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On decay properties of the linearized compressible Navier–Stokes equations around time-periodic flows in an infinite layer

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Abstract.

We investigate decay properties of solutions to the linearized compressible Navier–Stokes equation around time-periodic parallel flow. We show that if the Reynolds and Mach numbers are sufficiently small, solutions of the linearized problem decay in L^2 norm as an $n-1$ dimensional heat kernel. Furthermore, we prove that the asymptotic leading part of solutions is given by solutions of an $n-1$ dimensional linear heat equation with a convective term multiplied by time-periodic function.

§1. Introduction

We are concerned with the asymptotic behavior of solutions to the compressible Navier–Stokes equation with time-periodic external force and (or) time-periodic boundary conditions. We state the problem directly in dimensionless form.

We consider the system of equations

$$\partial_t \rho + \operatorname{div}(\rho v) = 0, \tag{1}$$

$$\rho(\partial_t v + v \cdot \nabla v) - \nu \Delta v - (\nu + \nu')\nabla \operatorname{div} v + \nabla P(\rho) = \rho \vec{g}, \tag{2}$$

in an n dimensional infinite layer $\Omega = \mathbb{R}^{n-1} \times (0,1)$:

$$\Omega_\ell = \{x = (x', x_n)\,; \quad x' = (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1},\ 0 < x_n < 1\}.$$

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Here $n \geq 2$; $\rho = \rho(x,t)$ and $v = (v^1(x,t), \ldots, v^n(x,t))$ denote the unknown density and velocity at time $t \geq 0$ and position $x \in \Omega$, respectively; P is the pressure, smooth function of ρ, where we assume $P'(1) > 0$, $\gamma = \sqrt{P'(1)}$; ν and ν' are the viscosity coefficients that are assumed to be constants satisfying $\nu > 0$, $\frac{2}{n}\nu + \nu' \geq 0$; $\vec{g}$ is a time-periodic external force of the form

$$\vec{g} = (g^1(x_n,t), 0, \ldots, 0, g^n(x_n)),$$

with g^1 being T-periodic function in time, where $T > 0$.

The system (1)–(2) is considered under the boundary condition

$$v|_{x_n=0} = V^1(t)\vec{e}_1, \quad v|_{x_n=1} = 0, \tag{3}$$

and the initial condition

$$(\rho, v)|_{t=0} = (\rho_0, v_0), \tag{4}$$

where V^1 is T-periodic function of time and $\vec{e}_1 = (1, 0, \ldots, 0) \in \mathbb{R}^n$. We note that the Reynolds number Re and Mach number Ma are given by $Re = \nu^{-1}$ and $Ma = \gamma^{-1}$, respectively.

One can see that if g^n is small enough then a time-periodic solution $(\rho_p, v_p) = (\rho_p(x_n), v_p^1(x_n,t)\vec{e}_1)$, so-called parallel flow, exists. More precisely, substituting $(\rho, v) = (\rho_p(x_n), v_p^1(x_n,t)\vec{e}_1)$ into (1)–(3), we have

$$\partial_t v_p^1 - \frac{\nu}{\rho_p}\partial_{x_n}^2 v_p^1 = \nu g^1, \tag{5}$$

$$\partial_{x_n}(P(\rho_p)) = \nu \rho_p g^n, \tag{6}$$

$$v_p^1|_{x_n=0} = V^1(t), \quad v_p^1|_{x_n=1} = 0. \tag{7}$$

Under the restrictions

$$1 = \int_0^1 \rho_p(x_n)\, dx_n,$$

and $|g^n|_\infty = \sup_{x_n \in [0,\ell]} |g^n(x_n)|$ is suitably small, problem (5)–(7) has a unique time-periodic solution $u_p = (\rho_p, v_p)$;

$$\rho_p = \rho_p(x_n),$$

$$v_p = (v_p^1(x_n, t), 0, \ldots, 0),$$

where ρ_p and v_p^1 satisfy

$$|\rho_p - 1|_\infty + |\partial_{x_n}\rho_p|_\infty + |P'(\rho_p) - \gamma^2|_\infty \leq \frac{C}{\gamma^2}|g^n|_\infty;$$

$$v_p^1(x_n, t) = (1-x_n)V^1(t) + \int_{-\infty}^{t} e^{-\nu A(t-s)}\{g^1(x_n, s) - (1-x_n)\partial_s V^1(s)\}\, ds.$$

Here A denotes the uniformly elliptic operator on $L^2(0,1)$ with domain $D(A) = H^2(0,1) \cap H_0^1(0,1)$ and

$$Av = -\frac{1}{\rho_p(x_n)}\partial_{x_n}^2 v,$$

for $v \in D(A)$. Let us say that v_p^1 is as smooth as we prescribe it by conditions on g.

We are interested in large time behavior of solutions to the problem (1)–(4) when the initial value (ρ_0, v_0) is sufficiently close to the value of time-periodic solution $u_p = (\rho_p, v_p)$ at some fixed time. Here we deal with the linearized problem around the time-periodic flow. Results presented in this proceedings were obtained as a joint work with Yoshiyuki Kagei in [1].

Lately stability of stationary parallel flows for compressible Navier–Stokes equation has been investigated in [3], [4], [5], [6], [7]. It was proved in [5], [6], that stationary parallel flow is asymptotically stable for sufficiently small initial disturbances if the Reynolds number and Mach number are sufficiently small. Furthermore, when $n \geq 3$ the disturbances behave in large time as solutions of the linearized problem, whose asymptotic leading parts are given by solutions of an $n-1$ dimensional linear heat equation with convective term. On the other hand, when $n = 2$ the asymptotic behavior is no longer described by the linearized problem; and it is described by a nonlinear diffusion equation, namely, by a 1-dimensional viscous Burgers equation.

In this paper we extend previously obtained results for stationary parallel flows to the case of general time-periodic parallel flows. Combining techniques used in [3], [4], [7] and [2] we treat the linearized problem around time-periodic parallel flow and establish decay estimates on solutions similar to those in the case of the stationary parallel flows. Whereas [3], [4], [7] are concerned with the stability of the stationary parallel flows, in [2] the diffusive stability of oscillations in reaction-diffusion systems is treated.

Problem (1)–(4) with $\vec{g} = (g^1(x_n,t), 0, \dots, 0, g^n(x_n))$ covers particularly interesting problem. Let us for a moment consider problem (1)–(4) together with $\vec{g} = (0, \dots, 0, g^n(x_n))$. This problem is a natural extension of Stokes' second problem from half space to infinite strip for compressible fluid. The motion of a fluid is caused by the periodic oscillation of a boundary plate. The study of the flow of a viscous fluid over an oscillating plate is not only of theoretical interest, but it also occurs in many applied problems and since Stokes (1851) it has received much attention under various settings.

Our main result in this paper reads as follows. Setting $\rho = \rho_p + \gamma^{-2}\phi$ and $v = v_p + w$ in (1)–(4) we arrive at problem:

$$\partial_t \phi + v_p^1 \partial_{x_1} \phi + \gamma^2 \mathrm{div}\,(\rho_p w) = f^0, \tag{8}$$

$$\partial_t w - \tfrac{\nu}{\rho_p}\Delta w - \tfrac{\nu+\nu'}{\rho_p}\nabla \mathrm{div}\, w + v_p^1 \partial_{x_1} w^n + (\partial_{x_n} v_p^1) w^n \vec{e}_1 + \tfrac{\nu}{\gamma^2\rho_p^2}(\partial_{x_n}^2 v_p^1)\phi\vec{e}_1 + \nabla\left(\tfrac{\tilde{P}'(\rho_p)}{\gamma^2\rho_p}\phi\right) = \vec{f}, \tag{9}$$

$$w|_{x_n=0} = w|_{x_n=1} = 0, \tag{10}$$

$$(\phi, w)|_{t=0} = (\phi_0, w_0), \tag{11}$$

in Ω. Here $\phi = \phi(x,t)$ and $w = (w^1(x,t), \dots, w^n(x,t))$ denote the unknowns. We are interested in the linearized problem, i.e., $(f^0, \vec{f}) = (0,0)$.

We write (8)–(11) in the form

$$\partial_t u + L(t)u = \vec{0}, \quad w|_{x_n=0,1} = 0, \quad u|_{t=s} = u_0, \tag{12}$$

where $u = {}^T(\phi, w)$; $L(t)$ is the linearized operator and $u_0 = {}^T(\phi_0, w_0)$.

We introduce the space Z_s defined by

$$\begin{aligned} Z_s = \{u = {}^T(\phi, w);\ \phi &\in C_{loc}([s,\infty); H^1), \\ \partial_{x'}^{\alpha'} w &\in C_{loc}([s,\infty); L^2) \cap L^2_{loc}([s,\infty); H_0^1)\ (|\alpha'| \le 1), \\ w &\in C_{loc}((s,\infty); H_0^1)\}, \end{aligned}$$

where the linearized problem can be uniquely solved. We will denote the solution operator for (12) by $\mathscr{U}(t,s)$.

§2. Main results

Theorem 1. *Let $s \geq 0$ be arbitrarily given. For any initial data $u_0 = {}^T(\phi_0, w_0)$ satisfying $u_0 \in H^1 \times L^2$ with $\partial_{x'} w_0 \in L^2$ there exists a unique solution $u_s(t) = \mathscr{U}(t,s)u_0$ of* (12) *in Z_s.*

Furthermore, $\mathscr{U}(t,s)u_0$ satisfies estimates

$$\|\mathscr{U}(t,s)u_0\|_2 \leq C\|u_0\|_2,$$

and

$$\|\partial_x Q_0 \mathscr{U}(t,s)u_0\|_2 + \|\partial_{x'} \widetilde{Q} \mathscr{U}(t,s)u_0\|_2 + (t-s)^{\frac{1}{2}} \|\partial_{x_n} \widetilde{Q} \mathscr{U}(t,s)u_0\|_2 \leq C\left\{\|u_0\|_{H^1 \times L^2} + \|\partial_{x'} w_0\|_2\right\},$$

for $0 < t - s \leq 4T$, $s \geq 0$. Here $\widetilde{Q}$ is $(n+1) \times (n+1)$ diagonal matrix defined as

$$\widetilde{Q} = \operatorname{diag}(0, 1, \ldots, 1).$$

Theorem 2. *There exist constants $\nu_0 > 0$ and $\gamma_0 > 0$ such that if $\nu \geq \nu_0$ and $\gamma^2/(2\nu + \nu') \geq \gamma_0^2$, then for any initial data $u_0 = {}^T(\phi_0, w_0)$ satisfying $u_0 \in (H^1 \times L^2) \cap L^1(\mathbb{R}^{n-1}; H^1(0,1) \times L^2(0,1))$ with $\partial_{x'} w_0 \in L^2$ the solution $u_s(t) = \mathscr{U}(t,s)u_0$ of problem* (12) *can be decomposed as*

$$\mathscr{U}(t,s)u_0 = \mathscr{U}^{(0)}(t,s)u_0 + \mathscr{U}^{(\infty)}(t,s)u_0,$$

where each term on the right-hand side has the following properties for $t - s \geq 4T$, $s \geq 0$.

(i)

$$\|\partial_{x'}^k \partial_{x_n}^l \mathscr{U}^{(0)}(t,s)u_0\|_2 \leq C(t-s)^{-\frac{n-1}{4} - \frac{k}{2}} \|u_0\|_{L^1(\mathbb{R}^{n-1}; H^1(0,1) \times L^2(0,1))},$$

$$\|\partial_{x'}^k \partial_{x_n}^l (\mathscr{U}^{(0)}(t,s)u_0 - \sigma_{t,s} u^{(0)}(t))\|_2 \leq C(t-s)^{-\frac{n-1}{4} - \frac{1}{2} - \frac{k}{2}} \|u_0\|_{L^1(\mathbb{R}^{n-1}; H^1(0,1) \times L^2(0,1))},$$

$k, l = 0, 1$. Here $u^{(0)}(t) = u^{(0)}(x_n, t)$ is a function T-periodic in t and $\sigma_{t,s} = \sigma_{t,s}(x')$ is a function whose Fourier transform in x' is given by

$$\mathscr{F}(\sigma_{t,s}) = e^{-(i\kappa_0 \xi_1 + \kappa_1 \xi_1^2 + \kappa'' |\xi''|^2)(t-s)} [\widehat{\phi}_0(\xi')],$$

where $\xi' = (\xi_1, \dots, \xi_{n-1})$. *Here* $[\widehat{\phi}_0(\xi')]$ *is a quantity given by*

$$[\widehat{\phi}_0(\xi')] = \int_0^1 \widehat{\phi}_0(\xi', x_n)\, dx_n,$$

with $\widehat{\phi}_0$ *being the Fourier transform of* ϕ_0 *in* x' *and* $\kappa_0 \in \mathbb{R}$, $\kappa_1 > 0$, $\kappa'' > 0$ *are some constants satisfying*

$$\kappa_1 = \tfrac{\gamma^2}{\nu} K, \ K > 0,$$

$$\kappa'' = \tfrac{\gamma^2}{\nu} K'', \ K'' > 0,$$

where $\xi_1 \in \mathbb{R}$ *and* $\xi'' = (\xi_2, \dots, \xi_{n-1}) \in \mathbb{R}^{n-2}$.

(ii)

$$\|\partial_x^l \mathscr{U}^{(\infty)}(t,s)u_0\|_2 \le Ce^{-d(t-s)}(\|u_0\|_{H^1 \times L^2} + \|\partial_{x'} w_0\|_2),$$

$l = 0, 1$, *for some positive constant* d.

Remark. Estimates similar to those in Theorems 1 and 2 hold for the case of stationary parallel flows. See [7].

§3. Idea of the proof

To obtain decay estimates as in [7], we consider the Fourier transform of (12) in $x' \in \mathbb{R}^{n-1}$. That can be written as

$$\frac{d}{dt} u + \widehat{L}_{\xi'}(t) u = 0, \quad u|_{t=s} = u_0, \tag{13}$$

on $H^1(0,1) \times L^2(0,1)$. Here $\widehat{L}_{\xi'}(t)$ is an operator on $H^1(0,1) \times L^2(0,1)$ with domain $D(\widehat{L}_{\xi'}(t)) = H^1(0,1) \times (H^2(0,1) \cap H_0^1(0,1))$ with a dual variable $\xi' \in \mathbb{R}^{n-1}$. We denote $\widehat{U}_{\xi'}(t,s)$ the solution operator for (13). The operator $\widehat{U}_{\xi'}(t,s)$ has different characters between cases $|\xi'| \ll 1$ and $|\xi'| \gg 1$. We thus decompose the solution operator $\mathscr{U}(t,s)$ associated with (12) into three parts:

$$\mathscr{U}(t,s) = \mathscr{F}^{-1}\left(\widehat{U}_{\xi'}(t,s)|_{|\xi'| \le r}\right) + \mathscr{F}^{-1}\left(\widehat{U}_{\xi'}(t,s)|_{r \le |\xi'| \le R}\right) + \mathscr{F}^{-1}\left(\widehat{U}_{\xi'}(t,s)|_{|\xi'| \ge R}\right),$$

for some $0 < r \ll 1 \ll R$, where $\mathscr{F}^{-1}$ denotes the inverse Fourier transform.

Since $\widehat{L}_{\xi'}(t)$ is periodic in t, we investigate the operator $\widehat{U}_{\xi'}(T) = \widehat{U}_{\xi'}(T,0)$ for $|\xi'| \leq r \ll 1$ as in [2] and $\widehat{U}_{\xi'}(T)$ can be regarded as a perturbation from monodromy operator $\widehat{U}_0(T) = \widehat{U}_{\xi'}(T)|_{\xi'=0}$ (see [8]). Spectrum of $\widehat{U}_0(T)$ consist of simple eigenvalue 1 and satisfies

$$\sigma(\widehat{U}_0(T)) \subset \{1\} \cup \{\lambda : |\lambda| \leq \delta_0\},$$

where δ_0 is a constant satisfying $0 < \delta_0 < 1$. Thus underlying space for initial data can be decomposed into two parts, where one part is one dimensional space and on the other part we have exponential decay of solutions.

We find that the spectrum of $\mathscr{U}(t,s)$ near 1 is given by that of operator $\widehat{U}_{\xi'}(T)$ with $|\xi'| \leq r \ll 1$, which is parametrized as $1 - i\kappa_0\xi_1 T - \kappa_1\xi_1^2 T - \kappa''|\xi''|^2 T + O(|\xi'|^3)T$ with some $\kappa_0 \in \mathbb{R}$, $\kappa_1 > 0$, provided Re and Ma are sufficiently small.

On the other hand, if $|\xi'| \geq R \gg 1$, we can derive the exponential decay property of the corresponding part of the solution operator $\mathscr{U}(t,s)$ by the Fourier transformed version of Matsumura–Nishida's energy method (e.g. see [4], [9]), provided that Re and Ma are sufficiently small.

As for the bounded frequency part $r \leq |\xi'| \leq R$, we employ a certain time-dependent decomposition argument and apply a variant of Matsumura-Nishida's energy method as in [7] to show the exponential decay.

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