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Deformations of product-quotient surfaces and reconstruction of Todorov surfaces via $\mathbb{Q}$-Gorenstein smoothing

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Abstract.

We consider the deformation spaces of some singular product-quotient surfaces $X = (C_1 \times C_2)/G$, where the curves C_i have genus 3 and the group G is isomorphic to $\mathbb{Z}_4$. As a by-product, we give a new construction of Todorov surfaces with $p_g = 1$, $q = 0$ and $2 \leq K^2 \leq 8$ by using $\mathbb{Q}$-Gorenstein smoothings.

§0. Introduction

In [To81], Todorov constructed some surfaces of general type with $p_g = 1$, $q = 0$ and $2 \leq K^2 \leq 8$ in order to give counterexamples of the global Torelli theorem. Todorov surfaces with $K^2 = 8 - k$ are double covers of a Kummer surface in $\mathbb{P}^3$ branched over a curve D, which is a complete intersection of the Kummer surface with a smooth quadric surface containing k of its nodes, and over the remaining $16 - k$ nodes. Surfaces with $K^2 = 2$, and $p_g = 1$ have been completely classified by Catanese and Debarre [CD89], while some examples were constructed by Todorov. C. Rito [Rito09] gave a detailed study of Todorov surfaces with an involution.

Recently, H. Park, J. Park and D. Shin constructed simply connected surfaces of general type with $p_g = 1$, $q = 0$ and $2 \leq K^2 \leq 8$ by considering $\mathbb{Q}$-Gorenstein smoothings of singular K3 surfaces with special configurations of cyclic quotient singularities, see [PPS1], [PPS2]. Their construction follows the method used by Lee and Park in the

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paper [LP07], where a simply connected surface of general type with $p_g = q = 0$ and $K^2 = 2$ is constructed via the $\mathbb{Q}$-Gorenstein smoothing of a singular rational surface. For more details about these kind of techniques, over a field of any characteristic, we refer the reader to the work of Lee and Nakayama [LN11].

Moreover, Bauer, Catanese, Grunewald and Pignatelli constructed many interesting examples of surfaces of general type with $p_g = 0$ by considering the minimal desingularization of singular product-quotient surfaces, see [BC04], [BCG08], [BCGP], [BP]. Similar methods are applied to surfaces of general type with $p_g = q = 1$ by Polizzi and others, see [Pol08], [Pol09], [CP09], [MP10]. These results motivated us to start the investigation of $\mathbb{Q}$-Gorenstein smoothings of singular product-quotient surfaces.

Let us recall that a projective surface S is called a *product-quotient surface* if there exists a finite group G, acting faithfully on two smooth curves C_1 and C_2 and diagonally on their product, so that S is isomorphic to the minimal desingularization of $X = (C_1 \times C_2)/G$. The surface X is called a *singular model of a product-quotient surface*, or simply a *singular product-quotient surface.*

This paper focuses on the case $g(C_1) = g(C_2) = 3$ and $G = \mathbb{Z}_4$. More precisely, we assume that there exist two simple $\mathbb{Z}_4$-covers $g_i \colon C_i \to \mathbb{P}^1$, both branched in four points. Then the singular product-quotient surface

$$X := (C_1 \times C_2)/\mathbb{Z}_4$$

contains precisely 16 cyclic quotient singularities; any of them is either of type $\frac{1}{4}(1, 1)$ or of type $\frac{1}{4}(1, 3)$. Note that $\frac{1}{4}(1, 3)$ is a rational double point, whereas $\frac{1}{4}(1, 1)$ is a singularity of class T, so both admit a *local* $\mathbb{Q}$-Gorenstein smoothing, see [KSB88] or [Man08, Sections 2–4]. The problem is to understand whether these local smoothings can be glued together in order to have a *global* $\mathbb{Q}$-Gorenstein smoothing of X. We will show that in some cases this is actually possible.

This paper is organized as follows.

In Section 1 we present some preliminaries and we set up notation and terminology. In particular, we recall the definitions of simple cyclic cover of a curve and of singular product-quotient surface and we explain how to compute their basic invariants.

In Section 2 we introduce the main objects that we want to study, namely the singular product quotient surfaces of the form $X = (C_1 \times C_2)/G$, where $g(C_1) = g(C_2) = 3$, $G = \mathbb{Z}_4$ and $C_i \to C_i/G$ is a simple cyclic cover for $i = 1, 2$.

Section 3 deals with the study of the singular product-quotient surface $Y = (C_1 \times C_2)/H$, where H is the unique subgroup of G isomorphic to $\mathbb{Z}_2$. By construction, Y contains exactly 16 ordinary double points as singularities. By using the infinitesimal techniques introduced in [Pin81] and [Cat89], we prove that Def(Y) is smooth at Y, of dimension 18 and ESDef(Y) is smooth at [Y], of dimension 8 (Proposition 3.6). Moreover, if $\mu\colon V \to Y$ is the minimal desingularization of Y, we have

$$\dim_{[V]} \mathrm{Def}(V) = 18, \quad h^1(\Theta_V) = 24,$$

hence Def(V) is singular at [V]; by [BW74] this implies that the sixteen (-2) curves of V do not have independent behavior in deformations.

In Section 4 we discuss three examples of singular product-quotient surface $X = (C_1 \times C_2)/G$ with different G-action.

- In the first example we have $\mathrm{Sing}(X) = 16 \times \frac{1}{4}(1, 3)$, so X contains only rational double points as singularities. We prove that Def(X) and ESDef(X) are both smooth at [X], of dimension 44 and 2, respectively (Propositions 4.4 and 4.2).

  The surface X satisfies $h^0(\omega_X) = 5$ and $K_X^2 = 8$; moreover it is not difficult to see that the canonical map $\phi_K\colon X \to \mathbb{P}^4$ is a birational morphism onto its image; by [Cat97, Proposition 6.2] it follows that the general deformation of X is isomorphic to a smooth complete intersection of bidegree (2, 4) in $\mathbb{P}^4$.

  Moreover we have

$$\dim_{[S]} \mathrm{Def}(S) = 44, \quad h^1(\Theta_S) = 50,$$

  hence Def(S) is singular at S. This means that the sixteen A_3-cycles of S do not have independent behavior in deformations.
- In the second example we have $\mathrm{Sing}(X) = 16 \times \frac{1}{4}(1, 1)$. We show that there exist a $\mathbb{Q}$-Gorenstein smoothing $\pi\colon \mathcal{X} \to T$ of X, whose base T has dimension 12, such that the general fibre X_t of π is a minimal surface of general type whose invariants are

$$p_g(X_t) = 1, \quad q(X_t) = 0, \quad K_{X_t}^2 = 8.$$

  Moreover X_t is isomorphic to a Todorov surface with $K^2 = 8$ (Theorem 4.6). By a slight modification of the construction, it is possible to obtain all Todorov surfaces with $2 \leq K^2 \leq 8$.

  This is related to the existence of complex structures on rational blow-downs of algebraic surfaces. More precisely, one can consider the rational blow-down $S(t)$ of t of the (-4)-curves in S, where $1 \leq t \leq 16$. This means that one considers the normal connected sum of S with t copies of $\mathbb{P}^2$, identifying a conic in each $\mathbb{P}^2$ with a (-4)-curve in S; then $S(t)$ is a symplectic 4-manifold. On can therefore raise the following:

Question. Is it possible to give a complex structure on $S(t)$ for $1 \leq t \leq 16$, and to describe $S(t)$ when such a complex structure exists?

Our results answer affirmatively this question when $10 \leq t \leq 16$; in these cases, indeed, one can give a complex structure to the rational blow-down $S(t)$, which make it isomorphic to a Todorov surface with $K^2 = t - 8$.

- In the third example, we have $\mathrm{Sing}(X) = 8 \times \frac{1}{4}(1, 1) + 8 \times \frac{1}{4}(1, 3)$. Rasdeaconu and Suvaina give an explicit construction of the minimal desingularization S of X, see [RS06, Section 3]; in fact, they prove that S is a simply connected, minimal elliptic surface with no multiple fibres.

  We show that there exists a $\mathbb{Q}$-Gorenstein smoothing of X, although $H^2(\Theta_X) \neq 0$ and all the natural deformations of the G-cover $u\colon X \to Q$ preserve the 8 singularities of type $\frac{1}{4}(1, 1)$, see Proposition 4.8. Indeed we prove that a general surface $\bar{X}$ in the subfamily of natural deformations of the G-cover of X can be deformed to a bidouble cover of $\mathbb{P}^1 \times \mathbb{P}^1$ branched over three smooth divisors of bidegree $(2, 2)$. By taking a general deformation of these three divisors we obtain a $\mathbb{Q}$-Gorenstein smoothing of X which smoothes all the singularities. More generally, by using the same method one can construct surfaces of general type with $p_g = 3$, $q = 0$ and $K^2 = k$ $(2 \leq k \leq 8)$ by first taking a $\mathbb{Q}$-Gorenstein smoothing of k singular points of type $\frac{1}{4}(1, 1)$ of $\bar{X}$ and then the minimal resolution of the remaining $8 - k$ singular points of the same type.

Notation and conventions.

We work over the field $\mathbb{C}$ of complex numbers.

By "surface" we mean a projective, non-singular surface S, and for such a surface $\omega_S = \mathcal{O}_S(K_S)$ denotes the canonical class, $p_g(S) = h^0(S, \omega_S)$ is the *geometric genus*, $q(S) = h^1(S, \omega_S)$ is the *irregularity* and $\chi(\mathcal{O}_S) = 1 - q(S) + p_g(S)$ is the *Euler–Poincaré characteristic.*

If X is any (possibly singular) projective scheme, we denote by $\mathrm{Def}(X)$ the base of the Kuranishi family of deformations of X and by $\mathrm{ESDef}(X)$ the base of the equisingular deformations of X. The tangent spaces to $\mathrm{Def}(X)$ and $\mathrm{ESDef}(X)$ at the point $[X]$ corresponding to X are given by $\mathrm{Ext}^1(\Omega^1_Y, \mathcal{O}_Y)$ and $H^1(\Theta_Y)$, respectively.

If L is a line bundle L on X, we use the notation L^n instead of $L^{\otimes n}$ if no confusion can arise.

If G is any finite abelian group, we denote by $\widehat{G}$ its dual group, namely the group of irreducible characters of G.

§1. Preliminaries

1.1. Simple cyclic covers of curves

Let Γ be a smooth, projective curve and $B \subset \Gamma$ an effective divisor such that $\mathcal{O}_\Gamma(B) = \mathcal{L}^n$ for some $\mathcal{L} \in \mathrm{Pic}(\Gamma)$. Therefore there exists a $\mathbb{Z}_n$-cover $g\colon C \to \Gamma$, totally branched over B, which is called a *simple cyclic cover.* We identify $\mathbb{Z}_n$ with the group of n-th roots of unity, namely $\mathbb{Z}_n = \langle \zeta \rangle$, where ζ is a primitive n-th root. The dual group $\widehat{\mathbb{Z}}_n$ is isomorphic to $\mathbb{Z}_n$, and it is generated by the character $\chi_1\colon \mathbb{Z}_n \to \mathbb{C}$ such that $\chi_1(\zeta) = \zeta^{-1}$. We will write χ_j instead of χ_1^j; then $\chi_j(\zeta) = \zeta^{-j}$. The group $\mathbb{Z}_n$ acts naturally on $g_*\mathcal{O}_C$, so there is a canonical splitting

$$g_*\mathcal{O}_C = \mathcal{O}_\Gamma \oplus \mathcal{L}^{-1} \oplus \ldots \oplus \mathcal{L}^{-(n-1)}, \tag{1}$$

where the summand $\mathcal{L}^{-j}$ is the eigensheaf $(g_*\mathcal{O}_C)^{\chi_j}$ corresponding to the character χ_j.

Similarly, $\mathbb{Z}_n$ acts naturally on $g_*\omega_C$ and $g_*\omega_C^2$, giving the following decompositions (see [Pa91] and [Cat89, Section 2]):

$$\begin{aligned} g_*\omega_C &= \omega_\Gamma \oplus (\omega_\Gamma \otimes \mathcal{L}) \oplus \ldots \oplus (\omega_\Gamma \otimes \mathcal{L}^{n-1}), \\ g_*\omega_C^2 &= (\omega_\Gamma^2(B) \otimes \mathcal{L}^{-1}) \oplus \omega_\Gamma^2(B) \oplus \ldots \oplus (\omega_\Gamma^2(B) \otimes \mathcal{L}^{n-2}). \end{aligned} \tag{2}$$

In the equations (2), the eigensheaves corresponding to χ_j are $\omega_\Gamma \otimes \mathcal{L}^j$ and $\omega_\Gamma^2(B) \otimes \mathcal{L}^j$, respectively.

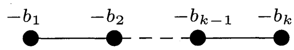
1.2. Cyclic quotient singularities, Hirzebruch Jung resolutions and singular product-quotient surfaces

Let n and q be natural numbers with $0 < q < n$, $(n, q) = 1$ and let ζ be a primitive n-th root of unity. Let us consider the action of the cyclic group $\mathbb{Z}_n = \langle \zeta \rangle$ on $\mathbb{C}^2$ defined by $\zeta \cdot (x,\, y) = (\zeta x,\, \zeta^q y)$. Then the analytic space $X_{n,q} = \mathbb{C}^2/\mathbb{Z}_n$ has a cyclic quotient singularity of type $\frac{1}{n}(1, q)$, and $X_{n,q} \cong X_{n',q'}$ if and only if $n = n'$ and either $q = q'$ or $qq' \equiv 1 \pmod{n}$. The exceptional divisor on the minimal resolution $\widetilde{X}_{n,q}$ of $X_{n,q}$ is a Hirzebruch–Jung string, that is to say, a connected union $E = \bigcup_{i=1}^k Z_i$ of smooth rational curves $Z_1, \ldots, Z_k$ with self-intersection ≤ -2, and ordered linearly so that $Z_i Z_{i+1} = 1$ for all i, and $Z_i Z_j = 0$ if

$|i-j| \geq 2$. More precisely, given the continued fraction

$$\frac{n}{q} = [b_1, \ldots, b_k] = b_1 - \cfrac{1}{b_2 - \cfrac{1}{\cdots - \cfrac{1}{b_k}}}, \quad b_i \geq 2,$$

the dual graph of E is



(cf. [Lau71, Chapter II]). Notice that a rational double point of type A_n corresponds to the cyclic quotient singularity $\frac{1}{n+1}(1,n)$.

Definition 1.1. *Let x be a cyclic quotient singularity of type $\frac{1}{n}(1,q)$. Then we set*

$$\begin{aligned} \mathfrak{h}_x &= 2 - \frac{2+q+q'}{n} - \sum_{i=1}^{k}(b_i - 2), \\ \mathfrak{e}_x &= k + 1 - \frac{1}{n}, \\ B_x &= 2\mathfrak{e}_x - \mathfrak{h}_x = \frac{1}{n}(q+q') + \sum_{i=1}^{k} b_i, \end{aligned}$$

where $1 \leq q' \leq n-1$ is such that $qq' \equiv 1 \pmod n$.

Definition 1.2. [BP] *We say that a projective surface S is a* product-quotient surface *if there exists a finite group G acting faithfully on two smooth projective curves C_1 and C_2 and diagonally on their product, so that S is isomorphic to the minimal desingularization of $X := (C_1 \times C_2)/G$. The surface X is called a* singular model of a product-quotient surface, *or simply a* singular product-quotient surface.

From this definition it follows that a singular product quotient surface contains a finite number of cyclic quotient singularities.

Proposition 1.3 (cf. [MP10], Section 3). *Let S be a product quotient surface, minimal desingularization of $X = (C_1 \times C_2)/G$. Then the invariants of S are*

(i) $K_S^2 = \frac{8(g(C_1)-1)(g(C_2)-1)}{|G|} + \sum_{x \in \mathrm{Sing}\, X} \mathfrak{h}_x$.

(ii) $e(S) = \frac{4(g(C_1)-1)(g(C_2)-1)}{|G|} + \sum_{x \in \mathrm{Sing}\, X} \mathfrak{e}_x$.

(iii) $q(S) = g(C_1/G) + g(C_2/G)$.

Set $\Gamma_i := C_i/G$ and let $g_i \colon C_i \to \Gamma_i$. The group G acts naturally on the sheaves $g_{i*}\mathcal{O}_{C_i}$, $g_{i*}\omega_{C_i}$, $g_{i*}\omega^2_{C_i}$. Assuming that G is *abelian*, we can write the following generalizations of (1) and (2):

$$
\begin{aligned}
g_{i*}\mathcal{O}_{C_i} &= \bigoplus_{\chi \in \widehat{G}} (g_{i*}\mathcal{O}_{C_i})^{\chi}, \\
g_{i*}\omega_{C_i} &= \bigoplus_{\chi \in \widehat{G}} (g_{i*}\omega_{C_i})^{\chi}, \\
g_{i*}\omega^2_{C_i} &= \bigoplus_{\chi \in \widehat{G}} (g_{i*}\omega^2_{C_i})^{\chi},
\end{aligned}
$$

where $(*)^{\chi}$ is the eigensheaf corresponding to the character $\chi \in \widehat{G}$.

§2. The main construction

Let us consider two smooth curves C_1, C_2 of genus 3, such that there are two *simple* $\mathbb{Z}_4$-covers $g_i \colon C_i \to \mathbb{P}^1$, both branched in 4 points. In the rest of the paper we write $G := \mathbb{Z}_4 = \langle \zeta \mid \zeta^4 = 1 \rangle$, where ζ is a primitive fourth root of unity; we also denote by H the subgroup of G defined by $H := \langle \zeta^2 \rangle \cong \mathbb{Z}_2$.

Now set $Z := C_1 \times C_2$ and consider the singular product-quotient surface

$$X := Z/G, \tag{3}$$

which has exactly 16 isolated singular points, corresponding to the fixed points of the G-action on Z. Let $\lambda \colon S \to X$ be the minimal resolution of singularities of X.

The G-cover g_i factors through the double cover $h_i \colon C_i \to E_i$, where $E_i := C_i/H$. Note that E_i is an elliptic curve and that the singular product-quotient surface

$$Y := Z/H \tag{4}$$

contains sixteen cyclic quotient singularities of type $\frac{1}{2}(1, 1)$, i.e. ordinary double points, as only singularities. Let us denote by $\mu \colon V \to Y$ the minimal desingularization of Y. We have a commutative diagram

$$(5)\qquad \begin{array}{ccccc}
V \xrightarrow{\ \mu\ } & Y & \xrightarrow{\qquad v \qquad} & & E_1 \times E_2\ , \\
& \Big\downarrow s & \nwarrow^{r} \qquad {}^{h}\nearrow & & \Big\downarrow t \\
& & Z & & \\
& & {}_{p}\swarrow \qquad \searrow_{g} & & \\
S \xrightarrow{\ \lambda\ } & X & \xrightarrow{\qquad u \qquad} & & \mathbb{P}^1 \times \mathbb{P}^1
\end{array}$$

where:

- $p\colon Z \to X$ and $r\colon Z \to Y$ are the natural projections, so $s\colon Y \to X$ is a double cover (more precisely, a G/H-cover) branched over the singular points of X;
- $g := g_1 \times g_2\colon Z \to \mathbb{P}^1 \times \mathbb{P}^1$ is a $G \times G$-cover branched on a divisor $B \subset \mathbb{P}^1 \times \mathbb{P}^1$ of product type and of bidegree (4, 4);
- $h := h_1 \times h_2\colon Z \to E_1 \times E_2$ is a $H \times H$-cover branched on a divisor $\Delta \subset E_1 \times E_2$ of product type and of bidegree (4, 4);
- $u\colon X \to \mathbb{P}^1 \times \mathbb{P}^1$ is a G-cover, whose branch locus coincides with B;
- $v\colon Y \to E_1 \times E_2$ is a H-cover, whose branch locus coincides with Δ;
- $t\colon E_1 \times E_2 \to \mathbb{P}^1 \times \mathbb{P}^1$ is a $G/H \times G/H$-cover whose branch locus is B and whose ramification locus is Δ.

Let us denote by B_i the branch locus of $g_i\colon C_i \to \mathbb{P}^1$ and by Δ_i the branch locus of $h_i\colon C_i \to E_i$. Both B_i and Δ_i consist of four points; clearly $B = B_1 \times B_2$ and $\Delta = \Delta_1 \times \Delta_2$. From the results of Section 1 we infer that

- there is a natural action of G on the sheaves $g_{i*}\mathcal{O}_{C_i}$, $g_{i*}\omega_{C_i}$, $g_{i*}\omega^2_{C_i}$, which gives decompositions:

$$(6)\qquad \begin{aligned}
g_{i*}\mathcal{O}_{C_i} &= \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{M}_i^{-1} \oplus \mathcal{M}_i^{-2} \oplus \mathcal{M}_i^{-3};\\
g_{i*}\omega_{C_i} &= \omega_{\mathbb{P}^1} \oplus (\omega_{\mathbb{P}^1} \otimes \mathcal{M}_i) \oplus (\omega_{\mathbb{P}^1} \otimes \mathcal{M}_i^2) \oplus (\omega_{\mathbb{P}^1} \otimes \mathcal{M}_i^3);\\
g_{i*}\omega^2_{C_i} &= \omega^2_{\mathbb{P}^1}(B_i) \oplus (\omega^2_{\mathbb{P}^1}(B_i) \otimes \mathcal{M}_i) \oplus (\omega^2_{\mathbb{P}^1}(B_i) \otimes \mathcal{M}_i^2)\\
&\qquad \oplus (\omega^2_{\mathbb{P}^1}(B_i) \otimes \mathcal{M}_i^{-1}),
\end{aligned}$$

  where $\mathcal{M}_i = \mathcal{O}_{\mathbb{P}^1}(1)$. Left to right, the direct summands are the four eigensheaves corresponding to the four characters χ_0, χ_1, χ_2, χ_3 of G;

- there is a natural action of H on the sheaves $h_{i*}\mathcal{O}_{C_i}$, $h_{i*}\omega_{C_i}$, $h_{i*}\omega_{C_i}^2$, which gives decompositions:

$$
\begin{aligned}
h_{i*}\mathcal{O}_{C_i} &= \mathcal{O}_{E_i} \oplus \mathcal{L}_i^{-1}, \\
h_{i*}\omega_{C_i} &= \omega_{E_i} \oplus (\omega_{E_i} \otimes \mathcal{L}_i), \\
h_{i*}\omega_{C_i}^2 &= \omega_{E_i}^2(\Delta_i) \oplus (\omega_{E_i}^2(\Delta_i) \otimes \mathcal{L}_i^{-1}),
\end{aligned} \tag{7}
$$

where $\mathcal{L}_i$ is a line bundle of degree 2 on C_i such that $\mathcal{L}_i^2 = \mathcal{O}_{E_i}(\Delta_i)$. Left to right, the direct summands correspond to the invariant and anti-invariant eigensheaves for the H-action, respectively.

§3. Deformations of the singular product-quotient surface $Y = Z/H$

Let us consider again the surface $Y = Z/H$ defined in Section 2, together with its minimal desingularization $\mu\colon V \to Y$. As we remarked in the previous section, we have

$$\mathrm{Sing}(Y) = 16 \times \frac{1}{2}(1,\, 1).$$

Proposition 3.1. *V is a minimal surface of general type whose invariants are*

$$
\begin{aligned}
p_g(V) = 5, \quad q(V) = 2, \quad K_V^2 = 16, \\
h^1(\Theta_V) = 24, \quad h^2(\Theta_V) = 16.
\end{aligned}
$$

Proof. The invariants $p_g(V)$, $q(V)$, K_V^2 can be computed by using Proposition 1.3. Since $p_g(V) > 0$ and $K_V^2 > 0$, it follows that V is a surface of general type. Let us denote by $H^0(*)^+$ and $H^0(*)^-$ the spaces of invariant and anti-invariant sections for the H-action and by $h^0(*)^+$ and $h^0(*)^-$ their dimensions. Since Y has only rational double points, Künneth formula and the third equality in (7) give

$$
\begin{aligned}
&H^0(\omega_V^2) = H^0(\omega_Y^2) = H^0(\omega_Z^2)^+ = H^0(\omega_{C_1}^2 \boxtimes \omega_{C_2}^2)^+ \\
&= (H^0(h_{1*}\omega_{C_1}^2)^+ \otimes H^0(h_{2*}\omega_{C_2}^2)^+) \oplus (H^0(h_{1*}\omega_{C_1}^2)^- \otimes H^0(h_{2*}\omega_{C_2}^2)^-) \\
&\cong \mathbb{C}^{20}.
\end{aligned}
$$

This shows that $h^0(\omega_V^2) = K_V^2 + \chi(\mathcal{O}_V)$, hence V is a minimal model.

Since Y is a normal surface, [BW74, Proposition 1.2] gives $\mu_*\Theta_V = \Theta_Y$. Therefore the argument in [BW74, Section 1] or [Cat89, p. 299] shows that there are two isomorphisms

$$H^1(\Theta_V) \cong H^1(\Theta_Y) \oplus H^1_E(\Theta_V), \quad H^2(\Theta_V) \cong H^2(\Theta_Y), \tag{8}$$

where $H^1_E(\Theta_V)$ denotes the local cohomology with support on the exceptional divisor $E \subset V$.

By the second isomorphism in (8), we have

$$H^2(\Theta_V)^* \cong H^2(\Theta_Y)^* = H^0(\Omega^1_Z \otimes \Omega^2_Z)^+ = T_1 \oplus T_2 \oplus T_3 \oplus T_4, \tag{9}$$

where

$$\begin{aligned}
T_1 &= H^0(h_{1*}\omega^2_{C_1})^+ \otimes H^0(h_{2*}\omega_{C_2})^+ = H^0(\omega^2_{E_1}(\Delta_1)) \otimes H^0(\omega_{E_2}), \\
T_2 &= H^0(h_{1*}\omega_{C_1})^+ \otimes H^0(h_{2*}\omega^2_{C_2})^+ = H^0(\omega_{E_1}) \otimes H^0(\omega^2_{E_2}(\Delta_2)), \\
T_3 &= H^0(h_{1*}\omega^2_{C_1})^- \otimes H^0(h_{2*}\omega_{C_2})^- \\
&= H^0(\omega^2_{E_1}(\Delta_1) \otimes \mathcal{L}_1^{-1}) \otimes H^0(\omega_{E_2} \otimes \mathcal{L}_2), \\
T_4 &= H^0(h_{1*}\omega_{C_1})^- \otimes H^0(h_{2*}\omega^2_{C_2})^- \\
&= H^0(\omega_{E_1} \otimes \mathcal{L}_1) \otimes H^0(\omega^2_{E_2}(\Delta_2) \otimes \mathcal{L}_2^{-1}).
\end{aligned} \tag{10}$$

Since $\dim T_i = 4$ for all $i \in \{1, 2, 3, 4\}$, we infer $h^2(\Theta_V) = h^2(\Theta_Y) = 16$. By Riemann–Roch we have $h^1(\Theta_V) - h^2(\Theta_V) = 10\chi(\mathcal{O}_V) - 2K^2_V = 8$, so it follows $h^1(\Theta_V) = 24$. Q.E.D.

Corollary 3.2. *We have*

$$h^1(\Theta_Y) = 8, \quad h^2(\Theta_Y) = 16.$$

Proof. Since $h^2(\Theta_Y) = h^2(\Theta_V)$, the first equality follows from Proposition 3.1. Furthermore, E is the disjoint union of sixteen (-2)-curves, hence [BW74, Section 1] implies $H^1_E(\Theta_V) \cong \mathbb{C}^{16}$. Using $h^1(\Theta_V) = 24$ and the first isomorphism in (8) we obtain $h^1(\Theta_Y) = 8$, which completes the proof. Q.E.D.

By using the local-to-global spectral sequence of $\mathcal{E}xt$-sheaves we obtain an exact sequence

$$0 \to H^1(\Theta_Y) \longrightarrow \mathrm{Ext}^1(\Omega^1_Y, \mathcal{O}_Y) \longrightarrow \mathcal{T}^1_Y \xrightarrow{\mathrm{ob}_Y} H^2(\Theta_Y), \tag{11}$$

where $\mathcal{T}^1_Y := H^0(\mathcal{E}xt^1(\Omega^1_Y, \mathcal{O}_Y))$. Notice that $\mathcal{T}^1_Y$ is a skyscraper sheaf supported on the sixteen nodes of Y, hence ob_Y is a linear map

$$\mathrm{ob}_Y \colon \mathbb{C}^{16} \to \mathbb{C}^{16}.$$

Thus its kernel and its cokernel have the same dimension.

Remark 3.3. The branch locus Δ of $v\colon Y \to E_1 \times E_2$ is a polarization of type (4, 4) on the abelian surface $E_1 \times E_2$, in particular $h^0(\Delta) = 16$. Since polarized abelian surfaces form a 3-dimensional family, it follows that the deformation space $\mathrm{Def}(Y)$ has dimension at least 18. Therefore we have

$$\dim \mathrm{Ext}^1(\Omega^1_Y, \mathcal{O}_Y) = \dim T_{[Y]}\mathrm{Def}(Y) \geq \dim_{[Y]} \mathrm{Def}(Y) \geq 18.$$

Proposition 3.4. *We have*

$$\dim \ker \mathrm{ob}_Y = \dim \mathrm{coker}\, \mathrm{ob}_Y = 10.$$

Proof. Notice that Remark 3.3 only gives $\dim(\ker \mathrm{ob}_Y) \geq 10$. In order to prove equality, we apply an argument used in [Cat89, Section 2].

Let us consider the dual map $\mathrm{ob}_Y^* : H^2(\Theta_Y)^* \to (\mathcal{T}^1_Y)^*$. We set

$$\Delta_1 = d'_1 + d'_2 + d'_3 + d'_4$$
$$\Delta_2 = d''_1 + d''_2 + d''_3 + d''_4$$

and we choose local coordinates (x, y) in Z vanishing at (d'_i, d''_j). Then the action of H with respect to these coordinates is given by $(x, y) \to (-x, -y)$.

By [Cat89] we have an isomorphism $(\mathcal{T}^1_Y)^* = (r_*\Omega^1_Z)^+/\Omega^1_Y$, therefore ob_Y^* can be seen as a map

$$\mathrm{ob}_Y^*\colon H^0(\Omega^1_Z \otimes \Omega^2_Z)^+ \to (r_*\Omega^1_Z)^+/\Omega^1_Y.$$

Near any of the ordinary double points of Y, the sheaf $(r_*\Omega^1_Z)^+$ is locally generated by xdx, xdy, ydx, ydy, whereas Ω^1_Y is locally generated by $d(x^2)$, $d(xy)$, $d(y^2)$; then $(r_*\Omega^1_Z)^+/\Omega^1_Y$ is locally generated by $xdy - ydx$, cf. [Cat89, Lemma 2.11].

Looking at (10) and making straightforward computations, one checks that

- the summand T_1 contributes expressions of type $\alpha_1\beta_1 ydx \otimes (dx \wedge dy)$;
- the summand T_2 contributes expressions of type $\alpha_2\beta_2 xdy \otimes (dx \wedge dy)$;
- the summand T_3 contributes expressions of type $\alpha_3\beta_3 xdx \otimes (dx \wedge dy)$;
- the summand T_4 contributes expressions of type $\alpha_4\beta_4 ydy \otimes (dx \wedge dy)$,

where $\alpha_i = \alpha_i(x^2)$ and $\beta_i = \beta_i(y^2)$ are pullbacks of local functions on E_i.

Since in the $\mathcal{O}_Y$-module $(r_*\Omega^1_Z)^+/\Omega^1_Y$ we have the relations

$$1/2(xdy - ydx) = xdy = -ydx \text{ and } xdx = ydy = 0,$$

it follows that the restriction of ob^*_Y to the subspace $T_3 \oplus T_4$ is zero, whereas the restriction of ob^*_Y to the subspace $T_1 \oplus T_2$ can be identified, up to a multiplicative constant, with the map

$$\phi\colon H^0(\omega^2_{E_1}(\Delta_1)) \oplus H^0(\omega^2_{E_2}(\Delta_2)) \to \bigoplus_{i,j=1}^{4} \mathbb{C}_{ij},$$

$$\phi(\sigma \oplus \tau) = \bigoplus_{i,j=1}^{4} (\mathrm{val}_{d'_i}(\sigma) - \mathrm{val}_{d''_j}(\tau)).$$

Here the valuation maps $\mathrm{val}_{d'_i}$ and $\mathrm{val}_{d''_j}$ are defined, as usual, by the short exact sequences

$$\begin{aligned} &0 \to H^0(\omega^2_{E_1}) \to H^0(\omega^2_{E_1}(\Delta_1)) \stackrel{\oplus \mathrm{val}_{d'_i}}{\longrightarrow} H^0(N_{\Delta_1}) \cong \oplus_{i=1}^4 \mathbb{C}_i, \\ &0 \to H^0(\omega^2_{E_2}) \to H^0(\omega^2_{E_2}(\Delta_2)) \stackrel{\oplus \mathrm{val}_{d''_j}}{\longrightarrow} H^0(N_{\Delta_2}) \cong \oplus_{j=1}^4 \mathbb{C}_j. \end{aligned} \tag{12}$$

Therefore we obtain

$$\begin{aligned} \ker \phi = \{\sigma \oplus \tau \,|\, &\mathrm{val}_{d'_1}(\sigma) = \mathrm{val}_{d'_2}(\sigma) = \mathrm{val}_{d'_3}(\sigma) = \mathrm{val}_{d'_4}(\sigma) \\ &= \mathrm{val}_{d''_1}(\tau) = \mathrm{val}_{d''_2}(\tau) = \mathrm{val}_{d''_3}(\tau) = \mathrm{val}_{d''_4}(\tau)\}. \end{aligned} \tag{13}$$

As E_i is an elliptic curve, we have $\omega^2_{E_i} = \omega_{E_i}$ and so (12) are the standard residue sequences for meromorphic 1-forms. By the Residue Theorem we get

$$\sum_{i=1}^4 \mathrm{val}_{d'_i}(\sigma) = \sum_{j=1}^4 \mathrm{val}_{d''_j}(\tau) = 0,$$

hence (13) implies that $\sigma \oplus \tau \in \ker \phi$ if and only if $\mathrm{val}_{d'_i}(\sigma) = \mathrm{val}_{d''_j}(\tau) = 0$ for all pairs (i, j). This yields $\ker \phi = H^0(\omega^2_{E_1}) \oplus H^0(\omega^2_{E_2}) \cong \mathbb{C} \oplus \mathbb{C}$.

Then $\ker \mathrm{ob}^*_Y = \ker \phi \oplus T_3 \oplus T_4 \cong \mathbb{C}^{10}$, hence $\dim \mathrm{coker}\, \mathrm{ob}_Y = 10$ and we are done. Q.E.D.

Corollary 3.5. *We have*

$$\dim \mathrm{Ext}^1(\Omega^1_Y, \mathcal{O}_Y) = 18.$$

Proof. Immediate from Corollary 3.2, Proposition 3.4 and exact sequence (11). Q.E.D.

Proposition 3.6. *The following holds*:
(i) Def(Y) *is smooth at* $[Y]$, *of dimension* 18;
(ii) ESDef(Y) *is smooth at* $[Y]$, *of dimension* 8.

Proof. By Remark 3.3 and Corollary 3.5 we have

$$18 = \dim \mathrm{Ext}^1(\Omega^1_Y, \mathcal{O}_Y) = \dim T_{[Y]}\mathrm{Def}(Y) \geq \dim_{[Y]} \mathrm{Def}(Y) \geq 18,$$

which proves ***(i)***.

On the other hand, if we move the branch loci $B_i \subset E_i$ the curve $\Delta \subset E_1 \times E_2$ remains of product type, so in this way we obtain a 8-dimensional family of *equisingular* deformations of Y; therefore the equisingular deformation space ESDef(Y) has dimension at least 8, and by Corollary 3.2 we have

$$8 = \dim H^1(\Theta_Y) = \dim T_{[Y]}\mathrm{ESDef}(Y) \geq \dim_{[Y]} \mathrm{ESDef}(Y) \geq 8.$$

This proves ***(ii)***. Q.E.D.

Summing up, Proposition 3.6 shows that the deformations of Y are unobstructed and that they are all obtained by deforming the pair (A, Δ), where A is an abelian surface and Δ a polarization of type $(4, 4)$. In particular, all the deformations preserve the action of H. Moreover, the equisingular deformations of Y are also unobstructed and are obtained by taking as A the product of two elliptic curves and by choosing the polarization Δ of product type.

Remark 3.7. Since Y has only rational double points, by [BW74] the dimension of Def(Y) equals the dimension of Def(V). Then

$$24 = h^1(\Theta_V) = \dim T_{[V]}\mathrm{Def}(V) > \dim_{[V]} \mathrm{Def}(V) = 18,$$

that is *Def*(V) is *singular* at $[V]$. By [BW74, Theorem 3.7], this means that the sixteen (-2)-curves of V do not have independent behavior in deformations.

§4. Deformations of the singular product-quotient surface $X = Z/G$

Let us consider now the surface $X = Z/G$ defined in Section 2 and its minimal resolution of singularities $\lambda\colon S \to X$. We must analyze several cases, according to the type of quotient singularities that X contains.

Throughout this section we set $Q := \mathbb{P}^1 \times \mathbb{P}^1$ and we denote by $\mathcal{O}_Q(a, b)$ the line bundle of bidegree (a, b) on Q.

The following exact sequence is the analogue of (11):

$$0 \to H^1(\Theta_X) \longrightarrow \mathrm{Ext}^1(\Omega^1_X, \mathcal{O}_X) \longrightarrow \mathcal{T}^1_X \xrightarrow{\mathrm{ob}_X} H^2(\Theta_X). \tag{14}$$

4.1. Example where Sing$(X) = 16 \times \frac{1}{4}(1, 3)$

Assume that, locally around each of the fixed points, the action of $G = \langle \zeta \,|\, \zeta^4 = 1 \rangle$ is given by $\zeta \cdot (x, y) = (\zeta x, \zeta^{-1} y)$. Therefore,

$$\mathrm{Sing}(X) = 16 \times \frac{1}{4}(1, 3).$$

In this case X contains only rational double points and we obtain

$$p_g(S) = 5, \quad q(S) = 0, \quad K_S^2 = 8.$$

Proposition 4.1. *S is a minimal surface of general type.*

Proof. S is of general type because $p_g(S) > 0$ and $K_S^2 > 0$. Since the action of G is twisted on the second factor and X has only rational double points, the Künneth formula and the third equality in (6) give

$$\begin{aligned} H^0(\omega_S^2) &= H^0(\omega_X^2) = H^0(\omega_Z^2)^G = H^0(\omega_{C_1}^2 \boxtimes \omega_{C_2}^2)^G \\ &= \bigoplus_{\chi \in \widehat{G}} (H^0(g_{1*}\omega_{C_1}^2)^\chi \otimes H^0(g_{2*}\omega_{C_2}^2)^\chi) = \mathbb{C}^{14}. \end{aligned}$$

This shows that $h^0(\omega_S^2) = K_S^2 + \chi(\mathcal{O}_S)$, hence S is a minimal surface. Q.E.D.

Proposition 4.2. *The following holds:*

(i) ob_X *is surjective;*
(ii) $h^1(\Theta_X) = 2, \quad h^2(\Theta_X) = 6, \quad h^1(\Theta_S) = 50, \quad h^2(\Theta_S) = 6.$
(iii) $\mathrm{ESDef}(X)$ *is smooth at* $[X]$*, of dimension* 2.

Proof. **(i)** Let us consider the dual map $\mathrm{ob}_X^* : H^2(\Theta_X)^* \to (\mathcal{T}^1_X)^*$. By Grothendieck duality (see [AK70, Chapter I]) and Künneth formula we obtain

$$
\begin{aligned}
H^2(\Theta_X)^* &= H^0(\Omega^1_Z \otimes \Omega^2_Z)^G \\
&= \bigoplus_{\chi \in \widehat{G}} \left[(H^0(g_{1*}\omega_{C_1})^\chi \otimes H^0(g_{2*}\omega^2_{C_2})^\chi) \right. \\
&\qquad \left. \oplus\, (H^0(g_{1*}\omega^2_{C_1})^\chi \otimes H^0(g_{2*}\omega_{C_2})^\chi) \right] \qquad (15) \\
&= U_1 \oplus U_2, \text{ where} \\
U_1 &= H^0(\omega_{\mathbb{P}^1} \otimes \mathcal{M}^2_1) \otimes H^0(\omega^2_{\mathbb{P}^1}(B_2) \otimes \mathcal{M}^2_2), \\
U_2 &= H^0(\omega^2_{\mathbb{P}^1}(B_1) \otimes \mathcal{M}^2_1) \otimes H^0(\omega_{\mathbb{P}^1} \otimes \mathcal{M}^2_2).
\end{aligned}
$$

This yields $h^2(\Theta_X) = 6$ and so $h^2(\Theta_S) = 6$. Now we set

$$
\begin{aligned}
B_1 &= b'_1 + b'_2 + b'_3 + b'_4 \\
B_2 &= b''_1 + b''_2 + b''_3 + b''_4
\end{aligned}
$$

and we choose local coordinates $(x,\, y)$ in Z vanishing at $(b'_i,\, b''_j)$. As in Section 3, we can interpret ob^*_X as a map

$$
\mathrm{ob}^*_X \colon H^0(\Omega^1_Z \otimes \Omega^2_Z)^G \to (p_*\Omega^1_Z)^G/\Omega^1_X,
$$

where $(p_*\Omega^1_Z)^G/\Omega^1_X$ is a skyscraper sheaf supported on the singular points of X and locally generated by $x^i y^{i+1} dx - y^i x^{i+1} dy$, for $i = 0, 1, 2$, see [Cat89].

A straightforward local computation shows that the summand U_1 in (15) contributes expressions of the form $\alpha_1\beta_1 x dy \otimes (dx \wedge dy)$ whereas the summand U_2 contributes expressions of the form $\alpha_2\beta_2 y dx \otimes (dx \wedge dy)$, where $\alpha_i = \alpha_i(x^2)$ and $\beta_i = \beta_i(y^2)$ are pullbacks of local functions on $\mathbb{P}^1$. Therefore the map ob^*_X can be identified, up to a multiplicative constant, with

$$
\begin{aligned}
\phi\colon H^0(\omega^2_{\mathbb{P}^1}(B_1) \otimes \mathcal{M}^2_1) &\oplus H^0(\omega^2_{\mathbb{P}^1}(B_2) \otimes \mathcal{M}^2_2) \\
&\to \bigoplus_{i,j=1}^{4} \mathbb{C}_{ij} \subset \bigoplus_{i,j=1}^{4} \mathbb{C}^{\oplus 3}_{ij} \cong (\mathcal{T}^1_X)^* \\
\phi(\sigma \oplus \tau) &= \bigoplus_{i,j=1}^{4} (\mathrm{val}_{b'_i}(\sigma) - \mathrm{val}_{b''_j}(\tau)),
\end{aligned}
$$

where the valuation maps are defined as in Section 3. Hence we obtain

$$
\begin{aligned}
\ker\phi = \{\sigma \oplus \tau \mid \mathrm{val}_{b'_1}(\sigma) &= \mathrm{val}_{b'_2}(\sigma) = \mathrm{val}_{b'_3}(\sigma) = \mathrm{val}_{b'_4}(\sigma) \\
&= \mathrm{val}_{b''_1}(\tau) = \mathrm{val}_{b''_2}(\tau) = \mathrm{val}_{b''_3}(\tau) = \mathrm{val}_{b''_4}(\tau)\}. \qquad (16)
\end{aligned}
$$

On the other hand, the valuation map $H^0(\omega^2_{\mathbb{P}^1}(B_i) \otimes \mathcal{M}_i^2) \to H^0(N_{B_i})$ can be identified with the residue map $H^0(\omega_{\mathbb{P}^1}(B_i)) \to H^0(N_{B_i})$ via the isomorphism $H^0(\omega^2_{\mathbb{P}^1}(B_i) \otimes \mathcal{M}_i^2) \cong H^0(\omega_{\mathbb{P}^1}(B_i))$. By the Residue Theorem we have

$$\sum_{i=1}^{4} \mathrm{val}_{b_i'}(\sigma) = \sum_{j=1}^{4} \mathrm{val}_{b_j''}(\tau) = 0,$$

so (16) implies that $\sigma \oplus \tau \in \ker \phi$ if and only if $\mathrm{val}_{b_i'}(\sigma) = \mathrm{val}_{b_j''}(\tau) = 0$ for all pairs (i, j). But there are no non-zero holomorphic 1-forms on $\mathbb{P}^1$, so $\ker \phi = 0$ and ob_X^* is injective. Therefore the obstruction map ob_X is surjective.

(ii) Let us denote by $F \subset S$ the exceptional divisor of $\lambda \colon S \to X$. Since S has only rational double points, we have

$$H^1(\Theta_S) \cong H^1(\Theta_X) \oplus H^1_F(\Theta_S), \quad H^2(\Theta_S) \cong H^2(\Theta_X).$$

By Riemann–Roch theorem we obtain

$$h^1(\Theta_S) - h^2(\Theta_S) = 10\chi(\mathcal{O}_S) - 2K_S^2 = 44,$$

then $h^1(\Theta_S) = 50$ since we have shown that $h^2(\Theta_S) = 6$, see part ***(i)***. Being F the union of sixteen disjoint A_3-cycles, we have $H^1_F(\Theta_S) \cong \mathbb{C}^{16\cdot 3} = \mathbb{C}^{48}$. Therefore $h^1(\Theta_X) = 2$.

(iii) The cover $u \colon X \to Q$ is a simple G-cover branched on the divisor $B = B_1 \times B_2$, which has bidegree (4, 4). By varying the branch loci $B_i \subset \mathbb{P}^1$ we obtain a 2-dimensional family of equisingular deformations of X. Then

$$2 = \dim H^1(\Theta_X) = \dim T_{[X]}\mathrm{ESDef}(X) \geq \dim_{[X]} \mathrm{ESDef}(X) \geq 2,$$

which implies the claim. Q.E.D.

Proposition 4.3. *The general deformation of the surface X is a canonically embedded, smooth complete intersection $S_{2,4}$ of type* (2, 4) *in $\mathbb{P}^4$.*

Proof. By [Cat97, Proposition 6.2] it is sufficient to check that the canonical map $\phi_K \colon X \to \mathbb{P}^4$ is a birational morphism onto its image. Since X has only Rational Double Points and $u \colon X \to Q$ is a *simple* G-cover, Hurwitz formula yields $K_X = u^*\mathcal{O}_Q(1, 1)$; but $|\mathcal{O}_Q(1, 1)|$ is base-point free, so $|K_X|$ is also base-point free and ϕ_K is a morphism.

It remains to show that ϕ_K separates two general points x, y on X. The decomposition of $u_*\omega_X$ with respect to the G-action is

$$u_*\omega_X = \omega_Q \oplus (\omega_Q \otimes L) \oplus (\omega_Q \otimes L^2) \oplus (\omega_Q \otimes L^3),$$

where $L = \mathcal{O}_Q(1, 1)$ and $\omega_Q \otimes L^i$ is the eigensheaf corresponding to the character χ_i. Therefore we obtain

$$H^0(u_*\omega_X) = H^0(\omega_Q \otimes L^2) \oplus H^0(\omega_Q \otimes L^3).$$

Now let $\{\tau\}$ be a basis of $H^0(\omega_Q \otimes L^2) = H^0(\mathcal{O}_Q)$ and let $\{\sigma_1, \sigma_2, \sigma_3, \sigma_4\}$ be a basis of $H^0(\omega_Q \otimes L^3) = H^0(\mathcal{O}_Q(1, 1))$. The four sections $\{\sigma_i\}$ provide an embedding $Q \hookrightarrow \mathbb{P}^3$, hence ϕ_K separates pairs of points which belong to the same fibre of $u\colon X \to Q$. Now let x, y be two points in the same (general) fibre of u. Then there exists $1 \leq a \leq 3$ such that $y = \zeta^a \cdot x$. Then

$$\sigma_i(y) = \zeta^a \sigma_i(x), \quad \tau(y) = \zeta^{2a}\tau(x),$$

that is

$$\begin{aligned} \phi_K(y) &= [\sigma_1(y)\colon \sigma_2(y)\colon \sigma_3(y)\colon \sigma_4(y)\colon \tau(y)] \\ &= [\sigma_1(x)\colon \sigma_2(x)\colon \sigma_3(x)\colon \sigma_4(x)\colon \zeta^a\tau(x)] \\ &\neq [\sigma_1(x)\colon \sigma_2(x)\colon \sigma_3(x)\colon \sigma_4(x)\colon \tau(x)] = \phi_K(x). \end{aligned}$$

Therefore ϕ_K also separates general pairs of points lying in the same fibre of $u\colon X \to Q$ and we are done. Q.E.D.

Now we can prove the following

Proposition 4.4. Def(X) *is smooth at* $[X]$*, of dimension* 44.

Proof. By using Proposition 4.2 and exact sequence (14) we obtain

$$\dim T_{[X]}\mathrm{Def}(X) = \dim \mathrm{Ext}^1(\Omega^1_X, \mathcal{O}_X) = 44. \tag{17}$$

On the other hand, by [Se06, Chapter 3] one knows that Def$(S_{2,4})$ is smooth, of dimension

$$h^0(N_{S_{2,4}/\mathbb{P}^4}) - \dim \mathrm{Aut}(\mathbb{P}^4) = h^0(\mathcal{O}_{S_{2,4}}(2)) + h^0(\mathcal{O}_{S_{2,4}}(4)) - 24 = 44.$$

Equality (17) and Proposition 4.3 yield

$$44 = \dim T_{[X]}\mathrm{Def}(X) \geq \dim_{[X]} \mathrm{Def}(X) = \dim_{[S_{2,4}]} \mathrm{Def}(S_{2,4}) = 44, \tag{18}$$

so we are done. Q.E.D.

Remark 4.5. Since X has only rational double points, by [BW74] the dimension of Def(X) equals the dimension of Def(S). So we infer

$$50 = h^1(\Theta_S) = \dim T_{[S]}\mathrm{Def}(S) > \dim_{[S]} \mathrm{Def}(S) = 44,$$

that is Def(S) is *singular* at $[S]$. By [BW74, Theorem 3.7], this means that the sixteen A_3-cycles of S do not have independent behavior in deformations.

Proposition 4.3 in particular shows that the general deformation of X does not preserve the G-action. Now we want to consider some particular deformations that preserve the quadruple cover $u\colon X \to Q$. According to [Pa91] we call them *natural deformations*, and we freely follow the notation of that paper everywhere. The building data of any totally ramified G-cover $u\colon X \to Q$ are

$$\begin{aligned} 4L_{\chi_1} &= 3D_{G,\chi_3} + D_{G,\chi_1} \\ 2L_{\chi_2} &= D_{G,\chi_1} + D_{G,\chi_3} \\ 4L_{\chi_3} &= D_{G,\chi_3} + 3D_{G,\chi_1}, \end{aligned} \tag{19}$$

see [Pa91, Proposition 2.1]. The G-cover $u\colon X \to Q$ defines a natural embedding i of X into the total space of the vector bundle $W = \bigoplus_{\chi \in \widehat{G}\setminus\{\chi_0\}} V(L_\chi^{-1})$. If w_χ is a local coordinate on $V(L_\chi^{-1})$ on an open set U and $\sigma_{G,\psi}$ is a local equation for $D_{G,\psi}$ on U, then $i(X)$ is defined by the equations

$$w_\chi w_{\chi'} = \Bigg(\prod_{\psi \in \{\chi_1, \chi_3\}} (\sigma_{G,\psi})^{\epsilon^{G,\psi}_{\chi,\chi'}} \Bigg) w_{\chi\chi'} \tag{20}$$

and the covering map is given by the composition $\pi \circ i$, where $\pi\colon W \to Q$ is the projection. Moreover, the integers $\epsilon^{G,\psi}_{\chi,\chi'}$ can be easily computed by using [Pa91, p. 196]:

$$\begin{array}{lllll} \epsilon^{G,\chi_1}_{\chi_0,\chi_0} = 0, & \epsilon^{G,\chi_1}_{\chi_0,\chi_1} = 0, & \epsilon^{G,\chi_1}_{\chi_0,\chi_2} = 0, & \epsilon^{G,\chi_1}_{\chi_0,\chi_3} = 0, & \epsilon^{G,\chi_1}_{\chi_1,\chi_1} = 0, \\ \epsilon^{G,\chi_1}_{\chi_1,\chi_2} = 0, & \epsilon^{G,\chi_1}_{\chi_1,\chi_3} = 1, & \epsilon^{G,\chi_1}_{\chi_2,\chi_2} = 1, & \epsilon^{G,\chi_1}_{\chi_2,\chi_3} = 1, & \epsilon^{G,\chi_1}_{\chi_3,\chi_3} = 1, \\ \epsilon^{G,\chi_3}_{\chi_0,\chi_0} = 0, & \epsilon^{G,\chi_3}_{\chi_0,\chi_1} = 0, & \epsilon^{G,\chi_3}_{\chi_0,\chi_2} = 0, & \epsilon^{G,\chi_3}_{\chi_0,\chi_3} = 0, & \epsilon^{G,\chi_3}_{\chi_1,\chi_1} = 1, \\ \epsilon^{G,\chi_3}_{\chi_1,\chi_2} = 1, & \epsilon^{G,\chi_3}_{\chi_1,\chi_3} = 1, & \epsilon^{G,\chi_3}_{\chi_2,\chi_2} = 1, & \epsilon^{G,\chi_3}_{\chi_2,\chi_3} = 0, & \epsilon^{G,\chi_3}_{\chi_3,\chi_3} = 0. \end{array} \tag{21}$$

Let us consider now a collection of sections

$$\{r_{G,\psi,\chi} \in H^0(\mathcal{O}_Q(D_{G,\psi}) \otimes L_\chi^{-1})\}_{\psi \in \{\chi_1,\chi_3\},\, \chi \in S_{G,\psi}},$$

where

$$S_{G,\chi_1} := \{\chi_0,\, \chi_1, \chi_2\}, \quad S_{G,\chi_3} := \{\chi_0,\, \chi_2, \chi_3\}.$$

Let $h_{G,\psi,\chi}$ be a local representative of $r_{G,\psi,\chi}$ on the open set U and define

$$\tau_{G,\psi} := \sum_{\substack{\psi \in \{\chi_1,\chi_3\} \\ \chi \in S_{G,\psi}}} h_{G,\psi,\chi} w_\chi.$$

Then the natural deformation of the G-cover $u\colon X \to Q$, associated to the collection of sections $\{r_{G,\psi,\chi}\}$, is the subvariety X' of W locally defined by

$$w_\chi w_{\chi'} = \Bigg(\prod_{\psi \in \{\chi_1, \chi_3\}} (\tau_{G,\psi})^{\epsilon^{G,\psi}_{\chi,\chi'}} \Bigg) w_{\chi\chi'},$$

together with the map $u'\colon X' \to Q$ obtained by restricting the projection $\pi\colon W \to Q$ to X'.

Coming back to our particular case, we have

$$D_{G,\chi_1} \in |\mathcal{O}_Q(4,\, 4)|, \quad D_{G,\chi_3} = 0,$$

$$L_{\chi_1} \cong \mathcal{O}_Q(1,\, 1), \quad L_{\chi_2} \cong \mathcal{O}_Q(2,\, 2), \quad L_{\chi_3} \cong \mathcal{O}_Q(3,\, 3),$$

and $B = D_{G,\chi_1}$. Since $D_{G,\chi_3} = 0$, the natural deformations of X are parameterized by the vector space

$$\bigoplus_{\chi \in S_{G,\chi_1}} H^0(\mathcal{O}_Q(D_{G,\chi_1}) \otimes L_\chi^{-1}) \tag{22}$$

$$= H^0(\mathcal{O}_Q(4,\, 4)) \oplus H^0(\mathcal{O}_Q(3,\, 3)) \oplus H^0(\mathcal{O}_Q(2,\, 2)) \cong \mathbb{C}^{50}.$$

4.2. Example where Sing$(X) = 16 \times \frac{1}{4}(1,1)$

Assume that, locally around each of the fixed points, the action of $G = \langle \zeta \mid \zeta^4 = 1 \rangle$ is given by $\zeta \cdot (x,\, y) = (\zeta x,\, \zeta y)$. In this case,

$$\mathrm{Sing}(X) = 16 \times \frac{1}{4}(1,1).$$

By using Proposition 1.3, we obtain

$$p_g(S) = 1, \quad q(S) = 0, \quad K_S^2 = -8,$$

hence S is not a minimal model.

Theorem 4.6. *The following holds*:

- ***(i)*** $h^2(\Theta_X) = 14$;
- ***(ii)*** *all natural deformations of* $u\colon X \to Q$ *preserve the* 16 *points of type* $\frac{1}{4}(1,\, 1)$;
- ***(iii)*** *there exists a* 12*-dimensional family of* $\mathbb{Q}$*-Gorenstein deformations of* X*, smoothing all the singularities. The general element* X_t *of this deformation is a smooth, minimal surface of general type with* $p_g(X_t) = 1$, $q(X_t) = 0$ *and* $K_{X_t}^2 = 8$;
- ***(iv)*** X_t *is isomorphic to a Todorov surface with* $K^2 = 8$.

Proof. ***(i)*** By using Grothendieck duality and Künneth formula as in Proposition 4.2 we obtain

$$\begin{aligned} H^2(\Theta_X)^* &= H^0(\Omega^1_Z \otimes \Omega^2_Z)^G \\ &= \bigoplus_{\chi \in \widehat{G}} \big[(H^0(g_{1*}\omega_{C_1})^{\chi} \otimes H^0(g_{2*}\omega^2_{C_2})^{\chi^{-1}}) \\ &\quad \oplus (H^0(g_{1*}\omega^2_{C_1})^{\chi} \otimes H^0(g_{2*}\omega_{C_2})^{\chi^{-1}})\big] \\ &= (H^0(\mathcal{O}_{\mathbb{P}^1}) \otimes H^0(\mathcal{O}_{\mathbb{P}^1}(2))) \oplus (H^0(\mathcal{O}_{\mathbb{P}^1}(1)) \otimes H^0(\mathcal{O}_{\mathbb{P}^1}(1))) \\ &\quad \oplus (H^0(\mathcal{O}_{\mathbb{P}^1}(1)) \otimes H^0(\mathcal{O}_{\mathbb{P}^1}(1))) \oplus (H^0(\mathcal{O}_{\mathbb{P}^1}(2)) \otimes H^0(\mathcal{O}_{\mathbb{P}^1})), \end{aligned}$$

which yields $h^2(\Theta_X) = 14$.

(ii) The G-cover $u \colon X \to Q$ is determined by the building data (19), with

$$D_{G,\chi_1} \in |\mathcal{O}_Q(4,\, 0)|, \quad D_{G,\chi_3} \in |\mathcal{O}_Q(0,\, 4)|,$$

$$L_{\chi_1} \cong \mathcal{O}_Q(1,\, 3), \quad L_{\chi_2} \cong \mathcal{O}_Q(2,\, 2), \quad L_{\chi_3} \cong \mathcal{O}_Q(3,\, 1).$$

The natural deformations of u are parameterized by the vector space

$$\begin{aligned} &\bigoplus_{\psi \in \{\chi_1, \chi_3\}} \Big(\bigoplus_{\chi \in S_{G,\psi}} H^0(\mathcal{O}_Q(D_{G,\psi}) \otimes L_{\chi}^{-1}) \Big) \\ &= H^0(\mathcal{O}_Q(4,\, 0)) \oplus H^0(\mathcal{O}_Q(0,\, 4)). \end{aligned} \tag{23}$$

Therefore they form a family of dimension 10, which is exactly the one obtained by keeping the branch divisor $B \subset Q$ of product type. In particular, all the natural deformations preserve the sixteen singular points of X.

(iii) For simplicity, set $w_i = w_{\chi_i}$ and $\tau_{G,\chi_i} = h_i w_0$. Writing $w_0 = 1$, the local equations defining the family of natural deformations of $u \colon X \to Q$ are the following:

$$\begin{aligned} &w_1^2 = h_3 w_2, \quad w_1 w_2 = h_3 w_3, \quad w_1 w_3 = h_1 h_3, \\ &w_2^2 = h_1 h_3, \quad w_2 w_3 = h_1 w_1, \quad w_3^2 = h_1 w_2. \end{aligned} \tag{24}$$

Relations (24) can be written in determinantal form in two different ways, namely

$$\textbf{(a)}\ \operatorname{rank} \begin{pmatrix} w_2 & w_3 & w_1 & h_1 \\ w_1 & w_2 & h_3 & w_3 \end{pmatrix} \leq 1,$$

$$\textbf{(b)}\ \operatorname{rank} \begin{pmatrix} h_3 & w_1 & w_2 \\ w_1 & w_2 & w_3 \\ w_2 & w_3 & h_1 \end{pmatrix} \leq 1.$$

In the sequel we will only consider the determinantal representation **(*b*)**. We can deform it by using the parameter $s \in H^0(L_{\chi_2}) = \mathbb{C}^9$, i.e.

$$\text{rank} \begin{pmatrix} h_3 & w_1 & w_2 \\ w_1 & w_2 + s & w_3 \\ w_2 & w_3 & h_1 \end{pmatrix} \leq 1. \tag{25}$$

It is no difficult to check that for general $s \neq 0$ one obtains a smooth surface, hence (25) provides a smoothing $\pi\colon \mathcal{X} \to T$ of X. This is actually a $\mathbb{Q}$-Gorenstein smoothing of X, since it is the globalization of the local $\mathbb{Q}$-Gorenstein smoothing of the quotient singularity $\frac{1}{4}(1, 1)$, see [Man08, Chapter 4]. Therefore the general fibre X_t of π is a surface of general type whose invariants are

$$p_g(X_t) = 1, \quad q(X_t) = 0, \quad K_{X_t}^2 = 8.$$

The canonical divisor K_X is big and nef (since $4K_X = u^*\mathcal{O}_Q(4, 4)$), so K_{X_t} is big and nef too, as X_t is obtained by a $\mathbb{Q}$-Gorenstein smoothing of X. This shows that X_t is a minimal model.

In order to give a more concrete description of X_t, let us look again at the double cover $v\colon Y \to E_1 \times E_2$ constructed in Section 3. By Proposition 3.6 we know that Def(Y) is smooth at $[Y]$ of dimension 18; moreover the general deformation Y_t of Y is a double cover $v_t\colon Y_t \to A_t$ of an abelian variety A_t, branched on a smooth divisor Ξ which is a polarization of type (4, 4). Let us compute the dimension of the subspace of Def(Y) consisting of surfaces for which it is possible to lift the natural involution $\iota_t\colon A_t \to A_t$ to an involution $\tilde{\iota}_t\colon Y_t \to Y_t$ such that $Y_t/\tilde{\iota}_t$ is smooth. By [BL04, Corollary 4.7.6], the divisor Ξ does not contain any of the 16 fixed points of ι_t. If we write locally the equation of the double cover $v_t\colon Y_t \to A_t$ as $z^2 = f(x, y)$ so that ι_t is given by $(x, y) \to (-x, -y)$, we see that ι_t lifts to Y_t if an only if the branch locus $f(x, y) = 0$ is ι_t-invariant; moreover in this case there is a unique lifting such that the quotient is smooth; it is locally given by $(x, y, z) \to (-x, -y, -z)$. By [BL04, Corollary 4.6.6], the divisors in $|\Xi|$ which are invariant under ι_t form a family of dimension $\frac{1}{2}h^0(\mathcal{O}_A(\Xi)) + 2 - 1 = 9$ and so, taking into account the three moduli of abelian surfaces, we obtain a 12-dimensional family $\{Y_t\}$ of deformations of Y which admit a lifting of ι_t.

One can further check that the lifted involution $\tilde{\iota}$ is fixed-point free and that the family $\{X_t\}$ constructed before can be obtained as $X_t = Y_t/\tilde{\iota}_t$.

(iv) Let us consider the Kummer surface $\text{Kum}(A_t) := A_t/\iota_t$. By ***(iii)*** a general fibre X_t of the $\mathbb{Q}$-Gorenstein smoothing of X is a double cover of $\mathrm{Kum}(A_t)$ branched over the 16 nodes of $\mathrm{Kum}(A_t)$ and the image D of the curve Ξ.

On the other hand, $\mathrm{Kum}(A_t)$ can be embedded in $\mathbb{P}^3$ as a quartic surface with 16 nodes and via this embedding the curve D is obtained by intersecting $\mathrm{Kum}(A_t)$ with a smooth quadric surface Φ which does not contain any of the nodes.

This shows that X_t belongs precisely to the family of surfaces with $p_g = 1$, $q = 0$ and $K^2 = 8$ constructed by Todorov in [To81]. Q.E.D.

Remark 4.7. Let us fix the abelian surface A and the embedding $\mathrm{Kum}(A) \hookrightarrow \mathbb{P}^3$. Then the choice of the deformation parameter $s \in H^0(L_{\chi_2})$ corresponds to the choice of the quadric surface $\Phi \in |\mathcal{O}_{\mathbb{P}^3}(2)|$. By [To81, Lemma 2.1] there is a quadric surface Φ_k in $\mathbb{P}^3$ which contains exactly k $(1 \leq k \leq 6)$ of the nodes of $\mathrm{Kum}(A)$ that are general position. This means that the pullback in A of the curve $D_k := \mathrm{Kum}(A) \cap \Phi_k$ is a polarization of type (4, 4) which contains exactly k of the fixed points of $\iota \colon A \to A$.

Therefore arguments similar to those used in the proof of Theorem *4.6*, part ***(ii)*** show that there exists a partial $\mathbb{Q}$-Gorenstein smoothing of X, whose general fibre X_t is isomorphic to the double cover of $\mathrm{Kum}(A)$ branched over the curve D_k and the remaining $16-k$ nodes of $\mathrm{Kum}(A)$. The surface X_t is not smooth, since it contains exactly k singular points of type $\frac{1}{4}(1, 1)$. Its minimal resolution of singularities is a Todorov surface with $K^2 = 8 - k$ $(1 \leq k \leq 6)$.

4.3. Example where $\mathrm{Sing}(X) = 8 \times \frac{1}{4}(1,3) + 8 \times \frac{1}{4}(1,1)$

We can also twist the action of G on Z in such a way that

$$\mathrm{Sing}(X) = 8 \times \frac{1}{4}(1,1) + 8 \times \frac{1}{4}(1,3).$$

By using Proposition 1.3, we obtain

$$p_g(S) = 3, \quad q(S) = 0, \quad K_S^2 = 0.$$

Rasdeaconu and Suvaina give an explicit construction of S in [RS06, Section 3], showing that it is a simply connected, minimal, elliptic surface with no multiple fibers. One can also prove that $H^2(\Theta_X) \neq 0$, see [LP11, Section 3].

Proposition 4.8. *The following holds*:

(i) *all natural deformations of X preserve the 8 points of type* $\frac{1}{4}(1, 1)$;

(ii) *there exists a family of* $\mathbb{Q}$*-Gorenstein deformations of* X*, smoothing all the singularities. The general element of this family is a smooth, minimal surface of general type with* $p_g = 3$*,* $q = 0$ *and* $K^2 = 8$*.*

Proof. ***(i)*** The abelian G-cover $u\colon X \to Q$ is determined by the building data (19), with

$$D_{G,\chi_1},\, D_{G,\chi_3}, \in |\mathcal{O}_Q(2,\, 2)|.$$

$$L_{\chi_1},\, L_{\chi_2}, L_{\chi_3} \cong \mathcal{O}_Q(2,\, 2).$$

The same argument of Theorem 4.6, part ***(ii)*** shows that the natural deformations of X are parameterized by the vector space

$$\begin{gathered} H^0(\mathcal{O}_Q(2,\, 2)) \oplus H^0(\mathcal{O}_Q(2,\, 2)) \\ \oplus H^0(\mathcal{O}_Q) \oplus H^0(\mathcal{O}_Q) \oplus H^0(\mathcal{O}_Q) \oplus H^0(\mathcal{O}_Q). \end{gathered}$$

Writing $w_i := w_{\chi_i}$ we have

$$h_1 = g_1 + c_1 w_1 + c_2 w_2, \quad h_3 = g_3 + d_2 w_2 + d_3 w_3,$$

where g_i a local equations of $D_{G,\,\chi_i}$ and $c_i,\, d_i \in \mathbb{C}$. Therefore the equations of the natural deformations of X are

$$\begin{aligned} w_1^2 &= (g_3 + d_2 w_2 + d_3 w_3) w_2, \\ w_1 w_2 &= (g_3 + d_2 w_2 + d_3 w_3) w_3, \\ w_1 w_3 &= (g_1 + c_1 w_1 + c_2 w_2)(g_3 + d_2 w_2 + d_3 w_3), \\ w_2^2 &= (g_1 + c_1 w_1 + c_2 w_2)(g_3 + d_2 w_2 + d_3 w_3), \\ w_2 w_3 &= (g_1 + c_1 w_1 + c_2 w_2) w_1, \\ w_3^2 &= (g_1 + c_1 w_1 + c_2 w_2) w_2. \end{aligned} \tag{26}$$

For a general choice of the parameters the morphism $\bar{u}\colon \bar{X} \to Q$ is *not* a Galois cover and an easy computation shows that its branch locus is of the form

$$D_{\bar{X}} = D_1 + \ldots + D_6$$

where the D_i belong to the pencil generated by $D_{G,\,\chi_1}$ and $D_{G,\,\chi_3}$. Then the singular locus of $D_{\bar{X}}$ is given by the 8 points $D_{G,\,\chi_1} \cap D_{G,\,\chi_3}$ and $\mathrm{Sing}(\bar{X})$ consists of the 8 points of type $\frac{1}{4}(1,\, 1)$ locally defined by setting

$$g_1 = g_3 = w_1 = w_2 = w_3 = 0$$

in (26).

(ii) We note that the set of natural deformations $\bar{X}$ of X which keep the G-action is parameterized by the vector space $H^0(\mathcal{O}_Q(2, 2)) \oplus H^0(\mathcal{O}_Q(2, 2))$. In fact, the action of the generator $i = \sqrt{-1}$ of G must be given by

$$w_1 \mapsto -iw_1, \quad w_2 \mapsto -w_2, \quad w_3 \mapsto iw_3$$

and substituting in (26) we obtain $c_1 = c_2 = d_1 = d_3 = 0$.

The G-cover $\bar{X} \to Q$ factors into two double covers

$$\bar{X} \to K \xrightarrow{p} Q$$

where K is a $K3$ surface with 8 ordinary double points and $p\colon K \to Q$ is a double cover branched over $D_{G,\chi_1} + D_{G,\chi_3}$. Let D_{G,χ_2} be a general member in the pencil induced by D_{G,χ_1} and D_{G,χ_3}. Let $\bar{D}_{G,\chi_2} = p^*D_{G,\chi_2}$ and $2\bar{D}_{G,\chi_i} = p^*D_{G,\chi_i}$ for $i = 1, 3$. Since D_{G,χ_2} is linearly equivalent to D_{G,χ_i} for $i = 1, 3$ and a $K3$ surface is simply connected, $\bar{D}_{G,\chi_2}$ is linearly equivalent to $\bar{D}_{G,\chi_1} + \bar{D}_{G,\chi_3}$. Note that both these curves have exactly 8 nodes. The double cover $\tilde{X}$ of K branched over $\bar{D}_{G,\chi_2}$ is deformation equivalent to $\bar{X}$, and $\tilde{X}$ can be realized as the bidouble cover of Q branched over D_{G,χ_1}, D_{G,χ_3} and D_{G,χ_2}. Therefore if one deforms D_{G,χ_2} to a general divisor of bidegree $(2, 2)$ we have a $\mathbb{Q}$-Gorenstein smoothing of $\tilde{X}$ which smoothes all the singularities. Since $\bar{X}$ is a deformation of X and $\tilde{X}$ is deformation equivalent to $\bar{X}$, we have a smooth projective surface in the deformation space of X which is a $\mathbb{Q}$-Gorenstein smoothing of $\tilde{X}$. Finally, we note that each deformation is a $\mathbb{Q}$-Gorenstein one. In fact, $\tilde{X}$ and $\bar{X}$ are double covers of the $K3$ surface K branched over $\bar{D}_{G,\chi_2}$ and $\bar{D}_{G,\chi_1} + \bar{D}_{G,\chi_3}$, respectively. Let $\mathcal{X} \to \Delta$ be a family of double covers of K obtained deforming the branch locus from $\bar{D}_{G,\chi_1} + \bar{D}_{G,\chi_3}$ to $\bar{D}_{G,\chi_2}$. By using the canonical divisor formula for a double cover, it is not hard to see that $K_{\mathcal{X}}$ is a $\mathbb{Q}$-Cartier divisor. Therefore the transitive property of $\mathbb{Q}$-Gorenstein deformations implies that X has a $\mathbb{Q}$-Gorenstein smoothing. Q.E.D.

Remark 4.9. By applying arguments similar to those used in Remark 4.7 and in [Lee10, Section 2], one can construct surfaces of general type with $p_g = 3$, $q = 0$ and $K^2 = k$ $(2 \leq k \leq 8)$ by first taking a $\mathbb{Q}$-Gorenstein smoothing of k singular points of type $\frac{1}{4}(1, 1)$ of $\bar{X}$ and then the minimal resolution of the remaining $8 - k$ singular points of the same type.

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