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Variation of mixed Hodge structures and the positivity for algebraic fiber spaces

Yujiro Kawamata

Abstract.

These are the lecture notes based on earlier papers with some additional new results. New and simple proofs are given for local freeness theorem and the semipositivity theorem. A decomposition theorem for higher direct images of dualizing sheaves of Kollár is extended to the sheaves of differential forms of arbitrary degrees in the case of a well prepared birational model. We will also prove the log versions of some of the results, i.e., the case where we allow horizontal boundary components.

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This is based on my papers [9] and [10] with some additional new results. New and simple proofs are given for local freeness theorem (proved in [8] in the case of the direct image sheaf and in [12] and [13] in the general case), and thereby the semipositivity theorem [8]. A decomposition theorem for higher direct images of dualizing sheaves of Kollár [12] is extended to the sheaves of differential forms of arbitrary degrees in the case of a well prepared birational model. We note that we have to consider differential forms of all degrees in order to use to the full power of the Hodge theory.

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An algebraic fiber space is the relative version of an algebraic variety. A good birational model of an algebraic variety is a smooth model. The corresponding good model of an algebraic fiber space is a *weak semistable model* found by Abramovich and Karu [1]. A *well prepared model* is a birational model which satisfies the same conditions as a weak semistable model except that all the fibers are reduced. One can obtain a weak semistable model from a well prepared model by a finite and flat base change.

From a well prepared model, we shall construct a good topological model by applying a method of log geometry ([6]). We associate a topological space $X^{\log}$, called a *log space*, to a log pair (X, B) consisting of a complex analytic variety with a boundary divisor. The log space $X^{\log}$ can be regarded as a compactification of the open space $X \setminus B$. It is a real analytic manifold with boundary and corners. This compactification is homeomorphic to a closed subset of $X \setminus B$, the complement of a tubular neighborhood of B. But because we have points of the log space $X^{\log}$ above the boundary divisor B, a complicated limiting argument becomes unnecessary in the proofs of our results. For example, the monodromy actions around the singular fibers become very easily handled on the log space, because the singularities of degenerate fibers disappear in this topological model, and the fibers over the boundary have stratifications.

We define the structure sheaf of a log space in a way that it contains the logarithms of local coordinates, and then the sheaves of differential forms that they contain logarithmic differentials. We prove our main result on the existence of a cohomological mixed Hodge complex on any fiber, a concept defined by Deligne [3]. As corollaries, we obtain E_1 and E_2 degenerations of spectral sequences with respect to the Hodge and weight filtrations respectively.

The construction of this paper is as follows. We recall some preliminary results in §§1 to 5, the weak semistable reduction theorem of Abramovich and Karu in §1, the construction of a topological space associated to a log scheme in §2, a criteria for the decomposition of an object in a derived category due to Deligne in §3, the definitions related to the cohomological mixed Hodge complex due also to Deligne in §4, and the Griffiths positivity theorem and the semi-simplicity theorem of the category of variations of polarized Hodge structures in §5.

We shall prove our results in §§6 to 8 on the existence of a mixed Hodge complex. We shall construct weight filtrations on the sheaves on the log spaces in §6. In order to compare the weight filtrations on the topological leval and the De Rham level, we introduce the sheaves of differential forms in §7 and prove Poincaré lemmas. Our construction is justified in §8 that the weight filtration and the Hodge filtration defined on a singular fiber define a cohomological mixed Hodge complex. The degenerations of spectral sequences follow from the general machinery explained in §4. The local freeness follows as a corollary.

In §9 we prove the semipositivity theorem. First we treat the case where the fiber space is weakly semistable and there is no horizontal boundary components following [8]. This is a consequence of the Griffiths positivity theorem for the open part and the boundary considerations. Then the theorem is extended to the case of a well prepared fiber space which may have horizontal boundary components. We prove decomposition theorems in the final §10 for the higher direct images of the constant sheaf and the sheaf of logarithmic differential forms.

We work over $\mathbf{C}$ in this paper. The topology of an algebraic variety we consider is the analytic topology of the underlying complex analytic space.

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§1. Weak semistable reduction

We recall a weak semistable reduction theorem of Abramovich and Karu [1]. A general reference for toroidal varieties is [11].

A *toroidal variety* (X, B) is a pair consisting of a normal variety and an effective reduced divisor such that each point $x \in X$ has a *toric local model* in the following sense: there is an complex analytic neighborhood $x \in U$ such that the pair $(U, B|_U)$ is complex analytically isomorphic to another pair $(U', B'|_{U'})$ which comes from a toric variety (X', B'), a normal variety with an action of an algebraic torus $X' \setminus B'$. We assume moreover that the pair is *strict* or *without self-intersection* in the sense that each irreducible component of B is normal.

A toroidal variety (X, B) is said to be *smooth* if X is smooth and B has only normal crossings. It is *quasi-smooth* if there exists a local toric model of each point which has only abelian quotient singularities.

A *toroidal morphism* $f : (X, B) \to (Y, C)$ between toroidal varieties is one which has a toric local model at each point $x \in X$ in the following sense: there is a toric morphism between local models $f' : (X', B') \to (Y', C)$, i.e., $f'|_{X' \setminus B'} : X' \setminus B' \to Y' \setminus C'$ is a surjective homomorphism of algebraic tori, and f' is equivariant under the torus actions.

A toric model at each point $x \in X$ is described by a rational polyhedral closed convex cone σ_x in a finite dimensional real vector space with an integral lattice. By gluing these cones, we construct a *fan* $\Delta_{X,B}$. Unlike the toric case, the fan is not embedded in a fixed real vector space, and the pair (X, B) cannot be reconstructed from the fan. But the local structure of the pair as well as its birational modifications are completely described by the fan.

The pair (X, B) is quasi-smooth if and only if each cone σ_x is simplicial, and smooth if and only if in addition that the lattice points on the edges of the cone generate a saturated subgroup of the lattice. A rational subdivision of the fan corresponds to a birational modification.

Example 1. Let $f : (X, B) \to (Y, C)$ be a toroidal morphism between smooth toroidal varieties. Take an arbitrary point $x \in X$ and its image $y = f(x)$. Then there exist local coordinates $(x_1, \dots, x_n)$ and $(y_1, \dots, y_m)$ around x and y respectively such that we can write $f^* y_i = \prod_j x_j^{r_{ij}}$, where the r_{ij} are non-negative integers such that $r_{ij} \neq 0$ for at most one i for each j.

More generally, if (X, B) is a quasi-smooth variety instead of a smooth one, then there is an analytic neighborhood $U \subset X$ of the given point $x \in X$ and a finite Galois toroidal covering $\pi : (X', B') \to (U, B|_U)$ from a smooth toroidal variety with a point $x' \in X'$ such that $\pi(x') = x$. There exist local coordinates $(x'_1, \dots, x'_n)$ around $x' \in X$ which are semi-invariant with respect to the Galois action. Therefore we have a similar local expression also in this case.

The following theorem of Abramovich and Karu gives a well prepared birational model of an algebraic fiber space:

Theorem 2 ([1]). *Let $f_0 : X_0 \to Y_0$ be a surjective morphism of projective varieties with geometrically connected fibers defined over a field of characteristic zero, and Z a closed subset of X_0. Then there exist a quasi-smooth projective toroidal variety (X, B), a smooth projective toroidal variety (Y, C), a projective morphism $f : X \to Y$ with geometrically connected fibers, and projective birational morphisms $\mu_X : X \to X_0$, $\mu_Y : Y \to Y_0$ such that $\mu_Y \circ f = f_0 \circ \mu_X$ and which satisfy the following conditions:*

(1) *$f : (X, B) \to (Y, C)$ is a toroidal morphism.*

(2) *All the fibers of f have the same dimension.*

(3) *$\mu_X^{-1}(Z) \subset B$.*

Moreover there exists another smooth projective toroidal variety (Y', C') and a finite surjective morphism $\pi : Y' \to Y$ such that $C' = \pi^{-1}(C)$, and satisfy the following condition:

(4) *The induced morphism $f' : (X', B') \to (Y', C')$ for the normalization (X', B') of the fiber product $(X, B) \times_Y Y'$ is still a toroidal morphism from a quasi-smooth projective toroidal variety to a smooth projective toroidal variety, such that all the fibers of f' are reduced.*

We note that the covering morphism π is flat because Y is smooth. The discriminant locus of π is a normal crossing divisor which properly contains C. Therefore π is not necessarily a toroidal morphism.

Idea of proof. The proof is based on the idea of De Jong's alteration. Because of the assumption on the characteristic of the base field, the ramification theory is simple and the singularities can be resolved.

First we prove the existence of a birational model of $f_0 : X_0 \to Y_0$ which is a toroidal morphism between smooth toroidal varieties. We proceed by induction on the relative dimension $\dim X_0 - \dim Y_0$. By using the generic projection, we decompose the morphism f_0 to two morphisms $g : X_0 \to X_1$ and $h : X_1 \to Y_0$ such that $\dim X_0 = \dim X_1 + 1$, and set $Z_1 = \emptyset$. We replace a birational model of $h : X_1 \to Y_0$ by the induction assumption, while we prepare the morphism $g : X_0 \to X_1$ by the semistable reduction using the moduli space of pointed curves.

Secondly we modify the toroidal model by using the fans associated to the toroidal structure. The morphism becomes equi-dimensional when we subdivide the fans Δ_X and Δ_Y of X and Y so that the images of 1-dimensional cones in Δ_X become 1-dimensional cones in Δ_Y. In order to make the fibers reduced, we apply the covering trick in [8]. Q.E.D.

We call the morphism $f : (X, B) \to (Y, C)$ a *well prepared birational model* of an algebraic fiber space $f_0 : X_0 \to Y_0$, and its finite base change $f' : (X', B') \to (Y', C')$ a *weak semistable model.* The latter is "weak" because the pair (X', B') has some mild singularities, i.e., abelian quotient singularities. But these singularities cause no trouble as explained in the later sections.

§2. Log space

The idea of the real oriented blowing-ups of the (quasi-)smooth toroidal pairs and the application to semistable degenerations of varieties can go back at least to a book [14]. A general reference of this section is [6] and [7].

A *log scheme* (X, M, α) consists of a scheme X, a sheaf of semi-groups with unit M, and a semi-group homomorphism $\alpha : M \to \mathcal{O}_X$ such that $\alpha^{-1}(\mathcal{O}_X^*) \cong \mathcal{O}_X^*$, where the semi-group structure of $\mathcal{O}_X$ is given by the multiplication.

A *morphism* of log schemes $(f, \phi) : (X, M, \alpha) \to (Y, N, \beta)$ consists of a morphism of schemes $f : X \to Y$ and a semi-group homomorphism $\phi : f^{-1}N \to M$ such that $\alpha \circ \phi = f^* \circ f^{-1}\beta$.

Example 3. (1) Let $T = \text{Spec } \mathbf{C}$, and $M_T = \mathbf{R}_{\geq 0} \times S^1$. We define $\alpha_T : M_T \to \mathbf{C}$ by $\alpha_T(r, \theta) = re^{i\theta}$. Then (T, M_T, α_T) is a log scheme.

(2) If (X, B) is a toroidal variety, then the subsheaf of $\mathcal{O}_X$ given by

$$M = \{h \in \mathcal{O}_X \mid h|_{X \setminus B} \in \mathcal{O}_X^*\}$$

defines a log structure on X.

Let (X, M, α) be a log scheme defined over $\mathbf{C}$. We define the *associated log space* $X^{\log}$ to be the set of all log morphisms $(f, \phi) : (T, M_T, \alpha_T) \to (X, M, \alpha)$ from the log scheme defined in the above example. Let $\rho : X^{\log} \to X$ be the natural map defined by $\rho(f, \phi) = \mathrm{Im}(f)$.

Example 4. If $(X, B) = (\mathbf{C}, 0)$, then $X^{\log} = \mathbf{R}_{\geq 0} \times S^1$ and $\rho(r, \theta) = re^{i\theta}$.

Indeed let t be a coordinate on X and assume that $f(T) = t_0$. If $t_0 \neq 0$, then the image of $t \in M_X$ is uniquely determined. If $t_0 = 0$, then the image of t is of the form $(0, \theta)$.

If (X, B) is a quasi-smooth toroidal variety, then the set $X^{\log}$ has a structure of a manifold with boundary and corners as shown in the following proposition. It is a kind of a compactification of the open variety $X \setminus B$:

Proposition 5. *Let (X, B) be a quasi-smooth toroidal variety. Then the following hold:*

(1) *The associated log space $X^{\log}$ has a structure of a real analytic manifold with boundary and corners, and $\rho : X^{\log} \to X$ is a proper continuous map of topological spaces.*

(2) *The complex manifold $X \setminus B$ is homeomorphic to a dense open subset of $X^{\log}$. There is a small analytic open neighborhood U of $B \subset X$ such that $X \setminus U$ is homeomorphic to $X^{\log}$.*

Proof. (1) If $(X, B) = (X_1 \times X_2, p_1^* B_1 + p_2^* B_2)$ for smooth toroidal varieties (X_i, B_i) with associated log spaces $X_i^{\log}$ for $i = 1, 2$, then we have $X^{\log} \cong X_1^{\log} \times X_2^{\log}$ and $\rho \cong \rho_1 \times \rho_2$.

If (X, B) is a quotient of a smooth toroidal variety (X', B') by a finite abelian group G acting on X' leaving B' preserved, then the associated log space $X^{\log}$ is the quotient of $(X')^{\log}$ by the induced group action of G which is fixed point free.

Since (X, B) is covered by standard open subsets as above and the set $X^{\log}$ is globally defined, the log spaces associated to this covering are glued to yield a manifold with boundary and corners.

(2) This is because the map ρ induces a bijection $\rho^{-1}(X \setminus B) \to X \setminus B$. Q.E.D.

Corollary 6. *Let $f : (X, B) \to (Y, C)$ be a well prepared toroidal morphism. Then the associated morphism of log spaces $f^{\log} : X^{\log} \to$*

$Y^{\log}$ is a locally trivial topological fiber bundle. In particular, the higher direct image sheaves $R^p f_*^{\log} \mathbf{Z}_{X^{\log}}$ are locally constant sheaves on $Y^{\log}$.

Proof. It follows from the fact that the induced morphism $f : X \setminus B \to Y \setminus C$ is a locally trivial topological fiber bundle. Q.E.D.

The important advantage of $X^{\log}$ over $X \setminus B$ is that the geometric objects are extended over the singular fibers so that we do not need limiting argument.

§3. Criteria for decompositions

We recall Deligne's criteria for an object of a derived category to be decomposed into its cohomology groups.

Proposition 7 ([2]). *Let $x \in D^b(A)$ be an object of a bounded derived category of an abelian category A. Then the following conditions are equivalent:*

(1) *$x \cong \bigoplus_i H^i(x)[-i]$ in $D^b(A)$.*

(2) *For an arbitrary exact functor $T : D^b(A) \to D(B)$ to the derived category of an arbitrary abelian category B, the Grothendieck–Verdier spectral sequence*

$$E_2^{p,q} = H^p(T(H^q(x))) \Rightarrow H^{p+q}(T(x))$$

degenerate at E_2.

Proof. The condition (1) implies that $H^k(T(x)) \cong \bigoplus_{i+j=k} H^i(T(H^j(x)))$, hence (2).

We prove the converse (2) $\Rightarrow$ (1). Let $T(y) = \bigoplus_j \mathrm{Hom}(H^i(x), y[j])[-j] \in D(\mathbf{Z}\text{-mod})$ for a fixed i. By assumption, the spectral sequence

$$E_2^{p,q} = \mathrm{Hom}(H^i(x), H^q(x)[p]) \Rightarrow \mathrm{Hom}(H^i(x), x[p+q])$$

degenerate at E_2. In particular, the edge homomorphism $E_\infty^i \to E_2^{0,i}$ given by

$$\mathrm{Hom}(H^i(x), x[i]) \to \mathrm{Hom}(H^i(x), H^i(x))$$

is surjective. Let $f_i : H^i(x)[-i] \to x$ be an arbitrary morphism which is mapped to the identity morphism of $H^i(x)[-i]$. Then the sum $\bigoplus_i f_i : \bigoplus_i H^i(x)[-i] \to x$ is a quasi-isomorphism. Q.E.D.

Theorem 8 ([2]). *Let $x \in D^b(A)$ be an object of a bounded derived category of an abelian category A, $u \in Hom(x, x[2])$, and n an integer. Assume that the induced morphisms $u^i : H^{n-i}(x) \to H^{n+i}(x)$ are isomorphisms for all $i \geq 0$. Then $x \cong \bigoplus_i H^i(x)[-i]$ in $D^b(A)$.*

Proof. We define primitive parts of $H^{n-i}(x)$ for $i \geq 0$ by

$$_0H^{n-i}(x) = \operatorname{Ker}(u^{i+1} : H^{n-i}(x) \to H^{n+i+2}(x)).$$

Then we have a Lefschetz decomposition

$$H^{n-i}(x) \cong \bigoplus_{k \geq 0} u^k {}_0H^{n-i-2k}(x),$$
$$H^{n+i}(x) \cong \bigoplus_{k \geq 0} u^{k+i} {}_0H^{n-i-2k}(x).$$

Let $T : D^b(A) \to D(B)$ be an arbitrary exact functor, and

$$E_2^{p,q} = H^p(T(H^q(x))) \Rightarrow H^{p+q}(T(x))$$

be the spectral sequence. We shall prove that the differentials $u_r^{p,q}$: $E_r^{p,q} \to E_r^{p+r,q-r+1}$ vanish for all $r \geq 2$ by induction on r. Assume that we already have $E_2^{p,q} \cong E_r^{p,q}$. Denote ${}_0E_r^{p,q} = {}_0E_2^{p,q} = H^p(T({}_0H^q(x)))$. We have the following commutative diagram

$$\begin{array}{ccc}
{}_0E_r^{p,n-i} & \xrightarrow{d_r^{p,n-i}} & E_r^{p+r,n-i-r+1} \\
{\scriptstyle u^{i+1}}\downarrow & & \downarrow{\scriptstyle u^{i+1}} \\
E_r^{p,n+i+2} & \xrightarrow{d_r^{p,n+i+2}} & E_r^{p+r,n+i-r+3}.
\end{array}$$

The left vertical arrow vanishes while the right vertical arrow is a monomorphism since $r \geq 2$. Therefore $d_r^{p,n-i}$ vanishes on ${}_0E_r^{p,n-i}$. Since $u^k d_r^{p,n-i} = d_r^{p,n-i+2k} u^k$, we have $d_r^{p,q} = 0$ for all p,q. Q.E.D.

§4. Hodge structure and Hodge complex

We recall the definitions concerning Hodge structures and Hodge complexes. The reference is [3].

A *Hodge structure* (HS) of weight n is a pair $(H_{\mathbf{Z}}, F)$, where $H_{\mathbf{Z}}$ is a finitely generated $\mathbf{Z}$-module and F is a decreasing filtration on $H_{\mathbf{C}} = H_{\mathbf{Z}} \otimes \mathbf{C}$ which induces a direct sum decomposition

$$H_{\mathbf{C}} = \bigoplus_{p+q=n} F^p(H_{\mathbf{C}}) \cap \overline{F^q(H_{\mathbf{C}})}.$$

F is called a *Hodge filtration.* We set $H^{p,q} = F^p(H_{\mathbf{C}}) \cap \overline{F^q(H_{\mathbf{C}})}$.

It is said to be *polarizable* if there is a non-degenerate bilinear form $Q : H_{\mathbf{R}} \times H_{\mathbf{R}} \to \mathbf{R}$ for $\mathbf{R} = H_{\mathbf{Z}} \otimes \mathbf{R}$ such that the following Hodge–Riemann index theorem holds:

$$i^w(-1)^{q+1}Q(H^{p,q}, \bar{H}^{p,q}) \gg 0,$$
$$Q(H^{p,q}, \bar{H}^{p',q'}) = 0 \text{ if } p \neq p',$$

for all $p + q = p' + q' = n$.

A *mixed Hodge structure* (MHS) is a triple consisting of a finitely generated $\mathbf{Z}$-module $H_{\mathbf{Z}}$, an increasing filtration W on $H_{\mathbf{Q}} = H_{\mathbf{Z}} \otimes \mathbf{Q}$, and a decreasing filtration F on $H_{\mathbf{C}} = H_{\mathbf{Z}} \otimes \mathbf{C}$ such that $(\mathrm{Gr}_n^W(H_{\mathbf{Q}}), F)$ becomes a Hodge $\mathbf{Q}$-structure of weight n, a Hodge structure tensorized by $\mathbf{Q}$, where F denotes the induced filtration from F by abuse of notation.

W is called a *weight filtration.* It is *graded polarizable* if each graded piece $\mathrm{Gr}_n^W(H_{\mathbf{Q}})$ is individually polarizable.

Example 9. Let X be a smooth projective variety of dimension n. Then the cohomology group $H^k(X, \mathbf{Z})$ carries a Hodge structure of weight k. If we take its primitive part, then we obtain a polarizable Hodge structure as follows.

Let $L \in H^2(X, \mathbf{Z})$ be the cohomology class of an ample divisor. For each integer $i \geq 0$, the cup product with L^i yields an isomorphism $L^i : H^{n-i}(X, \mathbf{Q}) \to H^{n+i}(X, \mathbf{Q})$. The *primitive part* ${}_0H^{n-i}(X, \mathbf{Q}) \subset H^{n-i}(X, \mathbf{Q})$ is defined as the kernel of the homomorphism given by the cup product with L^{i+1}. The *Lefschetz decomposition theorem* says that there is a direct sum decomposition

$$H^k(X, \mathbf{Q}) = \bigoplus_{i \geq 0} L^i{}_0H^{k-2i}(X, \mathbf{Q})$$

for $k \leq n$. The Hodge–Riemann index theorem says that the primitive part ${}_0H^k(X, \mathbf{Q})$ carries a polarizable $\mathbf{Q}$-Hodge structure of weight k. The polarization comes from the cup product:

$$Q(u, v) = (u \cup v \cup L^{n-k})[X]$$

for $u, v \in {}_0H^k(X, \mathbf{Q})$.

A *Hodge complex (HC)* of weight n is a triple $H = (H_{\mathbf{Z}}, H_{\mathbf{C}}, F)$, where $H_{\mathbf{Z}} \in D^+(\mathbf{Z})$ is a left bounded complex of abelian groups whose cohomology groups $H^k(H_{\mathbf{Z}})$ are of finite type, and $H_{\mathbf{C}} \in D^+F(\mathbf{C})$ is a left bounded complex of $\mathbf{C}$-modules with a decreasing filtration F such that the following conditions are satisfied:

(1) There is an isomorphism $H_{\mathbf{Z}} \otimes \mathbf{C} \cong H_{\mathbf{C}}$.

(2) The spectral sequence associated to the filtration F

$$_F E_1^{p,q} = H^{p+q}(\mathrm{Gr}_F^p(H_{\mathbf{C}})) \Rightarrow H^{p+q}(H_{\mathbf{C}})$$

degenerates at E_1.

(3) The cohomology groups $H^k(H_{\mathbf{Z}})$ with the filtrations on the $H^k(H_{\mathbf{C}})$ induced from F give Hodge structures of weight $n+k$ for all k.

If H is a Hodge complex of weight n, then the shifted complex $H[m]$ is a Hodge complex of weight $n+m$.

A *mixed Hodge complex* (MHC) is a quadruple $H = (H_{\mathbf{Z}}, H_{\mathbf{C}}, W, F)$, where $H_{\mathbf{Z}} \in D^+(\mathbf{Z})$ is a left bounded complex of abelian groups with an increasing filtration W on $H_{\mathbf{Q}} = H_{\mathbf{Z}} \otimes \mathbf{Q}$, and $H_{\mathbf{C}} \in D^+F_2(\mathbf{C})$ is a left bounded complex of $\mathbf{C}$-modules with an increasing filtration W and a decreasing filtration F such that the following conditions are satisfied:

(1) There is an isomorphism $(H_{\mathbf{Q}}, W) \otimes \mathbf{C} \cong (H_{\mathbf{C}}, W)$.

(2) $(\mathrm{Gr}_n^W(H_{\mathbf{Q}}), F)$ is a Hodge $\mathbf{Q}$-complex of weight n, a Hodge complex tensorized by $\mathbf{Q}$, where F denotes the induced filtration from F by abuse of notation.

Then it follows that the triple $(H^k(H_{\mathbf{Z}}), W[k], F)$ is a mixed Hodge structure.

A *cohomological Hodge complex* (CoHC) of weight n on a topological space X is a triple $H = (H_{\mathbf{Z}}, H_{\mathbf{C}}, F)$, where $H_{\mathbf{Z}} \in D^+(\mathbf{Z}_X)$ is a left bounded complex of constructible sheaves of abelian groups on X, and $H_{\mathbf{C}} \in D^+F(X)$ is a left bounded complex of sheaves of $\mathbf{C}$-modules on X with a decreasing filtration F such that the following conditions are satisfied:

(1) There is an isomorphism $H_{\mathbf{Z}} \otimes \mathbf{C} \cong H_{\mathbf{C}}$.

(2) The direct image complex $R\Gamma(X, H) = (R\Gamma(X, H_{\mathbf{Z}}), R\Gamma(X, H_{\mathbf{C}}, F))$ is a Hodge complex of weight n.

A *cohomological mixed Hodge complex* (CoMHC) on a topological space X is a quadruple $H = (H_{\mathbf{Z}}, H_{\mathbf{C}}, W, F)$, where $H_{\mathbf{Z}} \in D^+(\mathbf{Z}_X)$ is a left bounded complex of constructible sheaves of abelian groups on X with an increasing filtration W on $H_{\mathbf{Q}} = H_{\mathbf{Z}} \otimes \mathbf{Q}$, and $H_{\mathbf{C}} \in D^+F_2(X)$ is a left bounded complex of sheaves of $\mathbf{C}$-modules on X with an increasing filtration W and a decreasing filtration F such that the following conditions are satisfied:

(1) There is an isomorphism $(H_{\mathbf{Q}}, W) \otimes \mathbf{C} \cong (H_{\mathbf{C}}, W)$.

(2) $(\mathrm{Gr}_n^W(H_{\mathbf{Q}}), \mathrm{Gr}_n^W(H_{\mathbf{C}}), F)$ is a cohomological Hodge $\mathbf{Q}$-complex weight n on X.

In this case, it follows that the direct image complex

$$R\Gamma(X, H) = (R\Gamma(X, H_{\mathbf{Z}}), R\Gamma(X, H_{\mathbf{C}}), W, F)$$

becomes a mixed Hodge complex.

We summarize the above definitions as follows:

$$\begin{tikzcd} \mathrm{CoMHC} \arrow[r, "R\Gamma"] \arrow[d, "\mathrm{Gr}^W"'] & \mathrm{MHC} \arrow[r, "H^*"] \arrow[d, "\mathrm{Gr}^W"'] & \mathrm{MHS} \arrow[d, "\mathrm{Gr}^W"'] \\ \mathrm{CoHC} \arrow[r, "R\Gamma"] & \mathrm{HC} \arrow[r, "H^*"] & \mathrm{HS} \end{tikzcd}$$

Example 10. Let X be a smooth projective variety, B a normal crossing divisor, and $i : X \setminus B \to X$ the open immersion. Then the direct image sheaf $H = Ri_*\mathbf{Z}_{X\setminus B}$ carries a structure of a cohomological mixed Hodge complex. We can also write $H = R\rho_*\mathbf{Z}_{X^{\log}}$ for the real oriented blow-up $\rho : X^{\log} \to X$.

The weight filtration on $H_{\mathbf{Q}}$ on the $\mathbf{Q}$-level is given by the canonical filtration

$$W_q(H_{\mathbf{Q}}) = \tau_{\leq q}(Ri_*\mathbf{Z}_{X\setminus B})$$

where the truncation $\tau_{\leq q}(K^\bullet)$ of a complex $K^\bullet$ is defined by

$$\tau_{\leq q}(K^\bullet)_i = \begin{cases} K_i & \text{if } i < q, \\ \mathrm{Ker}(K_q \to K_{q+1}) & \text{if } i = q, \\ 0 & \text{if } i > q. \end{cases}$$

The $\mathbf{C}$-level complex is given by the sheaves of logarithmic differential forms $H_{\mathbf{C}} = \Omega_X^\bullet(\log B)$. We have a quasi isomorphism

$$Ri_*\mathbf{C}_{X\setminus B} \cong \Omega_X^\bullet(\log B)$$

by the Poincaré lemma. The weight filtration W on the $\mathbf{C}$-level is given by the order of log poles; $W_q(H_{\mathbf{C}})$ is a complex consisting of differential forms whose log poles have order at most q. The residue homomorphisms give an isomorphism of filtered complexes

$$(H_{\mathbf{Q}}, W) \otimes \mathbf{C} \cong (H_{\mathbf{C}}, W).$$

The Hodge filtration is given by the stupid filtration:

$$F^p(H_{\mathbf{C}}) = \Omega_X^{\geq p}(\log B).$$

For each non-negative integer t, let $B^{[t]}$ be the disjoint union of the irreducible components of intersections of t different irreducible components of B. We regard $B^{[0]} = X$, and $B^{[1]}$ is the normalization of B.

$B^{[t]}$ is $\dim X - t$ equi-dimensional. The following isomorphisms given by the residue homomorphisms are fundamental:

$$\mathrm{Gr}_q^W(\Omega_X^\bullet(\log B)) \cong \Omega_{B^{[q]}}^\bullet[-q],$$
$$F^p(\mathrm{Gr}_q^W(\Omega_X^\bullet(\log B))) \cong \Omega_{B^{[q]}}^{\geq p-q}[-q].$$

We denote by $F(-q)$ the shifted filtration defined by $F(-q)^p = F^{p-q}$. Then the triple

$$(\mathbf{Z}_{B^{[q]}}[-q], \Omega_{B^{[q]}}^\bullet[-q], F(-q)[-q])$$

is a cohomological Hodge complex of weight $-q+2q = q$, where the shift of F is counted twice because the opposite filtration $\bar{F}$ is also shifted. Therefore the quadruple

$$(Ri_*\mathbf{Z}_{X\setminus B}, \Omega_X^\bullet(\log B), W, F)$$

is a cohomological mixed Hodge complex.

Theorem 11 ([3]). *Let H be a cohomological mixed Hodge complex on a topological space X. Then the following hold:*

(1) *The spectral sequence associated to the Hodge filtration*

$${}_FE_1^{p,q} = H^{p+q}(Gr_F^p(H_{\mathbf{C}})) \Rightarrow H^{p+q}(H_{\mathbf{C}})$$

degenerates at E_1.

(2) *The spectral sequence associated to the weight filtration*

$${}_WE_1^{p,q} = H^{p+q}(Gr_{-p}^W(H_{\mathbf{Q}})) \Rightarrow H^{p+q}(H_{\mathbf{Q}})$$

degenerates at E_2.

§5. Semipositivity and semi-simplicity

Let H be a locally free sheaf on a complex manifold Y with a C^∞ hermitian metric h_H. Then there is a connection $\nabla_H : H \to H \otimes \Omega_Y^1$ which is compatible with the holomorphic structure of H and the metric h_H. Let Θ_H be the curvature form. It is a $\mathrm{Hom}(H,H)$ valued C^∞ differential forms of type $(1,1)$ given by the formula

$$\bar{\partial}\partial h_H(u,v) = -h_H(\nabla_H u, \nabla_H v) + h_H(\Theta_H u, v)$$

where u, v are holomorphic local sections of H and the equality holds as C^∞ differential form of type $(1,1)$. The first Chern class $c_1(H)$ is given by the C^∞ differential form $\frac{i}{2\pi}\mathrm{Tr}(\Theta)$.

Let F be a locally free subsheaf of H with an injection $i : F \to H$ and a surjection $p : H \to G$ to the quotient sheaf $G = H/F$. The hermitian metric h_H induces hermitian metrics h_F and h_G on F and G respectively. We can compare the curvature forms of these metrics:

Proposition 12. *Define the second fundamental form by*

$$b = p \circ \nabla_H \circ i \in \mathit{Hom}(F, G) \otimes \Omega^1_Y$$

and let u, v be local holomorphic sections of F. Then

$$h_F(\Theta_F u, v) = h_H(\Theta_H u, v) - h_G(b(u), b(v)).$$

Proof. Omitted. Q.E.D.

A *variation of mixed Hodge structures* (VMHS) over a complex manifold Y consists of a locally constant sheaf $H_{\mathbf{Z}}$, an increasing filtration W by locally constant subsheaves on a locally constant sheaf $H_{\mathbf{Q}} = H_{\mathbf{Z}} \otimes \mathbf{Q}$, and a decreasing filtration F by locally free subsheaves on a locally free sheaf $H_{\mathbf{C}} = H_{\mathbf{Z}} \otimes \mathcal{O}_Y$ which satisfy the following conditions:

(1) For each $y \in Y$, the data induced on the fiber $(H_{\mathbf{Z},y}, (H_{\mathbf{Q},y}, W), (H_{\mathbf{C}} \otimes \mathbf{C}_y, W, F))$ is a mixed Hodge structure.

(2) The connection, called the Gauss–Manin connection

$$\nabla : H_{\mathbf{C}} \to H_{\mathbf{C}} \otimes \Omega^1_Y$$

for which sections of $H_{\mathbf{Z}}$ are flat, satisfies the Griffiths transversality:

$$\nabla : F^p(H_{\mathbf{C}}) \to F^{p-1}(H_{\mathbf{C}}) \otimes \Omega^1_Y$$

for all p.

Example 13. Let $f : (X, B) \to (Y, C)$ be a well prepared toroidal morphism, and n an integer. We denote by $f^o : X \setminus f^{-1}(C) \to Y \setminus C$ the restriction. Let

$$\Omega^1_{X/Y}(\log) = \Omega^1_X(\log B)/f^*\Omega^1_Y(\log C),$$

$$\Omega^p_{X/Y}(\log) = \bigwedge^p \Omega^1_{X/Y}(\log).$$

Then $R^n f^o_* \mathbf{Z}_{X \setminus f^{-1}(C)}$ carries a structure of VMHS, and the associated Hodge to de Rham spectral sequence

$$E_1^{p,q} = R^q f^o_* \Omega^p_{X/Y}(\log) \Rightarrow R^{p+q} f^o_* (f^o)^{-1} \mathcal{O}_{Y \setminus C}$$

degenerates at E_1 ([3]), and gives the Hodge filtration.

If $W_q(H_{\mathbf{Q}}) = H_{\mathbf{Q}}$ and $W_{q-1}(H_{\mathbf{Q}}) = 0$, then our data is called a *variation of Hodge structures* (VHS) *of weight* q.

A VHS is said to be *polarizable* if there is a locally constant symmetric bilinear form $Q : H_{\mathbf{R}} \times H_{\mathbf{R}} \to \mathbf{R}_Y$ which induces polarizations of Hodge structures on fibers. A VMHS is said to be *graded polarizable* if each graded pieces $\mathrm{Gr}_q^W(H_{\mathbf{Q}})$ is a polarizable variation of Hodge structures.

Theorem 14 ([5]). *Let $(H_{\mathbf{Z}}, F, Q)$ be a polarized variation of Hodge structures on a complex manifold Y. Assume that $F^m(H_{\mathbf{C}}) \neq 0$ and $F^{m+1}(H_{\mathbf{C}}) = 0$. Then the Gauss–Manin connection induces a connection on $F^m(H_{\mathbf{C}})$ whose curvature is positive semi-definite.*

Proof. Let $i : F^m(H_{\mathbf{C}}) \to H_{\mathbf{C}}$ and $p : H_{\mathbf{C}} \to H_{\mathbf{C}}/F^m(H_{\mathbf{C}})$ be the natural homomorphisms. We introduce a hermitian metric h_H on $H_{\mathbf{C}}$ by $h_H(u,v) = Q(u,\bar{v})$. The *Gauss–Manin connection* ∇_H is flat, and $\Theta_H = 0$.

Let h_F and h_G be the induced hermitian metrics on the subsheaf $F^m(H_{\mathbf{C}})$ and the quotient sheaf $H_{\mathbf{C}}/F^m(H_{\mathbf{C}})$ respectively. We calculate the curvature Θ_F using the second fundamental form $b = p \circ \nabla_H \circ i$. By the Griffiths transversality, the image of b lies in $\mathrm{Gr}_{m-1}^W(H_{\mathbf{C}}) \otimes \Omega_Y^1$.

For holomorphic sections u, v of $F^m(H_{\mathbf{C}})$, we have

$$Q(\Theta_F u, \bar{v}) = Q(\Theta_H u, \bar{v}) - Q(b(u), \overline{b(v)})$$

by Proposition 12. Since the non-degenerate bilinear form Q has opposite signs on $\mathrm{Gr}_m^W(H_{\mathbf{C}})$ and $\mathrm{Gr}_{m-1}^W(H_{\mathbf{C}})$, we obtain the desired semipositivity. Q.E.D.

The following semi-simplicity theorem is very useful:

Theorem 15 ([3]). *Let Y be a complex manifold. Then the category of polarized variations of Hodge structures on Y is semi-simple in the sense that arbitrary injective homomorphism splits over $\mathbf{Q}$.*

Proof. Let $H_1 \to H$ be an injective homomorphism of variations of Hodge structures. We take the orthogonal complement $H_{2,\mathbf{Q}}$ of $H_{1,\mathbf{Q}}$ in $H_{\mathbf{Q}}$ with respect to the polarization Q. Then $H_2 = H \cap H_{2,\mathbf{Q}}$ is again a polarized variation of Hodge structures. Indeed, if $u = \sum u^{p,q} \in H_{2,\mathbf{C}}$, then $Q(u^{p,q}, v) = Q(u^{p,q}, v^{q,p}) = Q(u, v^{q,p}) = 0$ for all $v = \sum v^{p,q} \in H_{1,\mathbf{C}}$, hence $u^{p,q} \in H_{2,\mathbf{C}}$. Moreover we have $H_{1,\mathbf{C}} \cap H_{2,\mathbf{C}} = 0$ because of the Riemann–Hodge index theorem. Q.E.D.

§6. Weight filtration of the Q-level complex

Let $f : (X, B) \to (Y, C)$ be a weakly semistable toroidal model of an algebraic fiber space. Let $\rho_X : X^{\log} \to X$ and $\rho_Y : Y^{\log} \to Y$ be the associated log spaces with the induced continuous map $f^{\log} : X^{\log} \to Y^{\log}$ which is topologically locally trivial. Let $y \in Y$ be an arbitrary point, and $\bar{y} \in Y^{\log}$ a point above y. If y is contained in exactly s irreducible components of C, then $\rho_Y^{-1}(y)$ is homeomorphic to $(S^1)^s$, and $\bar{y}$ is parametrized by angles $(\theta_1, \dots, \theta_s)$.

Let $E = f^{-1}(y)$ and $D = (f^{\log})^{-1}(\bar{y})$ be the fibers. We shall put a weight filtration on a complex of sheaves $R(\rho_X)_*\mathbf{Z}_D$ on E in this section.

Let $\{E_i\}$ be the set of irreducible components of E. A *closed stratum* E_I of E is an irreducible component of the intersection of some of the E_i. Let $t = t(E_I) = \operatorname{codim}_E E_I$. We note that t is not necessarily equal to the number of the E_i which contain E_I. Let $D_I = \rho_X^{-1} E_I \cap D$. Then $\rho_X^{-1} E_I$ is homeomorphic to the product $D_I \times \rho_Y^{-1}(y)$. We denote $E^{[t]} = \coprod_{t(E_I)=t} E_I$ and $D^{[t]} = \coprod_{t(E_I)=t} D_I$.

Let G_I be the union of all the strata which is properly contained in E_I. Then (E_I, G_I) is a quasi-smooth toroidal variety. Let $\rho_I : E_I^{\log} \to E_I$ be the associated log space. We can see that $\rho_X : D_I \to E_I^{\log}$ is a t-times direct sum of oriented S^1 fiber bundles corresponding to the normal directions of E_I in E.

We recall the definition of a convolution of a complex of objects in a triangulated category [4]. Let

$$a_0 \to a_1 \to \cdots \to a_{n-1} \to a_n$$

be a complex of objects. If there exists a sequence of distinguished triangles

$$b_{k-1} \to a_{k-1} \to b_k \to b_{k-1}[1]$$

for $0 < k \le n$ with an isomorphism $b_n \to a_n$, then b_0 is said to be a *convolution* of the complex. A convolution may not exist and may not be unique if it exists.

We have a Mayor–Vietoris exact sequence

$$0 \to \mathbf{Z}_D \to \mathbf{Z}_{D^{[0]}} \to \mathbf{Z}_{D^{[1]}} \to \mathbf{Z}_{D^{[2]}} \to \cdots .$$

In other words, $\mathbf{Z}_D$ is a convolution of a complex

$$\mathbf{Z}_{D^{[0]}} \to \mathbf{Z}_{D^{[1]}} \to \mathbf{Z}_{D^{[2]}} \to \cdots .$$

Thus $R(\rho_X)_*\mathbf{Z}_D$ is a convolution of the following complex

$$R(\rho_X)_*\mathbf{Z}_{D^{[0]}} \to R(\rho_X)_*\mathbf{Z}_{D^{[1]}} \to R(\rho_X)_*\mathbf{Z}_{D^{[2]}} \to \cdots .$$

We define a weight filtration on $R(\rho_X)_*\mathbf{Z}_D$ as a convolution of canonical truncations:

Proposition 16. *For any integer q, the following complex*

$$\tau_{\le q}(R(\rho_X)_*\mathbf{Z}_{D^{[0]}}) \to \tau_{\le q+1}(R(\rho_X)_*\mathbf{Z}_{D^{[1]}}) \to \dots$$

has a convolution $W_q(R(\rho_X)_*\mathbf{Z}_D)$, where τ denotes the canonical filtration, i.e.,

$$H^p(\tau_{\le q+t}(R(\rho_X)_*\mathbf{Z}_{D^{[t]}})) = \begin{cases} H^p(R(\rho_X)_*\mathbf{Z}_{D^{[t]}}) & \text{if } p \le q+t, \\ 0 & \text{otherwise,} \end{cases}$$

which satisfies the following conditions:

(1) *$W_q(R(\rho_X)_*\mathbf{Z}_D) \cong 0$ for sufficiently small q, and $W_q(R(\rho_X)_*\mathbf{Z}_D) \cong R(\rho_X)_*\mathbf{Z}_D$ for sufficiently large q.*

(2) *There are distinguished triangles*

$$Gr_q^W(R(\rho_X)_*\mathbf{Z}_D)[-1] \to W_{q-1}(R(\rho_X)_*\mathbf{Z}_D) \to W_q(R(\rho_X)_*\mathbf{Z}_D) \\ \to Gr_q^W(R(\rho_X)_*\mathbf{Z}_D)$$

such that

$$Gr_q^W(R(\rho_X)_*\mathbf{Z}_D) \cong \bigoplus_{t\ge 0} R^{q+t}(\rho_X)_*\mathbf{Z}_{D^{[t]}}[-q-2t].$$

Proof. We denote $a_k = R(\rho_X)_*\mathbf{Z}_{D^{[k]}}$ and $a_k^q = \tau_{\le q+k}(R(\rho_X)_*\mathbf{Z}_{D^{[k]}})$. We define objects $\bar{a}_k^q$ by distinguished triangles

$$a_k^q \to a_k \to \bar{a}_k^q \to a_k^q[1].$$

Let b_k be the sequences of objects appearing in the process of convolutions of the a_k. We shall construct a convolution of the a_k^q, and denote by the b_k^q a sequence of objects appearing in the process of this convolution. We shall also construct a sequence of objects $\bar{b}_k^q$ from the objects $\bar{a}_k^q$ by distinguished triangles

$$\bar{b}_k^q \to \bar{a}_k^q \to \bar{b}_{k+1}^q \to \bar{b}_k^q[1].$$

We proceed by the descending induction on k for a fixed q. Assume that we have already constructed b_k^q and $\bar{b}_k^q$ for $k \ge k_0$. We have $H^i(a_k^q) = 0$ unless $i \le q+k$, and $H^i(\bar{a}_k^q) = 0$ unless $i \ge q+k+1$. Therefore $H^i(\bar{b}_k^q) = 0$ unless $i \ge q+k+1$ for $k \ge k_0$.

We have a commutative diagram

$$\begin{array}{ccccc}
a^q_{k_0-1} & \longrightarrow & a_{k_0-1} & \longrightarrow & \bar{a}^q_{k_0-1} \\
 & & \downarrow & & \\
b^q_{k_0} & \longrightarrow & b_{k_0} & \longrightarrow & \bar{b}^q_{k_0} \\
\downarrow & & \downarrow & & \downarrow \\
a^q_{k_0} & \longrightarrow & a_{k_0} & \longrightarrow & \bar{a}^q_{k_0}.
\end{array}$$

We have $\mathrm{Hom}(a^q_{k_0-1}, \bar{b}^q_{k_0}) = \mathrm{Hom}(a^q_{k_0-1}, \bar{b}^q_{k_0}[-1]) = 0$ by the comparison of the degrees. Therefore there exists a unique morphism $a^q_{k_0-1} \to b^q_{k_0}$ which makes the diagram commutative. Moreover the composition of morphisms $a^q_{k_0-1} \to b^q_{k_0} \to a^q_{k_0}$ coincides with the given morphism $a^q_{k_0-1} \to a^q_{k_0}$ by the same reason. Thus we construct a convolution of the a^q_k in a unique way. It follows that there exists a morphism $\bar{a}^q_{k_0-1} \to \bar{b}^q_{k_0}$ to make the whole diagram commutative. Then we construct $\bar{b}^q_{k_0-1}$ by the distinguished triangle above. The commutativity of the diagram in the next step is guaranteed by the octahedral axiom.

Now we prove the properties of the convolution. (1) is clear. (2) is also clear because any morphism

$$H^{q+k}(R(\rho_X)_*\mathbf{Z}_{D^{[k]}})[-q-k] \to H^{q+k+1}(R(\rho_X)_*\mathbf{Z}_{D^{[k+1]}})[-q-k-1]$$

is trivial. Q.E.D.

Corollary 17. *The monodromy actions of the group $\pi_1(\rho_Y^{-1}(y), \bar{y}) \cong \mathbf{Z}^s$ on the cohomology groups $H^p(D, \mathbf{Z}_D)$ are unipotent for all p.*

Proof. We have $H^p(D, \mathbf{Z}_D) = \mathbf{H}^p(E, R(\rho_X)_*\mathbf{Z}_D)$. The monodromy actions on the graded pieces $\mathbf{H}^p(E, \mathrm{Gr}^W_q(R(\rho_X)_*\mathbf{Z}_D))$ are trivial, because $\rho_X^{-1}(E_I)$ are just homeomorphic to the product spaces. By the spectral sequence

$$E_2^{p,q} = \mathbf{H}^{p+q}(E, \mathrm{Gr}^W_{-p}(R(\rho_X)_*\mathbf{Z}_D)) \Rightarrow H^{p+q}(D, \mathbf{Z}_D)$$

we conclude the proof. Q.E.D.

Example 18. Let $X = \mathrm{Spec}\,\mathbf{C}[x_1, x_2]$, $B = \mathrm{div}(x_1x_2)$, $Y = \mathrm{Spec}\,\mathbf{C}[y]$, $C = \mathrm{div}(y)$, and define $f : (X.B) \to (Y, C)$ by $f^*y = x_1x_2$. Then $f^{\log} : X^{\log} \to Y^{\log}$ is given by $(r_1, r_2, \theta_1, \theta_2) \mapsto (r_1r_2, \theta_1 + \theta_2)$. Let $\bar{y} = (0, \phi) \in Y^{\log}$. Then $(f^{\log})^{-1} = D_1 \cup D_2$, where $D_1 = \{(r_1, 0, \theta_1, \phi - \theta_1)\}$ and $D_2 = \{(0, r_2, \phi - \theta_2, \theta_2)\}$. There are homeomorphisms $D_1 \cong \mathbf{R}_{\geq 0} \times S^1$ and $D_2 \cong \mathbf{R}_{\geq 0} \times S^1$ given by $(r_1, 0, \theta_1, \phi - \theta_1) \mapsto (r_1, \theta_1)$

and $(0, r_2, \phi - \theta_2, \theta_2) \mapsto (r_2, \theta_2)$. Then the gluing of D_1 and D_2 is given by $(0, \theta) \mapsto (0, \phi - \theta)$ on S^1. In other words, the gluing is twisted by the argument ϕ of $\bar{y}$. If $\bar{y}$ goes around S^1, then the gluing is twisted correspondingly. This is the action of the monodromy.

§7. Sheaves on log spaces

Let (X, B) be a quasi-smooth toroidal variety, and $\rho : X^{\log} \to X$ the associated log space. We shall define the "structure sheaf" $\mathcal{O}_{X^{\log}}$ and the De Rham complex $\Omega^\bullet_{X^{\log}}$ in this section.

First we consider the case where $X = \mathbf{C}^n$ with coordinates $(x_1, \dots, x_n)$ and $B = \mathrm{div}(x_1 \dots x_r)$. Then we define

$$\mathcal{O}_{X^{\log}} = \sum_{k_1, \dots, k_r \in \mathbf{Z}_{\geq 0}} \rho^{-1}(\mathcal{O}_X) \prod_{i=1}^{r} (\log x_i)^{k_i}$$

where the symbols $\log x_i$ are regarded as holomorphic functions over the open subset $\rho^{-1}(X \setminus \mathrm{div}(x_i))$, while locally constant sections over the boundary $\rho^{-1}(\mathrm{div}(x_i))$. They are algebraically independent as long as they are symbols. We note that the right hand side of the above definition does not change if we replace the symbols $\log x_i$ by the shifts $\log x_i + c_i$ for arbitrary constants $c_i \in \mathbf{C}$. Thus $\mathcal{O}_{X^{\log}}|_{\rho^{-1}(X \setminus B)} = \mathcal{O}_X|_{X \setminus B}$, and the stalk $\mathcal{O}_{X^{\log}, \bar{x}}$ at a point $\bar{x} \in \rho^{-1}(B)$ is isomorphic to a polynomial ring $\mathcal{O}_{X,0}[t_{i_1}, \dots, t_{i_l}]$, where the t_i are independent variables corresponding to the symbols $\log x_i$ if $\rho(\bar{x})$ is contained in exactly l irreducible components $B_{i_1}, \dots, B_{i_l}$ of B.

Next if $\pi : (X, B) \to (X, B)/G = (X', B')$ is a quotient of the above pair (X, B) by a finite abelian group G, then we define $\mathcal{O}_{(X')^{\log}} = (\pi^{\log}_* \mathcal{O}_{X^{\log}})^G$. We note that the action of G on $X^{\log}$ is free, hence the stalks are isomorphic to polynomial rings for suitable number of variables.

In the general case, the structure sheaves for an open covering of $X^{\log}$ are glued together to yield the structure sheaf of $X^{\log}$ because they coincide with the usual structure sheaves when restricted to $X \setminus B$.

The sheaf of differentials is defined by the following formula:

$$\Omega^p_{X^{\log}} = \rho^{-1}(\Omega^p_X(\log B)) \otimes_{\rho^{-1}(\mathcal{O}_X)} \mathcal{O}_{X^{\log}}.$$

The differential $d : \Omega^p_{X^{\log}} \to \Omega^{p+1}_{X^{\log}}$ is defined by the rule $d(\log x) = dx/x$. It follows that the De Rham complex $\Omega^\bullet_{X^{\log}}$ does not have higher cohomologies:

Proposition 19 (Poincaré lemma [9]).

$$\begin{aligned}
&\mathbf{C}_{X^{\log}} \cong \Omega^{\bullet}_{X^{\log}},\\
&R\rho_*\Omega^p_{X^{\log}} \cong \Omega^p_X(\log B),\\
&R\rho_*\mathbf{C}_{X^{\log}} \cong \Omega^{\bullet}_X(\log B).
\end{aligned}$$

Let $f : (X, B) \to (Y, B)$ be a well prepared toroidal morphism. Then the sheaves of relative differential forms are defined similarly:

$$\Omega^p_{X^{\log}/Y^{\log}} = \rho^{-1}(\Omega^p_{X/Y}(\log)) \otimes_{\rho^{-1}(\mathcal{O}_X)} \mathcal{O}_{X^{\log}}.$$

In the following, we denote $\rho' = (\rho_X, f^{\log}) : X^{\log} \to X \times_Y Y^{\log}$ with the projections $q_1 : X \times_Y Y^{\log} \to X$ and $q_2 : X \times_Y Y^{\log} \to Y^{\log}$.

Proposition 20 (Poincaré lemma 2 [9]).

$$\begin{aligned}
&(f^{\log})^{-1}(\mathcal{O}_{Y^{\log}}) \cong \Omega^{\bullet}_{X^{\log}/Y^{\log}},\\
&R\rho'_*\Omega^p_{X^{\log}/Y^{\log}} \cong q_1^{-1}(\Omega^p_{X/Y}(\log)) \otimes_{q_2^{-1}\rho_Y^{-1}(\mathcal{O}_Y)} q_2^{-1}(\mathcal{O}_{Y^{\log}}),\\
&R\rho'_*(f^{\log})^{-1}(\mathcal{O}_{Y^{\log}}) \cong q_1^{-1}(\Omega^{\bullet}_{X/Y}(\log)) \otimes_{q_2^{-1}\rho_Y^{-1}(\mathcal{O}_Y)} q_2^{-1}(\mathcal{O}_{Y^{\log}}).
\end{aligned}$$

Now we define sheaves on the fiber D and its strata D_I. The restriction of ρ_X to D is denoted by ρ_D. Let y_i be the local coordinates at $y \in Y$ which define irreducible components of C passing through y. Then the symbols $\log y_i$ are replaced by 0 on D. we write this fact by $\sim$ in the following definition:

$$\begin{aligned}
&\mathcal{O}_D = (\mathcal{O}_{X^{\log}} \otimes_{\rho_X^{-1}(\mathcal{O}_X)} \rho_D^{-1}(\mathcal{O}_E))/\sim,\\
&\mathcal{O}_{D_I} = (\mathcal{O}_{X^{\log}} \otimes_{\rho_X^{-1}(\mathcal{O}_X)} \rho_D^{-1}(\mathcal{O}_{E_I}))/\sim,\\
&\Omega^p_D = \rho_D^{-1}\Omega^p_{E/\mathbf{C}}(\log) \otimes_{\rho_D^{-1}(\mathcal{O}_E)} \mathcal{O}_D = \Omega^p_{X^{\log}/Y^{\log}} \otimes_{\mathcal{O}_{X^{\log}}} \mathcal{O}_D,\\
&\Omega^p_{D_I} = \rho_D^{-1}\Omega^p_{E/\mathbf{C}}(\log) \otimes_{\rho_D^{-1}(\mathcal{O}_E)} \mathcal{O}_{D_I} = \Omega^p_{X^{\log}/Y^{\log}} \otimes_{\mathcal{O}_{X^{\log}}} \mathcal{O}_{D_I}.
\end{aligned}$$

Let

$$\Omega^p_{E/\mathbf{C}}(\log) = \Omega^p_{X/Y}(\log) \otimes \mathcal{O}_E.$$

Corollary 21 (Poincaré lemma 3 [9]).

$$\begin{aligned}
&\mathbf{C}_D \cong \Omega^{\bullet}_D, && \mathbf{C}_{D_I} \cong \Omega^{\bullet}_{D_I},\\
&R\rho_{D*}\Omega^p_D \cong \Omega^p_{E/\mathbf{C}}(\log), && R\rho_{D*}\Omega^p_{D_I} \cong \Omega^p_{E/\mathbf{C}}(\log) \otimes_{\mathcal{O}_E} \mathcal{O}_{E_I},\\
&R\rho_{D*}\mathbf{C}_D \cong \Omega^{\bullet}_{E/\mathbf{C}}(\log), && R\rho_{D*}\mathbf{C}_{D_I} \cong \Omega^{\bullet}_{E/\mathbf{C}}(\log) \otimes_{\mathcal{O}_E} \mathcal{O}_{E_I}.
\end{aligned}$$

The formation of the *canonical extension* is easily understood when we considers the structure sheaf of the log space as follows. Let (Y, C) be a smooth toroidal variety, $H_{\mathbf{Z}}$ a locally constant sheaf of free abelian groups on $Y \setminus C$, and let $H = H_{\mathbf{Z}} \otimes \mathcal{O}_{Y \setminus C}$ be the associated locally free sheaf on the open part $Y \setminus C$. Let $y \in C$ be an arbitrary point on the boundary, x_j the local coordinates corresponding to the local branches of C passing through y, and T_j the monodromy transformations of the local system $H_{\mathbf{Z}}$ around the branches of C corresponding to the x_j. We assume that the local monodromies T_j are unipotent for any y. Then the *canonical extension* $\tilde{H}$ of H is defined as a locally free sheaf on Y generated locally around y by the following type of local sections of H:

$$\exp(-\frac{1}{2\pi i}\sum_j \log T_j \log x_i)v$$

where the v are multivalued flat sections of $H_{\mathbf{Z}}$ and the power series for exp is finite because the $\log T_j$ are nilpotent. We note that the above expressions are single valued sections of H because the monodromy transformations are cancelled.

Proposition 22. *Let $H_{\mathbf{Z}}^{\log}$ be the local system on the log space $Y^{\log}$ obtained by extending the local system $H_{\mathbf{Z}}$. Then*

$$H_{\mathbf{Z}}^{\log} \otimes \mathcal{O}_{Y^{\log}} \cong \rho_Y^{-1}\tilde{H} \otimes_{\rho_Y^{-1}\mathcal{O}_Y} \mathcal{O}_{Y^{\log}}.$$

§8. Existence of a cohomological mixed Hodge complex

We shall prove the existence of a cohomological mixed Hodge complex on a singular fiber of a weakly semistable algebraic fiber space.

Theorem 23. *Let $f : (X, B) \to (Y, C)$ be a weakly semistable algebraic fiber space, i.e., a projective surjective toroidal morphism from a quasi-smooth toroidal variety to a smooth toroidal variety having connected, reduced and equi-dimensional geometric fibers. Let $f^{\log} : X^{\log} \to Y^{\log}$ be the induced continuous map between the associated log spaces $\rho_X : X^{\log} \to X$ and $\rho_Y : Y^{\log} \to Y$. Let $y \in Y$ be an arbitrary point, $\bar{y} \in \rho_Y^{-1}(y)$, $E = f^{-1}(y)$ and $D = (f^{\log})^{-1}(\bar{y})$. Denote $\rho_D = \rho_X|_D$. Then the following data is a cohomological mixed Hodge complex on E.*

(1) $H_{\mathbf{Z}} = R\rho_{D*}\mathbf{Z}_D$.

(2) $H_{\mathbf{C}} = \Omega^{\bullet}_{E/\mathbf{C}}(\log)$.

(3) *A weight filtration $W_q(R\rho_{D*}\mathbf{Q}_D)$ on $H_{\mathbf{Q}}$ is defined as a convolution of the following complex of objects:*

$$\tau_{\le q}(R\rho_{D*}\mathbf{Q}_{D^{[0]}}) \to \tau_{\le q+1}(R\rho_{D*}\mathbf{Q}_{D^{[1]}}) \to \tau_{\le q+2}(R\rho_{D*}\mathbf{Q}_{D^{[2]}}) \to \dots$$

where τ denotes the canonical filtration.

(4) *A weight filtration $W_q(\Omega^\bullet_{E/\mathbf{C}}(\log))$ on $H_{\mathbf{C}}$ is defined as a convolution of the following complex of objects*:

$$\begin{aligned} &W_q(\Omega^\bullet_{E/\mathbf{C}}(\log) \otimes_{\mathcal{O}_E} \mathcal{O}_{E^{[0]}}) \to W_{q+1}(\Omega^\bullet_{E/\mathbf{C}}(\log) \otimes_{\mathcal{O}_E} \mathcal{O}_{E^{[1]}}) \\ &\to W_{q+2}(\Omega^\bullet_{E/\mathbf{C}}(\log) \otimes_{\mathcal{O}_E} \mathcal{O}_{E^{[2]}}) \to \dots \end{aligned}$$

where W denotes the filtration determined by the order of log poles.

(5) *A Hodge filtration on $H_{\mathbf{C}}$ is the stupid filtration*:

$$F^p(\Omega^\bullet_{E/\mathbf{C}}(\log)) = \Omega^{\ge p}_{E/\mathbf{C}}(\log).$$

Proof. By the Poincaré lemma, we have $H_{\mathbf{Z}} \otimes \mathbf{C} \cong H_{\mathbf{C}}$, and

$$\begin{aligned} &\mathrm{Gr}^W_q(H_{\mathbf{Q}}) \cong \bigoplus_{t\ge 0} R^{q+t}\rho_{D*}\mathbf{Q}_{D^{[t]}}[-q-2t], \\ &\mathrm{Gr}^W_q(H_{\mathbf{C}}) \cong \bigoplus_{t\ge 0} \mathrm{Gr}^W_{q+t}(\Omega^\bullet_{E/\mathbf{C}}(\log) \otimes_{\mathcal{O}_E} \mathcal{O}_{E^{[t]}})[-t], \\ &\mathrm{Gr}^W_{q+t}(\Omega^\bullet_{E/\mathbf{C}}(\log) \otimes_{\mathcal{O}_E} \mathcal{O}_{E^{[t]}})[-t] \cong R^{q+t}\rho_{D*}\mathbf{C}_{D_I}[-q-2t]. \end{aligned}$$

We shall prove that

$$(\mathrm{Gr}^W_{q+t}(\Omega^\bullet_{E/\mathbf{C}}(\log) \otimes_{\mathcal{O}_E} \mathcal{O}_{E^{[t]}})[-t], F)$$

is a cohomological mixed Hodge complex of weight q on E.

We have an exact sequence

$$0 \to \Omega^1_{E_I}(\log G_I) \to \Omega^1_{E/\mathbf{C}}(\log) \otimes_{\mathcal{O}_E} \mathcal{O}_{E_I} \to \mathcal{O}^t_{E_I} \to 0$$

where the last arrow is given by the residue homomorphisms along the normal directions of E_I in E.

Let G_J be a closed stratum on E_I, an irreducible component of the intersection of some of the irreducible components of G_I. Let $s = \mathrm{codim}_{E_I} G_J$. When we derive a differential form on G_J from a section of $\Omega^\bullet_{E/\mathbf{C}}(\log) \otimes_{\mathcal{O}_E} \mathcal{O}_{E_I}$, we take residues s times in the normal directions of G_J in E_I and t' times in the directions of $\mathcal{O}^t_{E_I}$ of the above exact sequence, where we have $0 \le t' \le t$. In order to obtain the graded piece of degree $q+t$, we have to take residues $q+t$ times. Thus we have $q+t = s+t'$. The degree of the differential form drops also by $q+t$. Therefore we obtain

$$(\mathrm{Gr}^W_{q+t}(\Omega^\bullet_{E/\mathbf{C}}(\log) \otimes_{\mathcal{O}_E} \mathcal{O}_{E^{[t]}})[-t], F) \cong (\binom{t}{t'}\Omega^\bullet_{G^{[s]}}[-t-q-t], F(-q-t)).$$

Since

$$-t - q - t + 2(q + t) = q$$

the last term is a cohomological Hodge complex of weight q. Q.E.D.

The formula obtained in the above proof $q = s + t' - t$ says that the shift of the weight along the degeneration of fibers can be in both positive and negative directions, while the shift of the cohomology degree $s + t' + t$ is in one direction.

The following is an immediate corollary:

Corollary 24. *Assume the conditions of the theorem. Then the following hold*:

(1) *The spectral sequence associated to the Hodge filtration*

$${}_F E_1^{p,q} = H^q(E, \Omega^p_{E/\mathbf{C}}(\log)) \Rightarrow H^{p+q}(D, \mathbf{C})$$

degenerates at E_1.

(2) *The spectral sequence associated to the weight filtration*

$${}_W E_1^{p,q} = H^{p+q}(E, Gr^W_{-p}(R\rho_{D*}\mathbf{Q}_D)) \Rightarrow H^{p+q}(D, \mathbf{Q})$$

degenerates at E_2.

The upper semi-continuity theorem yields the local freeness theorem:

Corollary 25. *Assume the conditions of the theorem. Then the following hold*:

(1) *The higher direct image sheaves* $R^q f_* \Omega^p_{X/Y}(\log)$ *on* Y *are locally free for all* p, q.

(2) *Let* $\tilde{H}^k$ *be the canonical extension of the higher direct image sheaf* $R^k f^o_* \mathbf{C}_{X \setminus B} \otimes_{\mathbf{C}_{Y \setminus C}} \mathcal{O}_{Y \setminus C}$ *for any integer* k. *Then there is an isomorphism*

$$\tilde{H}^k \cong R^k f_* \Omega^\bullet_{X/Y}(\log).$$

Moreover

$$F^p(\tilde{H}^k) \cong R^k f_* \Omega^{\geq p}_{X/Y}(\log)$$

gives an increasing filtration by locally free subsheaves of the canonical extension.

Proof. (1) The rank of the cohomology groups $\bigoplus_{p+q=k} H^q(E, \Omega^p_{E/\mathbf{C}}(\log))$ for $E = f^{-1}(y)$ is independent of $y \in Y$. Hence the assertion follows from the upper semi-continuity theorem.

(2) We have

$$\begin{aligned}
&\rho_Y^{-1}\tilde{H}^k \otimes_{\rho_Y^{-1}\mathcal{O}_Y} \mathcal{O}_{Y^{\log}} \cong R^k f_*^{\log}\mathbf{C}_{X^{\log}} \otimes \mathcal{O}_{Y^{\log}} \\
&\cong \rho_Y^{-1} R^k f_* \Omega^\bullet_{X/Y}(\log) \otimes_{\rho_Y^{-1}\mathcal{O}_Y} \mathcal{O}_{Y^{\log}}, \\
&\rho_Y^{-1} F^p(\tilde{H}^k) \otimes_{\rho_Y^{-1}\mathcal{O}_Y} \mathcal{O}_{Y^{\log}} \cong F^p(R^k f_*^{\log}\mathbf{C}_{X^{\log}} \otimes \mathcal{O}_{Y^{\log}}) \\
&\cong \rho_Y^{-1} F^p(R^k f_* \Omega^\bullet_{X/Y}(\log)) \otimes_{\rho_Y^{-1}\mathcal{O}_Y} \mathcal{O}_{Y^{\log}}
\end{aligned}$$

hence the result. Q.E.D.

The above assertion can be derived from the nilpotent orbit theorem ([15]). Our proof is more geometric.

§9. Semipositivity theorem

A locally free sheaf F on a proper algebraic variety Y is said to be *numerically semipositive* if the tautological invertible sheaf $\mathcal{O}_{\mathbf{P}(F)}(1)$ on the associated projective space bundle is nef. It is equivalent to saying that, for all morphisms $\phi : \Gamma \to Y$ from a smooth projective curve Γ and for all quotient invertible sheaves G of the inverse images $\phi^* F$, the inequalities $\deg_\Gamma(G) \geq 0$ hold.

Theorem 26. *Let $f : (X, B) \to (Y, C)$ be a well prepared algebraic fiber space of relative dimension n, i.e., a projective surjective toroidal morphism from a quasi-smooth toroidal variety to a smooth toroidal variety having connected and equi-dimensional geometric fibers. Then the sheaves $f_*\Omega^k_{X/Y}(\log)$ and $R^k f_*\Omega^n_{X/Y}(\log)$ are numerically semipositive locally free sheaves for all non-negative integers k.*

If $f : (X, B) \to (Y, C)$ is weakly semistable, i.e., if all the geometric fibers are reduced, then $\Omega^n_{X/Y}(\log) = \omega_{X/Y}(B^h)$ is the relative dualizing sheaf twisted by the horizontal part of the boundary B. But the both sides are different in the general case.

Proof. At first, we consider the case where f is weakly semistable and there is no horizontal component of the boundary, i.e., $B = f^{-1}(C)$. The following proof is essentially in [8].

Let $H_t = R^t f_*\Omega^\bullet_{X/Y}(\log)$. It is a locally free sheaf on Y which is the canonical extension of a variation of Hodge structures on $Y \setminus C$. We denote $F_{n-k} = f_*\Omega^{n-k}_{X/Y}(\log)$ and $F_{n+k} = R^k f_*\Omega^n_{X/Y}(\log)$. They are locally free subsheaves of the H_t which coincide with the canonical extensions of the highest Hodge filtration for $t = n - k$ or $t = n + k$.

Let L be the cohomology class of an ample divisor on X. By the Lefschetz theorem, we have an isomorphism $L^k : H_{n-k} \to H_{n+k}$ which sends F_{n-k} to F_{n+k}. The primitive part

$$_0H_{n-k} = \operatorname{Ker}(L^{k+1} : H_{n-k} \to H_{n+k+2})$$

is the canonical extension of a polarized variation of Hodge structures on $Y \setminus C$. We note that F_{n-k} is contained in $_0H_{n-k}$.

We shall prove that F_t for $t = n - k$ is semipositive. Let $\phi : \Gamma \to Y$ and G as above. We assume first that the image of ϕ is general in the sense that $\Gamma^o = \phi^{-1}(Y \setminus C) \neq \emptyset$. The bilinear form $Q(u, v) = \int u \cup v \cup L^k$ induces a positive definite hermitian metric h_F on $F_t|_{Y \setminus C}$. Let h_G be the induced hermitian metric on $G|_{\Gamma^o}$. By the Griffiths semipositivity, the curvature form of h_Q is semi-positive.

We regard h_G as a hermitian metric on the whole space Γ with singularities along the boundary $\Gamma - \Gamma^o$, namely a *singular hermitian metric*. It can be written locally near a point $y \in \Gamma - \Gamma^o$ in a form $h_G = e^g h_0$ for a C^∞ hermitian metric h_0 and a locally integrable function g. Then we can express the curvature current Θ_G of h_G in a form $\Theta_G = \Theta_0 + \bar{\partial}\partial g$, where Θ_0 is the curvature from of h_0. The presentation of the canonical extension implies that the function g is the logarithm of a function which is expressed by C^∞ functions and logarithmic functions. Therefore the boundary contribution due to the singularities of h_G vanishes, and we have

$$\deg_\Gamma(G) = \int_{\Gamma_0} \frac{i}{2\pi} \Theta_G \geq 0.$$

Next we consider the case where $\phi(\Gamma)$ is contained in the boundary C. Let C_I be the irreducible component of the intersection of some of the irreducible components of C such that the image of ϕ is contained in C_I and general in the sense that $\Gamma^o = \phi^{-1}(C_I^o) \neq \emptyset$ where we set $C_I^o = C_I \setminus \bar{C}^{[s+1]}$ for the image $\bar{C}^{[s+1]}$ of $C^{[s+1]}$ in C_I with $s = \operatorname{codim}_Y C_I$.

For $y \in C_I^o$ and $\bar{y} \in \rho_Y^{-1}(y)$, we use the notation as in the previous sections such as $E = f^{-1}(y)$ and $D = (f^{\log})^{-1}(\bar{y})$. Moreover we denote $C_I^{o\,\log} = \rho_Y^{-1}(C_I^o)$, $E_I^o = f^{-1}(C_I^o)$ and $D_I = \rho_X^{-1}(E_I^o) = f^{\log\,-1}(C_I^{o\,\log})$. The mixed Hodge structure defined previously on $H^t(D, \mathbf{Z})$ varies continuously when $\bar{y}$ moves, and becomes a continuous variation of mixed Hodge structures $H_{\mathbf{Z}}^{\log} = R^t f_*^{\log} \mathbf{Z}_{D_I}$ on $C_I^{o\,\log}$.

Since the weight filtration W is flat along this variation, there exists a filtration, denoted by W again, on $H_{t,I}^o = H_t \otimes_{\mathcal{O}_Y} \mathcal{O}_{C_I^o}$ by locally free subsheaves on C_I^o such that

$$W_q(H_{\mathbf{Z}}^{\log}) \otimes \mathcal{O}_{C_I^{o\,\log}} \cong \rho_Y^{-1} W_q(H_{t,I}^o) \otimes_{\rho\mathcal{O}_{C_I^o}} \mathcal{O}_{C_I^{o\,\log}}.$$

The Hodge filtration F on $H^{\log}_{\mathbf{Z}}$ also comes from downstairs:

$$F^p(H^{\log}_{\mathbf{Z}} \otimes \mathcal{O}_{C^o_I{}^{\log}}) \cong \rho_Y^{-1} F^p(H^o_{t,I}) \otimes_{\rho\mathcal{O}_{C^o_I}} \mathcal{O}_{C^o_I{}^{\log}}.$$

We denote by $F^o_{t,I} = F^t(H^o_{t,I})$ the highest Hodge part, which coincides with $F_t \otimes_{\mathcal{O}_Y} \mathcal{O}_{C^o_I}$. The canonical extension $H_{t,I}$ (resp. $F_{t,I}$) of $H^o_{t,I}$ (resp. $F^o_{t,I}$) across the boundary $\bar{C}^{[s+1]}$ coincides with $H_t \otimes_{\mathcal{O}_Y} \mathcal{O}_{C_I}$ (resp. $F_t \otimes_{\mathcal{O}_Y} \mathcal{O}_{C_I}$).

Let q be the smallest integer such that the surjective homomorphism $\phi^* F_t \to G$ induces a non-zero homomorphism $W_q(F_{t,I}) \to G$. Then we have a non-zero homomorphism $\mathrm{Gr}^W_q(F_{t,I}) \to G$. The graded piece $\mathrm{Gr}^W_q(H_{t,I})$ is a direct sum of the canonical extensions of variations of (pure) Hodge structures on C_I defined by irreducible components of the intersection of some of the irreducible components of $E_I = f^{-1}(C_I)$. Therefore we infer that $\deg_\Gamma(G) \geq 0$ by the first part of this proof.

Now we consider the generalization to the case of well prepared algebraic fiber spaces without horizontal boundary components. Let $\pi_Y : Y' \to Y$ a finite surjective Galois base change morphism such that the induced algebraic fiber space $f' : X' \to Y'$ is weakly semistable. We denote by $\pi_X : X' \to X$ the induced finite morphism. We note that the ramification of π_Y occurs not only along the boundary divisor C, though we have $C' = \pi^{-1}C$. Our assertion follows from the following two lemmas.

Lemma 27. (1) $\pi_X^* \Omega^p_{X/Y}(\log) \cong \Omega^p_{X'/Y'}(\log)$ *for all* p.
(2) $R^q f_* \Omega^p_{X/Y}(\log)$ *is locally free for all* p, q.
(3) $\pi_Y^* R^q f_* \Omega^p_{X/Y}(\log) \cong R^q f'_* \Omega^p_{X'/Y'}(\log)$ *for all* p, q.

Proof. (1) The isomorphism is the pull-back homomorphism. We check that it is bijective. If $x \in X \setminus f^{-1}(C)$, then it is bijective in a neighborhood of x because f is smooth there. If $f(x) \in C$ but $f(x)$ is not contained in any irreducible component of the discriminant locus of π_Y other than those of C, then we have $\pi_X^* \Omega^p_X(\log B) \cong \Omega^p_{X'}(\log B')$ and $\pi_Y^* \Omega^p_Y(\log C) \cong \Omega^p_{Y'}(\log C')$, hence an isomorphism in a neighborhood of x. Therefore our homomorphism is bijective outside a codimension 2 subset, and we are done.

(2) Since $(\pi_{X*} \pi_X^* \Omega^p_{X/Y}(\log))^G \cong \Omega^p_{X/Y}(\log)$, we have

$$R^q f_* \Omega^p_{X/Y}(\log) \cong (\pi_{Y*} R^q f'_* \Omega^p_{X'/Y'}(\log))^G.$$

Since π_Y is flat, we have our assertion.

(3) This follows from the local freeness theorem and the upper semi-continuity theorem. Q.E.D.

Lemma 28. *Let $\pi : Y' \to Y$ be a finite surjective morphism of proper varieties, and let F be a locally free sheaf on Y. Then F is numerically semipositive if and only if $\pi^* F$ is numerically semipositive.*

Proof. The assertion is reduced to the case where the rank of F is equal to 1, and the latter is clear. Q.E.D.

We postpone the proof of the theorem in the case where there are horizontal boundary components to the end of the next section. Q.E.D.

§10. Weight spectral sequences

We use the following notation. $f : (X.B) \to (Y, C)$ is the given well prepared algebraic fiber space. We write $B = B^h + B^v$ for $B^v = f^{-1}(C)$ (h stands for horizontal and v the vertical). We denote by B_I^h an irreducible component of the intersection of some of the irreducible components of B^h, where I stands for a set of indices which determines B_I^h in a similar way as in the previous sections. Such B_I^h is called a *stratum* of X in this proof. For a fixed stratum B_I^h, we denote by $G_I = B^v \cap B_I^h$. Then the induced morphism $f_I : (B_I^h, G_I) \to (Y, C)$ is a well prepared algebraic fiber spaces without horizontal boundary components. In particular, we have $(B_\emptyset^h, G_\emptyset) = (X, B^v)$.

Let $\rho_{B_I^h} : B_I^{h\log} \to (B_I^h, G_I)$ be the real oriented blowing-up, and let $f_I^{\log} : B_I^{h\log} \to Y^{\log}$ be the induced morphism of topological spaces. The induced morphism $\sigma : X^{\log} \to B_\emptyset^{h\log}$ is the partial real oriented blowing-up along the horizontal boundary component such that $f^{\log} = f_\emptyset^{\log} \circ \sigma$.

We define a weight filtration W on a higher direct image $R\sigma_* \mathbf{Q}_{X^{\log}}$ on $B_\emptyset^{h\log}$ as the canonical filtration τ as before. Then we have

$$\mathrm{Gr}_q^W (R\sigma_* \mathbf{Q}_{X^{\log}}) \cong \bigoplus_{\mathrm{codim}\ B_I^h = q} \mathbf{Q}_{B_I^{h\log}}[-q]$$

where the $B_I^{h\log}$ are regarded as closed subspaces of $B_\emptyset^{h\log}$. There is an induced weight filtration, denoted again by W, on the higher direct image $Rf_*^{\log} \mathbf{Q}_{X^{\log}}$ on $Y^{\log}$ such that

$$\mathrm{Gr}_q^W (Rf_*^{\log} \mathbf{Q}_{X^{\log}}) \cong \bigoplus_{\mathrm{codim}\ B_I^h = q} Rf_*^{\log} \mathbf{Q}_{B_I^{h\log}}[-q].$$

In other words, we have distinguished triangles

$$\begin{aligned} &W_{q-1}(Rf_*^{\log} \mathbf{Q}_{X^{\log}}) \to W_q(Rf_*^{\log} \mathbf{Q}_{X^{\log}}) \\ &\to \mathrm{Gr}_q^W (Rf_*^{\log} \mathbf{Q}_{X^{\log}}) \to W_{q-1}(Rf_*^{\log} \mathbf{Q}_{X^{\log}})[1] \end{aligned}$$

so that $Rf_*^{\log}\mathbf{Q}_{X^{\log}}$ is a convolution of a complex of objects

$$\begin{aligned}\cdots \to & \bigoplus_{\operatorname{codim} B_I^h = q} Rf_*^{\log}\mathbf{Q}_{B_I^{h\,\log}}[-2q] \\ \to & \bigoplus_{\operatorname{codim} B_I^h = q-1} Rf_*^{\log}\mathbf{Q}_{B_I^{h\,\log}}[-2q+2] \to \\ \cdots \to & \bigoplus_{\operatorname{codim} B_I^h = 1} Rf_*^{\log}\mathbf{Q}_{B_I^{h\,\log}}[-2] \to Rf_*^{\log}\mathbf{Q}_{B_\emptyset^{h\,\log}}.\end{aligned}$$

We have a decomposition theorem for the constant sheaf when there is no horizontal components:

Theorem 29. *Let $f : (X.B) \to (Y, C)$ be a well prepared algebraic fiber space, and $f^{\log} : X^{\log} \to Y^{\log}$ the associated map of log spaces. Assume that there is no horizontal component of B, i.e., $B = f^{-1}(C)$. Then*

$$Rf_*^{\log}\mathbf{Q}_{X^{\log}} \cong \bigoplus_i R^i f_*^{\log}\mathbf{Q}_{X^{\log}}[-i].$$

Proof. Let $L \in H^2(X, \mathbf{Z})$ be the class of an ample divisor on X, and $y \in Y \setminus C$ a point. Then the fiber $f^{-1}(y)$ is a smooth projective variety and the Lefschetz decomposition theorem holds. Therefore we have a decomposition theorem for the fiber by Theorem 8. Since our sheaves are locally constant on the whole log space $Y^{\log}$, the Lefschetz decomposition theorem extends to singular fibers, and we are done. Q.E.D.

By the above theorem, we have

$$Rf_*^{\log}\mathbf{Q}_{B_I^{h\,\log}} \cong \bigoplus_p R^p f_*^{\log}\mathbf{Q}_{B_I^{h\,\log}}[-p].$$

With respect to this decomposition, the boundary morphisms of the above complex are the sums of the Gysin homomorphisms

$$R^p f_*^{\log}\mathbf{Q}_{B_I^{h\,\log}} \to R^{p+2} f_*^{\log}\mathbf{Q}_{B_J^{h\,\log}}$$

for immersions $B_I^h \subset B_J^h$ of codimension 1.

There is a spectral sequence associated to the weight filtration

(10.1)

$$E_1^{p,q} = H^{p+q}(\mathrm{Gr}_{-p}^W(Rf_*^{\log}\mathbf{Q}_{X^{\log}}) = \bigoplus_{\operatorname{codim} B_I^h = -p} R^{2p+q} f_*^{\log}\mathbf{Q}_{B_I^{h\,\log}}$$

$$\Rightarrow R^{p+q} f_*^{\log}\mathbf{Q}_{X^{\log}}$$

where the differentials $d_1^{p,q}$ are nothing but the Gysin homomorphisms above. It degenerates at E_2 by [3]; we have $d_r^{p,q} = 0$ for all $r \geq 2$.

Next we define a weight filtration on $\bigoplus_k \Omega^k_{X/Y}(\log)[-k]$ by the order of log poles along B^h so that we have

$$\mathrm{Gr}^W_q(\bigoplus_k \Omega^k_{X/Y}(\log)[-k]) \cong \bigoplus_k \bigoplus_{\mathrm{codim}\ B^h_I = q} \Omega^k_{B^h_I/Y}(\log)[-k-q]$$

where the B^h_I are regarded as closed subspaces of X whose boundaries have only vertical components. We note that the isomorphism does not hold for individual k. There is an induced weight filtration, denoted again by W, on the higher direct image $\bigoplus_k Rf_*\Omega^k_{X/Y}(\log)[-k]$ on Y such that

$$\mathrm{Gr}^W_q(\bigoplus_k Rf_*\Omega^k_{X/Y}(\log)[-k]) \cong \bigoplus_k \bigoplus_{\mathrm{codim}\ B^h_I = q} Rf_*\Omega^k_{B^h_I/Y}(\log)[-k-q].$$

Thus $\bigoplus_k Rf_*\Omega^k_{X/Y}(\log)[-k]$ is a convolution of a complex of objects

$$\begin{aligned} \cdots \to & \bigoplus_k \bigoplus_{\mathrm{codim}\ B^h_I = q} Rf_*\Omega^k_{B^h_I/Y}(\log)[-k-2q] \\ \to & \bigoplus_k \bigoplus_{\mathrm{codim}\ B^h_I = q-1} Rf_*\Omega^k_{B^h_I/Y}(\log)[-k-2q+2] \to \\ \cdots \to & \bigoplus_k \bigoplus_{\mathrm{codim}\ B^h_I = 1} Rf_*\Omega^k_{B^h_I/Y}(\log)[-k-2] \\ \to & \bigoplus_k Rf_*\Omega^k_{B^h_\emptyset/Y}(\log)[-k]. \end{aligned}$$

Now we prove the decomposition theorem for the sheaves of logarithmic differential forms when there is no horizontal components:

Corollary 30. *Let $f : (X.B) \to (Y, C)$ be a well prepared algebraic fiber space. Assume that there is no horizontal component of B, i.e., $B = f^{-1}(C)$. Then*

$$Rf_*\Omega^k_{X/Y}(\log) \cong \bigoplus_i R^if_*\Omega^k_{X/Y}(\log)[-i]$$

for all k.

Proof. We shall prove an isomorphism of their sum

$$\bigoplus_k Rf_*\Omega^k_{X/Y}(\log)[-k] \cong \bigoplus_i \bigoplus_k R^if_*\Omega^k_{X/Y}(\log)[-i-k]$$

By the E_1-degeneration of the spectral sequence associated to the Hodge filtration, we have Lefschetz type isomorphisms for the direct sum $\bigoplus_k Rf_*\Omega^k_{X/Y}(\log)[-k]$. Therefore we obtain the assertion. Q.E.D.

We remark that Kollár's result in [12] on the sheaf of top differential forms ω_X holds without the assumptions on the well preparedness of the morphism, because ω_X behaves well under the birational morphisms by the Grauert–Riemenschneider vanishing theorem.

By the above corollary, we have

$$\bigoplus_k Rf_*\Omega^k_{B^h_I/Y}(\log)[-k] \cong \bigoplus_i \bigoplus_k R^i f_*\Omega^k_{B^h_I/Y}(\log)[-i-k].$$

With respect to this decomposition, the boundary morphisms of the above complex are the sums of the Gysin homomorphisms

$$R^q f_*\Omega^k_{B^h_I/Y}(\log) \to R^{q+1} f_*\Omega^{k+1}_{B^h_J/Y}(\log)$$

for immersions $B^h_I \subset B^h_J$ of codimension 1. We note that underlying varieties of $B^h_\emptyset$ and X are the same but their boundaries are different.

There is a spectral sequence associated to the weight filtration

(10.2)

$$\begin{aligned} E_1^{p,q} &= \bigoplus_k H^{p+q-k}(\mathrm{Gr}^W_{-p}(Rf_*\Omega^k_{X/Y}(\log)) \\ &= \bigoplus_k \bigoplus_{\mathrm{codim}\ B^h_I = -p} R^{2p+q-k} f_*\Omega^k_{B^h_I/Y}(\log) \Rightarrow \bigoplus_k R^{p+q-k} f_*\Omega^k_{X/Y}(\log) \end{aligned}$$

where the differentials $d_1^{p,q}$ are nothing but the Gysin homomorphisms above. It degenerates at E_2 by [3]; we have $d_r^{p,q} = 0$ for all $r \geq 2$.

Proof of Theorem 26 *continued.* We continue the proof of the semi-positivity theorem in the case where there are horizontal boundary components. We use spectral sequences (10.1) and (10.2) associated to the weight filtrations, which will be denoted by ${}_IE_1^{p,q}$ and ${}_{II}E_1^{p,q}$. The second one ${}_{II}E_1^{p,q}$ is obtained from the first one ${}_IE_1^{p,q}$ by the decomposition with respect to the Hodge filtration:

$$\bigoplus_k \mathrm{Gr}^F_k({}_IE_1^{p,q}|_{Y\setminus C} \otimes \mathcal{O}_{Y\setminus C}) \cong {}_{II}E_1^{p,q}|_{Y\setminus C}.$$

They degenerate at the E_2-terms.

Let F denote either $f_*\Omega^k_{X/Y}(\log)$ or $R^k f_*\Omega^n_{X/Y}(\log)$. It has an induced weight filtration, and we have

$$\mathrm{Gr}^W(F) \cong \bigoplus_p F^{k+p}({}_{II}E_2^{p,k-p}) \quad \text{or} \quad \bigoplus_p F^{n+p}({}_{II}E_2^{p,k+n-p}).$$

Since $F^{k+p+1}({}_{II}E_2^{p,k-p}) = 0$ and $F^{n+p+1}({}_{II}E_2^{p,k+n-p}) = 0$, we deduce that $\mathrm{Gr}^W(F)$ is numerically semipositive, hence so is F. Q.E.D.

References

[1] D. Abramovich and K. Karu, Weak semistable reduction in characteristic 0, Invent. Math., **139** (2000), 241–273.

[2] P. Deligne, Théorème de Lefschetz et critères de dégénérescence de suites spectrales. (French), Inst. Hautes Études Sci. Publ. Math., **35** (1968), 259–278.

[3] P. Deligne, Théorie de Hodge. II. (French), Inst. Hautes Études Sci. Publ. Math., **40** (1971), 5–57.

[4] S. I. Gelfand and Y. I. Manin, Methods of Homological Algebra. Second ed., Springer Monogr. Math., Springer-Verlag, 2003.

[5] P. A. Griffiths, Periods of integrals on algebraic manifolds. III. (Some global differential-geometric properties of the period mapping), Inst. Hautes Études Sci. Publ. Math., **38** (1970), 125–180.

[6] K. Kato, Logarithmic structures of Fontaine–Illusie, In: Algebraic Analysis, Geometry, and Number Theory, Baltimore, MD, 1988, Johns Hopkins Univ. Press, Baltimore, MD, 1989, pp. 191–224.

[7] K. Kato and C. Nakayama, Log Betti cohomology, log étale cohomology, and log de Rham cohomology of log schemes over C, Kodai Math. J., **22** (1999), 161–186.

[8] Y. Kawamata, Characterization of abelian varieties, Compositio Math., **43** (1981), 253–276.

[9] Y. Kawamata, On algebraic fiber spaces, In: Contemporary Trends in Algebraic Geometry and Algebraic Topology, Tianjin, 2000, Nankai Tracts Math., **5**, World Sci. Publ., River Edge, NJ, 2002, pp. 135–154.

[10] Y. Kawamata, Semipositivity theorem for reducible algebraic fiber spaces, preprint, arXiv:0911.1670.

[11] G. Kempf, F. F. Knudsen, D. Mumford and B. Saint-Donat, Toroidal Embeddings. I, Lecture Notes in Math., **339**, Springer-Verlag, 1973.

[12] J. Kollár, Higher direct images of dualizing sheaves. II, Ann. of Math. (2), **124** (1986), 171–202.

[13] N. Nakayama, Hodge filtrations and the higher direct images of canonical sheaves, Invent. Math., **85** (1986), 217–221.

[14] U. Persson, On Degenerations of Algebraic Surfaces, Mem. Amer. Math. Soc., **11**, no. 189, Amer. Math. Soc., Providence, RI, 1977.

[15] W. Schmid, Variation of Hodge structure: the singularities of the period mapping, Invent. Math., **22** (1973), 211–319.

Graduate School of Mathematical Sciences
University of Tokyo
Komaba, Meguro, Tokyo, 153-8914
Japan
E-mail address: `kawamata@ms.u-tokyo.ac.jp`