Advanced Studies in Pure Mathematics 64, 2015
Nonlinear Dynamics in Partial Differential Equations
pp. 319–326

A vector fields approach to smoothing and decaying estimates for equations in anisotropic media

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Abstract.

It is well known that the vector fields

$$\Omega = x \wedge D = (\Omega_{ij})_{i<j}, \qquad \Omega_{ij} = x_i D_j - x_j D_i$$

commute with the Laplacian $-\Delta$. Hence we have

$$Pu = f \quad \Rightarrow \quad P(\Omega u) = \Omega f,$$

where P is a function of $-\Delta$, and in this way we can control the growth/decaying order of solution u to the equation $Pu = f$. This fact was actually used to induce some decaying estimates for the wave equation ([3]) in a context of nonlinear analysis, and smoothing estimates for the Scrödinger equation ([6]) in a critical case. In this article, we will discuss how to trace this idea for equations with the Laplacian $-\Delta$ replaced by general elliptic (pseudo-)differential operators.

§1. Introduction

Let $-\Delta$ be the Laplacian on $\mathbf{R}^n$ and let $P = p(-\Delta)$, where p is a function ($p(s) = s, \sqrt{s}$, etc.). As a general setting, let us consider the equation $Pu = f$ or its non-linear version $Pu = F(u)$, or even its time revolution version

$$\begin{cases} (D_t - P)\, u(t,x) = F(u(t,x)) \\ \qquad\qquad u(0,x) = \varphi(x). \end{cases}$$

Received February 2, 2012.
2010 *Mathematics Subject Classification.* 35E15, 35L45, 35S10.
Key words and phrases. Scrödinger equation, Maxwell equation, elastic equations, anisotropic media.

Let us try to work with them on Sobolev spaces H^s with the norm

$$\|g\|_{H^s} = \left(\int |\Lambda^s g(x)|^2\, dx\right)^{1/2}; \quad \Lambda = \sqrt{1-\Delta}$$

or weighted L^2 spaces L^2_k with the norm

$$\|g\|_{L^2_k} = \left(\int \left|\langle x\rangle^k g(x)\right|^2 dx\right)^{1/2}; \quad \langle x\rangle = \left(1+|x|^2\right)^{1/2}.$$

Assume that the statement

$$Pu = f \in L^2 \quad \Rightarrow \quad u \in H^m$$

is true for example. Then, since $[\Lambda^s,\, P] = 0$, we have automatically a general statement

$$Pu = f \in H^s \quad \Rightarrow \quad u \in H^{m+s},$$

which is sometimes called *lifting property*, while in general we do not have the statement

$$Pu = f \in L^2_k \quad \Rightarrow \quad u \in L^2_{m+k}$$

since $[\langle x\rangle^k,\, P] \neq 0$.

On the other hand, rotational vector fields

$$\Omega_{ij} = x_i D_{x_j} - x_j D_{x_i}, \quad x = (x_1, \ldots, x_n)$$

satisfies $[\Delta,\, \Omega_{ij}] = 0$ and we have the statement

$$Pu = f \quad \Rightarrow \quad P(\Omega u)(t,x) = \Omega f$$

for $\Omega = x\wedge D = (\Omega_{ij})_{i<j}$. In this way we can control the growth/decaying order of solution u to the equation $Pu = f$. Even for the non-linear equation, we can apply this idea and have the statement

$$Pu = F(u) \quad \Rightarrow \quad P(\Omega u)(t,x) = F'(u)\Omega u,$$

where we use the chain rule relation $\Omega F(u) = F'(u)\Omega u$. Note that this relation is justified since Ω is a differential operator of order one.

The idea of using vector fields Ω is actually applied to inducing decaying estimates for the wave equation $\Box u = F$ with 0-initial data:

$$|u(x,t)| \leq C(t+|x|)^{-(n-1)/2} \sup_{0\leq s\leq t} \langle s\rangle^a \sum_{|\alpha|\leq M} \|Z^\alpha F(\cdot,s)\|_{L^2},$$

where Z is Ω_{ij} or other type of relevant vector fields. We have a time global existence result for semi-linear wave equations (Klainerman [3]) by this type of estimate. Smoothing estimates for the Scrödinger equation of the type

$$\left\| \langle x \rangle^{-3/2} \Omega e^{-it\Delta} \varphi \right\|_{L^2(\mathbf{R}_t \times \mathbf{R}^n_x)} \leq C \left\| \langle D \rangle^{1/2} \varphi \right\|_{L^2(\mathbf{R}^n_x)},$$

suggested by Hoshiro [2], can be also given by the same idea ([6]), from which we obtain a time global existence result for Scrödinger equations with derivative non-linearity ([5]).

Let us use the idea of vector fields to more general elliptic operators:

$$\begin{aligned} &a(D) = F^{-1} a(\xi) F\,; \quad a(\xi) \in C^\infty(\mathbf{R}^n \setminus 0), \\ &a(\xi) > 0, \quad a(\lambda \xi) = \lambda^2 a(\xi) \quad (\lambda > 0). \end{aligned}$$

Note that $a(D) = -\Delta$ when $a(\xi) = |\xi|^2$. Such generalized situation naturally arises in many important equations of physics. For example the equation $D_t - \sqrt{a(D)} = f$ is reduced from Maxwell system in anisotropic media (6×6 system)

$$(D_t - A(D_x))\, U = 0,$$

where

$$A(D_x) = \frac{1}{i} \begin{pmatrix} 0 & \varepsilon^{-1} \operatorname{curl} \\ -\mu^{-1} \operatorname{curl} & 0 \end{pmatrix};$$

$$\varepsilon = \begin{pmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_3 \end{pmatrix}, \quad \mu = \begin{pmatrix} \nu & 0 & 0 \\ 0 & \nu & 0 \\ 0 & 0 & \nu \end{pmatrix}$$

or elastic wave equations in anisotropic media (3×3 system)

$$\left(D_t^2 - A(D_x)\right) U = 0,$$

where

$$A(D_x) = (A_{ij}(D_x)); \quad A_{ij}(D_x) = \sum_{p,q=1}^{3} c_{ijpq} D_{x_p} D_{x_q},$$

assuming that the system is hyperbolic in the time direction and $c_{ijpq} = c_{jipq} = c_{ijqp} = c_{pqij}$. But then we come across a natural question:

Question. Does a vector fields corresponding to $a(D)$ exists like $x \wedge D$ to $-\Delta$? If not, what should be the substitution?

This short article is a trial to answer this question, and after stating some useful theorems (Theorems 1 and 2), an answer will be given which says the existence of a vector field which does not commute with $a(D)$ but can control the growth/decaying order.

§2. Canonical transform

As a first step to answer our question, we introduce an idea of using canonical transform.

For the homogeneous diffeomorphism $\psi : \mathbf{R}^n \setminus 0 \to \mathbf{R}^n \setminus 0$, we set

$$Iu(x) = \int_{\mathbf{R}^n} \int_{\mathbf{R}^n} e^{i(x\cdot\xi - y\cdot\psi(\xi))} u(y)\, dy d\xi,$$
$$I^{-1}u(x) = \int_{\mathbf{R}^n} \int_{\mathbf{R}^n} e^{i\left(x\cdot\xi - y\cdot\psi^{-1}(\xi)\right)} u(y)\, dy d\xi$$

$(x \in \mathbf{R}^n)$. Then we have the relation

$$a(D) = I \cdot \sigma(D) \cdot I^{-1}, \quad a(\xi) = (\sigma \circ \psi)(\xi).$$

In particular, if we take

$$\sigma(\eta) = |\eta|^2, \quad \psi(\xi) = \sqrt{a(\xi)} \frac{\nabla a(\xi)}{|\nabla a(\xi)|},$$

then we have $a(\xi) = (\sigma \circ \psi)(\xi)$, hence

$$a(D) = I \cdot (-\Delta) \cdot I^{-1}$$

under the assumption that the Gaussian curvature of

$$\Sigma_a = \{\xi; a(\xi) = 1\}$$

never vanishes. (Note that the Gauss map $\nabla a/|\nabla a| : \Sigma_a \to S^{n-1}$ is a global diffeomorphism by the curvature assumption, and the existence of the inverse ψ^{-1} is guaranteed.)

Then the transformed operator

$$\Omega = I \cdot (x \wedge D) \cdot I^{-1}$$

is expected to be a candidate of the solution to our question. By computation, we have

$$\Omega = x\psi'(D)^{-1} \wedge \psi(D)$$

and it surely satisfies

$$[a(D), \Omega] = 0. \tag{1}$$

But this Ω is not a family of vector fields, and unfortunately we cannot have the chain rule relation

$$\Omega F(u) = F'(u)\Omega u \tag{2}$$

which is needed for the nonlinear analysis.

§3. Set of classical orbits

We investigate more properties of the operator

$$\Omega = x\psi'(D)^{-1} \wedge \psi(D), \quad \psi(\xi) = \sqrt{a(\xi)}\frac{\nabla a(\xi)}{|\nabla a(\xi)|} \tag{3}$$

to find a vector field as a good substitution of it.

Let $\{(x(t), \xi(t)) : t \in \mathbf{R}\}$ be the classical orbit associated to $a(D)$, that is, the solution of the ordinary differential equation

$$\begin{cases} \dot{x}(t) = (\nabla a)(\xi(t)), & \dot{\xi}(t) = 0, \\ x(0) = 0, & \xi(0) = k, \end{cases}$$

and consider the set of the path of all classical orbits

$$\begin{aligned} \Gamma_a &= \{(x(t), \xi(t)) : t \in \mathbf{R},\, k \in \mathbf{R}^n \setminus 0\} \\ &= \{(\lambda \nabla a(\xi), \xi) : \lambda \in \mathbf{R},\, \xi \in \mathbf{R}^n \setminus 0\} \\ &= \{(x, \xi) \in T^*\mathbf{R}^n \setminus 0 : x \wedge \nabla a(\xi) = 0\}. \end{aligned}$$

For example, in the Laplacian case $a(\xi) = |\xi|^2$, we have

$$\Gamma_a = \{(x, \xi) \in T^*\mathbf{R}^n \setminus 0 : x \wedge \xi = 0\}.$$

We know the following result established in [4].

Theorem 1. *Let $k \in \mathbf{R}$. Suppose that $\sigma(x, \xi)$ satisfies*

$$\left|\partial_x^\alpha \partial_\xi^\gamma \sigma(x, \xi)\right| \le C_{\alpha\gamma} \langle x \rangle^{1-|\alpha|} \langle \xi \rangle^{1-|\gamma|},$$

for all α, γ and vanishes outside $|\xi| \ge C > 0$. Assume the structural condition

$$(x, \xi) \in \Gamma_a \quad \Rightarrow \quad \sigma(x, \xi) = 0.$$

Then we have

$$\|\sigma(X, D)g\|_{L^2_k(\mathbf{R}^n)} \le C\Big(\|\Omega g\|_{L^2_k(\mathbf{R}^n)} + \|g\|_{L^2_k(\mathbf{R}^n)}\Big),$$

where Ω is the operator given by (3).

Note that

$$\Gamma_a = \{(x,\xi) \in \mathbf{R}^n \times (\mathbf{R}^n \setminus 0) : \Omega(x,\xi) = 0\}$$

with the symbol $\Omega(x,\xi)$ of the operator Ω, hence $\Omega(x,\xi)$ is an example of $\sigma(x,\xi)$ in Theorem 1 which satisfies the structural condition.

§4. Geometric structure

Another straightforward example of $\sigma(x,\xi)$ which satisfies the structural condition in Theorem 1 is

$$\sigma(x,\xi) = x \wedge \nabla a(\xi),$$

which also commutes with $a(D)$ but is not a vector field. We will construct a vector field which satisfy the structural condition in Theorem 1 by considering a geometric structure of Γ_a.

For $a(\xi)$, the *dual* function $a^*(\xi) \in C^\infty(\mathbf{R}^n \setminus 0)$ is uniquely determined, which satisfies the same property as $a(\xi)$ and

$$\Sigma_a^* = \Sigma_{a^*}, \quad \Sigma_{a^*}^* = \Sigma_a.$$

Here we have used the notation

$$\Sigma_q = \{\xi \in \mathbf{R}^n \setminus 0 : q(\xi) = 1\}, \quad \Sigma_q^* = \left\{\frac{1}{2}\nabla q(\xi) : \xi \in \Sigma_q\right\}.$$

Moreover,

$$\frac{1}{2}\nabla a : \Sigma_a \to \Sigma_{a^*}$$

is a C^∞-diffeomorphism and

$$\frac{1}{2}\nabla a^* : \Sigma_{a^*} \to \Sigma_a$$

is its inverse. Hence we have

$$\begin{aligned}(x,\xi) \in \Gamma_a \quad &\Rightarrow \quad x \wedge \nabla a(\xi) = 0 \\ &\Rightarrow \quad \nabla a^*(x) \wedge \xi = 0.\end{aligned}$$

Then we have

$$\begin{aligned}\Gamma_a &= \{(\lambda \nabla a(\xi), \xi) : \xi \in \mathbf{R}^n \setminus 0, \lambda \in \mathbf{R}\} \\ &= \{(\lambda x, \nabla a^*(x)) : x \in \mathbf{R}^n \setminus 0, \lambda \in \mathbf{R}\},\end{aligned}$$

and the operator with the symbol

$$\sigma(x,\xi) = \nabla a^*(x) \wedge \xi$$

also satisfies the structural conditions of Theorem 1. Note that

$$\sigma(X, D) = \nabla a^*(x) \wedge D$$

is a vector field!

In the case $a(\xi) = |\xi A|^2$, where A is a positive definite symmetric matrix, we have $a^*(\xi) = \left|\xi A^{-1}\right|^2$. We remark that the operator with the symbol

$$\tau(x, \xi) = \frac{a^*(x)}{|\nabla a^*(x)|^2} |\nabla a^*(x) \wedge \xi|^2$$

is the homogeneous extension of the Laplace–Beltrami operator of the surface Σ_a^*. That means, $\nabla a^*(x) \wedge D$ is a vector field along the surface Σ_a^* in other word.

§5. Replacement argument

Now we are in a position to give a complete answer to our question. Let $\mathfrak{X}$ be the vector field whose symbol is

$$\mathfrak{X}(x, \xi) = \kappa(x) \wedge \xi \kappa'(x)^{-1}, \quad \kappa(x) = \sqrt{a^*(x)} \frac{\nabla a^*(x)}{|\nabla a^*(x)|}. \tag{4}$$

Note that $\mathfrak{X}(x, \xi)$ satisfy

$$\Gamma_a = \{(x, \xi) \in \mathbf{R}^n \times (\mathbf{R}^n \setminus 0) : \mathfrak{X}(x, \xi) = 0\},$$

and $\mathfrak{X}(x, \xi)$ is an example of $\sigma(x, \xi)$ in Theorem 1 which satisfies the structural condition. Then we have the following result if we change the role of x and ξ in the proof of Theorem 1.

Theorem 2. *Let $k \in \mathbf{R}$. Suppose that $\sigma(x, \xi)$ satisfies*

$$\left|\partial_x^\alpha \partial_\xi^\gamma \sigma(x, \xi)\right| \leq C_{\alpha\gamma} \langle x \rangle^{1-|\alpha|} \langle \xi \rangle^{1-|\gamma|},$$

for all α, γ and vanishes outside $|x| \geq C > 0$. Assume the structural condition

$$(x, \xi) \in \Gamma_a \quad \Rightarrow \quad \sigma(x, \xi) = 0.$$

Then we have

$$\|\sigma(X, D) g\|_{L_k^2(\mathbf{R}^n)} \leq C\Big(\|\mathfrak{X} g\|_{L_k^2(\mathbf{R}^n)} + \|g\|_{L_k^2(\mathbf{R}^n)}\Big),$$

where $\mathfrak{X}$ is the vector field given by (4).

Roughly speaking, we have the following equivalence:

$$\|\mathfrak{X}g\|_{L^2_k(\mathbf{R}^n)} + \|g\|_{L^2_k(\mathbf{R}^n)} \sim \|\Omega g\|_{L^2_k(\mathbf{R}^n)} + \|g\|_{L^2_k(\mathbf{R}^n)}$$

as a corollary of Theorems 1 and 2. In this way, we can anytime replace the operator Ω in (3) by the vector field $\mathfrak{X}$ in (4) at an estimate level, and vice versa. We use Ω for the commutativity (1), and $\mathfrak{X}$ for the chain rule (2). Theorems 1 and 2 guarantee such replacement argument.

§6. Works to be done

Further applications of the idea explained here will be expected. We end this article by listing our ongoing/future works:

- Application to non-linear problems: We expect to establish decaying estimates and some time global existence result for semi-linear Maxwell system and elastic wave equations in an anisotropic media (cf. Georgiev–Lucent–Ziliotti [1]).
- Generalization to the case of variable coefficients: We need more serious consideration of canonical transform and geometric structure.

References

[1] V. Georgiev, S. Lucente and G. Ziliotti, Decay estimates for hyperbolic systems, Hokkaido Math. J., **33** (2004), 83–113.

[2] T. Hoshiro, On weighted L^2 estimates of solutions to wave equations, J. Anal. Math., **72** (1997), 127–140.

[3] S. Klainerman, Uniform decay estimates and Lorentz invariance of the classical wave equation, Comm. Pure Appl. Math., **38** (1985), 321–332.

[4] M. Ruzhansky and M. Sugimoto, A smoothing property of Schrödinger equations in the critical case, Math. Ann., **335** (2006), 645–673.

[5] M. Ruzhansky and M. Sugimoto, Structural resolvent estimates and derivative nonlinear Schrödinger equations, Comm. Math. Phys., **314** (2012), 281–304.

[6] M. Sugimoto, A smoothing property of Schrödinger equations along the sphere, J. Anal. Math., **89** (2003), 15–30.

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