvariety. It is known as the *bifurcation set*.



Images of stable perturbations of codimension 1 germs of maps from the plane to 3-space

Mather's point of view was global. For him the nice dimensions were characterised by the equivalent property that (n, p) is a nice pair if and only if whenever N and P are C^∞ manifolds of dimension n and p respectively, then the set of stable mappings is dense in $C^\infty_{\mathrm{pr}}(N, P)$, where the sub-index pr means proper maps. Mather carried out long calculations to determine the nice dimensions, which were published in [38]. Curiously, the nice dimensions are also characterised by the fact that every stable germ in these dimensions is weighted homogeneous, in appropriate coordinates.

When the bifurcation set B is a proper analytic subvariety of a smooth space, it does not separate it topologically (remember we're working in $\mathbb{C}^d$). That is, any two points u_1 and u_2 in its complement can be joined by a path $\gamma(t)$ which does not meet B. Because f_{u_1} and f_{u_2} are locally stable, each germ of the unfolding

$$(x, t) \mapsto (f_{\gamma(t)}(x), t)$$

is trivial; so f_{u_1} and f_{u_2} are locally isomorphic and globally C^∞-equivalent. Thus, to each complex germ of finite codimension we can associate a *stable perturbation* (any one of the mappings f_u for $u \notin B$) which is independent of the choice of u, at least up to diffeomorphism. Some care must be taken to define the domain of f_u; it is more than a germ, but not a global mapping $\mathbb{C}^n \to \mathbb{C}^p$. The situation is analogous to the construction of the Milnor fibre, in which several choices of neighbourhoods must be made, but in which the final result is nevertheless independent of the choices. Details may be found in [30].

§**6. Stable Images and Discriminants**

6.1. Review of the Milnor fibre

In the theory of isolated hypersurface singularities a key role is played by the Milnor fibre. Here is a very brief description.

(1) Let f be a complex analytic function defined on some neighbourhood of 0 in $\mathbb{C}^{n+1}$, and suppose it has isolated singularity at 0. Then by the curve selection lemma, there exists $\varepsilon > 0$ such that for ε' with

$0 < \varepsilon' \le \varepsilon$, the sphere of radius ε' centred at 0 is transverse to $f^{-1}(0)$. Let B_ε be the closed ball centred at 0 and with radius ε. Then from the transversality it follows that $f^{-1}(0) \cap B_\varepsilon$ is homeomorphic (indeed, diffeomorphic except at 0) to the cone on its boundary $f^{-1}(0) \cap S_\varepsilon$. The ball B_ε is a *Milnor ball* for the singularity.

(2) By an argument involving properness, one can show that for suitably small $\eta > 0$, all fibres $f^{-1}(t)$ with $|t| < \eta$ are transverse to S_ε. Let D_η be the closed ball in $\mathbb{C}$ with radius η and centre 0, and let $D_\eta^* = D_\eta \smallsetminus \{0\}$.

(3) By the Ehresmann fibration theorem,

$$f| : B_\varepsilon \cap f^{-1}(D_\eta^*) \to D_\eta^*$$

is a C^∞-locally trivial fibration. It is known as the Milnor fibration. Up to fibre-homeomorphism, it is independent of the choice of ε.

(4) Its fibre is called the *Milnor fibre* of f. It has the homotopy type of a wedge of n-spheres, whose number μ, the *Milnor number of* f, is equal to the dimension of the Jacobian algebra of f,

$$\mathcal{O}_{\mathbb{C}^{n+1},0}\,/J_f.$$

The argument for the last statement is based on two facts:

(1) if $\dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^{n+1},0}\,/J_f = 1$ (in which case f is said to have a 'non-degenerate" critical point), then by the holomorphic Morse lemma, f is right-equivalent to $x \mapsto x_1^2 + \cdots + x_{n+1}^2$. An explicit calculation now shows that the Milnor fibre is diffeomorphic to the unit ball sub-bundle of the tangent bundle of S^n. This has S^n as a deformation-retract.

(2) f can be perturbed so that the critical point at 0 splits into non- degenerate critical points. There are exactly μ of them, and each contributes one sphere to the wedge.

The dimension of the Jacobian algebra plays a second, completely different, role in the theory. The quotient by which we measure instability,

$$\frac{\{\frac{d}{dt}f_t|_{t=0} : f_0 = f\}}{\{\frac{d}{dt}f \circ \varphi_t|_{t=0}\}}$$

is the self-same Jacobian algebra, and indeed the Jacobian ideal itself is the extended tangent space for right-equivalence. The analogue of Theorem 5.22 shows that one can construct a versal deformation of f (versal for right-equivalence, that is) by taking $g_1, \ldots, g_\mu \in \mathcal{O}_{\mathbb{C}^{n+1},0}$ whose images in the Jacobian algebra span it as vector space, and defining

$$F(x, u_1, \ldots, u_\mu) = f(x) + \sum_j u_j g_j.$$

The Milnor fibration extends to a fibration over the complement of the discriminant Δ in the base-space $S = \mathbb{C}^\mu$; taking its associated cohomology bundle we obtain a holomorphic vector bundle of rank μ over the μ-dimensional space S. It is equipped with a canonical flat connection, the *Gauss-Manin connection.*

The objective now is to show that many of these same ingredients can be found in the theory of singularities of mappings.

6.2. Image and Discriminant Milnor Number

We have already seen, in Example 5.24, that the *real* image of each codimension 1 germ f of mappings from surfaces to 3-space grows a 2-dimensional homotopy-sphere when f is suitably perturbed.

Proposition 6.1. *(1) If $f : (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$ is a map-germ of finite codimension, then the image of a stable perturbation of f has the homotopy type of a wedge of n-spheres.*

(2) Suppose that $f : (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ is a map-germ of finite codimension, with $n \geq p$. Then the discriminant (= set of critical values) of a stable perturbation of f has the homotopy-type of a wedge of $(p-1)$-spheres.

Terminology The number of spheres in the wedge is called the *image Milnor number*, μ_I, in case (1), and the *discriminant Milnor number*, μ_Δ, in case (2).

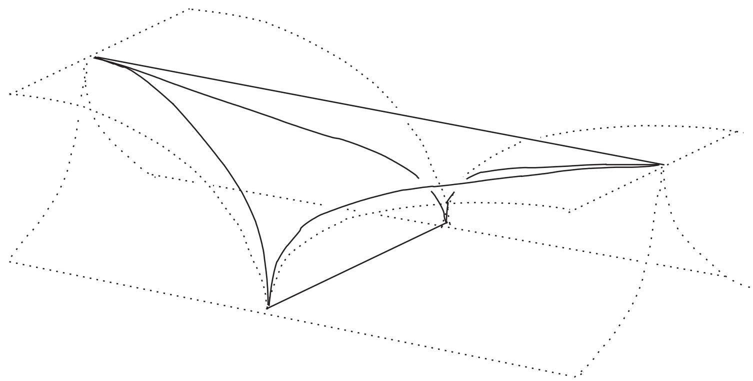
Proof of 6.1 Both statements are consequences of a fibration theorem of Lê Dũng Tráng ([52]), that says, in effect, that if (X, x_0) is a p-dimensional complete intersection singularity and $\pi : (X, x_0) \to (\mathbb{C}, 0)$ is a function with isolated singularity, in a suitable sense, then the analogue of the Milnor fibre of π (i.e. the intersection of a non-zero level set with a Milnor ball around x_0) has the homotopy-type of a wedge of spheres of dimension $p-1$. To apply this theorem here, we take, as X, the germ of the image in case (1), or discriminant, in case (2), of a 1-parameter stabilisation of f: that is, an unfolding $F : (\mathbb{C}^n \times \mathbb{C}, S \times \{0\}) \to (\mathbb{C}^p \times \mathbb{C}, 0)$ with $F(x,u) = (\tilde{f}(x,u), u) = (f_u(x), u)$ such that f_u is stable for $u \neq 0$. Then $(X, 0)$ is a hypersurface singularity, and thus a complete intersection. We take, as π, the projection to the parameter space. Thus $\pi^{-1}(u)$ is the image (or discriminant) of f_u. That π has isolated singularity is a consequence of the facts that

(i) f_0 has isolated instability at 0, and

(ii) f_u is stable for $u \neq 0$.

For (i) and (ii) imply that the unfolding F is (locally) trivial away from

$F^{-1}(0,0) \cap \Sigma_F$, so that away from $(0,0)$, the vector field $\partial/\partial u$ in the target of π lifts to a vector field tangent to X. $\square$



Discriminant (shown with dotted lines) of stable perturbation of the bi-germ

$$\begin{cases} (u,v,w) & \mapsto & (u,v,w^3-uw) \\ (x,y,z) & \mapsto & (x,y^3+xy,z) \end{cases}$$

The solid lines outline a 2-cycle carrying the vanishing homology of the discriminant.

Siersma proves in [50] that the number of spheres in the wedge is counted by the sum of the Milnor numbers of the isolated critical points of the defining equation g of the image/discriminant which move off the image/discriminant as f (and with it g) is deformed. The proof can be understood as follows. Let $g_u : B_\varepsilon \to \mathbb{C}$ be a reduced defining equation for the image/discriminant of f_u, varying analytically with u for $u \in (\mathbb{C}, 0)$. We apply Morse theory. Up to homotopy, the space B_ε is obtained from $g_u^{-1}(0)$ by progressively thickening it: considering

$$|g_u|^{-1}([0,\eta])$$

and increasing η. Away from critical points of $|g_u|$, this thickening does not change the homotopy type. Changes in homotopy-type occur only when η passes through a critical value of $|g_u|$. The critical points of $|g_u|$ off $g_u^{-1}(0)$ are the same as those of g_u, and each has index equal to the ambient dimension, because of the complex structure. Thus, the contractible space B_ε is obtained from $g_u^{-1}(0)$ by gluing in cells of dimension p. It follows by a standard Mayer-Vietoris type argument that

$g^{-1}(0)$ is homotopy-equivalent to the wedge of the boundaries of these cells. We can assume that g_u has only non-degenerate critical points off $g_u^{-1}(0)$; so the number of cells is the sum of their Milnor numbers.
This counting procedure is essential for the proofs of the following theorems.

Theorem 6.2. ([10]) *Let $f : (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ be a map-germ of finite codimension, with $n \geq p$ and (n, p) nice dimensions. Then*

$$\mu_\Delta(f) \geq \mathcal{A}_e - codim(f)$$

with equality if f is weighted homogeneous.

Theorem 6.3. *Let $f : (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$ ($n = 1$ or 2) have finite codimension. Then*

$$(1)\ \mu_I(f) \geq \mathcal{A}_e - codim(f), \text{ and} \tag{6.1}$$

(2) Equality holds if f is weighted homogeneous.

Theorem 6.3 was proved for $n = 2$ by de Jong and van Straten in [11]; another proof, also inspired by de Jong and van Straten, was given in [42], and an analogous proof for the case $n = 1$ was given in [43].

Many examples ([7],[24],[23],[45]) support the "Mond conjecture" that (6.1) should hold for all n for which $(n, n+1)$ are nice dimensions, but it remains unproven. Part of the difficulty in proving the conjecture lies in the fact that we do not have an effective method for computing image Milnor numbers. The method used in the next subsection to compute discriminant Milnor numbers when $n \geq p$ would prove the conjecture, if it could be shown to work (as indeed all the examples suggest that it does) for maps from n-space to $(n+1)$-space.

6.3. Sections of stable images and discriminants

To explain the method for computing discriminant Milnor numbers, we begin by simplifying our initial description of $T^1(f)$, using an idea of Jim Damon ([8], [9]). If $F : V \to W$ and $i : Y \to W$ are two maps, the *fibre product of V and Y over W*, denoted by $V \times_W Y$, is the space

$$V \times_W Y = \{(v, y) \in V \times Y : F(v) = i(y).\}$$

A *fibre square* is the commutative diagram which results,

$$\begin{tikzcd} V \arrow[r, "F"] & W \\ V \times_W Y \arrow[u, "\pi_V"] \arrow[r, "f:=\pi_Y"'] & Y \arrow[u, "i"'] \end{tikzcd} \tag{6.2}$$

where π_Y and π_V are the restrictions to $V \times_W Y$ of the projections $V \times Y \to Y$ and $V \times Y \to V$. If V, W and Y are smooth spaces and $i \pitchfork F$ then $V \times_W Y$ is smooth also. We say that the map $\pi_Y : V \times_W Y \to Y$, which here we will denote by f, is the *pull-back of F by i*, or *transverse pull-back* in the case where $i \pitchfork F$, and write $f = i^*(F)$. There is no canonical choice of coordinate system on $V \times_W Y$, however, so the map $i^*(F)$ is really defined only up to right-equivalence.

We use the term *standard fibre square* for a fibre square

$$\begin{tikzcd} (X \times S, (x_0, s_0)) \arrow[r, "F"] & (Y \times S, (y_0, s_0)) \\ (X, x_0) \arrow[u, "j"] \arrow[r, "f"] & (Y, y_0) \arrow[u, "i"'] \end{tikzcd} \tag{6.3}$$

in which F is an unfolding of f, and i and j are standard immersions, with $i(y) = (y, 0)$ and $j(x) = (x, 0)$. Every germ $f : (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ of finite singularity type can be obtained by transverse pull-back from a stable germ: simply construct a stable unfolding $F : (\mathbb{C}^n \times \mathbb{C}^d, 0) \to (\mathbb{C}^p \times \mathbb{C}^d, 0)$ and then recover f from F by means of a standard fibre square.

Example 6.4. Take $V = \mathbb{C}^2$, $W = \mathbb{C}^3$, $Y = \mathbb{C}^3$, and let $F(v_1, v_2) = (v_1, v_2^2, v_1v_2)$ and $i(y_1, y_2, y_3) = (p(y_1, y_2), y_2, y_3)$. Then $i \pitchfork F$ and $V \times_W Y$ is equal to

$$\{(v_1, v_2, y_1, y_2, y_3) : \ v_1 = p(y_1, y_2), \ v_2^2 = y_2, \ v_1v_2 = y_3.\}$$

The three equations defining $V \times_W Y$ allow us to dispense with the coordinates v_1, y_2 and y_3, retaining y_1, v_2 as coordinates on $V \times_W Y$. With respect to these coordinates, the maps π_V and π_Y are then given by

$$\begin{aligned} \pi_Y(y_1, v_2) &= (y_1, v_2^2, v_2 p(y_1, v_2^2)) \\ \pi_V(y_1, v_2) &= (p(y_1, v_2^2), v_2) \end{aligned}$$

Exercises 6.5. (1) Show that if V, W and Y are smooth and $i \pitchfork F$ then $V \times_Y W$ is smooth of dimension $\dim V + \dim Y - \dim W$.

(2) Show that if f is obtained by transverse pull-back from F then

(a) the set of critical points of f is the preimage by π_X of the set of critical points of F;

(b) the set of critical values of f is the preimage by i of the set of critical values of F;

(c) the local algebras $Q(f)$ and $Q(F)$ are isomorphic.

(3) Let $F(u, v, y) = (u, v, y^4 + uy^2 + vy)$ and let $f(x, y) = (x, xy + y^4)$. Find $i : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ such that $i^*(F) \simeq_{\mathcal{A}} f$.

(4) Let $f(x,y)=(x,y^3+x^ky)$. Find a stable germ F and a germ i such that $f\simeq_{\mathcal{A}} i^*(F)$.

(5) Let f be the germ of type H_2 given by $(x,y)\mapsto(x,y^3,xy+y^5)$ and let $F(a,b,c,y)=(a,b,c,y^3+ay,by^2+cy)$. Find $i:\mathbb{C}^3\to\mathbb{C}^5$ such that $i^*(F)\simeq_{\mathcal{A}} f$.

(6) Find $F:(\mathbb{C},\{0,0'\})\to(\mathbb{C}^2,0)$ and $i:(\mathbb{C}^2,0)\to(\mathbb{C}^2,0)$, with F a stable bi-germ, such that i^*F is $\mathcal{A}$-equivalent to

$$\begin{cases} s & \mapsto & (s,s^2) \\ t & \mapsto & (t,-t^2) \end{cases}$$

Suppose that f is obtained from the stable map F by transverse pull-back by i. The main theorem of this section, 6.9, shows how to recover the module T^1_f in terms of the interaction of F and i.

Before stating it, we need a definition.

Definition 6.6. If $D\subset W$ is an analytic subvariety, $\mathrm{Der}(-\log D)$ is the $\mathcal{O}_W$-module (sheaf) of germs of vector fields on W tangent to D at its smooth points.

It is easy to show that if D is the variety of zeros of an ideal I then

$$\mathrm{Der}(-\log D)=\{\chi\in\theta_W:\chi\cdot g\in I \text{ for all } g\in I\},$$

and in particular if D is a hypersurface with equation h then

$$\mathrm{Der}(-\log D)=\{\chi\in\theta_W:\chi\cdot h=\alpha h \text{ for some } \alpha\in\mathcal{O}_W\}.$$

If D is any complex space and $x\in D$, the *isosingular locus of D at w*, $\mathcal{I}_{D,w}$, is the germ of the set of points

$$\{x\in(D,w): \text{ the germs } (D,w) \text{ and } (D,x) \text{ are isomorphic}\}.$$

Theorem 6.7. (Ephraim, [15]) *Let $D\subset W$. Then $\mathcal{I}_{D,w}$ is non-singular, and $T_w\mathcal{I}_{D,w}=\{\chi(w):\chi\in Der(-\log D)_w\}$.*

□

The inclusion of right hand side in left in 6.7 is clear: if $\chi\in\mathrm{Der}(-\log D)_w$ and $\chi(w)\neq 0$ then the integral flow of χ preserves D, and thus induces a family of isomorphisms of D. Evidently the integral curve of χ through w is contained in $\mathcal{I}_{D,w}$, and so its tangent vector $\chi(w)$ is contained in $T_w\mathcal{I}_{D,w}$.

The vector space in 6.7 is known as the *logarithmic tangent space to D at w*; we denote it by $T^{\log}_w D$. If Y is a smooth space and $i:Y\to W$ a map, we say i is *logarithmically transverse to D at $y_0\in Y$* if

$$d_{y_0}i(T_{y_0}Y)+T^{\log}_{i(y_0)}D=T_{i(y_0)}W. \tag{6.4}$$

Each of the three vector spaces in (6.4) is the evaluation at y_0 of the stalk of a sheaf of $\mathcal{O}_Y$-modules: the three sheaves are, respectively, $ti(\theta_Y)$, $i^*(\mathrm{Der}(-\log D))$ and $\theta(i)$.

Proposition 6.8. *The equality* (6.4) *holds if and only if*

$$\frac{\theta(i)}{ti(\theta_Y) + i^*(Der(-\log D))} = 0. \tag{6.5}$$

Proof. This is just Nakayama's Lemma: reducing the module on the left of (6.5) modulo $\mathfrak{m}_{Y,y_0}\theta(i)$ we get the quotient of the space on the right of (6.4) by the space on the left. Q.E.D.

The module on the left of (6.5) thus measures the failure of logarithmic transversality of i to D. For reasons which will become clear later, we will denote it by $T^1_{\mathcal{K}_D}i$. Its $\mathbb{C}$-vector-space dimension is finite if and only if i is logarithmically transverse to D outside y_0.

Let $F : (V, 0) \to (W, 0)$ be a stable map-germ, and let D be its discriminant.

Theorem 6.9. (J.N.Damon,[9]) *If f is obtained from the stable map F by transverse pull back by $i : Y \to W$ then $T^1(f)$ and $T^1_{\mathcal{K}_D}i$ are isomorphic as $\mathcal{O}_Y$-modules.*

To prove this we need

Lemma 6.10. *Let $F : (V, 0) \to (W, 0)$ be a map-germ of finite $\mathcal{A}_e$-codimension, and let D be its discriminant. Let $\chi \in \theta_{W,w_0}$. Then $\chi \in Der(-\log D)_{w_0}$ if and only if it can be lifted to a vector field $\tilde{\chi}$ on (V, v_0) – that is, if and only if there exists $\tilde{\chi} \in \theta_{V,v_0}$ such that*

$$tF(\tilde{\chi}) = \omega F(\chi).$$

Proof. Suppose that $\chi \in \theta_W$ has lift $\tilde{\chi} \in \theta_V$. By integrating χ and $\tilde{\chi}$ we obtain flows Ψ_t, Φ_t on W and V respectively, such that

$$F \circ \Phi_t = \Psi_t \circ F. \tag{6.6}$$

Suppose $y \in D$, and let $x \in \Sigma_F$ satisfy $y = F(x)$. For every t, (6.6) shows that the germs

$$F : (V, \Phi_t(x)) \to (W, \Psi_t(y)) \quad \text{and} \quad F : (V, x) \to (W, y)$$

are left-right equivalent. Since x is a critical point of F, so is $\Phi_t(x)$, and therefore $\Psi_t(y)$, which is equal to $F(\Phi_t(x))$, lies in D. That is, we have shown that the flows Φ_t and Ψ_t preserve Σ_F and D respectively. It follows that the vector fields $\tilde{\chi}$ and χ are tangent to Σ_F and D respectively. In particular, $\chi \in \mathrm{Der}(-\log D)$.

Reciprocally, if $\chi \in \mathrm{Der}(-\log D)$ then we can certainly lift $\chi|_D$ to a vector field $\tilde{\chi}_0$ on Σ_F. For Σ_F is the normalisation[4] of D, and vector fields lift to the normalisation, by a theorem of Seidenberg[5]. Suppose $\tilde{\chi}_0$ is the restriction to Σ_F of a vector field $\tilde{\chi}_1 \in \theta_V$. We have no guarantee that $\tilde{\chi}_1$ is a lift of χ – i.e. that $tF(\tilde{\chi}_1) = \omega F(\chi)$ – only that this equality holds on Σ_F. But J_F is radical, so the fact that $\tilde{\chi}|_{\Sigma_F}$ is a lift of $\chi|_D$ means that $tF(\tilde{\chi}_1) - \omega F(\chi) \in J_F\theta(F)$. By Cramer's rule, $J_F\theta(F) \subset tF(\theta_V)$, and thus there exists a vector field $\xi \in \theta_V$ such that

$$tF(\tilde{\chi}_1 - \xi) = \omega F(\chi), \tag{6.7}$$

showing that χ is liftable. Q.E.D.

Proof of 6.9. We show first that we can assume that f, i, j and F form a standard fibre square as in (6.3). To see this, we may suppose that i is an immersion, for if we replace i by the immersion $i_1(y) = (i(y), y) \in X \times Y$, and F by $F_1 = F \times \mathrm{id}_Y : X \times Y \to Z \times Y$, then the discriminant of F_1 is equal to $D \times Y$, the pulled-back map $i_1^*(F_1) : (X \times Y) \times_{Z \times Y} Y \to Y$ is isomorphic (even the same as) the previous pull-back $i^*(F)$, and (by Exercise 6.11 below)

$$\frac{\theta(i_1)}{ti_1(\theta_Y) + i_1^*(\mathrm{Der}(-\log D \times Y))} \simeq \frac{\theta(i)}{ti(\theta_Y) + i^*(\mathrm{Der}(-\log D))}.$$

With this assumption, by 2.2 we can choose coordinates $y_1, \ldots, y_p$ on Y and $y_1, \ldots, y_p, u_1, \ldots, u_d$ on $Z \times Y$ so that i becomes the standard immersion $i(y) = (y, 0)$. Of course this changes the map f, but the new T^1_f is isomorphic to the old. As $F \pitchfork i$, the map $(u_1 \circ F, \ldots, u_d \circ F)$ is a submersion, so its d component functions form part of a coordinate system on X, with respect to which F is an unfolding of f.

---

[4] Σ_F is Cohen Macaulay, essentially by Corollary 2.36 – see the argument in the proof of 6.14 below. Hence it is normal if and only if it is non-singular in codimension 1 (i.e. it set of singular points has codimension at least 2 in Σ_F). Because j^1F is transverse to the stratification $\{\Sigma^k : k \in \mathbb{N}\}$ of $L(n,p)$,

$$(\Sigma_F)_{\mathrm{Sing}} = j^1F^{-1}((\overline{\Sigma_1})_{\mathrm{Sing}}) = j^1F^{-1}(\overline{\Sigma_2});$$

it therefore has codimension in Σ_F equal to codim Σ^2 – codim Σ^1, which is greater than 1.

[5] Sketched argument: the normalisation is unique up to isomorphism, so any automorphism of D lifts to an automorphism of its normalisation Σ_F; given a vector field on D, integrate it to get a 1-parameter family of automorphisms Ψ_t, lift the Ψ_t to a 1-parameter family of automorphisms Φ_t of Σ_F, then differentiate Φ_t with respect to t and set $t = 0$ to get a vector field on Σ_F lifting χ.

Now that we are in the situation of the standard fibre square, we revert to the notation of 6.3 in which the parameter space is denoted by S. We denote by $\theta_{X\times S/S}$ the $\mathcal{O}_{X\times S}$-submodule of $\theta_{X\times S}$ consisting of vector fields on $X \times S$ with zero component in the S direction, and, similarly, by $\theta(F/S)$ the $\mathcal{O}_{X\times S}$-submodule of $\theta(F)$ consisting of vector fields along F with zero component in the S direction. We define $\theta_{Y\times S/S}$ and $\mathrm{Der}(-\log D)/S$ analogously.

Let $\pi : Y \times S \to S$ be projection. Consider the following diagram.

(6.8)
$$\begin{tikzcd}
0 & & 0 & 0 & & \\
C_\bullet \arrow[u] & 0 \arrow[r] & t\pi(\mathrm{Der}(-\log D)) \arrow[r] \arrow[u] & \theta(\pi) \arrow[r] \arrow[u] & 0 & \\
B_\bullet \arrow[u] & 0 \arrow[r] & \mathrm{Der}(-\log D) \arrow[r] \arrow[u, "t\pi"] & \theta_{Y\times S} \arrow[r] \arrow[u, "t\pi"] & \frac{\theta(F)}{tF(\theta_{X\times S})} \arrow[r] \arrow[u] & 0 \\
A_\bullet \arrow[u] & 0 \arrow[r] & \mathrm{Der}(-\log D)/S \arrow[r] \arrow[u] & \theta_{Y\times S/S} \arrow[r] \arrow[u] & \frac{\theta(F/S)}{tF(\theta_{X\times S/S})} \arrow[r] \arrow[u] & 0 \\
0 \arrow[u] & & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

Each column is exact. This is obvious for the first two columns; for the third, it is an easy calculation that the homomorphism

$$\frac{\theta(F/S)}{tF(\theta_{X\times S/S})} \to \frac{\theta(F)}{tF(\theta_{X\times S})},$$

induced by the inclusion $\theta(F/S) \hookrightarrow \theta(F)$, is an isomorphism. Each row in the diagram is a complex, and thus (6.8) is a short exact sequence of complexes. Let us give the columns the indices $2, 1, 0$. The long exact sequence of homology contains the portion

$$\cdots \to H_1(B_\bullet) \to H_1(C_\bullet) \to H_0(A_\bullet) \to H_0(B_\bullet) \to \cdots$$

However, $B_\bullet$ is exact, by Lemma 6.10, and thus $H_1(C_\bullet) \simeq H_0(A_\bullet)$. Evaluating these homology modules, we obtain

$$\frac{\theta(\pi)}{t\pi(\mathrm{Der}(-\log D))} \simeq \frac{\theta(F/S)}{tF(\theta_{X\times S/S}) + \omega F(\theta_{Y\times S/S})}. \tag{6.9}$$

Reducing each side modulo $\mathfrak{m}_S$ preserves the isomorphism. On the right we now have $T^1(f)$. It remains to show that what we have on the left is isomorphic to $T^1_{\mathcal{K}_D} i$. We have

$$\theta(\pi) = \sum_{j=1}^{d} \mathcal{O}_{Y\times S} \frac{\partial}{\partial s_j}$$

so

$$\frac{\theta(\pi)}{\mathfrak{m}_S\, \theta(\pi)} = \sum_{j=1}^{d} \mathcal{O}_Y \frac{\partial}{\partial s_j}.$$

Also

$$\theta(i) = \sum_{k=1}^{p} \mathcal{O}_Y \frac{\partial}{\partial y_k} \oplus \sum_{j=1}^{d} \mathcal{O}_Y \frac{\partial}{\partial s_j}$$

so

$$\frac{\theta(i)}{ti(\theta_Y)} \simeq \sum_{j=1}^{d} \mathcal{O}_Y \frac{\partial}{\partial s_j}.$$

So $\theta(i)/ti(\theta_Y)$ can be identified with $\theta(\pi)/\,\mathfrak{m}_S\,\theta(\pi)$. Using this identification, we have

$$\frac{\theta(i)}{ti(\theta_Y) + i^*\mathrm{Der}(-\log D)} \simeq \frac{\theta(\pi)}{t\pi(\mathrm{Der}(-\log D)) + \mathfrak{m}_S\,\theta(\pi)}.$$

$\square$

Exercises 6.11. (i) Show that, as stated in the proof of 6.9, the natural map

$$\frac{\theta(F/S)}{tF(\theta_{X\times S/S})} \to \frac{\theta(F)}{tF(\theta_{X\times S})}$$

is an isomorphism.
(ii) Suppose that $D = D_0 \times S \subset Z \times S$, $i : Y \to Z \times S$, and let $\pi : Z\times S \to Z$ be projection and $j_0 : Z \to Z\times S$ be the inclusion $z \mapsto (z, 0)$. Show that

$$T^1_{\mathcal{K}_D} i \simeq \frac{\theta(\pi\circ i)}{t(\pi\circ i)(\theta_Y) + (\pi\circ i)^*(\mathrm{Der}(-\log D_0))}.$$

Hint: $\mathrm{Der}(-\log D)$ contains all the vector fields $\partial/\partial s_i$. You can choose the remaining generators for $\mathrm{Der}(-\log D)$ in $\theta_{Z\times S/S}$.

The module in the denominator of (6.5) is the (extended) tangent space to the orbit of i under a variant of contact equivalence introduced by Damon in [8] and called $\mathcal{K}_D$-*equivalence*, though we will not make use of this here. It was the key to his proof of 6.9 in [9], where he showed that if i_t is a deformation of i then the family $i_t^*(F)$ is $\mathcal{A}$-trivial if and only if i_t is $\mathcal{K}_D$-trivial.

Definition 6.12. *Let $f, g : (Y, y_0) \to (W, w_0)$ and let $(D, w_0) \subset (W, w_0)$. We say f is $\mathcal{K}_D$-equivalent to g if there exists diffeomorphisms $\Phi : (Y \times W, (y_0, w_0)) \to (Y \times W, (y_0, w_0))$ and $\varphi : (Y, y_0) \to (Y, y_0)$ such that*

(a) Φ *lifts* φ, *i.e.* $\pi_Y \circ \Phi = \varphi \circ \pi_Y$;
(b) $\Phi(Y \times D) = Y \times D$,
(c) $\Phi(graph(f)) = graph(g)$.

In the usual version of contact equivalence, $D = \{y_0\}$.

The advantage of the quotient (6.5) over the expression (4.5) is that in (6.5) all the objects are finite modules over the same ring, $\mathcal{O}_Y$, whereas the first summand in the denominator in (4.5) is an $\mathcal{O}_{\mathbb{C}^n,S}$-module while the second is only an $\mathcal{O}_{\mathbb{C}^p,0}$ module. This makes (6.5) algebraically much simpler to work with.

Definition 6.13. If D is a divisor (hypersurface) in W, we say D is a *free divisor* if $\mathrm{Der}(-\log D)$ is a locally free $\mathcal{O}_W$-module.

Proposition 6.14. ([29, 6.13]) *If $F : (\mathbb{C}^n, S) \to (\mathbb{C}^p, y_0)$ $(n \geq p)$ is stable then the discriminant D of F is a free divisor.*

Proof. Let us write $(\mathbb{C}^n, S) =: X$, $(\mathbb{C}^p, y_0) =: Y$. The proof uses the exact sequence

$$0 \longrightarrow \mathrm{Der}(-\log D) \longrightarrow \theta_Y \xrightarrow{\bar{\omega}F} \frac{\theta(F)}{tF(\theta_X)} \longrightarrow 0 \tag{6.10}$$

which already appeared (in the special case that F is a parameterised unfolding) as the complex $B_\bullet$ of (6.8). We may assume that F is not a trivial unfolding of a lower-dimensional germ; freeness of $\mathrm{Der}(-\log D)$ under this assumption implies freeness in general, since the discriminant of a trivial unfolding $F \times \mathrm{id}_S$ is equal to the product of S with the discriminant of F.

From the assumption, it follows that all of members of $\mathrm{Der}(-\log D)$ vanish at y_0. For if $\chi \in \mathrm{Der}(-\log D)_{y_0}$ and $\chi(y_0) \neq 0$, then lifting χ to $\tilde{\chi}$ in θ_X (using 6.10), the Thom-Levine Lemma 3.5 implies that the integral flows of χ and $\tilde{\chi}$ give a 1-parameter trivialisation of F.

Let $\chi_1, \ldots, \chi_\ell$, with $\chi_i = \sum_j \chi_i^j \partial/\partial y_j$, be a minimal set of generators of $\mathrm{Der}(-\log D)$, and let χ be the $p \times \ell$ matrix of coefficients χ_i^j. Then

$$\mathcal{O}_Y^\ell \xrightarrow{\chi} \theta_Y \xrightarrow{\bar{\omega}F} \frac{\theta(F)}{tF(\theta_X)} \longrightarrow 0 \tag{6.11}$$

is exact. Since all of the entries in χ vanish at y_0, (6.11) extends to a minimal free resolution of $\theta(F)/tF(\theta_Y)$. But such a free resolution must have length 1. We prove this in two steps:
Step 1: We show that $\theta(F)/tF(\theta_X)$ is Cohen-Macaulay of dimension $p-1$. The support of this module is the critical set Σ_F of F, the set of points where F is not a submersion. We deduce Cohen-Macaulayness, using Theorem 2.34, from a classical theorem of Buchsbaum and Rim, [4, Cor 2.7], which implies that the cokernel of a $p \times n$ $(p \le n)$ matrix M of indeterminates is Cohen-Macaulay [6]. To apply Theorem 2.34 we ned only check that the codimension of the support of $\theta(F)/tF(\theta_X)$ is $n-p+1$. It is no greater than that, because Σ_F is the preimage, under the map $j^1F : X \to L(n,p)$, of the set of matrices of rank $< p$ in $L(n,p)$, which has codimension $n-p+1$. Thus, its dimension is *at least* $\dim \Sigma_F = p-1$. To prove equality, we first deduce, from the $\mathcal{K}$-finiteness of F, that

$$\dim_{\mathbb{C}} \frac{\mathcal{O}_X}{J_F + F^*(\mathfrak{m}_{Y,0})\,\mathcal{O}_X} < \infty;$$

this means that the restriction of F to its critical set $V(J_F)$ is finite. The argument is that

$$\frac{\mathcal{O}_X}{J_F + F^*(\mathfrak{m}_{Y,0})\,\mathcal{O}_X}$$

and

$$\frac{\theta(F)}{tF(\theta_X) + F^*\mathfrak{m}_{Y,0}\theta(F)} \tag{6.12}$$

have the same support, namely $F^{-1}(0) \cap V(J(F))$, and Theorem 4.10 implies that (6.12) has finite complex vector space dimension (in fact $\le p$), so its support is just $\{0\}$. So the dimension of $V(J_F)$ is no greater than the dimension of its image in $D \subset Y$. This image is a closed variety, by finiteness, and cannot be all of Y, by Sard's theorem. Therefore it

[6]In fact Buchsbaum and Rim prove that the cokernel is Cohen-Macaulay if and only if the quotient of the polynomial ring in the indeterminantes by the ideal $I(M)$ of maximal minors of M is Cohen-Macaulay, and this latter condition was proved by Northcott in [46]

has dimension no *greater* than $p-1$. Hence $\dim V(J_F) = \dim D \leq p-1$, and $\theta(F)/tF(\theta_X)$ is Cohen Macaulay of dimension $p-1$ as required. For future use we note that J_F must in fact be radical: the condition on the codimension of the support of $\theta(F)/tF(\theta_X)$ guarantees that $\mathcal{O}_X\,/J_F$ also is Cohen-Macaulay; a Cohen-Macaulay space is reduced if and only if it is generically reduced (see e.g.[29, page 50]), so one can check reducedness at a generic point, i.e. by a local calculation, and this is easily done for example at a fold point.
Step 2: Because $\theta(F)/tF(\theta_X)$ has depth $p-1$ over $\mathcal{O}_X$, and is finite over $\mathcal{O}_Y$, its $\mathcal{O}_Y$-depth is also $p-1$. Therefore by the Auslander-Buchsbaum theorem (see e.g. [39, Theorem 19.1] or [14, Theorem 19.9]) its projective dimension (the length of a minimal free resolution) is 1. It follows that the kernel of $\overline{\omega F}$ is free.

Q.E.D.

Let h be an equation of the discriminant D of F. We define $\mathrm{Der}(-\log h)$ as the $\mathcal{O}_Y$-module of germs of vector fields which annihilate h; that is, which are tangent not only to $D = h^{-1}(0)$, but to all level sets of h. Clearly $\mathrm{Der}(-\log h)$ is a submodule of $\mathrm{Der}(-\log D)$, but it depends on the choice of equation h, and is not determined by D alone.

Theorem 6.15. ([10]) *If $f : (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$, with $n \geq p$ and (n,p) nice dimensions, and f is obtained from the stable map-germ F by transverse pull back by i, then*

$$\mu_\Delta(f) = dim_{\mathbb{C}}\ \frac{\theta(i)}{ti\big(\theta_{\mathbb{C}^p,0}\big) + i^*\big(Der(-\log h)\big)}. \tag{6.13}$$

The proof of this result uses conservation of multiplicity, and depends in an essential way both on the fact that $\Delta(F)$ is a free divisor, and on a striking property of the nice dimensions – that in the nice dimensions all stable germs are weighted homogeneous, with respect to suitable coordinates. This property of the nice dimensions can be checked by inspection of Mather's list of stable types in [38]. In fact it characterises the nice dimensions, a fact which surely deserves explanation. We denote the module on the right of (6.13) by $T^1_{\mathcal{K}_h}i$; its denominator is the extended tangent space to the orbit of i under the subgroup $\mathcal{K}_h$ of $\mathcal{K}_D$ (*cf* Definition 6.12) in which the diffeomorphisms of $Y \times W$ preserve all the level sets of h, not just D.

The inequality in Theorem 6.2 follows immediately from Theorems 6.15 and 6.9. Equality holds in the quasihomogeneous case because the presence of an Euler vector field means that $T^1_{\mathcal{K}_D}i = T^1_{\mathcal{K}_h}i$.

6.4. Open questions

(1) We reiterate the "Mond conjecture" described at the end of Subsection 6.2. The fact that it is still open is an embarrassing gap in the theory. The argument used to prove (6.15) in [10] very nearly proves that if f is obtained by transverse fire product as $i^*(F)$, then $\mu_I(f) = \dim_{\mathbb{C}} T^1_{\mathcal{K}_h} i$. All that is missing is a proof of conservation of multiplicity. In fact conservation of multiplicity is equivalent to the equality $\mu_I(f) = \mathcal{A}_e - \operatorname{codim}(f)$ for weighted homogeneous f, for which there is abundant empirical evidence. This open question invites the uncovering of some as yet unrecognised algebraic structure.

(2) A famous theorem of Lê and Ramanujan states that (provided the ambient dimension is not 3) a μ-constant family of isolated hypersurface singularities is topologically trivial. Do the image and discriminant Milnor numbers have an equally crucial role in determining the topology of map germs?

(3) A stable perturbation of a finitely determined real map-germ $(\mathbb{R}^n, S) \to (\mathbb{R}^{n+1}, 0)$ is *maximal* if it exhibits all of the 0-dimensional stable singularities present in its complexification. It is a *good real perturbation* if the real image has n-th homology of rank $\mu_I(f)$ (so that inclusion of real image in complex image induces an isomorphism on H_n). Is it true that every good real perturbation is maximal? This is the case in all known examples. The same question is also open, concerning maps $(\mathbb{R}^n, S) \to (\mathbb{R}^p, 0)$ with $n \geq p$, with "discriminant" replacing "image" and μ_Δ replacing μ_I.

§7. Multiple points in the source

The multiple point spaces of a germ $(\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ with $n < p$ play an important rôle in the study of its geometry, as well as the topology of the image of a stable perturbation ([42], [31], [19], [20]).

The k'th *source multiple point space* D^k of a finite proper map between topological spaces is the closure of the set of k-tuples of pairwise distinct points having the same image under the map. The k'th *target multiple point space* $M_k(f)$ is the set of points having k or more distinct preimages, counting multiplicity. When $f\colon X \to Y$ is a finite analytic map of complex manifolds, the space $M_k(f)$ has a natural analytic structure as the subspace of Y defined by the $(k-1)$'st Fitting ideal $\mathrm{Fitt}_{k-1}(f_*\mathcal{O}_X)$ of the pushforward $f_*\mathcal{O}_X$ (see [51], [44], [27]). This structure is particularly good when X is Cohen-Macaulay, Y is smooth and $\dim Y = \dim X + 1$. One might hope for an analogous formula giving equations for $D^k(f)$ in X^k, in terms of f itself. No such formula is known in general, though for $k = 2$ the ideal defined, in terms of local coordinates $x_1, \ldots, x_n$ and $y_1, \ldots, y_p$ on X and Y, by

$$\mathcal{I}_2 := (f \times f)^* I_{\Delta_p} + \mathrm{Fitt}_0(I_{\Delta_n}/(f \times f)^* I_{\Delta_p}) \tag{7.1}$$

where I_{Δ_n} and I_{Δ_p} are the ideal sheaves defining the diagonals Δ_n in $\mathbb{C}^n \times \mathbb{C}^n$ and Δ_p in $\mathbb{C}^p \times \mathbb{C}^p$, gives $D^2(f)$ a scheme structure with many desirable qualities: if f is dimensionally correct – that is, if $D^2(f)$ has the expected dimension, $2n-p$, then $D^2(f)$ is Cohen Macaulay. If moreover f is finitely determined (for left-right equivalence), or, equivalently, has isolated instability, then provided its dimension is greater than 0, $\mathcal{I}_2$ is radical.

If the corank of f (the dimension of $\mathrm{Ker}\, df_0$), is equal to 1, much more is possible. An explicit list of generators for the ideal defining $D^k(f)$ in $(\mathbb{C}^n)^k$ is given in [31], where it is shown that a finite corank 1 map-germ $f\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ with $n < p$ is stable if and only if each $D^k(f)$ is smooth of dimension $p - k(p-n)$, or empty, for all $k \geq 2$. Moreover, it is finitely $\mathcal{A}$-determined if and only if D^k is an ICIS of dimension $p - k(p-n)$ or empty for those k with $p - k(p-n) \geq 0$, and D^k consists at most of only the origin if $p - k(p-n) < 0$ (see, e.g., [30], [19] for other results).

We will say that f is *dimensionally correct* if for each k, $D^k(f)$ satisfies these dimensional requirements, including the requirement that when $p - k(p-n) < 0$, $D^k(f)$ consists at most of the origin.

7.1. Multiple point spaces

Given a map $f : X \to Y$, we define ${}^oD^k(f)$ as the set

$$\{(x_1, \ldots, x_k) \in X^k | f(x_1) = \cdots = f(x_k), x_i \neq x_j \text{ if } i \neq j\} \tag{7.2}$$

and define the k'th *source multiple point space of* f, $D^k(f)$, by

$$D^k(f) = \text{closure } {}^oD^k(f) \tag{7.3}$$

(where the closure in taken in X^k) *provided* ${}^oD^k(f)$ *is not empty.* We extend this definition to germs of maps by taking the limit over representatives; if $f \in \mathcal{E}^0_{n,p}$ is finite, the local conical structure guarantees that we obtain in this way a well defined germ at $\mathbf{0} \in (\mathbb{C}^n)^k$. We give $D^k(f)$ an analytic structure as follows. First, choose a stable unfolding $F : X \times \mathbb{C}^d \to Y \times \mathbb{C}^d$ and give $D^k(F)$ its reduced structure. Because F is an unfolding, $D^k(F)$ embeds naturally in $X^k \times \mathbb{C}^d$, with defining ideal $\mathcal{I}_k(F)$. Define

$$\mathcal{I}_k(f) = \mathcal{I}_k(F)|_{\mathbf{u}=0}.$$

It is straightforward to check that this is independent of the choice of stable unfolding, and is compatible with unfolding in the sense that for *any* germ of unfolding $F : (\mathbb{C}^n \times \mathbb{C}^d, 0) \to (\mathbb{C}^p \times \mathbb{C}^d, 0)$ of f, the diagram

$$\begin{tikzcd} D^k(f), 0 \arrow[r] \arrow[d] & D^k(F), 0 \arrow[d] \\ \{0\} \arrow[r] & \mathbb{C}^d, 0 \end{tikzcd} \tag{7.4}$$

in which the vertical arrows are projections to the base and the horizontal arrows are inclusions, is a fibre square.

This definition of $\mathcal{I}_k(f)$ is canonical, but gives no hint as to how $\mathcal{I}_k(f)$ is to be calculated. But suppose that $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ has corank 1. Then with respect to suitable coordinates it can be written in the form

$$f(\mathbf{x}, y) = (\mathbf{x}, f_n(\mathbf{x}, y), \ldots, f_p(\mathbf{x}, y)) \tag{7.5}$$

where $(\mathbf{x}, y)$ are suitable coordinates on $\mathbb{C}^n$. That is, we write f explicitly as an unfolding of a map-germ in the single variable y. Now any k points $(\mathbf{x}_1, y_1), \ldots, (\mathbf{x}_k, y_k)$ sharing the same image must have equal $\mathbf{x}$ coordinates, and so $D^k(f)$ embeds naturally in $\mathbb{C}^{n-1} \times \mathbb{C}^k$. We take coordinates $\mathbf{x}, y_1, \ldots, y_k$ on $\mathbb{C}^{n-1} \times \mathbb{C}^k$ and look for equations defining $D^k(f)$ in $\mathbb{C}^{n-1} \times \mathbb{C}^k$.

The following analysis will be applied to each of the component functions $f_j, j = n \ldots, p$ of f. To spare notation for the moment, let h be any function of x, y. The map

$$(x, y_1, \ldots, y_k) \mapsto \big(h(x, y_1), \ldots, h(x, y_p)\big) \tag{7.6}$$

is equivariant with respect to the symmetric group actions on the source permuting the y_i and on the target permuting the $f_j(x, y_i)$. The set $\mathcal{E}^{S_k}$ of equivariant maps $\mathbb{C}^{n-1+k} \to \mathbb{C}^k$ is a module over the ring $\mathcal{O}^{S_k}$ of invariant functions on the source, generated (although we will not *need* this fact) by the gradient vectors of the generators of $\mathcal{O}^{S_k}$ ([48]). The ring $\mathcal{O}^{S_k}$ is generated over $\mathcal{O}_{\mathbb{C}^{n-1},0}$ by the sums of powers $\rho_1 = y_1 + \cdots + y_k, \ldots, \rho_k = y_1^k + \cdots + y_k^k$, and so every equivariant mapping can be written as a linear combination, over $\mathcal{O}^{S_k}$, of the maps

$$\begin{array}{rcl} m_1(y_1, \ldots, y_k) & = & (1, \ldots, 1) \\ m_2(y_1, \ldots, y_k) & = & (y_1, \ldots, y_k) \\ \cdots & \cdots & \cdots \\ m_{k-1}(y_1, \ldots, y_k) & = & (y_1^{k-1}, \ldots, y_k^{k-1}) \end{array} \tag{7.7}$$

Thus there exist invariant functions $\alpha_0^k, \alpha_1^k, \ldots, \alpha_{k-1}^k$ such that

$$\begin{pmatrix} h(x,y_1) \\ \vdots \\ h(x,y_k) \end{pmatrix} = \alpha_0^k \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} + \alpha_1^k \begin{pmatrix} y_1 \\ \vdots \\ y_k \end{pmatrix} + \cdots + \alpha_{k-1}^k \begin{pmatrix} y_1^{k-1} \\ \vdots \\ y_k^{k-1} \end{pmatrix} \tag{7.8}$$

Solving for the α_i^k by Cramer's rule gives

$$\alpha_\ell^k(x, y_1, \ldots, y_k) = \tag{7.9}$$

$$\frac{\begin{vmatrix} 1 & y_1 & \cdots & y_1^{\ell-1} & h(x,y_1) & y_1^{\ell+1} & \cdots & y_1^{k-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & y_k & \cdots & y_k^{\ell-1} & h(x,y_k) & y_k^{\ell+1} & \cdots & y_k^{k-1} \end{vmatrix}}{\begin{vmatrix} 1 & y_1 & \cdots & y_1^{k-1} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & y_k & \cdots & y_k^{k-1} \end{vmatrix}}.$$

In fact we do not the statement of Poenaru referred to above to see that the α_ℓ^k are regular (analytic): the numerator in (7.9) vanishes whenever $y_i = y_\ell$ for any i, ℓ, and thus is divisible in $\mathcal{O}$ by $\prod_{i<\ell}(y_i - y_\ell)$, i.e. by the Vandermonde determinant, which is the denominator in (7.9). In other words the system of equations (7.8) has analytic solutions. As can be seen from (7.9), they are S_k-invariant. They are also unique, since the Vandermonde determinant vanishes only along a hypersurface.

Let $I_k(h)$ be the ideal generated by the α_ℓ^k for $\ell = 1, \ldots, k-1$.

Remark 7.1. The ideal $I_k(h)$ is also generated over $\mathcal{O}_{\mathbb{C}^{n-1}\times\mathbb{C}^k,0}$ by the $k-1$ functions $R_i(h)$, for $i = 2, \ldots, k$, which are defined iteratively by

$$R_2(h)(\mathbf{x}, y_1, y_2) = \frac{h(\mathbf{x}, y_2) - h(\mathbf{x}, y_1)}{y_2 - y_1} \tag{7.10}$$

$$R_i(h)(\mathbf{x}, y_1, \ldots, y_{i+1}) =$$

$$\frac{R_{i-1}(h)(\mathbf{x}, y_1, \ldots, y_{i-1}, y_{i+1}) - R_{i-1}(h)(\mathbf{x}, y_1, \ldots, y_{i-1}, y_i)}{y_{i+1} - y_i}.$$

Theorem 7.2. ([31]) *If f as in* (7.5) *is dimensionally correct then* $\mathcal{I}_k(f) = I_k(f_{n+1}) + \cdots + I_k(f_p)$.

Thus we have $(k-1)(p-n+1)$ explicit equations for $D^k(f)$.

Exercise 7.3. (1) Find equations for $D^2(f)$ and $D^3(f)$ when f is the map-germ given by
(a) $f(x_1, x_2, x_3, y) = (x_1, x_2, x_3, y^3 + x_1 y, x_2 y^2 + x_3 y)$ (stable map-germ of type $\sum^{1,1,0}$).
(b) $f(x, y) = (x, y^2, y^3 + x^{k+1} y)$ (type S_k in [41] – here $D^3(f) = \emptyset$)
(c) $f(x, y) = (x, y^3, xy + y^5)$ (type H_2 in [41])
(d) $f(x, y) = (x, y^3, xy + y^{3k-1})$ (type H_k in [41]).

(2) In 1(a), check that $D^k(f)$ is smooth whenever non-empty.

(3) For 1(b),1(c) and 1(d), check that $D^2(f)$ has isolated singularity.

(4) For 1(c) and 1(d), check that $D^3(f)$ is zero-dimensional. What is $\dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}\times\mathbb{C}^3,0} / \mathcal{I}_3(f)$ in these two cases? Your answer should be divisible by 6.

(5) Suppose that $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$ has corank 1 and is finitely determined.
(a) Show that $\dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^{n-1}\times\mathbb{C}^{n+1},0} / \mathcal{I}_{n+1}(f)$ is divisible by $(n+1)!$, and
(b) use Theorem 2.20 to show that if f_t is a stable perturbation of f, then the image of f_t contains

$$\frac{1}{(n+1)!}\left(\dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^{n-1}\times\mathbb{C}^{n+1},0} / \mathcal{I}_{n+1}(f)\right)$$

ordinary $(n+1)$-tuple points (at each of which it is locally isomorphic to the union of the $n+1$ coordinate hyperplanes in $(\mathbb{C}^{n+1}, 0)$).

Theorem 7.4. ([31]) *Let $f : (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$, with $n < p$, have corank 1 .*

(a) *f is stable if and only if for each k with $\geq (k-1)p$, $D^k(f)$ is smooth of dimension $kn - (k-1)p$, and $D^k(f) = \emptyset$ if $kn < (k-1)p$.*
(b) *f is finitely determined if and only if for each k with $kn \geq (k-1)p$, $D^k(f)$ is an isolated complete intersection singularity of dimension $kn-(k-1)p$, and $D^k(f) = \{0\}$ or $\emptyset$ if $kn < (k-1)p$.*

As a result of the two parts of Theorem 7.4, it follows that when f_t is a stable perturbation of a finitely determined corank 1 germ f, then $D^k(f_t)$ is a smoothing, and therefore a Milnor fibre, of the ICIS $D^k(f)$.

Exercise 7.5. Find the Milnor numbers of $D^2(f)$ and $D^3(f)$ for the map germs of type S_k, H_2 and H_k in Exercise 7.3.

Now in (7.8) subtract the first row from each of the others. Omitting the first row in the resulting equation gives

$$\begin{pmatrix} h(x,y_2)-h(x,y_1) \\ \vdots \\ h(x,y_k)-h(x,y_1) \end{pmatrix} \tag{7.11}$$

$$= \begin{pmatrix} (y_2-y_1) & \cdots & (y_2^{k-1}-y_1^{k-1}) \\ \vdots & \vdots & \vdots \\ (y_k-y_1) & \cdots & (y_k^{k-1}-y_1^{k-1}) \end{pmatrix} \begin{pmatrix} \alpha_1^k \\ \vdots \\ \alpha_k^{k-1} \end{pmatrix}$$

The determinant of the new matrix of coefficients on the right is still $\mathrm{Vdm}(y_1,\ldots,y_k)$ (check this!) It follows that

$$I_k(h) \supseteq \big(h(x,y_2)-h(x,y_1),\ldots,h(x,y_k)-h(x,y_1)\big) \tag{7.12}$$

and

$$y_1,\ldots,y_k \text{ are pairwise distinct } \Longrightarrow \tag{7.13}$$

$$I_k(h) = \big(h(x,y_2)-h(x,y_1),\ldots,h(x,y_k)-h(x,y_1)\big).$$

By contrast, the restriction of $I_k(h)$ to the set $\{y_1=\cdots=y_k\}$ reduces to an ideal of partial derivatives.

For $k>\ell$ we define $D^k_\ell(f)$ to be the image in $D^k(f)$ of $D^\ell(f)$ under the composite $\pi_\ell^{\ell+1}\circ\cdots\circ\pi^k_{k-1}$. Then we have set-theoretic equalities

$$f^{(k)}(D^k(f)) = M_k(f), \quad f^{-1}M_k(f) = D^k_1(f)$$

for all $k\geq 1$.

7.2. Computing the homology of the image

Let $f : X \to Y$ be a finite map. For each $k\geq 2$ there are projections $D^k(f)\to D^{k-1}(f)$ defined by forgetting one of the copies of X. When f has corank 1 then each of the spaces $D^k(f)$ is an ICIS, and if f_t is a stable perturbation of f then $D^k(f_t)$ is a Milnor fibre of $D^k(f)$, and in particular smooth. Because $D^{k+1}(f)$ can be re-interpreted as the double-point space of the projection $D^k(f)\to D^{k-1}(f)$ (this is the "priniciple of iteration" discussed by Kleiman in [28] and attributed to Salomonsen), these projections are all themselves stable maps. Thus we obtain the rather rich structure of a "simplicial stable map". In any case, even when f has higher corank, these give rise to maps on the vanishing homology of the $D^k(f_t)$ when f_t is a perturbation of f; there is thus a rich structure of homology groups and homomorphisms associated to a perturbation. It turns out that from this one can obtain information about the homology of the image of the perturbation.

The symmetric group S_k acts on $D^k(f_t)$ by restriction of its action on X^k, permuting the copies of X. This action reflects the gluing which takes place when the domain of f_t is mapped to the image, and it is therefore no surprise that in the computation of the homology of the image, this action should play a rôle. In fact it is the *alternating* part of the homology which enters into the calculation of $H_*(\mathrm{image}(f_t))$. This was first observed in [19] at the level of rational homology. For any map $f : X \to Y$, we define

$$\mathrm{Alt}_k H_q(D^k(f);\mathbb{Q})$$
$$= \{[c] \in H_q(D^k(f);\mathbb{Q}) : \sigma_*([c]) = \mathrm{sign}(\sigma)[c] \text{ for all } \sigma \in S_k\},$$

and refer to it as the *alternating part* of $H_q(D^k(f);\mathbb{Q})$. Later the construction was greatly clarified by Goryunov in [20], by the introduction of the *alternating chain complex*. The description here differs from Goryunov's only in that it uses singular homology in place of cellular homology.

7.3. The alternating chain complex

Let D^k be any space on which the symmetric group S_k acts, and let $C_\ell(D^k)$ be the usual free abelian group of singular ℓ-chains in D^k. The symmetric group S_k acts on $C_\ell(D^k)$: if $\sigma \in S_k$ then $\sigma_\#(\sum_j n_j\Delta_j) = \sum_j n_j \sigma \circ \Delta_j$, where the Δ_j are singular ℓ-simplices in D^k. A chain $c \in C_\ell(D^k)$ is *alternating* if for each $\sigma \in S_k$, $\sigma_\#(c) = \mathrm{sign}(\sigma)c$. We denote the set of alternating ℓ-chains (with integer coefficients) on D^k by $C_\ell^{\mathrm{Alt}}(D^k)$. It is, evidently, a subgroup of $C_\ell(D^k)$, and therefore free abelian. The $C_\ell^{\mathrm{Alt}}(D^k)$ form a complex under the usual boundary map; we call its homology the *alternating homology of* D^k, and denote it by $H_*^{\mathrm{Alt}}(D^k)$.

Proposition 7.6.

$$H_*^{Alt}(D^k) \otimes_{\mathbb{Z}} \mathbb{Q} \simeq Alt_k H_q(D^k;\mathbb{Q}).$$

Proof. **Exercise** Q.E.D.

We will use this as a heuristic guide to later constructions. In particular, if $D^k = D^k(f_t)$, where f_t is a stable perturbation of a corank 1 map-germ $(\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$, then $D^k(f_t)$ is the Milnor fibre of an ICIS of dimension $p - k(p - n)$ provided $p - k(p - n) \geq 0$, and empty if $p - k(p - n) < 0$; thus $H_q(D^k(f);\mathbb{Q}) = 0$ unless $q = 0$ or $q = p - k(p - n)$. Now if $p - k(p - n) > 0$, $D^k(f)$ is connected and so S_k acts trivially on $H_0(D^k(f);\mathbb{Q})$, and it follows that $\mathrm{Alt}_k H_0(D^k(f);\mathbb{Q}) = 0$. Thus

Proposition 7.7. *If f_t is a stable perturbation of a corank 1 map-germ $(\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$, then*

$$Alt_k(H_q(D^k(f_t); \mathbb{Q}) = 0 \textit{ if } q \neq p - k(p-n).$$

In other words, *for all k $Alt_k H_*(D^k(f_t); \mathbb{Q})$ is concentrated in middle dimension.*

Let us return to the situation of a map $f : X \to Y$. For each $k \in \mathbb{N}$ let π^k be the projection $D^k(f) \to D^{k-1}(f)$ defined by

$$\pi^k(x_1, \ldots, x_k) = (x_1, \ldots, x_{k-1}).$$

Proposition 7.8. $\pi^k_\#(C_\ell^{Alt}(D^k(f)) \subset C_\ell^{Alt}(D^{k-1}(f))$.

Proof. Define an embedding $i : S_{k-1} \hookrightarrow S_k$ by

$$i(\sigma)(j) = \begin{cases} \sigma(j) & \text{if } j < k \\ k & \text{if } j = k \end{cases}$$

Then for $\sigma \in S_{k-1}$, as maps on $D^k(f)$, we have

$$\sigma \circ \pi^k = \pi^k \circ i(\sigma).$$

The sign of $i(\sigma)$ is the same as the sign of σ; it follows that if $c \in C_\ell^{\mathrm{Alt}}(D^k(f))$ then for any $\sigma \in S_{k-1}$,

$$\sigma_\#(\pi^k_\#(c)) = \pi^k_\# i(\sigma)_\#(c) = \pi^k_\#(\mathrm{sign}(i(\sigma))c) = \mathrm{sign}(\sigma)\pi^k_\#(c).$$

Thus $\pi^k_\#(c) \in C_\ell^{\mathrm{Alt}}(D^{k-1}(f))$. Q.E.D.

Proposition 7.9. *$\pi^{k-1}_\# \circ \pi^k_\# = 0$ on $C_\bullet^{Alt}(D^k(f))$, and $f_\# \pi^2_\# = 0$ on $C_\bullet^{Alt}(D^2(f))$.*

Proof. Let $\sigma \in S_k$ be the transposition $(k-1\ k)$. Clearly $\pi^{k-1} \circ \pi^k = \pi^{k-1} \circ \pi_k \circ \sigma$, and it follows that for $c \in C_\ell^{\mathrm{Alt}}(D^k(f))$,

$$(\pi^{k-1} \circ \pi^k)_\#(c) = (\pi^{k-1} \circ \pi_k)_\#(\sigma_\#(c))$$

$$= (\pi^{k-1} \circ \pi^k)_\#(-c) = -(\pi^{k-1} \circ \pi^k)_\#(c).$$

Since $C_\ell^{\mathrm{Alt}}(D^{k-2}(f)$ is free abelian, this proves that $(\pi^{k-1} \circ \pi^k)_\#(c) = 0$.

The second statement is proved by essentially the same argument. Q.E.D.

Suppose that $c_2 \in C_\ell^{\mathrm{Alt}}(D^2(f))$ is a cycle in $C_\bullet^{\mathrm{Alt}}(D^2(f))$. Then $\pi^2_\#(c_2)$ is also closed in $C_\bullet(X)$. *Now let us make the assumption that $H_\ell(X) = 0$.* This is certainly justified if X is the (contractible) domain of a stable perturbation of a corank 1 map-germ. The assumption also tallies with the evidence provided by Propositions 7.6 and 7.7 in the case of a stable perturbation of a corank 1 map-germ, for these suggest (though they do not prove) that if $H_\ell^{\mathrm{Alt}}(D^k(f_t)) \neq 0$ then $H_\ell^{\mathrm{Alt}}(D^{k-1}(f_t)) = 0$. We make it now in order to motivate a later more formal construction.

We will refer to this assumption as the *Vanishing Assumption*.

Under this assumption, since $\pi^2_\#(c_2)$ is a cycle, it must also be a boundary: there exists $c_1 \in C_{\ell+1}(X)$ such that $\partial c_1 = \pi^2_\#(c_2)$. Then $f_\#(c_1)$ is a cycle in the image of f, for $\partial f_\#(c_1) = f_\#(\partial c_1) = f_\# \pi^2_\#(c_2)$, and this is equal to 0 by 7.9.

Conclusion: From the alternating homology class $[c_2] \in H_\ell^{\mathrm{Alt}}(D^2(f))$, under the assumption that $H_\ell(X) = 0$, we have constructed a homology class $[f_\#(c_1)] \in H_{\ell+1}(Y)$.

Warning: We have not constructed a map from $H_\ell^{\mathrm{Alt}}(D^2(f))$ to $H_{\ell+1}(Y)$; there was an element of arbitrariness in the choice of c_1. In fact if c_1' is any other choice of $\ell+1$-chain on X such that $\partial c_1' = \pi^2_\#(c_2)$ then $c_1 - c_1'$ represents a homology class in $H_{\ell+1}(X)$, and thus the homology classes of $f_\#(c_1)$ and $f_\#(c_1')$ in $H_{\ell+1}$ differ by an element of $f_* H_{\ell+1}(X)$. Our construction in fact yields a map $H_\ell^{\mathrm{Alt}}(D^2) \to H_{\ell+1}(Y)/f_* H_{\ell+1}(X)$.

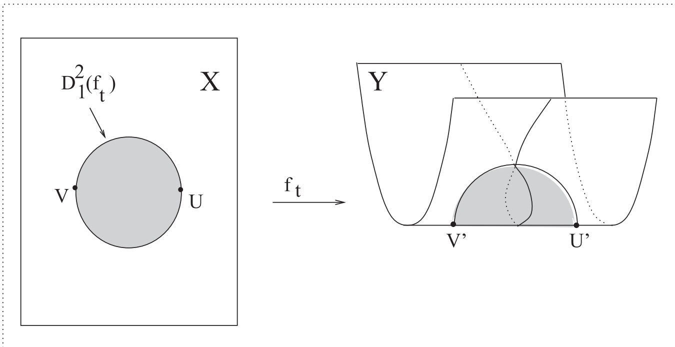
Example 7.10. We return to the map considered in Example 5.24. Here the domain X is contractible, so the imprecision in the choice of the cycle $f_\#(c_2)$ does not arise. We are interested in the stable perturbation $f_t : \mathbb{R}^2 \to \mathbb{R}^3$, defined by $f_t(x, y) = (x, y^2, y^3 + x^2 + ty)$, of the singularity $f = f_0$ of type S_1. We have

$$D^2(f_t) = \{(x, y_1, y_2) : y_1 + y_2 = 0 = x^2 + y_1^2 + y_1 y_2 + y_2^2 + t\};$$

The projection $\pi^2(x, y_1, y_2) = (x, y_1)$ (with inverse $(x, y) \mapsto (x, y, -y)$) maps this isomorphically to the conic

$$D_1^2(f_t) := \{(x, y) \in \mathbb{C}^2 : x^2 + y^2 + t = 0\},$$

with the involution $\sigma(x, y_1, y_2) = (x, y_2, y_1)$ now induced by $(x, y) \mapsto (x, -y)$.



The involution on $D^2(f_t)$ has two fixed points, U and V, where f_t is locally equivalent to the germ parameterising the cross-cap, studied in Example 4.6. Let a be a 1-simplex running from U to V on the upper arc of D^2, and let $b = \sigma \circ a$. Then the alternating homology $H_1^{\mathrm{Alt}}(D^2(f_t)$ is generated by $a - b$. Since here $D^2(f_t)$ is embedded in the domain X of f_t, we identify $a - b \in C_1^{\mathrm{Alt}}(D^2(f_t))$ with its image in $C_1(X)$.Taking as c_2 a suitable triangulation of the interior of the shaded disc, we have $\partial c_2 = a - b$. As can be seen in the picture, $f_\#(c_2)$ forms a bubble whose homology class generates $H_2(Y)$.

In fact this picture accurately represents the topology of the complexified map $\mathbb{C}^2 \to \mathbb{C}^3$. Here $D^2(f_t) \simeq D^2_1(f_t)$ is the complex Milnor fibre of an A_1 singularity, and is diffeomorphic to a cylinder. However its alternating homology is generated by the cycle shown in the real picture, and from there on the construction is the same.

Exercise 7.11. (1) Check that our map

$$H_\ell^{\mathrm{Alt}}(D^2(f)) \to H_{\ell+1}(Y)/f_* H_{\ell+1}(X)$$

is well-defined in the sense that if c_2 and c_2' represent the same alternating homology class in $H_\ell^{\mathrm{Alt}}(D^2(f))$ then the resulting homology classes are equal in $H_{\ell+1}(Y)/f_* H_{\ell+1}(X)$.
(2) Show that if we dispense with the Vanishing Assumption (that $H_\ell(X) = 0$), our construction yields a map

$$\ker\left[\pi^2_* : H_2^{\mathrm{Alt}}(D^2(f)) \to H_2(X)\right] \to H_{\ell+1}(Y).$$

(3) Under the Vanishing Assumption (to simplify notation) let F_ℓ be the image of $H_\ell^{\mathrm{Alt}}(D^2)$ in $H_{\ell+1}(Y)/f_* H_{\ell=1}(X)$, and let $\bar{F}_\ell$ be the preimage of F_ℓ in $H_{\ell+1}(Y)$. Show that if we assume also that $H_{\ell-1}^{\mathrm{Alt}}(D^2(f)) = 0$, the construction of the last two pages can be extended to give a map $H_{\ell-1}^{\mathrm{Alt}}(D^3(f)) \to H_{\ell+1}(Y)/\bar{F}_\ell$. The scheme of the argument is shown in

the following diagram, in which we begin with an alternating $(\ell-1)$-cycle a_3 on $D^3(f)$ and successively choose $a_2 \in C_\ell^{\mathrm{Alt}} D^2(f)$ and $a_1 \in C_{\ell+1}(X)$.

$$
\begin{tikzcd}
 & \ell-2 & \ell-1 & \ell & \ell+1 \\
D^3(f) \arrow[d, "\pi^3"'] & 0 & a_3 \arrow[l, mapsto, "\partial"'] \arrow[d, mapsto] & & \\
D^2(f) \arrow[d, "\pi^2"'] & 0 & \pi^3_\# a_3 \arrow[l, mapsto, "\partial"'] \arrow[d, mapsto] & a_2 \arrow[l, mapsto, "\partial"'] \arrow[d, mapsto] & \\
X \arrow[d, "f"'] & & \pi^2_\# \pi^3_\# a_3 = 0 & \pi^2_\# a_2 \arrow[l, mapsto, "\partial"'] \arrow[d, mapsto] & a_1 \arrow[l, mapsto, "\partial"'] \arrow[d, mapsto] \\
Y & & & f_\# \pi^2_\# a_2 = 0 & f_\# a_1 \arrow[l, mapsto, "\partial"']
\end{tikzcd}
\tag{7.14}
$$

(4) How can one adapt the construction if the Vanishing Assumptions are dropped?

7.4. The image computing spectral sequence

The rather complicated combinatorics of the previous constructions are all bundled up together in a spectral sequence which was first described in [19] and later developed and extended in [20], [21] and [22]. The main theorems of [20] on this topic are the following. We give the first in approximate form in order not to hide its statement in a technical fog.

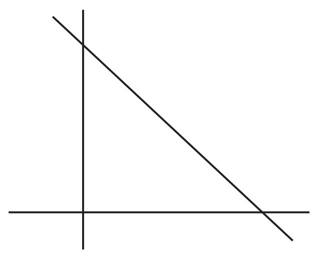
Theorem 7.12. *Let $f : X \to Y$ be a finite surjective map of topological spaces. Then there is a spectral sequence with $E^1_{pq} = H_p^{Alt}(D^q(f))$, converging to $H_{p+q-1}(Y)$.*

This means in particular that *all* of the homology of the image comes either from the homology of X, or from the alternating homology of the multiple point spaces.

Exercise 7.13. (1) Viewing $\mathbb{RP}^2$ as the image of the upper unit disc under the map which identifies opposite points on the boundary, find an alternating homology class in $H_0^{\mathrm{Alt}}(D^2(f))$ which gives rise to a generator of $H_1(\mathbb{RP}^2) \simeq \mathbb{Z}/2\mathbb{Z}$. Generalise this to $\mathbb{RP}^n$, taking care to distinguish between the case n even and n odd.

(2) Let X be the disjoint union of 3 real lines and let $f : X \to \mathbb{R}^2$ be the map

$$\begin{cases} u & \mapsto & (u,0) \\ v & \mapsto & (0,v) \\ w & \mapsto & (w,1-w) \end{cases}$$



(a) Where does the generator $[c]$ of the first homology of the image of f come from? In other words, find an alternating cycle in some $D^k(f)$ giving rise to $[c]$.
(b) Does complexifying f into a map from the disjoint union of three complex lines into $\mathbb{C}^2$ make any difference?

(3) Generalising the previous exercise, consider the map from the disjoint union of $n+2$ copies of $\mathbb{R}^n$ to $\mathbb{R}^{n+1}$, mapping the j'th copy of $\mathbb{R}^n$ to the coordinate plane $\{x_j = 0\}$ for $j = 1, \ldots, n+1$ and mapping the last copy of $\mathbb{R}^n$ by

$$(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n, (1 - \sum_i x_i)).$$

The image, Y, is the boundary of an $n+1$ simplex, and topologically a sphere. Where does the n- cycle generating $H_n(Y)$ come from?

Corollary 7.14. *Suppose that* $f_t : X_t \twoheadrightarrow Y_t$ *is a stable perturbation of a corank 1 map germ* $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+r}, 0)$.

(a) *If* $r \geq 2$, *then*

$$H_q(Y_t) = \begin{cases} H^{Alt}_{n-(k-1)r}(D^k(f_t)) & \text{if } q = n-(k-1)(r-1) \\ & \text{for some } k \\ \mathbb{Z} & \text{if } q = 0 \\ 0 & \text{otherwise} \end{cases}$$

(b) *If* $r = 1$, *then* $H_q(Y) = 0$ *if* $q \neq 0, n$, *and there is a filtration on* $H_n(Y_t)$ *such that the associated graded module is isomorphic to* *the direct sum*

$$\bigoplus_{k=2}^{n+1} H^{Alt}_{n-k+1}(D^k(f_t)).$$

If D^k is an S_k-invariant ICIS of dimension r with S_k-invariant Milnor fibre D^k_t, let us refer to the rank of $H^{\mathrm{Alt}}_r(D^k_t)$ as the *alternating Milnor number* of D^k. Then we have

Corollary 7.15. *In the situation of 7.14(2), the image Milnor number of f is the sum of the alternating Milnor numbers of the ICISs $D^k(f)$ for $k = 2, \ldots, n+1$.*

If $f : (\mathbb{C}^n, S) \to (\mathbb{C}^{n+r}, 0)$ is no longer assumed to have corank 1, then we know very little about its multiple point spaces $D^k(f)$ and those of a stable perturbation f_t. In particular, $D^k(f)$ is not in general an ICIS, and, if the dimensions $(n, n+r)$ are such that there may be corank 2 stable singularities of maps $\mathbb{C}^n \to \mathbb{C}^{n+r}$, then $D^k(f_t)$ is not in general a smoothing of $D^k(f)$. Nevertheless, Kevin Houston showed in [21] that the conclusion of Corollary 7.14 still holds. The main step in the proof is the following.

Theorem 7.16. *Let f_t be a stable perturbation of a finitely determined map-germ $(\mathbb{C}^n, 0) \to (\mathbb{C}^{n+r}, 0)$. Then $H^{Alt}_q(D^k(f_t)) = 0$ if $q \neq dim\, D^k(f_t)$.*

7.5. Open questions:

(1) Theorem 7.16 is proved by a rather complicated argument using equivariant stratified Morse theory. This remarkable theorem has not received the attention it deserves, in part because the published version (in [21]) is hard to read and suffers from some unfortunate typography. It would be a worthwhile project to write a clearer account.

Houston's heuristic motivation for the theorem is illuminating. The difficulty in describing $D^k(f)$ is entirely due to the need to remove the diagonals, by which $D^k(f)$ differs from the simple minded scheme $(X/Y)^k =$

$$= \{(x_1, \ldots, x_k) \in X^k : f(x_i) = f(x_j) \text{ for all } i, j\}.$$

Away from these diagonals, $(X/Y)^k$ is a complete intersection, defined in X^k by the $(k-1)p$ equations $f_k(x_1) = f_k(x_i)$ for $1 \leq k \leq p$ and $2 \leq i \leq k$. Indeed, if f is finitely determined, then $(X/Y)^k$ is non-singular away from the diagonals, since at all genuine k-tuple points, the corresponding multi-germ of f is stable. In the alternating chain complexes $C^{\mathrm{Alt}}_\bullet(f)$

and $C_\bullet^{\mathrm{Alt}}(D^k(f_t))$, the support of no chain can contain a simplex c lying entirely in $\{x_i = x_j\}$, since the transposition (i,j) leaves c fixed. It follows that for the alternating homology, $D^k(f)$ ought to behave like an ICIS, and new cycles should appear only in middle dimension.

(2) How can one compute the "alternating Milnor number" of $D^k(f)$ when f has corank > 1?

(3) How can one compute the image Milnor number of a map-germ $(\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$? An answer to (2), together with Corollary 7.15, would provide a method; beyond this, there is only the conjectural equality $\mu_I(f) = \dim_{\mathbb{C}} T^1_{\mathcal{K}_h} i$.

(4) How can we find equations for $D^3(f)$, and higher multiple point spaces, when f has corank greater than 1?

§8. Multiple points in the target

By the Preparation Theorem, if $f : (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$ is a finite map-germ then $\mathcal{O}_{\mathbb{C}^n,0}$ is a finite module over $\mathcal{O}_{\mathbb{C}^{n+1},0}$. A *presentation* of $\mathcal{O}_{\mathbb{C}^n,S}$ as $\mathcal{O}_{\mathbb{C}^{n+1},0}$ -module is an exact sequence

$$\mathcal{O}^p_{\mathbb{C}^{n+1},0} \xrightarrow{\lambda} \mathcal{O}^q_{\mathbb{C}^{n+1},0} \xrightarrow{\alpha} \mathcal{O}_{\mathbb{C}^n,S} \longrightarrow 0\ . \tag{8.1}$$

From a presentation one can learn a great deal about the geometry of the map f. Indeed in principle one can learn everything, since from the presentation one can obtain an equation for the image, and from this equation once can, in principle, determine the f itself, up to isomorphism, since it is the normalisation of its image. Other information, in the form of the Fitting Ideals, can be derived more immediately. We return to this after first developing an algorithm for finding a presentation.

Note that $\mathcal{O}_{\mathbb{C}^n,S} = \oplus_{x\in S}\,\mathcal{O}_{\mathbb{C}^n,x}$, and so if λ_x is the matrix of a presentation of $\mathcal{O}_{\mathbb{C}^n,x}$ over $\mathcal{O}_{\mathbb{C}^{n+1},0}$, then the block diagonal matrix $\oplus_{x\in S}\lambda_x$ presents $\mathcal{O}_{\mathbb{C}^n,S}$ over $\mathcal{O}_{\mathbb{C}^{n+1},0}$. So it is enough to develop a procedure to find each local presentation λ_x. In what follows we take $x = 0 \in \mathbb{C}^n$.

8.1. Procedure for finding a presentation:

Nakayama's Lemma and the Preparation Theorem tell us that if $g_1, \ldots, g_m \in \mathcal{O}_{\mathbb{C}^n,0}$ project to a $\mathbb{C}$-basis for the local algebra $Q(f)$, then $g_1, \ldots, g_m$ form a minimal set of generators for $\mathcal{O}_{\mathbb{C}^n,0}$ over $\mathcal{O}_{\mathbb{C}^{n+1},0}$. The structure of $\mathcal{O}_{\mathbb{C}^n,0}$ as $\mathcal{O}_{\mathbb{C}^{n+1},0}$-module is determined by the relations between these generators. The fact that the g_i generate $\mathcal{O}_{\mathbb{C}^n,0}$ over $\mathcal{O}_{\mathbb{C}^{n+1},0}$ is equivalent to the surjectivity of

$$\mathcal{O}^m_{\mathbb{C}^{n+1},0} \xrightarrow{\mathbf{g}} \mathcal{O}_{\mathbb{C}^n,0} \ ,$$

where $\mathbf{g}$ sends the i-th basis vector e_i to g_i. The module of relations between the g_i is the kernel of $\mathbf{g}$, and because $\mathcal{O}_{\mathbb{C}^{n+1},0}$ is Noetherian, it is finitely generated, say by r elements. Thus there is an $m \times r$ matrix λ over $\mathcal{O}_{\mathbb{C}^{n+1},0}$ such that

$$\mathcal{O}^r_{\mathbb{C}^{n+1},0} \xrightarrow{\lambda} \mathcal{O}^m_{\mathbb{C}^{n+1},0} \xrightarrow{\mathbf{g}} \mathcal{O}_{\mathbb{C}^n,0} \longrightarrow 0 \tag{8.2}$$

is exact (with λ sending the i-th basis vector of $\mathcal{O}^r_{\mathbb{C}^{n+1},0}$ to the i-th generator of $\ker \mathbf{g}$). Because the g_i form a minimal generating set for $\mathcal{O}_{\mathbb{C}^n,0}$, all entries in λ lie in the maximal ideal of $\mathcal{O}_{\mathbb{C}^{n+1},0}$. Thus (8.2) is the beginning of a minimal free resolution of $\mathcal{O}_{\mathbb{C}^n,0}$ over $\mathcal{O}_{\mathbb{C}^{n+1},0}$. The Auslander-Buchsbaum theorem (see e.g. [39, Theorem 19.1] or [14, Theorem 19.9]) tells us that if p is the length of such a free resolution (the *projective dimension* of $\mathcal{O}_{\mathbb{C}^n,0}$ over $\mathcal{O}_{\mathbb{C}^{n+1},0}$), then $p + \operatorname{depth}_{\mathcal{O}_{\mathbb{C}^{n+1},0}} \mathcal{O}_{\mathbb{C}^n,0} = \operatorname{depth}_{\mathcal{O}_{\mathbb{C}^{n+1},0}} \mathcal{O}_{\mathbb{C}^{n+1},0}$; it follows that $p = 1$. In other words, λ may be chosen injective. This forces r to be equal to m; for tensoring the exact sequence

$$0 \longrightarrow \mathcal{O}^r_{\mathbb{C}^{n+1},0} \xrightarrow{\lambda} \mathcal{O}^m_{\mathbb{C}^{n+1},0} \xrightarrow{\mathbf{g}} \mathcal{O}_{\mathbb{C}^n,0} \longrightarrow 0$$

with the field of fractions of $\mathcal{O}_{\mathbb{C}^{n+1},0}$ (the field $\mathcal{M} = \mathcal{M}_{\mathbb{C}^{n+1},0}$ of meromorphic functions), we retain exactness while killing $\mathcal{O}_{\mathbb{C}^n,0}$, and thus get an exact sequence

$$0 \longrightarrow \mathcal{M}^r \longrightarrow \mathcal{M}^m \longrightarrow 0 \ .$$

To find a matrix λ, one can use the following procedure:

(1) Choose a projection $\pi : \mathbb{C}^{n+1} \to \mathbb{C}^n$ such that $\pi \circ f$ is finite. A suitable projection always exists. In practice this usually means selecting n of the $n+1$ component functions of f, though in principle it may be that none of these coordinate projections is finite. *In what follows we will assume that coordinates are chosen so that* $\pi(y_1, \ldots, y_{n+1}) = (y_1, \ldots, y_n)$.

(2) Then $\mathcal{O}_{\mathbb{C}^n,0}$ (source) is free over $\mathcal{O}_{\mathbb{C}^n,0}$ (target); let $g_0, \ldots, g_d$ be a basis. Once again, by Nakayama's Lemma it is sufficient that the g_i form a $\mathbb{C}$-vector-space basis for $\mathcal{O}_{\mathbb{C}^n,0} / (\pi \circ f)^* \, \mathfrak{m}_{\mathbb{C}^n,0}$, which is finite dimensional by finiteness of $\pi \circ f$. *One of the g_i at least must be a unit in $\mathcal{O}_{\mathbb{C}^n,0}$; we take $g_0 = 1$.*

(3) Find $\lambda_j^i \in \mathcal{O}_{\mathbb{C}^{n+1},0}$ such that

$$
\begin{array}{rcccccccc}
f_{n+1} & = & \lambda_0^0 g_0 & + & \cdots & + & \lambda_0^m g_m \\
g_1 f_{n+1} & = & \lambda_1^0 g_0 & + & \cdots & + & \lambda_1^m g_m \\
\cdots & = & \cdots & \cdots & \cdots & \cdots & \cdots \\
g_m f_{n+1} & = & \lambda_m^0 g_0 & + & \cdots & + & \lambda_m^m g_m
\end{array}
\tag{8.3}
$$

Since $f_{n+1} = y_{n+1} \circ f$, (8.3) can be rewritten as

$$
\begin{array}{rcccccc}
0 & = & (\lambda_0^0 - y_{n+1}) g_0 & + & \cdots & + & \lambda_0^m g_m \\
0 & = & \lambda_1^0 g_0 & + & \cdots & + & \lambda_1^m g_m \\
\cdots & = & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & = & \lambda_m^0 g_0 & + & \cdots & + & (\lambda_m^m - y_{n+1}) g_m
\end{array}
\tag{8.4}
$$

Thus the columns of the matrix

$$
\begin{pmatrix}
\lambda_0^0 - y_{n+1} & \lambda_1^0 & \cdots & \lambda_m^0 \\
\lambda_0^1 & \lambda_1^1 - y_{n+1} & \cdots & \lambda_m^1 \\
\cdots & \cdots & \cdots & \cdots \\
\lambda_0^m & \lambda_1^m & \cdots & \lambda_m^m - y_{n+1}
\end{pmatrix}
\tag{8.5}
$$

are relations between the g_i.

Proposition 8.1. (8.5) *is the matrix of a presentation of* $\mathcal{O}_{\mathbb{C}^n,0}$ *over* $\mathcal{O}_{\mathbb{C}^{n+1},0}$. *In other words, the columns of* (8.5) *generate all the relations among the* g_i *over* $\mathcal{O}_{\mathbb{C}^{n+1},0}$.

Proof. A useful trick is described in [44, 2.2]: embed $\mathbb{C}^n$ as the hyperplane $\{t = 0\}$ in $\mathbb{C}^n \times \mathbb{C}$, and define $F : \mathbb{C}^n \times \mathbb{C} \to \mathbb{C}^{n+1}$ by

$$F(x, t) = (f_1(x), \ldots, f_n(x), f_{n+1}(x) - t).$$

Write S for $\mathbb{C}^n \times \mathbb{C}$ (source) and T for $\mathbb{C}^{n+1}$ (target). Then

$$\mathcal{O}_{S,0} / F^* \mathfrak{m}_{T,0} = \frac{\mathcal{O}_{S,0}}{(f_1, \ldots, f_n, f_{n+1} - t)} \simeq \frac{\mathcal{O}_{\mathbb{C}^n,0}}{(f_1, \ldots, f_n)}$$

so $g_0, \ldots, g_m$ form an $\mathcal{O}_{T,0}$-basis for $\mathcal{O}_{S,0}$, and thus determine an $\mathcal{O}_T$-isomorphism $\mathcal{O}_{T,0}^{m+1} \xrightarrow{\varphi} \mathcal{O}_{S,0}$. In the diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{O}_{S,0} \arrow[r, "t"] & \mathcal{O}_{S,0} \arrow[r] & \mathcal{O}_{\mathbb{C}^n,0} \arrow[r] & 0 \\ & \mathcal{O}_{T,0}^{m+1} \arrow[u, "\varphi"] \arrow[r, "{[t]_G^G}"] & \mathcal{O}_{T,0}^{m+1} \arrow[u, "\varphi"] & & \end{tikzcd} \tag{8.6}$$

$[t]_G^G$ is the matrix of the $\mathcal{O}_{T,0}$-linear map $\mathcal{O}_{S,0} \xrightarrow{t} \mathcal{O}_{S,0}$ (multiplication by t), with respect to the basis $g_0, \ldots, g_m$ of $\mathcal{O}_{S,0}$. We have

$$tg_i = (f_{n+1} - y_{n+1})g_i = \lambda_i^0 g_0 + \cdots + (\lambda_i^i - y_{n+1})g_i + \cdots + \lambda_i^m g_m$$

and thus $[t]_G^G$ is equal to the matrix (8.5). From the commutativity of (8.6) it follows that the cokernel of (8.5) is indeed isomorphic to $\mathcal{O}_{\mathbb{C}^n,0}$ as claimed. Q.E.D.

The presentation obtained above is not necessarily minimal, since in general

$$\dim_{\mathbb{C}} \frac{\mathcal{O}_{\mathbb{C}^n,0}}{f^* \mathfrak{m}_{\mathbb{C}^{n+1},0}} < \dim_{\mathbb{C}} \frac{\mathcal{O}_{\mathbb{C}^n,0}}{(\pi \circ f)^* \mathfrak{m}_{\mathbb{C}^n,0}}.$$

Nevertheless it is always injective, since the determinant of (8.5) is not zero – as can easily be seen, it is a monic polynomial of degree $m+1$ in $\mathbb{C}\{y_1, \ldots, y_n\}[y_{n+1}]$.

8.2. Fitting ideals

From the square matrix λ one can extract a great deal of information about the geometry of f.

Definition 8.2. Let $R^p \xrightarrow{\lambda} R^q \xrightarrow{g} M \longrightarrow 0$ be a presentation of the R-module M. The k'th *Fitting ideal of M as R-module*, $\mathrm{Fitt}_k^R(M)$, or simply $\mathrm{Fitt}_k(M)$ if it is clear which ring we are talking about, is the ideal generated by the $(q-k) \times (q-k)$ minors of λ, provided $p \geq q-k$, and is defined to be 0 if $p < q-k$ and R if $q-k \leq 0$.

Exercise 8.3. The Fitting ideals are independent of the choice of presentation of M. Prove this by showing

(1) If

$$R^a \xrightarrow{\alpha} R^q \xrightarrow{g} M \longrightarrow 0$$

and

$$R^b \xrightarrow{\beta} R^q \xrightarrow{g} M \longrightarrow 0$$

are presentations of the same module with respect to the same set of generators, then

$$\min_{q-k}(\alpha) = \min_{q-k}(\beta).$$

(2) If $R^s \xrightarrow{\mu} R^t \xrightarrow{h} M \longrightarrow 0$ is another presentation of the same module M, then $g+h : R^{q+t} \to M$ is surjective. For each basis vector e_i in R^t there exists $c_i \in R^q$ such that $g(c_i) = h(e_i)$, and thus $(c_i, -e_i) \in \ker(g+h)$. Show that the kernel of $g+h$ is generated by such pairs $(c_i, -e_i)$ together with pairs $(c, 0)$ with $c \in \ker g$, so that there is a presentation of the form

$$R^{p+t} \xrightarrow{\nu} R^{q+t} \xrightarrow{g+h} M \longrightarrow 0 \tag{8.7}$$

with

$$\nu = \begin{pmatrix} \lambda & -c \\ 0 & I_t \end{pmatrix}.$$

Clearly

$$\min_{q+t-k}(\nu) = \min_{q-k}(\lambda).$$

By symmetry, the kernel of $g+h$ is also generated by pairs $(0, d)$ with $d \in \ker h$ and pairs (e_j, d_j) where e_j is the j'th basis vector of R^p and $g(e_j) = -h(d_j)$. By 1, the ideals of $(q-k+t)$-minors are the same.

The Fitting ideals tell us a great deal about the geometry of f. We give two versions of this, first, one from algebraic geometry:

Proposition 8.4. $V(Fitt_k^R(M)) =$

$$\{x \in \mathit{Spec}\ R : M_p \text{ needs more than } k \text{ generators over } R\}.$$

In analytic geometry there are always two ways of looking at the same object. Let $\mathcal{S}$ be a coherent sheaf on the analytic space X. Define the ideal sheaf $\mathcal{F}_k(\mathcal{S})$ as the sheaf associated to the presheaf

$$U \mapsto \mathrm{Fitt}_k^{\Gamma(U,\mathcal{O}_X)}\Gamma(U, \mathcal{S});$$

Proposition 8.5. $V(\mathcal{F}_k(\mathcal{S})) =$

$$\{x \in X : \mathcal{S}_x \text{ needs more than } k \text{ generators over } \mathcal{O}_{X,x}\}.$$

Proof. From the presentation

$$\mathcal{O}_{X,x}^p \xrightarrow{\lambda} \mathcal{O}_{X,x}^q \longrightarrow \mathcal{S}_x \longrightarrow 0$$

tensoring with $\mathbb{C} = \mathcal{O}_{X,x} \,/\, \mathfrak{m}_{X,x}$ over $\mathcal{O}_{X,x}$ we obtain the exact sequence

$$\mathbb{C}^p \xrightarrow{\lambda(x)} \mathbb{C}^q \longrightarrow \mathcal{S}_x/\mathfrak{m}_{X,x}\,\mathcal{S}_x \longrightarrow 0\ ,$$

where $\lambda(x)$ is the $q \times p$ matrix over $\mathbb{C}$ obtained by evaluating λ at x. Now $\dim_{\mathbb{C}}\mathcal{S}_x/\mathfrak{m}_{X,x}\,\mathcal{S}_x$ is the minimum number of generators need by $\mathcal{S}_x$ as $\mathcal{O}_{X,x}$-module. If $x \in V(\mathrm{Fitt}_k(\mathcal{S}_x))$, then all $(q-k)\times(q-k)$ minors of $\lambda(x)$ vanish, and this means that the rank of $\lambda(x)$ is less than $q-k$, and, in turn, that $\dim_{\mathbb{C}}\mathcal{S}_x/\mathfrak{m}_{X,x}\,\mathcal{S}_x > k$. Q.E.D.

By coherence, we have

Proposition 8.6. $Fitt_k^{\mathcal{O}_{X,x}}(\mathcal{S}_x) = \big(\mathcal{F}_k(\mathcal{S})\big)_x$.

Corollary 8.7. *Let* $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$ *be finite and analytic. Then*

$$V(Fitt_k^{\mathcal{O}_{\mathbb{C}^{n+1},0}}(\mathcal{O}_{\mathbb{C}^n,0})) = \{y \in \mathbb{C}^{n+1} : \textstyle\sum_{x\in f^{-1}(y)} mult_x(f) > k\}$$
$$= \{y \in \mathbb{C}^{n+1} : y \textit{ has } \geq k+1 \textit{ preimages, counting multiplicity}\}.$$

In particular, $\det\lambda$ defines the image of f, and the ideal of submaximal minors of λ defines the set of double points.

Definition 8.8. *The k'th target multiple point space of* f, $M_k(f)$, *is the space* $V(Fitt_k^{\mathcal{O}_{\mathbb{C}^{n+1},0}}(\mathcal{O}_{\mathbb{C}^n,0}))$.

Example 8.9. (1) Let $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ be defined by

$$f(x,y) = (x, y^3, xy + y^5).$$

Take $\pi(Y_1,Y_2,Y_3) = (Y_1,Y_2)$; then $\mathcal{O}_{\mathbb{C}^2,0}$ (source) is generated over $\mathcal{O}_{\mathbb{C}^2,0}$ (target) by the classes of $1, y, y^2$. We have

$$\begin{array}{lclcrcrcr} f_3 & = & xy+y^5 & = & 0\cdot 1 & + & Y_1\cdot y & + & Y_2\cdot y^2 \\ g_1 f_3 & = & xy^2+y^6 & = & Y_2^2\cdot 1 & + & 0\cdot y & + & Y_1\cdot y^2 \\ g_2 f_3 & = & xy^3+y^7 & = & Y_1Y_2\cdot 1 & + & Y_2^2\cdot y & + & 0\cdot y^2 \end{array}$$

so as matrix of the presentation we obtain

$$\begin{pmatrix} -Y_3 & Y_2^2 & Y_1Y_2 \\ Y_1 & -Y_3 & Y_2^2 \\ Y_2 & Y_1 & -Y_3 \end{pmatrix}.$$

(2) Let $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ be defined by $f(x_1, x_2) = (x_1^2, x_2^2, x_1x_2)$, and as before take $\pi(Y_1, Y_2, Y_3) = (Y_1, Y_2)$. Then $\mathcal{O}_{\mathbb{C}^2,0}$ (source) is generated over $\mathcal{O}_{\mathbb{C}^2,0}$ (target) by $1, x_1, x_2, x_1x_2$. We have

$$\begin{array}{lcrcrcrcr}
f_3 & = & 0 \cdot 1 & + & 0 \cdot x_1 & + & 0 \cdot x_2 & + & 1 \cdot x_1x_2 \\
g_1 f_3 & = & 0 \cdot 1 & + & 0 \cdot x_1 & + & Y_1 \cdot x_2 & + & 0 \cdot x_1x_2 \\
g_2 f_3 & = & 0 \cdot 1 & + & Y_2 \cdot x_1 & + & 0 \cdot x_2 & + & 0 \cdot x_1x_2 \\
g_3 f_3 & = & Y_1Y_2 \cdot 1 & + & 0 \cdot x_1 & + & 0 \cdot x_2 & + & 0 \cdot x_1x_2
\end{array}$$

giving presentation matrix

$$\begin{pmatrix} -Y_3 & 0 & 0 & Y_1Y_2 \\ 0 & -Y_3 & Y_2 & 0 \\ 0 & Y_1 & -Y_3 & 0 \\ 1 & 0 & 0 & -Y_3 \end{pmatrix}.$$

Row and column operations transform this to

$$\begin{pmatrix} 0 & 0 & 0 & Y_3^2 - Y_1Y_2 \\ 0 & -Y_3 & Y_2 & 0 \\ 0 & Y_1 & -Y_3 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

This is now the matrix of a presentation with respect to different set of generators (**Exercise:** which?), of which one is, according to the first column, superfluous. Deleting it gives the minimal presentation

$$\begin{pmatrix} 0 & 0 & Y_3^2 - Y_1Y_2 \\ -Y_3 & Y_2 & 0 \\ Y_1 & -Y_3 & 0 \end{pmatrix}.$$

The determinant here is a square: this corresponds to the fact that f is a double covering of its image.

Exercise 8.10. Find a presentation for $\mathcal{O}_{\mathbb{C}^n,0}$ over $\mathcal{O}_{\mathbb{C}^{n+1},0}$ when

(a) $n = 1$ and $f(x) = (x^2, x^5)$;
(b) $n = 1$ and $f(x = (x^2, x^{2k+1})$;
(c) $n = 2$ and $f(x, y) = (x, y^2, yp(x, y^2))$;
(d) $n = 2$ and $f(x, y) = (x, y^3, xy + y^{3k-2})$;
(e) $n = 6$ and $f(a, b, c, d, x_1, x_2) =$
$(a, b, c, d, x_1^2 + ax_2, x_2^2 + bx_1, x_1x_2 + cx_1 + dx_2)$.

Exercise 8.11. Show that if $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$ is finite and generically k-to-1 onto its image, and if λ is the matrix of a presentation of $\mathcal{O}_{\mathbb{C}^n,0}$ over $\mathcal{O}_{\mathbb{C}^{n+1},0}$, then $\det \lambda$ is the k'th power of a reduced equation for the image.

Proposition 8.12. ([44]) *Let* $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$ *be finite and generically 1-1, and let* λ *be the* $(m+1)\times(m+1)$ *matrix of a presentation of* $\mathcal{O}_{\mathbb{C}^n,0}$ *over* $\mathcal{O}_{\mathbb{C}^{n+1},0}$*, with respect to generators* $g_0 = 1, g_1, \ldots, g_m$*. Then the ideal* $Fitt_1(\mathcal{O}_{\mathbb{C}^n,0})$ *is generated by the* $m \times m$ *minors of the matrix* λ' *obtained from* λ *by deleting its first row.*

Proof. The sub-$\mathcal{O}_{\mathbb{C}^{n+1},0}$-module of $\mathcal{O}_{\mathbb{C}^n,0}$ generated by 1 can be identified with $\mathcal{O}_{D,0}$, where D is the image of f. Now λ' is a presentation of the $\mathcal{O}_{\mathbb{C}^{n+1},0}$-module $\mathcal{O}_{\mathbb{C}^n,0} \,/\, \mathcal{O}_{D,0}$, and the ideal generated by the $m \times m$ minors of λ' is the 0-th Fitting ideal of this module. A theorem of Buchsbaum and Eisenbud ([3]) asserts that provided the codimension of the support of $\mathcal{O}_{\mathbb{C}^n,0} \,/\, \mathcal{O}_{D,0}$ is at least 2 (in fact its greatest possible value), then

$$\mathrm{Fitt}_0^{\mathcal{O}_{\mathbb{C}^{n+1},0}}\big(\mathcal{O}_{\mathbb{C}^n,0} \,/\, \mathcal{O}_{D,0}\big) = \mathrm{Ann}_{\mathcal{O}_{\mathbb{C}^{n+1},0}}\big(\mathcal{O}_{\mathbb{C}^n,0} \,/\, \mathcal{O}_{D,0}\big).$$

The proof is completed by showing that because $\mathcal{O}_{\mathbb{C}^n,0}$ is a *ring*, all $m \times m$ minors of λ lie in $\mathrm{Ann}_{\mathcal{O}_{\mathbb{C}^{n+1},0}}\big(\mathcal{O}_{\mathbb{C}^n,0} \,/\, \mathcal{O}_{D,0}\big)$. I leave the details as a guided exercise. Q.E.D.

Exercise 8.13. Let m^i_j be the $m \times m$ minor determinant of λ obtained by omitting row i and column j.

(a) Use Cramer's rule to show that for all i, j, k,

$$m^i_j g_k = m^k_j g_i \tag{8.8}$$

and in particular

$$m^i_j = m^0_j g_i. \tag{8.9}$$

(b) Because $g_i g_j$ lies in $\mathcal{O}_{\mathbb{C}^n,0}$ and $\mathcal{O}_{\mathbb{C}^n,0}$ is generated over $\mathcal{O}_{\mathbb{C}^{n+1},0}$ by the g_k, there exist $\Gamma^k_{ij} \in \mathcal{O}_{\mathbb{C}^{n+1},0}$ such that $g_i g_j = \sum_k \Gamma^k_{ij} g_k$, with $\Gamma^k_{ij} \in \mathcal{O}_{\mathbb{C}^{n+1},0}$. Use (a) to show that

$$m^i_j g_k = \sum_\ell \Gamma^\ell_{ik} m^\ell_j.$$

The details of the proof of 8.12 can be found in [44, Theorem 3.4].

Exercise 8.14. *(1) Find equations for the double-point locus,* C*, of the image of the map-germ* f *of type* H_2*, given by* $f(x, y) = (x, y^3, xy + y^5)$.

(2) Show that C *is the image of the map* $t \mapsto (t^4, t^3, t^5)$.

(3) Check that $f^*(Fitt_1^{\mathcal{O}_{\mathbb{C}^{n+1},0}}(\mathcal{O}_{\mathbb{C}^n,0}))$ *is a principal ideal in* $\mathcal{O}_{\mathbb{C}^n,0}$.

(4) Find the pre-image in $\mathbb{C}^2$ *of* C*, and show that it has a singularity of type* A_6 *at* 0.

(5) Show that the set of real points on this curve is just 0.

(6) Can you reconcile the conclusions of (2) and (5)?

The argument in the proof of 8.1 serves to prove another result:

Proposition 8.15. *([6], [44]) The matrix* λ *of a presentation of* $\mathcal{O}_{\mathbb{C}^n,0}$ *over* $\mathcal{O}_{\mathbb{C}^{n+1},0}$ *can be chosen symmetric.*

Proof. We replace the diagram (8.6) by a second diagram in which the two isomorphisms of $\mathcal{O}_{\mathbb{C}^n\times\mathbb{C},0}$ (source) with $\mathcal{O}_{\mathbb{C}^{n+1},0}$ (target) are no longer assumed to be the same. Write $\mathcal{O}_{S,0} := \mathcal{O}_{\mathbb{C}^n\times\mathbb{C},0}$ (source), and $\mathcal{O}_{T,0} := \mathcal{O}_{\mathbb{C}^{n+1},0}$ (target). Because $\mathcal{O}_{S,0}$ is a Gorenstein ring, and is finite over $\mathcal{O}_{T,0}$ (target), there is a perfect symmetric $\mathcal{O}_{T,0}$-bilinear pairing $(\cdot,\cdot) : \mathcal{O}_{S,0}\times\mathcal{O}_{S,0} \to \mathcal{O}_{T,0}$. This is a consequence of local duality. In [49], Scheja and Storch show that as $\mathcal{O}_S$-module, $\mathrm{Hom}_{\mathcal{O}_{T,0}}(\mathcal{O}_{S,0},\mathcal{O}_{T,0})$ is cyclic. Picking an $\mathcal{O}_{S,0}$-generator Φ, and setting

$$(s_1, s_2) = \Phi(s_1 s_2)$$

gives a perfect pairing, and for each $\mathcal{O}_T$-basis $G := g_0, \ldots, g_m$ for $\mathcal{O}_{S,0}$ there is therefore a dual basis $\check{G} := \check{g}_0, \ldots, \check{g}_m$ with the property that $(\check{g}_i, g_j) = \delta_{ij}$. Let $\check{\varphi}$ be the $\mathcal{O}_{T,0}$ isomorphism $\mathcal{O}_{T,0}^{m+1} \to \mathcal{O}_{S,0}$ determined by the basis $\check{G}$. Then the matrix $\big[t\big]_G^{\check{G}}$ is symmetric (**Exercise**), and, by the argument of the proof of 8.1, is the matrix of a presentation of $\mathcal{O}_{\mathbb{C}^n,0}$ over $\mathcal{O}_{\mathbb{C}^{n+1},0}$. Q.E.D.

Corollary 8.16. *Let* $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$ *be finite and generically 1-1. Then* $f^*\mathit{Fitt}_1(\mathcal{O}_{\mathbb{C}^n,0})$ *is a principal ideal.*

Proof. Choose a symmetric presentation λ, with respect to generators $g_0 = 1, \ldots, g_m$. Then in the language of the proof of 8.12, $\mathrm{Fitt}_1(\mathcal{O}_{\mathbb{C}^n,0})$ is generated by $(m_0^0, \ldots, m_m^0)$, and so $f^*\mathrm{Fitt}_1(\mathcal{O}_{\mathbb{C}^n,0})$ is generated by $f^*(m_0^0), \ldots, f^*(m_m^0)$. It follows by (8.9) and the symmetry of λ that $f^*\mathrm{Fitt}_1(\mathcal{O}_{\mathbb{C}^n,0})$ is generated by $f^*(m_0^0)$. Q.E.D.

Because $\mathrm{Fitt}_1^{\mathcal{O}_{\mathbb{C}^{n+1},0}}(\mathcal{O}_{\mathbb{C}^n,0}) = \mathrm{Ann}_{\mathcal{O}_{\mathbb{C}^{n+1},0}}(\mathcal{O}_{\mathbb{C}^n,0}\,/\,\mathcal{O}_{D,0})$, the ideal $\mathrm{Fitt}_1^{\mathcal{O}_{\mathbb{C}^{n+1},0}}(\mathcal{O}_{\mathbb{C}^n,0})\,\mathcal{O}_{D,0}$ is known as the *conductor* ideal of the ring homomorphism $\mathcal{O}_{D,0} \to \mathcal{O}_{\mathbb{C}^n,0}$. We denote it by $\mathcal{C}$. In fact $\mathcal{C}$ is also an ideal in $\mathcal{O}_{\mathbb{C}^n,0}$; it is the largest ideal of $\mathcal{O}_{D,0}$ which is also an ideal in $\mathcal{O}_{\mathbb{C}^n,0}$. The last corollary shows that as an ideal in $\mathcal{O}_{\mathbb{C}^n,0}$, $\mathcal{C}$ is principal. One can find a generator by picking a symmetric presentation λ, but there is an easier method, due, with a rather sophisticated proof, to Ragni Piene ([47]), and, with a simpler proof, to Bill Bruce and Ton Marar ([2]):

Theorem 8.17. ([2]) *Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$ be finite and generically 1-1. Let h be a reduced equation for its image, and let*

$$r_i := \frac{\partial(f_1, \ldots, \widehat{f_i}, \ldots, f_{n+1})}{\partial(x_1, \ldots, x_n)}$$

be the minor determinant of the matrix of the derivative df obtained by omitting row i. Then $(\partial h/\partial Y_i) \circ f$ is divisible by r_i in $\mathcal{O}_{\mathbb{C}^n,0}$, and the quotient generates the conductor ideal $\mathcal{C}$.

Exercise 8.18. (1) Show that the quotient r_i in 8.17 is independent of i.
(2) Find a generator for the conductor when f is the map of Exercise 7.3(a).
(3) Show that in this case $D^2_1(f)$ is isomorphic to the product $\mathbb{C} \times D_2$, where D_2 is the image of the stable map of Example 4.6. This has an explanation! What is it?

In a similar vein to 8.12, ([44, Theorem 4.1]) shows:

Theorem 8.19. *Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$ be finite and generically 1-1, and let λ be a symmetric presentation of $\mathcal{O}_{\mathbb{C}^n,0}$ over $\mathcal{O}_{\mathbb{C}^{n+1},0}$, with respect to generators $g_0 = 1, g_1, \ldots, g_m$. Then $Fitt_2(\mathcal{O}_{\mathbb{C}^n,0})$ is generated by the $(m-1) \times (m-1)$ minors of the matrix obtained from λ by deleting its first row and column.*

The variety of zeros of the ideal of submaximal minors of an $m \times m$ matrix can have codimension no greater than 3, and if the codimension is 3 then the variety in question is Cohen Macaulay, by Theorem 2.34 and a theorem of Jozefiak ([25]). Thus

Corollary 8.20. *Suppose, in 8.19, that $V(Fitt_2(\mathcal{O}_{\mathbb{C}^n,0}))$ has codimension 2. Then it is Cohen-Macaulay.*

By conservation of multiplicity (see Subsection 2.4) we obtain

Corollary 8.21. *If $n = 2$, and f satisfies the hypotheses of 8.20, then the number of triple points in the image of a stable perturbation of f is equal to $\dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^3,0} / Fitt_2(\mathcal{O}_{\mathbb{C}^2,0})$.*

8.3. Open questions

(1) Do the Fitting ideals give a reasonable analytic structure to the multiple point spaces? And are these spaces well-behaved in the case of finitely determined map-germs $(\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$? How do

they behave under deformation? In particular, if F is an unfolding of f on parameter space S, then is $M_k(F)$ Cohen Macaulay (and therefore flat over S)? Some partial answers are known, see [44],[27], [26], but for maps of corank greater than 1, nothing is known about the behaviour of $\mathrm{Fitt}_k^{\mathcal{O}_{\mathbb{C}^{n+1},0}}(\mathcal{O}_{\mathbb{C}^n,0})$ under deformation when $k > 2$. Recent improvements in computing power make more calculations possible, and new examples might clarify these questions. In particular, does a version of 8.21 hold for higher Fitting ideals? For example, is it true that if $f : (\mathbb{C}^3, 0) \to (\mathbb{C}^4, 0)$ is finite and generically 1-1, and $\operatorname{codim}(V(\mathrm{Fitt}_3(\mathcal{O}_{\mathbb{C}^3,0})) = 4$, then the number of quadruple points in the image of a stable perturbation of f is equal to $\dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^4,0}/\mathrm{Fitt}_3(\mathcal{O}_{\mathbb{C}^3,0}))$?

(2) One of the most famous open problems is the Lê Conjecture that if $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ has corank 2 then it cannot be injective. Do the Fitting ideals give any handle on this question? It seems not, since they do not distinguish between genuine double points, with two distinct preimages, and points with a non-immersive preimage. If there were some way of incorporating the involution on $D^2(f)$ into the picture, it might be possible to make some progress on this surprisingly intractable problem.

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