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Orbitally stable standing-wave solutions to a coupled non-linear Klein–Gordon equation

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Abstract.

We outline some results on the existence of standing-wave solutions to a coupled non-linear Klein–Gordon equation. Standing-waves are obtained as minimizers of the energy under a two-charges constraint. The ground state is stable. The standing-waves are stable provided a non-degeneracy condition is satisfied.

§1. Introduction

Let (X, d) be a metric space and let $\{U_t \,|\, t \geq 0\}$ be a family of operators on X such that

$$U_{t+s} = U_t \circ U_s.$$

We define some dynamical properties of the pair (X, U): a subset $S \subset X$ is said *invariant* if for every $t \geq 0$ and $\Phi \in S$, there holds

$$U_t(\Phi) \in S.$$

A subset $S \subset X$ is said *stable* if for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\Phi \in B(S, \delta) \Rightarrow U_t(\Phi) \in B(S, \varepsilon) \text{ for every } t \geq 0,$$

where

$$B(S, \delta) := \{\Phi \in X \mid \operatorname{dist}(\Phi, S) < \delta\}.$$

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Finally, a state $\Phi \in X$ is said *orbitally stable* if there exists a finite-dimensional manifold $S \subset X$ stable and invariant such that $\Phi \in S$. In evolution equations, X plays the role of a space of initial data where the Cauchy problem is locally well-posed; $U_t(\Phi)$ is defined as the solution of the evolution equation with initial datum Φ, at the time t.

A well-known example of orbitally stable state is provided as standing-wave solution to the non-linear Schrödinger equation

$$\text{(NLS)} \quad i\partial_t v(t,x) + \Delta_x v(t,x) + |v(t,x)|^{p-2} v(t,x) = 0, \ 2 < p < 2 + \frac{4}{N}$$

by H. Cazenave and P. L. Lions in [9]. Therein $X = H^1_{\mathbb{C}}(\mathbb{R}^N)$ and Φ is the initial value of a standing-wave solution to (NLS)

$$v(t,x) = e^{-i\omega t} u(x) \tag{1}$$

where $\omega \in \mathbb{R}$, $u \in H^1(\mathbb{R}^N)$ and

$$\Delta u + \omega u + |u|^{p-2} u = 0.$$

It is easy to check that v solves (NLS) if and only if u solves the elliptic equation above.

In [9], they prove that the manifold

$$S := \left\{ \lambda u(\cdot + y) \mid (\lambda, y) \in S^1 \times \mathbb{R}^N \right\} \tag{2}$$

is invariant and stable, where

$$S^1 := \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}.$$

In fact, it can be shown that the homeomorphism relation

$$S^1 \times \mathbb{R}^N \cong S$$

also holds. Thus, $\Phi = v(0,\cdot) = u$ is orbitally stable. Since then, their results have been extended to more general non-linearities and other evolution equations, as in [4] (NLS, $N \geq 3$), [26] (NLS, $N \geq 1$), [20], [21], [23] (NLS + NLS, $N = 1$), [15], [22] (multiple NLS, $1 \leq N \leq 3$), [1], [2], [10] (coupled NLS and Korteweg–de Vries equation, $N = 1$). In the above references, the stable manifold S is defined according to the non-linearity—scalar equations or coupled equations. Moreover, having a family of operators defined for every $t \geq 0$ requires the equations above to be globally well-posed—this is not always the case, starting from (NLS), when $p \geq 2 + 4/N$.

In higher order evolution equations, as the non-linear Klein–Gordon

$$\text{(NLKG)} \qquad \partial_t^2 v - \Delta_x v + v - |v|^{p-2} v = 0,\ 2 < p < \frac{2N}{N+2}$$

the first derivative must be taken into account. Then, the most suitable candidate to be a stable manifold is

$$\text{(3)} \qquad \Gamma := \{\lambda(u(\cdot + y), -i\omega u(\cdot + y)) \mid (\lambda, y) \in S^1 \times \mathbb{R}^N\}.$$

Among the references on the orbital stability of standing-wave solutions to (NLKG) we include the joint works of M. Grillakis, J. Shatah and W. Strauss, [13], [14]. For coupled non-linear Klein–Gordon equations, we note [28] along with some counterexamples in [24], [27]. In our work [11], we address standing-wave solutions to the coupled non-linear Klein–Gordon equation

$$\text{(CNLKG)} \qquad \begin{aligned} \partial_t^2 u_1 - \Delta_x u_1 + m_1^2 u_1 - \gamma\mu |u_1|^{\gamma-2} |u_2|^\gamma u_1 + \partial_{z_1} G(u) = 0 \\ \partial_t^2 u_2 - \Delta_x u_2 + m_2^2 u_2 - \gamma\mu |u_2|^{\gamma-2} |u_1|^\gamma u_2 + \partial_{z_2} G(u) = 0 \end{aligned}$$

where $m_j > 0$ for $j = 1, 2$. We discuss stability results of the manifold Γ and the stability of the ground state.

§2. Hypotheses on the non-linearity

Let G be a continuously differentiable non-negative, real-valued function on $\mathbb{C} \times \mathbb{C}$ such that there are two powers

$$2 < p \le q < 2^*,\ 2^* = \frac{2N}{N-2}$$

and a constant $c \ge 0$ such that

$$\text{(4)} \qquad |DG(z)| \le c(|z|^{p-1} + |z|^{q-1}),\ G(0) = 0.$$

In other words, $|DG|$ is a *combined power-type*. Moreover, let γ be such that

$$\text{(5)} \qquad 2 < 2\gamma < 2 + \frac{4}{N},\ 2\gamma < p.$$

We define

$$F(z) = -\mu |z_1 z_2|^\gamma + G(z).$$

From assumptions (4) and (5) it follows that

$$|F(z)| \le d(|z|^{2\gamma} + |z|^q)$$

for some $d \geq 0$; thus, for every $u \in H^1(\mathbb{R}^N;\mathbb{R}^2)$, $F(u)$ is in $L^1(\mathbb{R}^N)$. From the sub-critical growth assumption, (NLKG) is locally well-posed in $H^1 \times L^2$, [12]. We suppose that (CNLKG) is locally well-posed in

$$X := H^1(\mathbb{R}^N;\mathbb{C}^2) \times L^2(\mathbb{R}^N;\mathbb{C}^2),$$

even if we expect that it follows from the same techniques used in [12]. From the additional assumption

$$V(z) := \frac{1}{2}\left(m_1^2|z_1|^2 + m_2^2|z_2|^2\right) + F(z) \geq 0 \tag{6}$$

local solutions extend to $[0,+\infty)$. We require G to satisfy the symmetry

$$G(z) = G(|z_1|,|z_2|). \tag{7}$$

That gives arise to conserved quantities on solutions to (CNLKG), namely, the energy, charges and momenta [3, §2]. We define below the energy and the charges (momenta are zero on standing-waves) as functions on the space X. When we write a state $\Phi \in X$ component-wise, we use the notation $\Phi := (\phi, \phi_t)$;

$$X \ni \Phi \mapsto \mathbf{E}(\Phi) := \frac{1}{2}\sum_{j=1}^{2}\int_{\mathbb{R}^N}\left(|\phi_t^j|^2 + |D\phi_j|^2 + V(\phi)\right)$$

$$X \ni \Phi \mapsto \mathbf{C}_j(\Phi) := -\mathrm{Im}\int_{\mathbb{R}^N}\phi_t^j\overline{\phi}_j,$$

for $j = 1,2$. Finally, we assume that

$$\int_{\mathbb{R}^N} G(u_1^*, u_2^*) \leq \int_{\mathbb{R}^N} G(u_1,u_2) \tag{8}$$

for every $u_j \geq 0$. In the inequality above, u_j^* is the Steiner symmetrization taken with respect to any linear subspace of $\mathbb{R}^N$. We refer to §3.7 in [17] for definitions and properties of the Steiner symmetrization. In the scalar case, such inequality holds for every $G\colon \mathbb{R}^+ \to \mathbb{R}$ and $u \geq 0$. In higher dimensions, a counterexample can be produced by taking $u_1 \in L^2_+(\mathbb{R}^N)$ symmetrically decreasing and with compact support, and $y \in \mathbb{R}^N$ such that $u_2 := u_1(\cdot + y)$ and u_1 have supports disjoint from each other. Thus

$$u_1^* = u_1, \;\; u_2^* = u_1.$$

Hence, the function $G_0(z) = |z_1 z_2|$ fails to satisfy inequality (8). In our assumptions, the coupling term has negative sign. Thus, from [17, Theorem 3.4] and [17, (v) p.81], it follows that F fulfills (8) as G does.

We conclude this section with an example of non-linearity G in $C^1(\mathbb{C}^2, \mathbb{R}^+)$ and a pair (m_1, m_2) in $(0, +\infty)^2$ satisfying assumptions (4), (6), (7) and (8)

$$G(z) = |z|^r - c|z_1 z_2|^s + |z|^t,\ 2 < t < 2s < r < 2^*$$

where $c > 0$ is chosen in such a way that $G \geq 0$. From these assumptions it follows that there exists a pair (m_1, m_2) such that $V \geq 0$.

§3. The variational characterisation

If $v_j := e^{-i\omega_j t} u_j$ is a solution to (CNLKG), then (u, ω) is a solution to the non-linear elliptic system

$$\begin{aligned} -\Delta u_1 + (m_1^2 - \omega_1^2) u_1 + \partial_{z_1} F(u) = 0 \\ -\Delta u_2 + (m_2^2 - \omega_2^2) u_2 + \partial_{z_2} F(u) = 0. \end{aligned} \tag{9}$$

We define the energy functional

$$\begin{aligned} E \colon H^1(\mathbb{R}^N; \mathbb{R}^2) \times \mathbb{R}^2 \to \mathbb{R} \\ (v, \alpha) \mapsto \frac{1}{2} \sum_{j=1}^{2} \left(\|Dv_j\|_{L^2}^2 + m_j^2 \|v_j\|_{L^2}^2 + \alpha_j^2 \|v_j\|_{L^2}^2 \right) + \int_{\mathbb{R}^N} F(v) \end{aligned}$$

and

$$\begin{aligned} C_j \colon H^1(\mathbb{R}^N; \mathbb{R}^2) \times \mathbb{R}^2 \to \mathbb{R} \\ (v, \alpha) \mapsto \alpha_j \|v_j\|_{L^2}^2, \quad 1 \leq j \leq 2. \end{aligned}$$

Given $C \in \mathbb{R}^2$, we define the following closed and differentiable submanifold

$$M_C = \{(v, \alpha) \mid C_j(v, \alpha) = C_j\} \subset H^1(\mathbb{R}^N; \mathbb{R}^2) \times \mathbb{R}^2$$

of co-dimension two. There are several benefits in searching for minima of E over M_C: firstly if v is a standing-wave solution to (CNLKG), then

$$\mathbf{E}(v(t, \cdot), \partial_t v(t, \cdot)) = E(u, \omega),\ \mathbf{C}_j(v(t, \cdot), \partial_t v(t, \cdot)) = C_j(u, \omega)$$

for $j = 1, 2$. Secondly, one can check with small effort that critical points of E over M_C are classic solutions to (9), for example as in [3, Theorem 2.6] in the scalar case, or [11, Proposition 2.2] in the coupled case. We seek solutions to

$$E(u, \omega) = \inf_{M_C} E =: m_C \tag{10}$$

for every C such that $C_1C_2 \neq 0$ and $C_j > 0$ for every $1 \leq j \leq 2$. We note

$$K_C := \{(u, \omega) \mid E(u, \omega) = m_C\}.$$

The assumption $C_j > 0$ is just a technical restriction which can be removed by observing that

$$E(u, \omega) = E(u, -\omega_1, \omega_2) = E(u, \omega_1, -\omega_2) = E(u, -\omega)$$

and that $C_j(\cdot, \omega)$ is an odd function of ω. We do not consider the semi-trivial and the completely trivial case $C_1 = 0, C_2 > 0$ and $C_1 = C_2 = 0$, even if both are interesting from the point of view of the orbital stability. The semi-trivial case is interesting from the point of view of the existence of minima as well, while in the completely trivial case the minima are $(0, \omega_1, 0, \omega_2)$ for any choice of ω_1 and ω_2.

§4. Main results

In [11, Theorem 1.1], we prove that minimising sequences of E over M_C exhibit a concentration behaviour. One of the consequences is the stability of some subsets of X.

Theorem 1. *Given a minimising sequence $(u_n, \omega_n)_{n\geq 1}$ for E over M_C, there exists a minimum (u, ω) and $(y_n)_{n\geq 1} \subset \mathbb{R}^N$ such that, up to extract a subsequence,*

$$u_n^j = u_j(\cdot + y_n) + o(1) \text{ in } H^1(\mathbb{R}^N), \quad \omega_n \to \omega \text{ in } \mathbb{R}^2$$

for $1 \leq j \leq 2$.

The proof of the theorem above is carried out as in the scalar case [3]: we define the functional and constraint

$$H^1(\mathbb{R}^N) \ni u \mapsto J(u) := \frac{1}{2}\int_{\mathbb{R}^N} |Du|^2 + \int_{\mathbb{R}^N} F(u)$$
$$N_\rho := \{u \in H^1(\mathbb{R}^N; \mathbb{R}^2) \mid \|u_j\|_{L^2}^2 = \rho_j\}$$

and show in [11, Theorem 4.1] that a concentration result holds:

Theorem 2. *Let $(u_n)_{n\geq 1}$ be a minimising sequence for J over N_ρ. Then, there exists $u \in N_\rho$ and a sequence $(y_n)_{n\geq 1}$ such that*

$$u_n = u(\cdot + y_n) + o(1) \text{ in } H^1(\mathbb{R}^N)$$
$$J(u) = \inf_{N_\rho} J.$$

The two previous statements can be regarded as consequences of the concentration-compactness Theorem of P. L. Lions [18], [19]. However, we prefer to consider the following alternative classification, provided in [7], using the same terminology (concentration, dichotomy, vanishing) as in [18]: Given a bounded sequence $(g_n)_{n\geq 1}$ in $L^2(\mathbb{R}^N)$, we say that there is a *concentration* if there exists a sequence $(y_n)_{n\geq 1}$ and $g \in L^2$ such that

$$\text{(C)} \qquad g_n(\cdot + y_n) \to g \text{ in } L^2(\mathbb{R}^N),$$

a *dichotomy*, if

$$\text{(D)} \qquad g_n(\cdot + y_n) \rightharpoonup g \text{ in } L^2(\mathbb{R}^N)$$

and

$$0 < \|g\|_{L^2} < \liminf_{n\to+\infty} \|g_n\|_{L^2}.$$

If neither of (C) or (D) holds, $(g_n)_{n\geq 1}$ is said to *vanish*. In this case, for every sequence $(z_n)_{n\geq 1}$

$$\text{(V)} \qquad g_n(\cdot + z_n) \rightharpoonup 0 \text{ in } L^2(\mathbb{R}^N).$$

The proof of the theorem above is carried out as follows: we show that if $(u_n)_{n\geq 1}$ is a minimising sequence for J over N_ρ, then there exists $(y_n)_{n\geq 1} \subset \mathbb{R}^N$ and $u_1, u_2 \neq 0$ such that

$$u_n^1(\cdot + y_n) \rightharpoonup u_1,\; u_n^2(\cdot + y_n) \rightharpoonup u_2 \text{ in } L^2(\mathbb{R}^N).$$

The sequence $(y_n)_{n\geq 1}$ is the same for each component. This is due to the fact that

$$\liminf_{n\to\infty} \int_{\mathbb{R}^N} |u_n^1 u_n^2|^\gamma > 0$$

and [19, Lemma I.1]. Thus, (V) does not occur for any of the sequences $(u_n^j)_{n\geq 1}$. Then, in order to prove that (C) holds for each $j = 1, 2$, we need to rule out the case

$$\|u_j\|_{L^2} < \liminf_{n\to\infty} \|u_n^j\|_{L^2}$$

for some $j = 1, 2$. Up to a normalization, the sequences

$$v_n^j := u_n^j - u_j,\; u_j$$

lie in two constraints, namely N_τ and $N_{\rho-\tau}$. By applying techniques already set up in [4], [5], [6], we can show that

$$J(u_n) = J(u) + J(v_n) + o(1).$$

We define

$$I(\rho) := \inf_{N_\rho} J$$

and prove that I satisfies the strictly sub-additivity property, that is

$$I(\rho) < I(\tau) + I(\rho - \tau),\ 0 < \tau_j \leq \rho_j,\ \tau \neq \rho \tag{11}$$

and obtain a contradiction. In literature, the inequality above is achieved either by direct computation [1], [21] of I (non-linearities are provided explicitly), or by showing the existence of a minimiser and obtaining a strict inequality using rescaling arguments as in [4]. In our case, we use the following argument based on the properties of the Steiner symmetrization (to this purpose we need assumption (8)): suppose that we are given a pair

$$(u, v) \in N_\tau \times N_{\rho-\tau}$$

of functions such that u_j and v_j have supports disjoint from each other and u and v are a suitably good approximation of $I(\tau)$ and $I(\rho - \tau)$, respectively. Then there exists a constant D depending only on ρ and τ such that

$$\|Dw_j^*\|_{L^2}^2 < \|Du_j\|_{L^2}^2 + \|Dv_j\|_{L^2}^2 - D, \tag{12}$$

where

$$w_j := u_j + v_j$$

and w_j^* is the symmetrically decreasing rearrangement of w_j. In dimension $N = 1$ (check also [2], [8]) the equality is

$$\|Dw_j^*\|_{L^2}^2 \leq \|Du_j\|_{L^2}^2 + \|Dv_j\|_{L^2}^2 - \frac{3}{4} \min\left\{\|Du_j\|_{L^2}^2, \|Dv_j\|_{L^2}^2\right\}.$$

When $N \geq 3$, (12) is obtained with a contradiction argument which envolves the one-dimensional inequality and several rearrangements. We show that the correction term D is the bounded away from zero, [11, Proposition 3.1].

In order to state the stability results of [11], preliminary notation is required. Given two complex vectors $z, w \in \mathbb{C}^2$, we define

$$\mathbb{C}^2 \ni (z \cdot w)_j := z_j w_j$$

the component-wise product. Given $(u, \omega) \in K_C$, we define the following subsets of X:

$$\Gamma(u, \omega) := \left\{ \begin{array}{c} \lambda \cdot (u(\cdot + y), -i\omega \cdot u(\cdot + y)) \\ (\lambda, y) \in \mathbb{T}^2 \times \mathbb{R}^N \end{array} \right\}$$

and

$$\Gamma_C := \bigcup_{(u,\omega)\in K_C} \Gamma(u,\omega),$$

where $\mathbb{T}^2 = S^1 \times S^1$. The manifold Γ_C is called *ground state.*

Theorem 3 (Theorem 1.2 of [11]). *Given a sequence*

$$(\Phi_n)_{n\geq 1} \subset X$$

then $\mathrm{dist}(\Phi_n, \Gamma_C) \to 0$ *if and only if*

$$\mathbf{E}(\Phi_n) \to m_C, \quad \mathbf{C}_j(\Phi_n) \to C_j.$$

for $1 \leq j \leq 2$.

In other words, the theorem states that the function

$$\mathbf{V}\colon X \to \mathbb{R},$$

$$\Phi \mapsto (\mathbf{E}(\Phi) - m_C)^2 + \sum_{j=1}^{2} (\mathbf{C}_j(\Phi) - C_j)^2$$

is a Lyapunov function for Γ_C, that is,

$$\mathrm{dist}(\Phi_n, \Gamma_C) \to 0 \iff \mathbf{V}(\Phi_n) \to 0.$$

A definition of Lyapunov function for a subset $\Gamma \subset X$ is in [3, Definition 2.4]. A proof of the theorem above in the scalar case can be found in [3, §3.1]. We give an alternative proof to this fact, based on the following property: let

$$\phi \in H^1_{\mathbb{C}}(\mathbb{R}^N)$$

be such that $\operatorname{ess\,inf}_\Omega |\phi| > 0$ for every bounded subset $\Omega \subset \mathbb{R}^N$, and

$$\int_{\mathbb{R}^N} |D\phi|^2 = \int_{\mathbb{R}^N} |D|\phi||^2.$$

Then there exists $\lambda \in S^1$ such that

$$\phi(x) = \lambda |\phi(x)|$$

for every x in $\mathbb{R}^N$ (in a similar result, known as Convex Inequality for Gradients [17, Theorem 7.8], it is supposed that $|\mathrm{Im}(\phi)| > 0$ everywhere). We show this in [11, Lemma 6.1] for ϕ in $H^1(\mathbb{R}^N, \mathbb{R}^m)$ and $m \geq 1$.

Given $(u,\omega) \in K_C$, we define the subset

$$S(u,\omega) = \{(u(\cdot + y), \omega) \mid y \in \mathbb{R}^N\} \subset H^1(\mathbb{R}^N; \mathbb{R}^2) \times \mathbb{R}^2.$$

Theorem 4 (Theorem 1.3 of [11]). *The ground state is stable. If* $(u, \omega) \in K_C$ *and there exists* $\delta > 0$ *such that*

$$B(S(u, \omega), \delta) \cap S(v, \alpha) = \emptyset$$

for every (v, α) *such that* $\Gamma(u, \omega) \neq \Gamma(u, \alpha)$*, then* $\Gamma(u, \omega)$ *is stable.*

The problem of the stability of $\Gamma(u, \omega)$ is more challenging than the stability of Γ_C, even in scalar non-linear Schrödinger equation. In the work of H. Cazenave and P. L. Lions, [9], the non-linearity is a pure power: in this special case, positive solutions are unique up a translation, from a well-known result of [16]. Moreover, pure powers enjoy special rescalings with the result that Γ is equal to the ground state. So, Γ is stable because the ground state is stable.

In our case, as in [4], [3], the choice of the non-linear term is very general, so it is not easy to conclude that $\Gamma(u, \omega)$ is stable from the stability of the ground state. This explains the non-degeneracy condition stated in the theorem above.

We wish to account a recent work of Masataka Shibata, [25], on the scalar non-linear Schrödinger equation, where (12) is replaced by a simple strict inequality. This is combined to a careful study of the function I in order to obtain (11). To achieve this purpose he defines an *ad hoc* rearrangement for a two-bumps function.

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