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Dissipative structure of the coupled kinetic-fluid models

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Abstract.

We present a study of dissipative structures for a class of the coupled kinetic-fluid models with partial relaxations at the linearized level. It is a generalization of several known results in the decoupled case that is either for the kinetic model or for the symmetric hyperbolic system. Precisely, a partially dissipative linearized system is of the type (p, q) if the real parts of all eigenvalues in terms of the frequency variable k admit an upper bound $-|k|^{2p}/(1+|k|^2)^q$ up to a common positive constant. It is well known that a symmetric hyperbolic system with partial relaxation is of the type $(1, 1)$ if the so-called Shizuta–Kawashima conditions are satisfied. In the current study of the coupled kinetic-fluid models, we postulate more general conditions together with some concrete examples to include the case $(1, 2)$ investigated also in [14] and the new case $(2, 3)$. Thus, the coupled kinetic-fluid models may exhibit more complex dissipative structures.

§1. Model and problem

Consider

$$u_t + \xi \cdot \nabla_x u + \mathcal{L}u + B_1^T \mathrm{e} \cdot v = 0, \tag{1}$$

$$v_t + \sum_{j=1}^{n} A^j v_{x_j} + Lv + B_2 \mathrm{e}[u] = 0. \tag{2}$$

The unknowns are $u = u(t, x, \xi) \in \mathbb{R}$ for the kinetic part and $v = v(t, x) \in \mathbb{R}^{m_1}$ for the fluid part, where $t \geq 0$, $x \in \mathbb{R}^n$, $\xi \in \mathbb{R}^n$ and $n \geq 1$, $m_1 \geq 1$ are integers. In the kinetic equation, $\mathcal{L}$ is a linear, nonnegative definite, self-adjoint operator from $L^2(\mathbb{R}^n_\xi)$ to itself which only acts on

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the velocity variable. In the fluid part, A^j $(j = 1, 2, \cdots, n)$ are real, symmetric, $m_1 \times m_1$ matrices, and L is a real, nonnegative definite, $m_1 \times m_1$ matrix. In both coupling terms, $\mathrm{e} = (\mathrm{e}_1, \mathrm{e}_2, \cdots, \mathrm{e}_{m_2})^T$ for a integer $m_2 \geq 1$ is a column vector function of the only variable ξ with each $\mathrm{e}_\ell = \mathrm{e}_\ell(\xi) \in L^2(\mathbb{R}^n_\xi)$ $(\ell = 1, 2, \cdots, m_2)$, $\mathrm{e}[f]$ for a given function $f = f(\xi)$ of the velocity variable denotes

$$\mathrm{e}[f] = \Big(\mathrm{e}_1[f], \mathrm{e}_2[f], \cdots, \mathrm{e}_{m_2}[f]\Big)^T,$$
$$\mathrm{e}_\ell[f] = \int_{\mathbb{R}^n} e_\ell(\xi) f(\xi)\, d\xi, \quad \ell = 1, 2, \cdots, m_2,$$

and B_1, B_2 are real $m_2 \times m_1$ and $m_1 \times m_2$ matrices, respectively.

Specifically, as will be pointed out later on through some concrete examples, the linearized collision operator $\mathcal{L}$ in the kinetic equation could be either the relaxation operator, Fokker–Planck operator, Boltzmann-type operator, cf. [5], while the linearized fluid equation could correspond to either the Maxwell system or the compressible Euler system, cf. [3], [6] and [2].

The goal of the paper is to determine the dissipative structure of the above coupled kinetic-fluid models under some conditions, which can induce the explicit time decay rate of solutions in the energy space, as studied in [4], [13] and [3].

§2. Basic assumption

We now postulate the first assumption on $\mathcal{L}$ and the velocity vector-valued function e in the kinetic equation.

(A1): $\mathcal{L}$ is a linear, nonnegative-definite, self-adjoint operator from $L^2(\mathbb{R}^n_\xi)$ to itself, with $\ker \mathcal{L} \neq \{0\}$. The set

$$\{\mathrm{e}_\ell = \mathrm{e}_\ell(\xi) \in L^2(\mathbb{R}^n),\ 1 \leq \ell \leq m_2\}$$

is orthonormal such that the subset $\{\mathrm{e}_1, \cdots, \mathrm{e}_{m_0}\}$ of the first m_0 elements is an orthonormal basis of $\ker \mathcal{L}$ and

$$\text{(3)} \qquad \mathrm{span}\{\mathrm{e}_\ell,\ 1 \leq \ell \leq m_2\} = \mathrm{span}\{\mathrm{e}_\ell, \xi_j \mathrm{e}_\ell,\ 1 \leq \ell \leq m_0, 1 \leq j \leq n\}.$$

There is a constant $\lambda_{\mathcal{L}} > 0$ such that

$$\int_{\mathbb{R}^n} f\mathcal{L}f d\xi \geq \lambda_{\mathcal{L}} \int_{\mathbb{R}^n} |\{\mathbf{I} - \mathbf{P}_{\mathcal{L}}\}f|^2\, d\xi$$

for any $f = f(\xi) \in L^2(\mathbb{R}^n)$, where $\mathbf{I}$ is the identity, and $\mathbf{P}_{\mathcal{L}}$ is the orthogonal projection from $L^2(\mathbb{R}^n)$ to $\ker \mathcal{L}$ with respect to $\{\mathrm{e}_1, \mathrm{e}_2, \cdots, \mathrm{e}_{m_0}\}$, explicitly given by

$$\mathbf{P}_{\mathcal{L}} f = \sum_{\ell=1}^{m_0} \mathrm{e}_\ell(f)\mathrm{e}_\ell.$$

Remark 1. One can replace ξ in the free transport operator $\partial_t + \xi \cdot \nabla_x$ of (1) by $V(\xi) : \mathbb{R}^n \to \mathbb{R}^n$. In the case of $\partial_t + V(\xi) \cdot \nabla_x$, the identity (3) should be replaced by

$$\mathrm{span}\{\mathrm{e}_\ell,\ 1 \le \ell \le m_2\} = \mathrm{span}\{\mathrm{e}_\ell, V_j(\xi)\mathrm{e}_\ell,\ 1 \le \ell \le m_0, 1 \le j \le n\}.$$

This kind of extension can include both classical and relativistic cases; for the latter, $V(\xi) = \xi/\sqrt{1 + |\xi|^2}$.

The second assumption is postulated on matrices A^j and L in the fluid equation.

(A2): A^j $(j = 1, 2, \cdots, n)$ are constant real symmetric $m_1 \times m_1$ matrices, and L is a constant real $m_1 \times m_1$ matrix, not necessarily symmetric.

§3. Moment equation and partially dissipative assumption

By applying the Fourier transform with respect to the space variable x, we write (1), (2) as

$$\begin{aligned} &\hat{u}_t + i|k|\xi \cdot \kappa \hat{u} + \mathcal{L}\hat{u} + B_1^T \mathrm{e} \cdot \hat{v} = 0, \\ &\hat{v}_t + i|k| A_\kappa^L \hat{v} + L\hat{v} + B_2 \mathrm{e}[\hat{u}] = 0, \end{aligned}$$

where

$$\kappa = (\kappa_1, \kappa_2, \cdots, \kappa_n) = \frac{k}{|k|} \text{ for } 0 \ne k \in \mathbb{R}^n, \quad A_\kappa^L = \sum_{j=1}^n A^j \kappa_j.$$

Set $w = \mathrm{e}[u]$. One can derive the evolution equation of $w = w(t, x)$ as

$$\hat{w}_t + i|k| A_\kappa^{\mathcal{L}} \hat{w} + L^{\mathcal{L}} \hat{w} + B_1 \hat{v} = \widetilde{R}(\hat{u}).$$

Here, the notations are explained as follows. $A_\kappa^{\mathcal{L}}$ is a real symmetric matrix, given by, for $1 \le j, \ell \le m_2$,

$$(A_\kappa^{\mathcal{L}})_{j\ell} = \begin{cases} \mathrm{e}_j(\kappa \cdot \xi \mathrm{e}_\ell) = \displaystyle\int_{\mathbb{R}^n} \kappa \cdot \xi \mathrm{e}_j \mathrm{e}_\ell \, d\xi & \text{if either } 1 \le j \le m_0, \\ \qquad 1 \le \ell \le m_2, \text{ or } m_0 + 1 \le j \le m_2, 1 \le \ell \le m_0; \\ 0 \qquad \text{otherwise.} \end{cases}$$

$L^{\mathcal{L}}$ is a constant real symmetric matrix, given by, for $1 \le j, \ell \le m_2$,

$$(L^{\mathcal{L}})_{j\ell} = \begin{cases} \mathrm{e}_j(\mathcal{L}\mathrm{e}_\ell) = \displaystyle\int_{\mathbb{R}^n} \mathrm{e}_j \mathcal{L}\mathrm{e}_\ell \, d\xi & \text{if } m_0 + 1 \le j, \ell \le m_2; \\ 0 & \text{otherwise,} \end{cases}$$

and hence $L^{\mathcal{L}}$ satisfies

$$w^T \cdot L^{\mathcal{L}} w \ge \lambda_{\mathcal{L}} \sum_{\ell=m_0+1}^{m_2} |w_\ell|^2,$$

for any $w \in \mathbb{R}^{m_2}$. $R(\hat{u}) = \big(R_1(\hat{u}), \cdots, R_{m_2}(\hat{u})\big)^T$ is a column vector-valued function, given by, for $1 \le \ell \le m_2$,

$$R_\ell(\hat{u}) = \begin{cases} 0 \quad \text{if } 1 \le \ell \le m_0; \\ -\mathrm{e}_\ell\big(i|k|\kappa \cdot \xi\{\mathbf{I} - \mathbf{P}_{m_2}\}\hat{u} + \mathcal{L}\{\mathbf{I} - \mathbf{P}_{\mathcal{L}}\}\hat{u}\big) \\ \qquad\qquad\qquad\qquad \text{if } m_0 + 1 \le \ell \le m_2, \end{cases}$$

where $\mathbf{P}_{m_2}$ is an orthonormal projection from $L^2(\mathbb{R}^n)$ to $\mathrm{span}\{\mathrm{e}_\ell,\, 1 \le \ell \le m_2\}$.

Therefore, by setting $U = (w, v)^T$, we arrive at

$$\hat{U}_t + i|k|A_\kappa \hat{U} + \widetilde{L}\hat{U} = \widetilde{R}(\hat{u}) \tag{4}$$

with

$$A_\kappa = \begin{pmatrix} A_\kappa^{\mathcal{L}} & 0 \\ 0 & A_\kappa^L \end{pmatrix}, \ \widetilde{L} = \begin{pmatrix} L^{\mathcal{L}} & B_1 \\ B_2 & L \end{pmatrix}, \ \widetilde{R}(\hat{u}) = \begin{pmatrix} R(\hat{u}) \\ 0 \end{pmatrix}.$$

Notice that A is a real symmetric matrix, the real matrix $\widetilde{L}$ is not necessarily symmetric, and $\widetilde{R}(\hat{u})$ is the moment function of elements in $(\ker \mathcal{L})^{\perp}$.

In order to achieve the desired goal, the key problem is reduced to analyze the finite-dimensional symmetric hyperbolic system with relaxations (4) by postulating some additional conditions as in [14]. Beside two assumptions **(A1)** and **(A2)**, we also require the assumption

(A3): $\widetilde{L}$ is nonnegative definite, i.e.,

$$U^T \cdot \widetilde{L}U \ge 0, \quad \forall U \in \mathbb{R}^{m_2+m_1}.$$

Proposition 1. *The coupled linear system* (1)–(2) *is partially dissipative under the assumptions* **(A1), (A2)** *and* **(A3)**.

It is straightforward to prove the above proposition, cf. [7].

§4. Modeling

Before discussing (4), we list several coupled kinetic-fluid models whose dissipative structures have been well established individually. Recall as in [14] the following

Definition 1. *A linearized homogeneous system* (1)–(2) *is partially dissipative of the type* (p,q) *with* $q \geq p > 0$ *if there are constants* $\lambda > 0$, C *such that*

$$\|\mathcal{F}\{\mathbb{S}(t)(u_0, v_0)\}\|_Y \leq Ce^{-\frac{\lambda|k|^{2p}}{(1+|k|^2)^q}t}\|\mathcal{F}\{(u_0, v_0)\}\|_Y, \quad \forall t \geq 0, k \in \mathbb{R}^n,$$

where $\mathbb{S}(t)$ *is the linear solution operator,* (u_0, v_0) *is initial data and* $\|\cdot\|_Y$ *is a properly chosen norm.*

Recall also for the fluid equations in the decoupled situation:

- Type $(1,1)$: This is a standard type, e.g., the Euler system with damping [10] and the electro-magneto-fluid system [15]. A general theory was established in [12].
- Type $(1,2)$: This is a new type, e.g., the Euler–Maxwell system with damping [4], [13] and the Timoshenko system [8], [9]. A general theory has been recently given in [14].

It can be seen from the following examples that some of either kinetic or coupled kinetic-fluid models which are partially dissipative expose the above similar property.

Model 0. Boltzmann equation, cf. [11]: Type $(1,1)$. The linearized version takes the form of

$$u_t + \xi \cdot \nabla_x u + \mathcal{L}u = 0. \tag{5}$$

It is the first equation of the decoupled system (1)–(2) when the coupling matrices B_1 and B_2 vanishes; see also [5] for a general choice of $\mathcal{L}$.

Model 1. Vlasov–Euler–Fokker–Planck system, cf. [2]: Type $(1,1)$. For the model studied in [2], $\mathcal{L}$ takes the linearized self-adjoint Fokker–Planck operator, the fluid part consists of the incompressible Euler system, and the kinetic and fluid equations are coupled through the frictional forcing. Notice that the result in [2] is easily extent to the case when the Euler system is compressible.

Model 2. Vlasov–Maxwell–Boltzmann system of two-species, cf. [6]: Type $(1,2)$. The linearized system take the form of the kinetic equations

$$\partial_t u_\pm + \xi \cdot \nabla_x u_\pm \mp E \cdot \xi M^{1/2} = \mathcal{L}_\pm u, \tag{6}$$

coupled with the Maxwell equations. Here M is a normalized global Maxwellian and the kinetic unknown $u = (u_+, u_-)^T$ is a vector-valued function; refer to [6] for more notations and the complete analysis of the system structure.

Model 3. Vlasov–Maxwell–Boltzmann system of one-species, cf. [3]: Type $(2,3)$. It is a model simplified from (6) to describe the motion of only one species of electrons with the other species of ions fixed as a background profile; see [3] for more details.

Model 4. Vlasov–Maxwell–Fokker–Planck system of one-species: Type $(1,2)$. The system has the same form as in **Model 3** with $\mathcal{L}$ replaced by the linearized self-adjoint Fokker–Planck operator.

In the decoupled case when $B_1 = 0, B_2 = 0$, let us discuss a little about the kinetic equation (5). A sufficient condition to assure that the equation (5) is partially dissipative of type $(1,1)$ was given in [11] by using thirteen moments as well as the compensating function method. Inspired by [1] and [5], one can postulate a rank-type condition to achieve the same goal. We point out that this kind of the rank-type condition, on one hand, is indeed a sufficient condition to assure the existence of the compensation function and on the other hand, provides a convenient way of constructing the compensation function as explicitly given in [5].

Theorem 1. *Under the assumption* **(A1)** *and the rank condition* $\mathbf{(R1)}_0$:

$$\operatorname{rank}\begin{bmatrix} E^{\mathcal{L}} \\ E^{\mathcal{L}} A_\kappa^{\mathcal{L}} \\ \vdots \\ E^{\mathcal{L}} (A_\kappa^{\mathcal{L}})^{m_2-1} \end{bmatrix} = m_2,$$

where $E^{\mathcal{L}}$ is a diagonal $m_2 \times m_2$ matrix $\operatorname{diag}\{0, \cdots, 0, 1, \cdots, 1\}$ *with the first m_0 entries of the diagonal vanishing, the equation* (5) *is partially dissipative of type* $(1,1)$.

§5. Main result: a sufficient condition for type $(2,3)$

Since the system structure of (1)–(2) is equivalent with that of (4), let us start with the general system

$$\hat{U}_t + i|k|A_\kappa \hat{U} + L\hat{U} = 0, \tag{7}$$

where $\hat{U} = \hat{U}(t,k)$ is the Fourier transform of $U = U(t,x) \in \mathbb{R}^m$ with $t \geq 0$, $x \in \mathbb{R}^n$ and $k \in \mathbb{R}^n$, and for brevity of presentation, we have used the same notations A_κ and L as before. Suppose the following condition

(A): Let A_κ be defined by $A_\kappa = \sum_{j=1}^n \kappa_j A^j$, where for each $j = 1, 2, \cdots, n$, $\kappa_j = k_j/|k|$ when $k \neq 0$ and A^j is a real symmetric $m \times m$ matrix; L is a real, nonnegative definite $m \times m$ matrix, not necessarily symmetric, with the nontrivial kernel.

In what follows, associated with a real $m \times m$ matrix L, we use

$$L^{\mathrm{sy}} = \frac{L + L^T}{2}, \quad L^{\mathrm{asy}} = \frac{L - L^T}{2}$$

to denote the symmetric part and anti-symmetric part, respectively and use $\mathbf{P}_L$ to denote the projection from $\mathbb{R}^m$ to the linear subspace $\ker L$. Suppose further the conditions

(S-K$_1$): There are a real symmetric $m \times m$ matrix S and a real anti-symmetric $m \times m$ matrix K_1 such that

$$\begin{aligned}
&L^{\mathrm{sy}} + (SL)^{\mathrm{sy}} + (K_1 A_\kappa)^{\mathrm{sy}} \geq 0, \\
&\ker\left(L^{\mathrm{sy}} + (SL)^{\mathrm{sy}} + (K_1 A_\kappa)^{\mathrm{sy}}\right) \subseteq \ker L, \\
&i(SA_\kappa)^{\mathrm{asy}} \geq 0 \quad \text{on } \ker L^{\mathrm{sy}}, \\
&P_{L^{\mathrm{sy}}} K_1 L^{\mathrm{asy}} = 0.
\end{aligned}$$

(K$_2$): There is a real anti-symmetric $m \times m$ matrix K_2 such that

$$L^{\mathrm{sy}} + (SL)^{\mathrm{sy}} + (K_1 A_\kappa)^{\mathrm{sy}} + (K_2 A_\kappa)^{\mathrm{sy}} > 0.$$

Then, one has

Theorem 2. *Under the conditions* **(A)**, **(SK$_1$)** *and* **(K$_2$)**, *system* (7) *is partially dissipative of the type* $(2, 3)$.

Refer to [7] for the complete proof of the above theorem, and an example can be given by Model 3 mentioned before, for which S, K_1 and K_2 can be explicitly constructed in terms of [3] so as to satisfy all the conditions. Finally, we point out that whenever K_1 is identical to zero, type $(2, 3)$ can be improved to be type $(1, 2)$ as studied in [14].

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