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Allen–Cahn equation as a long-time modulation to a reaction-diffusion system

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Abstract.

We examine a two-component reaction-diffusion system on the real axis with quadratic nonlinearity. Using semigroup estimates, we obtain a solution to our nonlinear system for long-time. For appropriate initial data, we show that a slowly-varying, scaled solution of the Allen–Cahn equation will estimate the solution of our nonlinear system for long-time. We additionally extend this work to $\mathbb{R}^d$.

§1. Introduction

Modulation equations approximate the dynamics of an original system in an attracting set. Modulation equations are essential in understanding complicated systems near the threshold of instability [2].

This paper expands results of [3], sharpening an assumption on the nonlinearity, and producing sharper stability estimates. These results are also in a more general function space.

We study the following reaction-diffusion system:

$$\partial_t u_1 = \epsilon^2 u_1 + \partial_x^2 u_1 + g(u), \tag{1.1a}$$

$$\partial_t u_2 = -\nu u_2 + \partial_x^2 u_2 + h(u), \tag{1.1b}$$

where $0 < \epsilon \ll 1$, $\nu > 0$, $t \geq 0$, $x \in \mathbb{R}$, $u(x,t) = (u_1(x,t), u_2(x,t))^T \in \mathbb{R}^2$, and the nonlinearities $h(u)$ and $g(u)$ satisfy,

$$(g(u), h(u)) = u^T \left(\begin{pmatrix} 0 & c_{12} \\ c_{21} & c_{22} \end{pmatrix}, \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix} \right) u + O(||u||^3), \tag{1.2}$$

for c_{12}, c_{21}, c_{22}, d_{11}, d_{12}, d_{21}, and d_{22} constant.

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§2. Semigroup estimates

First, we analyze only the linear components of the system (1.1). For the u_1 component, we solve $\partial_t \phi = \mathcal{L}_1 \phi = \epsilon^2 \phi + \partial_x^2 \phi$. A solution to this is $\phi = S_1(t)\phi(0)$, where $S_1(t) = e^{\mathcal{L}_1 t}$. For the u_2 component, we solve $\partial_t \phi = \mathcal{L}_2 \phi = -\nu\phi + \partial_x^2 \phi$, where $\phi = S_2(t)\phi(0)$, for $S_2(t) = e^{\mathcal{L}_2 t}$. We have the following semigroup estimates.

Proposition 2.1. *There exists $C > 0$ independent of ϵ and $t > 0$ such that for any $\phi \in L^1$,*

$$||S_1(t)\phi||_{H^1} \leq Ce^{\epsilon^2 t}\left(t^{-1/4} + t^{-3/4}\right)||\phi||_{L^1}, \tag{2.1}$$

$$||S_2(t)\phi||_{H^1} \leq Ce^{-\nu t}\left(t^{-1/4} + t^{-3/4}\right)||\phi||_{L^1}. \tag{2.2}$$

Also for any $\phi \in H^1$,

$$||S_1(t)\phi||_{H^1} \leq e^{\epsilon^2 t}||\phi||_{H^1}, \tag{2.3}$$

$$||S_2(t)\phi||_{H^1} \leq e^{-\nu t}||\phi||_{H^1}. \tag{2.4}$$

Sketch of Proof. For (2.1) and (2.2), L^2 to L^1 estimates are used. The proofs of (2.3) and (2.4) are standard. Q.E.D.

§3. Reduction of long-time dynamics

If $v = (v_1, v_2)^T$ solves (1.1) absent the nonlinear terms, we can apply (2.3) and (2.4) to v for $t \in [0, T_0/\epsilon^2]$ for fixed $T_0 = 0(1)$. If the initial data is $O(\epsilon^\alpha)$ in H^1 norm, then at $t = T_0/\epsilon^2$, v has the representation:

$$v(x, T_0/\epsilon^2) = (A(x), B(x))^T, \tag{3.1}$$

where $||A(x)||_{H^1} = O(\epsilon^\alpha)$ and $||B(x)||_{H^1} = O(\epsilon^\alpha e^{-C/\epsilon^2})$. This linear reduction is close to the correct representation for a solution to (1.1). But now the u_2 component is forced by the nonlinearity, so it is not exponentially decaying. We formalize this with the following theorem:

Theorem 3.1. *Fix $C_0 > 0$, then there exists T_0, $C_f > C_0$, and $\epsilon_0 > 0$ such that for all $\epsilon \in (0, \epsilon_0)$ the following holds: let $||u_0||_{H^1} \leq C_0\epsilon$ where $u_0 = (u_1(x,0), u_2(x,0))^T$, then the solution u of (1.1) at a time $t = T_0/\epsilon^2$ can be written as*

$$u(x, T_0/\epsilon^2) = \left(\epsilon A(x), \epsilon^2 B(x)\right)^T, \tag{3.2}$$

where $||A||_{H^1} \leq C_f$ and $||B||_{H^1} \leq C_f$.

Proof. From (1.2) we observe for u small,

$$|g(u)| \leq C\left(|u_1 u_2| + |u_2^2|\right) + O(||u||^3), \tag{3.3}$$
$$|h(u)| \leq C||u||^2 + O(||u||^3). \tag{3.4}$$

For (1.1a), we solve for u_1 by variation of constants and apply the H^1 norm to $u_1(x,t)$ and the semigroup estimates (2.1) and (2.3),

$$\begin{aligned} ||u_1(x,t)||_{H^1} \leq& ||S_1(t)u_1(0)||_{H^1} + ||\int_0^t S_1(t-s)g(u(s))ds||_{H^1} \\ \leq& C\epsilon e^{\epsilon^2 t} + C\int_0^t \psi_1(t-s)||g(u(s))||_{L^1}ds, \end{aligned} \tag{3.5}$$

where we define $\psi_1(t) = e^{\epsilon^2 t}\left(t^{-1/4} + t^{-3/4}\right)$. Substituting (3.3) above, we estimate $\left|\left||u_1 u_2| + |u_2|^2\right|\right|_{L^1}$ with Holder's and Young's Inequality and $\left|\left|||u||^3\right|\right|_{L^1}$ with the Sobolev Embedding Theorem,

$$\left|\left||u_1 u_2| + |u_2|^2\right|\right|_{L^1} \leq C(||u_2||_{H^1}^{3/2} + ||u||_{H^1}^3), \tag{3.6}$$
$$\left|\left|||u||^3\right|\right|_{L^1} \leq C||u||_{L^3}^3 \leq C||u||_{H^1}^3. \tag{3.7}$$

Applying the above to (3.5), we have

$$||u_1(x,t)||_{H^1} \leq C\epsilon e^{\epsilon^2 t} + C\int_0^t \psi_1(t-s)(||u_2(s)||_{H^1}^{3/2} + M(\tau)^3)ds, \tag{3.8}$$

where we define $M_1(\tau) = \sup_{t\leq\tau}||u_1(t)||_{H^1}$, $M_2(\tau) = \sup_{t\leq\tau}||u_2(t)||_{H^1}$, and $M = M_1 + M_2$, for $\tau \leq T_0/\epsilon^2$. We solve for u_2 in (1.1a) by variation of constants and apply the H^1 norm to $u_2(x,t)$ and the semigroup estimates (2.2) and (2.4),

$$\begin{aligned} ||u_2(x,t)||_{H^1} \leq& \left|\left|S_2(t)u_2(0) + \int_0^t S_2(t-s)h(u(s))ds\right|\right|_{H^1} \\ \leq& C\epsilon e^{-\nu t} + \int_0^t \psi_2(t-s)||h(u(s))||_{L^1}ds \\ \leq& C(\epsilon e^{-\nu t} + M(\tau)^2), \end{aligned} \tag{3.9}$$

where $\psi_2(t) = e^{-\nu t}\left(t^{-1/4} + t^{-3/4}\right)$. Of note, we omit an $M(\tau)^3$ from (3.9) since it does not have a leading order contribution. This $M(\tau)^3$ term would result in a $M(\tau)^{9/2}$ in the subsequent equation (3.10) below, but we again omit this term since it does not have a leading order contribution. We substitute (3.9) into (3.8) and apply the $\sup_{t\leq\tau}$:

$$\begin{aligned} M_1 \leq& C\sup_{t\leq\tau}\left(\epsilon e^{\epsilon^2 t} + C\int_0^t \psi_1(t-s)(\epsilon^{3/2}e^{-3\nu s/2} + M(\tau)^3)ds\right) \\ \leq& C\left(\epsilon + \epsilon^{3/2}\int_0^\tau \psi_1(\tau-s)e^{-3\nu s/2}ds\right) \\ \leq& C(\epsilon + \epsilon^{-3/2}M^3), \end{aligned} \tag{3.10}$$

since $\int_0^\tau \psi_1(\tau-s)ds \leq C\epsilon^{-3/2}$. Applying the $\sup_{t\leq\tau}$ to (3.9) we have:

$$M_2 \leq C(\epsilon + M^2). \tag{3.11}$$

Summing (3.11) and (3.10) implies $M \leq C_f(\epsilon + M^2 + \epsilon^{-3/2}M^3)$, where $C_f > C_0$. We take the corresponding equality and define

$$w(M) \equiv C_f(\epsilon + M^2 + \epsilon^{-3/2}M^3) - M. \tag{3.12}$$

At leading order, $w(M)$ has two positive roots at $C_f\epsilon$ and $\epsilon^{3/4}/\sqrt{C_f}$. Depending on the size of the initial data, either $M < C_f\epsilon$ or $M(0) > \epsilon^{3/4}/\sqrt{C_f}$ for long-time. From an assumption of Theorem 3.1, $M(0) \leq C_0\epsilon$, so $M < C_f\epsilon$. Applying this to (3.9), we have

$$||u_2(x, T_0/\epsilon^2)||_{H^1} \leq C_f\epsilon^2. \tag{3.13}$$

Using the above estimate in (3.10), it follows that

$$||u_1(x, T_0/\epsilon^2)||_{H^1} \leq C_f\epsilon. \tag{3.14}$$

To finish the proof, we define

$$\epsilon A(x) = u_1(x, T_0/\epsilon^2), \tag{3.15}$$

$$\epsilon^2 B(x) = u_2(x, T_0/\epsilon^2). \tag{3.16}$$

Q.E.D.

Remark 3.1. We can extend Theorem 3.1 to the case when the first component of the nonlinearity $g(u)$ is controlled by $C(|u_1u_2| + |u_2|^2) + O(||u||^\beta)$, for $\beta \geq 5/2$.

§4. Approximation by the Allen–Cahn equation

Motivated by Theorem 3.1, for $A, B \in \mathbb{R}$, we make the ansatz $u = \left(\epsilon A(X,T), \epsilon^2 B(X,T)\right)^T$, for $X = \epsilon x$ and $T = \epsilon^2 t$. Formally, plugging this into (1.1),

$$\partial_T A = \quad \partial_X^2 A + A + \epsilon^{-3} g((\epsilon A, \epsilon^2 B)), \tag{4.1}$$

$$\epsilon^2 \partial_T B = \quad -\nu B + \epsilon^2 \partial_X^2 B + \epsilon^{-2} h((\epsilon A, \epsilon^2 B)). \tag{4.2}$$

Using the information about g and h from (1.2), we have:

$$g((\epsilon A, \epsilon^2 B)) = (c_{21} + c_{12})\epsilon^3 AB + c_{111}\epsilon^3 A^3 + O(\epsilon^4), \tag{4.3}$$

$$h((\epsilon A, \epsilon^2 B)) = d_{11}\epsilon^2 A^2 + O(\epsilon^3), \tag{4.4}$$

where c_{111} is the first entry in the 3-tensor of the cubic part of g. Plugging these into (4.1) and (4.2), at leading order we have,

$$\partial_T A = \partial_X^2 A + A + (c_{21} + c_{12})AB + c_{111}A^3, \tag{4.5}$$

$$0 = -\nu B + d_{11}A^2. \tag{4.6}$$

With the second line, we express B in terms of A, where $B = d_{11}A^2/\nu$. Substituting this into the system above, we have the Allen–Cahn system:

$$\partial_T A = \partial_X^2 A + A + \gamma A^3, \tag{4.7}$$

where $\gamma = d_{11}(c_{21} + c_{12})/\nu + c_{111}$. To begin a rigorous reduction, we define the ansatz to our nonlinear system (1.1) as

$$\Phi_\epsilon[A_0](x,t) = \begin{pmatrix} \epsilon A(\epsilon x, \epsilon^2 t) \\ \epsilon^2 B(\epsilon x, \epsilon^2 t) \end{pmatrix}, \tag{4.8}$$

where A solves (4.7), $A|_{t=0} = A_0$ is the initial data, and $B = d_{11}A^2/\nu$. The function Φ_ϵ maps the initial data forward, both scaling space and time. We define the following residuals for $v = (v_1, v_2)^T$ where $v_1, v_2 \in H^1((0,T_0); L^2(\mathbb{R})) \cap L^2((0,T_0); H^2(\mathbb{R}))$:

$$Res_1(v) \quad = -\partial_t v_1 + \epsilon^2 v_1 + \partial_x^2 v_1 + g(v_1, v_2), \tag{4.9}$$

$$Res_2(v) \quad = -\partial_t v_2 - \nu v_2 + \partial_x^2 v_2 + h(v_1, v_2). \tag{4.10}$$

The next proposition details bounds on these residuals for our ansatz.

Proposition 4.1. *Define* $\Phi_\epsilon[A_0]$ *by* (4.8) *where* A *solves* (4.7) *and* $\sup_{T\in[0,T_0]} ||A(T)||_{H^2} < \infty$, *then we have the following estimates:*

$$\sup_{t\in[0,T_0/\epsilon^2]} ||Res_1(\Phi_\epsilon[A_0])||_{H^1} \le C\epsilon^4, \tag{4.11}$$

$$\sup_{t\in[0,T_0/\epsilon^2]} ||Res_2(\Phi_\epsilon[A_0])||_{H^1} \le C\epsilon^3. \tag{4.12}$$

The above follows from A solving (4.7), $B = d_{11}A^2/\nu$, and using the given expansions of g and h above.

The following theorem is our main result.

Theorem 4.1. *For all* $K, d > 0$, *there exists* C_1, ϵ_0, *and* $T_0 > 0$ *such that for all* $\epsilon \in (0, \epsilon_0)$, *the following holds: let* A *be a solution of Allen–Cahn with* $\sup_{t\in[0,T_0]} ||A(t)||_{H^2} \le K$, *and* $u_0 = (u_1(0), u_2(0))^T \in H^1$ *an initial condition for* (1.1) *with*

$$||u_1(0) - \epsilon A(\epsilon x, 0)||_{H^1} \le d\epsilon^2, \tag{4.13}$$

$$||u_2(0) - \epsilon^2 B(\epsilon x, 0)||_{H^1} \le d\epsilon^3, \tag{4.14}$$

then there exists a unique solution u *of* (1.1) *with* $u|_{t=0} = u_0$ *such that*

$$\sup_{t\in[0,T_0/\epsilon^2]} ||u_1(t) - \epsilon A(\epsilon x, \epsilon^2 t)||_{H^1} \le C_1\epsilon^2, \tag{4.15}$$

$$\sup_{t\in[0,T_0/\epsilon^2]} ||u_2(t) - \epsilon^2 B(\epsilon x, \epsilon^2 t)||_{H^1} \le C_1\epsilon^3. \tag{4.16}$$

Proof. We define the error of $\Phi_\epsilon[A_0]$ as a solution of (1.1) as $R = (R_1, R_2)^T = \left(\epsilon^{-2}\left(u_1 - \epsilon A(\epsilon x, \epsilon^2 t)\right), \epsilon^{-3}\left(u_2 - \epsilon^2 B(\epsilon x, \epsilon^2 t)\right)\right)^T$. Plugging these errors into (1.1), we have the following system,

$$\partial_t R_1 = \epsilon^2 R_1 + \partial_x^2 R_1 + N_1(u), \tag{4.17}$$

$$\partial_t R_2 = -\nu R_2 + \partial_x^2 R_2 + N_2(u). \tag{4.18}$$

Lemma 4.1. *The following* H_1 *bounds on* N_1 *and* N_2 *hold:*

$$||N_1(u)||_{H^1} \le C\epsilon^2(||R||_{H^1} + \epsilon||R||^2_{H^1}) + C\epsilon^2, \tag{4.19}$$

$$||N_2(u)||_{H^1} \le C(||R_1||_{H^1} + \epsilon||R_2||_{H^1} + \epsilon||R||^2_{H^1}) + C. \tag{4.20}$$

Proof. We now sketch some of the proof. We substitute for $N_1(u)$ using (4.17) and the form of R above, so

$$||N_1(u)||_{H^1} = ||\frac{1}{\epsilon^2}\left(\left(\epsilon^2+\partial_x^2-\partial_t\right)(\epsilon A)+g\left(\epsilon^2 R_1+\epsilon A, \epsilon^3 R_2+\epsilon^2 B\right)\right)||_{H^1}$$

$$= ||\frac{1}{\epsilon^2}\left(Res_1(\Phi_\epsilon(A))+G\left(g\right)\right)||_{H^1}$$

$$\le C_{Res_1}\epsilon^2+\frac{1}{\epsilon^2}||G\left(g\right)||_{H^1}, \tag{4.21}$$

where we define

$$G\left(g\right) \equiv g(\epsilon^2 R_1+\epsilon A, \epsilon^3 R_2+\epsilon^2 B)-g(\epsilon A, \epsilon^2 B). \tag{4.22}$$

Using our knowledge about g to thoroughly analyze the differences contained in G, we arrive at the following estimate:

$$||G(g)||_{H^1} =||g(\epsilon^2 R_1+\epsilon A, \epsilon^3 R_2+\epsilon^2 B)-g(\epsilon A, \epsilon^2 B)||_{H^1}$$

$$\le C\epsilon^4(||R_1||_{H^1}+||R_2||_{H^1}+\epsilon(||R_1||_{H^1}+||R_2||_{H^1})^2), \tag{4.23}$$

from which we conclude the first estimate in this lemma. Here, we use the fact that $||W^2||_{H^1} \le ||W||_{H^1}^2$, since $W \in L^\infty$ for any $W \in H^1$, which follows from the Sobolev Embedding Theorem. The estimate on N_2 follows similarly, by examining the difference of two h terms. Q.E.D.

For (4.17) and (4.18) we solve by variation of constants,

$$R_1(t) =S_1(t)R_1(0)+\int_0^t S_1(t-s)N_1(u(s))ds, \tag{4.24}$$

$$R_2(t) =S_2(t)R_2(0)+\int_0^t S_2(t-s)N_2(u(s))ds. \tag{4.25}$$

We define $\tilde{M}_1(\tau)=\sup_{t\le\tau}||R_1(t)||_{H^1}$, $\tilde{M}_2(\tau)=\sup_{t\le\tau}||R_2(t)||_{H^1}$, and $\tilde{M}(\tau) = \tilde{M}_1(\tau)+\tilde{M}_2(\tau)$ for $\tau \le T_0/\epsilon^2$. We apply $\sup_{t\le\tau}$ to (4.24) and (4.25),

$$\tilde{M}_1(\tau) \le Ce^{T_0}+CT_0e^{T_0}(\tilde{M}_1(\tau)+\tilde{M}_2(\tau)+\epsilon\tilde{M}(\tau)^2), \tag{4.26}$$

$$\tilde{M}_2(\tau) \le C+C(\tilde{M}_1(\tau)+\epsilon\tilde{M}_2(\tau)+\epsilon\tilde{M}(\tau)^2+C). \tag{4.27}$$

Picking ϵ_0 small enough such that $C\epsilon \le 1/2$, it follows that

$$\tilde{M}_2(\tau) \le C+C\left(\tilde{M}_1(\tau)+\epsilon(\tilde{M}_1(\tau)+\tilde{M}_2(\tau))^2\right). \tag{4.28}$$

Plugging the above bound into (4.26) and picking $T_0 > 0$ small enough so that $CT_0e^{T_0} \leq 1/2$, we arrive at the following estimate:

$$\tilde{M}_1(\tau) \leq C + C(\epsilon(\tilde{M}_1(\tau) + \tilde{M}_2(\tau))^2). \tag{4.29}$$

Substituting (4.29) into (4.28), we have

$$\tilde{M}_2(\tau) \leq C + C(\epsilon(\tilde{M}_1(\tau) + \tilde{M}_2(\tau))^2). \tag{4.30}$$

Finally, we sum (4.29) and (4.30), so $\tilde{M} \leq C_1(1+\epsilon\tilde{M}^2)$, where $C_1 \geq 2d$. With this inequality, we solve the corresponding equality,

$$\tilde{w}(\tilde{M}) = C_1(1 + \epsilon\tilde{M}^2) - \tilde{M}. \tag{4.31}$$

At leading order, the roots are $\tilde{M} = C_1$ and $\tilde{M} = 1/(C_1\epsilon)$. Similar to Theorem 3.1, initial data bounds imply $\tilde{M}(0) \leq 2d$, so $\tilde{M} \leq C_1$. Q.E.D.

§5. Higher spatial dimensions

We must change spaces for our results to hold for $x \in \mathbb{R}^d$. We need the new space to control L^∞, so we require $kp > d$. The obvious space is the Sobolev space H^k, with $k > d/2$. We want $p = 2$ to maintain Plancherel's Theorem and other befitting properties of the Fourier transform in L^2. In H^k, for $k > 2/d$, we still have L^q controlled for $q > p$, which is needed in the proof of Theorem 3.1.

We require new L^1 semigroup estimates; otherwise the proof of Theorem 3.1 will fail. Short and long-time estimates are necessary to avoid integrating near 0. With the next estimate replacing (2.1), the results of this paper will follow for $x \in \mathbb{R}^d$:

$$||S_1(t)\phi||_{H^k} \leq C\frac{e^{\epsilon^2 t}}{(t+1)^{d/2-1/4}}(||\phi||_{H^k} + ||\phi||_{L^1}). \tag{5.1}$$

§6. Conclusion

This work demonstrates new results showing that a scaled solution of the Allen–Cahn system accurately approximates a solution to the nonlinear reaction-diffusion system (1.1) for long-time. We build on previous results by providing a sharper representation of the nonlinear term g, which leads to sharper estimates. For Theorem 4.1, we are sharper to a higher order of ϵ in the assumption and result with respect to the second component. We also work in a more general function space.

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References

[1] L. Hörmander, The Analysis of Linear Partial Differential Operators 14, Springer-Verlag, 1985.

[2] G. Schneider, A new estimate for the Ginzburg–Landau approximation on the real axis, J. Nonlinear Sci., **4** (1994), 23–34.

[3] G. Schneider, Bifurcation theory for dissipative systems on unbounded cylindrical domains—An introduction to the mathematical theory of modulation equations, ZAMM Z. Angew. Math. Mech., **81** (2001), 507–522.

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