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Schrödinger equations on compact symmetric spaces and Gauss sums

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Abstract.

In this article, we will give a reciprocity formula for generalized Gauss sums and its application to Schrödinger equations on certain compact symmetric spaces.

§1. Introduction

According to quantum physics, it is impossible to determine the future position of a particle. Nonetheless, under some conditions, it is possible to predict the position of a free particle on a compact symmetric space in the future. The purpose of this article is to state a reciprocity formula for generalized Gauss sums and to apply it to future prediction of a free particle.

Let $M = U/K$ be a compact symmetric space with even root multiplicities and let $A \subset M$ be a maximal torus. We assume a certain condition on the weight lattice and the coroot lattice associated with the maximal torus A. (We will give this condition later in Section 2.) We consider the following Cauchy problem for the Schödinger equation on M.

$$(1) \qquad (\mathrm{Sch})_M \begin{cases} \sqrt{-1}\partial_t \psi + \Delta_M \psi = 0, & t \in \mathbb{R}, \\ \psi(0,x) = \delta_o(x), & x \in M. \end{cases}$$

Here $\delta_o(x)$ denotes the Dirac's delta function with singularity at $o = eK \in U/K$ and Δ_M denotes the Laplace-Beltrami operator on M with respect to the U-invariant metric. In addition, for simplicity, we denote $\frac{\partial}{\partial t}$ by ∂_t. (We consider Δ_M to be a non-positive operator as in the case

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of $\mathbb{R}^n$.) Let $E_M(t,x)$ be the fundamental solution to the Schödinger equation corresponding to a free particle on M, namely, the solution to the above equation $(\mathrm{Sch})_M$.

Then in [K1] we proved the following.

Theorem 1.1. (*See Theorem* 1.1 *in* [K1].) *There exists a positive number* c *such that the following* (I) *and* (II) *hold.*

(I) *In the case when* t/c *is a rational number. Let us put* $\dfrac{t}{c} = \dfrac{p}{q} \in \mathbb{Q}$, *where* $p,\ q \in \mathbb{Z}$, $q > 0$ *and* p *and* q *are coprime. Then there exists a finite subset* $\mathcal{G}_q$ *of* A *depending on* q *such that the support of* $E_M(\frac{cp}{q}, \cdot)$ (*as a distribution on* M) *is given by*

$$\mathrm{Supp}E_M\left(\frac{cp}{q}, \cdot\right) = \{\, k \cdot a \in M \mid k \in K, a \in \mathcal{G}_q \,\}.$$

(*We will construct the above finite set* $\mathcal{G}_q$ *explicitly in Section* 2.)

(II) *In the case when* t/c *is an irrational number. The support of* $E_M(t, \cdot)$ *is given by*

$$\mathrm{Supp}\, E_M(t, \cdot) = \mathrm{SingSupp}\, E_M(t, \cdot) = M.$$

Here $\mathrm{SingSupp}E_M(t, \cdot)$ *denotes the singular support of* $E_M(t, \cdot)$.

Here we remark that the above finite set $\mathcal{G}_q$ is given in terms of *generalized Gauss sums.*

So in Section 2, we will define generalized Gauss sums, and in Section 4, we will explain how we can predict the position of a free particle on $M = U/K$ at a future rational time.

§2. Generalized Gauss sums

In this section, we will define a generalized Gauss sum and construct the finite subset $\mathcal{G}_q$in Theorem 1.1.

Let us first introduce several notations on symmetric spaces.

Notation 2.1. *Let* $\mathfrak{u}$ *and* $\mathfrak{k}$ *be the Lie algebras of* U *and* K *respectively, and let* $\mathfrak{u} = \mathfrak{k} \oplus \mathfrak{m}$ *be the corresponding Cartan decomposition. We take a maximal abelian subalgebra* $\mathfrak{a} \subset \mathfrak{m}$ *such that* $\exp \mathfrak{a} = A$. *We put* $d := \dim \mathfrak{a}$ $(= \mathrm{rank} M)$. *We fix an Ad-U-invariant inner product* $\langle \cdot, \cdot \rangle$ *on* $\mathfrak{u}$, *which induces inner products on* $\mathfrak{m}$ *and on* $\mathfrak{a}$. *We denote these induced inner products by the same* $\langle \cdot, \cdot \rangle$. *For* $\alpha \in \mathfrak{a}$, *let*

$\mathfrak{u}_\alpha = \{ X \in \mathfrak{u}^{\mathbb{C}} \mid [H, X] = \sqrt{-1}\langle \alpha, H\rangle X, \text{ for } \forall\, H \in \mathfrak{a} \}$. *$\alpha$ is called a restricted root if* $\mathfrak{u}_\alpha \neq \{0\}$. *We put* $m_\alpha = \dim_{\mathbb{C}} \mathfrak{u}_\alpha$ *for a restricted root* α *and call it the multiplicity of* α. *In addition, we write* $\alpha^\vee = 2\alpha/\langle \alpha, \alpha\rangle$ *for* α. *Let* $\gamma_1, \cdots, \gamma_d$ *be the simple roots.* (*For the details, see for example* [H-1] *or* [Tak].)

Assumption 2.2. *Here we assume the following even multiplicity condition* (EMC).

$$\text{(EMC)} \ :\ m_\alpha \textit{ is even for any restricted root } \alpha.$$

Now we introduce two lattices Γ and Γ_0 of the d-dimensinal vector space $\mathfrak{a}$ as follows.

$$\Gamma := \{ H \in \mathfrak{a} \mid \exp(H) \in K \}, \tag{2}$$

$$\Gamma_0 := \mathbb{Z}{\gamma_1}^\vee \oplus \cdots \oplus \mathbb{Z}{\gamma_d}^\vee. \tag{3}$$

Then A is written as $A \cong \mathfrak{a}/\Gamma$, and under (EMC) we have $\pi\Gamma_0 = \Gamma$. The lattice Γ_0 is called the coroot lattice.

Let Λ be the dual lattice of Γ_0, then Λ is written as

$$\Lambda := \mathbb{Z}\lambda_1 \oplus \cdots \oplus \mathbb{Z}\lambda_d,$$

where $\lambda_1, \cdots, \lambda_d$ be the fundamental weights which correspond to the simple roots $\gamma_1, \cdots, \gamma_d$ (namely $\lambda_1, \cdots, \lambda_d$ satisfy $\langle \lambda_i, \gamma_j^\vee\rangle = \delta_{ij}$.)

Moreover, for $q \in \mathbb{Z}, q > 0$, let

$$\Gamma_0[q] := \{ \ell_1{\gamma_1}^\vee + \cdots + \ell_d{\gamma_d}^\vee \in \Gamma_0 \mid 0 \le \ell_1, \cdots, \ell_d \le q-1 \}, \tag{4}$$

$$\Lambda[q] := \{ \ell_1\lambda_1 + \cdots + \ell_d\lambda_d \in \Lambda \mid 0 \le \ell_1, \cdots, \ell_d \le q-1 \}. \tag{5}$$

In addition to (EMC), we assume the following condition on the coroot lattice Γ_0 and the weight lattice Λ.

Assumption 2.3. *There exists a constant* $c > 0$ *such that* $c\Lambda \subset \Gamma_0$.

So we take a constant c_0 as

$$c_0 := \min\{ c > 0 \mid c\Lambda \subset \Gamma_0 \}. \tag{6}$$

Remark 2.4. (i) If M is irreducible, then the above constant c_0 coincides with the dual Coxeter number of the corresponding root system. In particular, such M satisfies Assumption 2.3.

(ii) There is another constant $\widetilde{c} > 0$ such that $c_0\widetilde{c} \in \mathbb{Z}$ and $\widetilde{c}\Gamma_0 \subset \Lambda$ for the above constant c_0. So we take the least positive number c_1 among such constants $\widetilde{c}$ and put $L := c_0 c_1 \in \mathbb{Z}$.

Now we define a generalized Gauss sum.

Definition 2.5. *For p, $q \in \mathbb{Z}$, $q > 0$, p and q coprime, for $\mu \in \Gamma_0$, we define a Gauss sum associated with Λ and Γ_0 by*

$$(7) \qquad G(p,q;\mu;\Lambda,\Gamma_0) := \sum_{\lambda \in \Lambda[q]} \exp \frac{2\pi\sqrt{-1}}{q}\{p\langle c_0\lambda, \lambda\rangle - \langle \lambda, \mu\rangle\}.$$

For simplicity, we call the above sum (7) a generalized Gauss sum.

Example 2.6. *In the case $d = 1$ and $A \cong \mathbf{S}^1 \cong \mathbb{R}/2\pi\mathbb{Z}$, $\Gamma = 2\pi\mathbb{Z}$, $\Gamma_0 = 2\mathbb{Z}$, and $\Lambda = \frac{1}{2}\mathbb{Z}$. Therefore, $c_0 = 4$. In this case, the Gauss sum is given by*

$$(8) \qquad G(p,q;\mu;\Lambda,\Gamma_0) = G(p,q,2k,\frac{1}{2}\mathbb{Z},2\mathbb{Z}) = \sum_{\ell=0}^{q-1} e^{\frac{2\pi\sqrt{-1}}{q}\{p\ell^2 - k\ell\}},$$

for p, $q \in \mathbb{Z}$ with p and q coprime, and for $\mu = 2k \in \Gamma_0 = 2\mathbb{Z}$. We note that the above sum (8) is called a quadratic Gauss sum in the theory of cyclotomic fields. (See [L].) In this sense, $G(p,q;\mu;\Lambda,\Gamma_0)$ can be regarded as a generalization of a quadratic Gauss sum.

Finally, we define a finite subset $\mathcal{G}_q$ of the maximal torus A by

$$(9) \qquad \mathcal{G}_q := \left\{ \left[\frac{\pi\mu}{q}\right]_\Gamma \in \mathfrak{a}/\Gamma \cong A \,|\, G(p,q;\mu;\Lambda,\Gamma_0) \neq 0 \right\}.$$

We note that $\mathcal{G}_q$ does not depend on p such that p and q are coprime. We also note that a necessary and sufficient condition for $G(p,q;\mu;\Lambda,\Gamma_0) \neq 0$ is given by Proposition 2.9 in [K1].

§3. Expression of E_M at a rational time

In this section, we will state some results of [K1].

For the constant c_0, let

$$(10) \qquad c := \frac{\pi c_0}{2}.$$

Then for the above constant c and the finite subset $\mathcal{G}_q$, we have Theorem 1.1 in Introduction.

Let $E_A(t,a)$ be the solution to

$$(11) \qquad \sqrt{-1}\partial_t\psi + \Delta_A\psi = 0, \quad \psi(0,a) = \delta_A(a), \quad t \in \mathbb{R}, \; a \in A,$$

where Δ_A denotes the flat Laplacian on A and δ_A denotes the Dirac's delta funtion on A.

Then the fundamental solution $E_M(t,x)$ is given by the following.

Theorem 3.1. (*See Theorem* 4.5 *in* [K1].)

$$\begin{aligned} E_M(t,x) &= const. e^{\sqrt{-1}\langle\rho,\rho\rangle t} D_M E_A(t,a), \\ (\, a = &\ a \text{ radial component of } x\ ), \end{aligned} \tag{12}$$

where $\rho = \dfrac{1}{2} \sum_{\alpha\,:\,\text{positive root}} m_\alpha \alpha$ *and where* D_M *is a differential operator on* A *of order* $\frac{1}{2}(\dim M - \operatorname{rank} M)$. *(*$D_M$ *is called a shift operator of Heckman and Opdam.)*

We note that if we assume (EMC) there exists such an operator D_M. In addition to the above theorem, we have the following.

Theorem 3.2. (*See Theorem* 2.11 *in* [K1].)

$$E_A\left(\frac{cp}{q}, a\right) = q^{-d} \sum_{\mu\in\Gamma_0[q]} \overline{G(p,q;\mu;\Lambda,\Gamma_0)}\, \delta_A\left(\left[H - \frac{\pi}{q}\mu\right]_\Gamma\right), \tag{13}$$

for $a = [H]_\Gamma \in A \cong \mathfrak{a}/\Gamma$.

The above two theorems assert that the support of the fundamental solution E_M at $t = \frac{cp}{q}$ is the K-orbit of the above defined finite subset $\mathcal{G}_q$.

§4. Reciprocity formula and future prediction

In this section, we give a reciprocity formula for the generalized Gauss sum.

As is well known in algebraic number theory, the quadratic Gauss sum $G(p,q,k)$ satisfies the following reciprocity formula.

Theorem 4.1. (*Reciprocity formula for quadratic Gauss sums.*)

$$G(p,q,k) = \frac{1}{2}\left|\frac{q}{2p}\right|^{\frac{1}{2}} e^{\frac{\sqrt{-1}\pi}{4pq}(|pq|-2k^2)}\, G(-q,4p,-2k). \tag{14}$$

The above formula is due to Cauchy, Dirichlet, Kronecker, etc.

In Section 2, we introduced the sum $G(p,q;\mu;\Lambda,\Gamma_0)$. There is another reason to call $G(p,q;\mu;\Lambda,\Gamma_0)$ a generalized Gauss sum. In fact, $G(p,q;\mu;\Lambda,\Gamma_0)$ satisfies the following.

Theorem 4.2. (*Reciprocity formula for generalized Gauss sums.*)

$$G(p,q;\mu;\Lambda,\Gamma_0) = \left|\frac{q}{2c_0 p}\right|^{\frac{d}{2}} \frac{|\Lambda/c_1\Gamma_0|\,\mathrm{Vol}(A)}{(2\pi)^d} e^{\frac{\sqrt{-1}\pi}{4c_0 pq}(c_0 d|pq|-2|\mu|^2)} \times G(-q,4Lp;-2\mu;\Lambda,\Gamma_0),$$

where c_0 *is the constant given by* (6) *and* c_1 *and* L *are given in Remark* 2.4 *(ii).*

The above reciprocity formula enables us to determine the location of a free particle at a future rational time. In fact, by making use of Theorem 4.2 , Theorem 3.1, and Theorem 3.2, we get the information (position, amplitude, etc) of a free particle at $t = \frac{cp}{q}$ from its information at $t = -\frac{c}{4L} \times \frac{q}{p}$ and vice versa. (See the figure below.) Here $c = \frac{\pi c_0}{2}$ is the positive constant given by (10). This constant c is the same as the constant c in Theorem 1.1.

$$E_M\left(\frac{cp}{q}, x\right) \iff \{\, G(p,q;\mu;\Lambda,\Gamma_0)\,\}_{\mu\in\Lambda}$$

$$\text{(Reciprocity formula)} \Updownarrow$$

$$E_M\Big(-\frac{c}{4L} \times \frac{q}{p}, x\Big) \iff \{\, G(q,-4Lp;-2\mu;\Lambda,\Gamma_0)\,\}_{\mu\in\Lambda}$$

§5. Some remarks

(I) The phenomenon described in Theorem 1.1 is similar to Huygens' principle for the modified wave equation on odd dimensional symmetric spaces with even root multiplicities in the sense that the support of the fundamental solution becomes a lower dimensional subset. For the references on the modified wave equation, see Branson–Olafsson–Pasquale [BOP], Branson–Olafsson–Schlichtkrull [BOS], Chalykh–Veselov [Cha-Ves], Gonzalez [Gonz], Helgason [H-3], [H-4], Helgason–Schlichtkrull [H-Sch], Olafsson–Schlichtkrull [Olaf-Sch], and Solomatina [Sol]. However, this phenomenon is quite different from Huygens' principle in the sense that the support of the fundamental solution E_M becomes lower dimensional only at a rational time.

(II) The even multiplicity condition (EMC) plays an essential role in the proofs of Theorem 1.1 and Theorem 3.1. In fact, (EMC) guarantees the existence of the shift operator D_M in Theorem 3.1. This fact is due to Heckman and Opdam. For the details of their results, see Heckman [Heck], Heckman–Schlichtkrull [Heck-Sch], and Opdam [Opd-1], [Opd-2], [Opd-3].

(III) Another generalization of Gauss sums and its properties are studied by Turaev from the point of view of link invariants on three dimensional manifolds. For the details, see [Tur].

(IV) The examples of compact symmetric space which satisfy (EMC) and Assumption 2.3 are given in the following list.

(i) $M = SO(2m+2)/SO(2m+1)(\cong \mathbf{S}^{2m+1})$. The odd dimensional sphere.
(ii) $M = SU(2m)/Sp(m)$.
(iii) $M = E_6/F_4$.
(iv) $M = (U \times U)/\triangle U \ (\cong U)$ for any compact simple Lie group U. Here $\triangle U$ denotes the diagonal set $\{ (u,u) \in U \times U \mid u \in U \}$.

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