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A note on quadratic residue curves on rational ruled surfaces

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Abstract.

Let Σ be a smooth projective surface, let $f' : S' \to \Sigma$ be a double cover of Σ and let $\mu : S \to S'$ be the canonical resolution of S'. Put $f = f' \circ \mu$. An irreducible curve D on Σ is said to be a splitting curve with respect to f if f^*D is of the form $D^+ + D^- + E$, where $D^+ \neq D^-$, $D^- = \sigma_f^* D^+$, σ_f being the covering transformation of f and all irreducible components of E are contained in the exceptional set of μ. In this article, we consider "reciprocity" concerning splitting curves when Σ is a rational ruled surface.

§0. Introduction

Let Σ be a smooth projective surface and let Z' be a normal projective surface with finite surjective morphism $f' : Z' \to \Sigma$ of degree 2. Let $\mu : Z \to Z'$ be the canonical resolution (see [4] for the canonical resolution) of Z' and put $f := f' \circ \mu$. We denote the involution on Z induced by the covering transformation of f' by σ_f. The branch locus $\Delta_{f'}$ of f' is the subset of Σ consisting of points x such that f' is not locally isomorphic over x. Similarly we define the branch locus Δ_f of f. Note that $\Delta_{f'} = \Delta_f$. In [10], we introduce a notion "*a splitting curve with respect to* f" as follows:

Definition 0.1. Let D be an irreducible curve on Σ. We call D a splitting curve with respect to f if f^*D is of the form

$$f^*D = D^+ + D^- + E,$$

where $D^+ \neq D^-$, $\sigma_f^* D^+ = D^-$, $f(D^+) = f(D^-) = D$ and $\mathrm{Supp}(E)$ is contained in the exceptional set of μ. If the double cover $f : Z \to \Sigma$ is

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uniquely determined by its branch locus Δ_f and D is a splitting curve with respect to f, we say that "Δ_f *is a quadratic residue curve mod* D".

Remark 0.1. One can similarly define a splitting divisor with respect to a double cover or a quadratic residue divisor for higher dimensional cases.

We here recall our notation introduced in [10]. Suppose that $f : Z \to \Sigma$ is uniquely determined by Δ_f. For an irreducible curve D on Σ, we put

$$(\Delta_f/D) = \begin{cases} 1 & \text{if } \Delta_f \text{ is a quadratic residue curve mod } D \\ -1 & \text{if } \Delta_f \text{ is not a quadratic residue curve mod } D. \end{cases}$$

Remark 0.2. Note that any double cover is determined by its branch locus if there exists no element of order 2 in $\mathrm{Pic}(\Sigma)$. This condition is satisfied if Σ is simply connected, for example.

In [10], we studied splitting quartics Q with respect to a double cover, $f_C : Z_C \to \mathbb{P}^2$, branched along a smooth conic C. Our key idea in [10] is that we consider a double cover $f'_Q : Z'_Q \to \mathbb{P}^2$ in order to determine the value of (C/Q). In other words, we showed that a kind of "reciprocity" holds between C and Q ([10, Theorem 2.1]). Our purpose of this article is to prove "reciprocity" for some curves on rational ruled sufaces. More precisely we consider a generalization of Theorem 1.2 in [10], which is a "reciprocity" between sections and trisections on rational ruled surfaces. Note that our proof of [10, Theorem 2.1] is based on [10, Theorem 1.2]. Let us explain our setting.

Let Σ_d (d: even) be the Hirzebruch surface of degree d. Throughout this article, we fix the following notation:

- Δ_0: the section of Σ_d with $\Delta_0^2 = -d$.
- F: a fiber of the ruling of Σ_d.
- B_i $(i = 1, 2)$: irreducible curves on Σ_d such that $B_i \sim (2g_i + 1)(\Delta_0 + dF)$ $(i = 1, 2, g_i \in \mathbb{Z}_{\geq 0})$.

Also we always assume that

$(*)$ neither singular point of B_1 nor B_2 is in $B_1 \cap B_2$.

Let $p'_i : S'_i \to \Sigma_d$ be the double cover of Σ_d with branch curve $\Delta_0 + B_i$ and let $\mu_i : S_i \to S'_i$ be its canonical resolution and put $p_i := p'_i \circ \mu_i$. The ruling $\Sigma_d \to \mathbb{P}^1$ induces a hyperelliptic fibration of genus g_i on S_i, which we denote by $\varphi_i : S_i \to \mathbb{P}^1$. Since φ_i has a canonical section O_i arising from Δ_0, one can consider the Mordell–Weil group $\mathrm{MW}(\mathcal{J}_{S_i})$ of the Jacobian of the generic fiber $S_{i,\eta}$. For an irreducible curve C not contained in any fiber of φ_i, $s(C)$ denote the element of $\mathrm{MW}(\mathcal{J}_{S_i})$ determined by C as in [8, §3]. Then we have

Proposition 0.1. *Suppose that*

- B_2 *has only nodes (resp. at worst simple singularities) if* $g_2 \geq 2$ *(resp.* $g_2 = 1$*), and*
- B_1 *is a splitting curve with respect to* p_2*; and* $p_2^* B_1$ *is of the form* $B_1^+ + B_1^-$*.*

If $s(B_1^+)$ *is 2-divisible, then* B_2 *is a splitting curve with respect to* p_1.

Proposition 0.2. *Suppose that* B_1 *has at worst simple singularities and* $\mathrm{MW}(\mathcal{J}_{S_1}) = \{0\}$*. If* B_2 *is a splitting curve with respect to* p_1*, then we have the following:*

- B_1 *is a splitting curve with respect to* p_2 *and* $p_2^* B_1$ *is of the form* $B_1^+ + B_1^-$*.*
- $s(B_1^{\pm})$ *is 2-divisible in* $\mathrm{MW}(\mathcal{J}_{S_2})$*.*

Remark 0.3. (i) The condition $\mathrm{MW}(\mathcal{J}_{S_1}) = \{0\}$ can be replaced by more geometric condition (see Remark 1.1).

(ii) For $x \in B_1 \cap B_2$, we denotes the intersection multiplicity between B_1 and B_2 at x by $I_x(B_1, B_2)$. Note that if there exists a point $x \in B_1 \cap B_2$ such that $I_x(B_1, B_2)$ is odd, then B_1 (resp. B_2) is not a splitting curve with respect to p_2 (resp. p_1). Hence under the conditions of Propositions 0.1 and 0.2, we may assume that $I_x(B_1, B_2)$ is even for $\forall x \in B_1 \cap B_2$.

From Propositions 0.1 and 0.2, we have the following theorem, which is a generalization of [10, Theorem 1.2]:

Theorem 0.1. *Let* B_1 *and* B_2 *be as before. If* $g_1 = 0$ *and* $I_x(B_1, B_2)$ *is even for all* $x \in B_1 \cap B_2$*, then*

$$(\Delta_0 + B_1/B_2) = (-1)^{\varepsilon(s(B_1^+))}$$

where, for an element $s \in \mathrm{MW}(\mathcal{J}_{S_2})$*,* $\varepsilon(s)$ *is defined as follows:*

$$\varepsilon(s) = \begin{cases} 0 & \text{if } \exists s_o \in \mathrm{MW}(\mathcal{J}_{S_2}) \text{ such that } s = 2s_o \\ 1 & \text{if } \nexists s_o \in \mathrm{MW}(\mathcal{J}_{S_2}) \text{ such that } s = 2s_o. \end{cases}$$

§1. Preliminaries

1.1. Summary on cyclic covers and double covers

Let $\mathbb{Z}/n\mathbb{Z}$ be a cyclic group of order n. We call a $(\mathbb{Z}/n\mathbb{Z})$- (resp. a $(\mathbb{Z}/2\mathbb{Z})$-) cover by an n-cyclic (resp. a double) cover. We here summarize some facts about cyclic and double covers.

Fact: Let Y be a smooth projective variety and let B be a reduced divisor on Y. If there exists a line bundle $\mathcal{L}$ on Y such that $B \sim n\mathcal{L}$, then we can construct a hypersurface X in the total space, L, of $\mathcal{L}$ such that

- X is irreducible and normal, and
- $\pi := \mathrm{pr}|_X$ gives rise to an n-cyclic cover, where pr is the canonical projection $\mathrm{pr} : L \to Y$.

(See [1] for the above fact.)

As we see in [9], cyclic covers are not always realized as a hypersurface of the total space of a certain line bundle. As for double covers, however, the following lemma holds.

Lemma 1.1. *Let $f : X \to Y$ be a double cover of a smooth projective variety with $\Delta_f = B$, then there exists a line bundle $\mathcal{L}$ such that $B \sim 2\mathcal{L}$ and X is obtained as a hypersurface of the total space, L, of $\mathcal{L}$ as above.*

Proof. Let φ be a rational function in $\mathbb{C}(Y)$ such that $\mathbb{C}(X) = \mathbb{C}(Y)(\sqrt{\varphi})$. By our assumption, the divisor of φ is of the form

$$(\varphi) = B + 2D,$$

where D is a divisor on Y. Choose $\mathcal{L}$ as the line bundle determined by $-D$. This implies our statement. Q.E.D.

By Lemma 1.1, note that any double cover X over Y is determined by the pair $(B, \mathcal{L})$ as above. In particular, if there exists no 2-torsion in $\mathrm{Pic}(Y)$, then $\mathcal{L}$ is uniquely determined by B as $2\mathcal{L} \sim 2\mathcal{L}'$ implies $\mathcal{L} \sim \mathcal{L}'$.

1.2. Review on the Mordell–Weil groups for fibrations over curves

In this section, we summarize some results on the Mordell–Weil groups given by Shioda in [7, 8].

Let S be a smooth algebraic surface with fibration $\varphi : S \to C$ of genus g (≥ 1) curves over a smooth curve C. Throughout this article, we always assume that

- φ has a section O and
- φ is relatively minimal, i.e., no (-1) curve is contained in any fiber.

Let S_η be the generic fiber of φ and let $\mathbb{C}(C)$ be the rational function field of C. S_η is regarded as a curve of genus g over $\mathbb{C}(C)$.

Let $\mathcal{J}_S := J(S_\eta)$ be the Jacobian variety of S_η. We denote the set of rational points over $\mathbb{C}(C)$ by $\mathrm{MW}(\mathcal{J}_S)$. By our assumption, $\mathrm{MW}(\mathcal{J}_S)$

is not empty and it is well-known that $\mathrm{MW}(\mathcal{J}_S)$ has the structure of an abelian group.

Let $\mathrm{NS}(S)$ be the Néron–Severi group of S and let $\mathrm{Tr}(\varphi)$ be the subgroup of $\mathrm{NS}(S)$ generated by O and irreducible components of fibers of φ. Under these notation, we have:

Theorem 1.1. *If the irregularity of S is equal to C, then we have*

$$\mathrm{MW}(\mathcal{J}_S) \cong \mathrm{NS}(S)/\,\mathrm{Tr}(\varphi).$$

In particular, $\mathrm{MW}(\mathcal{J}_S)$ *is finitely generated.*

See [7, 8] for a proof.

Let $p_i : S_i \to \Sigma_d$ $(i = 1, 2)$ be the double covers of Σ_d with branch loci $\Delta_0 + B_i$ $(i = 1, 2)$ as in the Introduction. Then we have

Lemma 1.2. *There exists no unramified cover of S_i. In particular,* $\mathrm{Pic}(S_i)$ *has no torsion element.*

Proof. By Brieskorn's results on the simultaneous resolution of rational double points([2, 3]), we may assume that B_i is smooth. Since the linear system $|B_i|$ is base point free, it is enough to prove our statement for one special case. Chose an affine open set $U = \Sigma_d \setminus (\Delta_0 \cup F)$ of Σ_d isomorphic to $\mathbb{C}^2$ with a coordinate (t, x) so that a curve $x = 0$ gives rise to a section linear equivalent to $\Delta_0 + dF$. Choose B_i whose defining equation in U is

$$B_i : f_{B_i}(t, x) = x^{2g_i+1} - \Pi_{i=1}^{(2g_i+1)d}(t - \alpha_i) = 0,$$

where α_i $(i = 1, \dots, (2g_i + 1)d)$ are distinct complex numbers. Note that

- B_i is smooth,
- singular fibers of φ are over α_i $(i = 1, \dots, (2g_i + 1)d)$, and
- all the singular fibers are irreducible rational curves with unique singularity whose local analytic equation is given by $v^2 - u^{2g_i+1} = 0$.

Suppose that there exists an unramified cover $\gamma : \widehat{S}_i \to S_i$, $\deg \gamma \geq 2$, and let $\hat{g} : \widehat{S}_i \to \mathbb{P}^1$ be the fibration induced by φ_i. As γ is unramified, $\gamma^*(O_i)$ consists of disjoint $\deg \gamma$ sections. Choose one of them, $\widehat{O}_i$, in $\gamma^* O_i$. Let $\widehat{S}_i \overset{\rho_1}{\to} C \overset{\rho_2}{\to} \mathbb{P}^1$ be the Stein factorization. Then $\deg(\rho_2 \circ \rho_1)|_{\widehat{O}_i} = \deg \hat{g}|_{\widehat{O}_i} = 1$. Hence $\deg \rho_1 = \deg \rho_2 = 1$ and $\hat{g}$ has a connected fiber.

On the other hand, since all the singular fibers of φ_i are simply connected, all fibers over α_i $(i = 1, \dots, (2g_i + 1)d)$ are disconnected. This leads us to a contradiction. Q.E.D.

Corollary 1.1. *The irreguarity $h^1(S_i, \mathcal{O}_{S_i})$ of S_i is 0. In particular,*

$$\mathrm{MW}(\mathcal{J}_{S_i}) \cong \mathrm{NS}(S_i)/\operatorname{Tr}(\varphi_i),$$

where $\operatorname{Tr}(\varphi_i)$ *denotes the subgroup of* $\mathrm{NS}(S_i)$ *introduced as above.*

Proof. By Lemma 1.2, we infer that $H^1(S_i, \mathbb{Z}) = \{0\}$. Hence $h^1(S_i, \mathcal{O}_{S_i}) = 0$. Q.E.D.

Remark 1.1. By Corollary 1.1, $\mathrm{MW}(\mathcal{J}_{S_i}) = \{0\}$ if and only if $\mathrm{NS}(S_i) = \operatorname{Tr}(\varphi_i)$. We use this geometric condition in our proof of Proposition 0.2.

§2. Proof of Proposition 0.1

Let us start with the following lemma:

Lemma 2.1. *$f : X \to Y$ be the double cover of Y determined by $(B, \mathcal{L})$ as in Lemma 1.1. Let Z be a smooth subvariety of Y such that (i) $\dim Z > 0$ and (ii) $Z \not\subset B$. We denote the inclusion morphism $Z \hookrightarrow Y$ by ι. If there exists a divisor B_1 on Z such that*

- $\iota^* B = 2B_1$ *and*
- $\iota^* \mathcal{L} \sim B_1$,

then the preimage $f^* Z$ splits into two irreducible components Z^+ and Z^-.

Proof. Let $f|_{f^{-1}(Z)} : f^{-1}(Z) \to Z$ be the induced morphism. $f^{-1}(Z)$ is realized as a hypersurface in the total space of $\iota^* L$ as in usual manner (see [1, Chapter I, §17], for example). Our condition implies that $f^*(Z)$ is reducible. Since $\deg f = 2$, our statement follows. Q.E.D.

Lemma 2.2. *Let Y be a smooth projective variety, let $\sigma : Y \to Y$ be an involution on Y, let R be a smooth irreducible divisor on Y such that $\sigma|_R$ is the identity, and let B be a reduced divisor on Y such that $\sigma^* B$ and B have no common component.*

If there exists a σ-invariant divisor D on Y (i.e., $\sigma^* D = D$) such that

- *$B + D$ is 2-divisible in* $\operatorname{Pic}(Y)$, *and*
- *R is not contained in* $\operatorname{Supp}(D)$,

then there exists a double cover $f : X \to Y$ with branch locus $B + \sigma^* B$ such that R is a splitting divisor with respect to f (see Remark 0.1 for a splitting divisor and a quadratic residue divisor).

Moreover, if there is no 2-torsion in $\operatorname{Pic}(Y)$, *then $B + \sigma^* B$ is a quadratic residue divisor* $\operatorname{mod} R$.

Proof. Since Y is projective, there exists a divisor D_o on Y such that

(1) R is not contained in $\mathrm{Supp}(D_o)$, and
(2) $B + D \sim 2D_o$.

Hence $B + \sigma^*B \sim 2(D_o + \sigma^*D_o - D)$. Let $f : X \to Y$ be a double cover determined by $(Y, B + \sigma^*B, D_o + \sigma^*D_o - D)$ and let $\iota : R \hookrightarrow Y$ denote the inclusion morphism. Since $\sigma|_R = \mathrm{id}_R$,

$$\iota^*B = \iota^*\sigma^*B, \qquad \iota^*(D_o - D) = \iota^*(\sigma^*D_o - D),$$

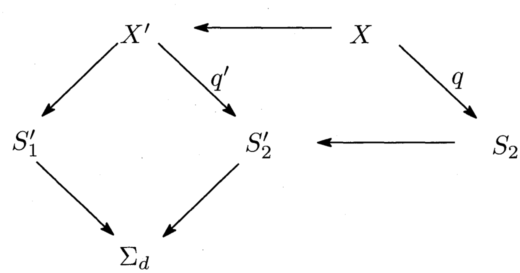
we have

$$\begin{aligned} \iota^*B &\sim \iota^*(2D_o - D) \\ &= \iota^*D_o + \iota^*(\sigma^*D_o - D) \\ &= \iota^*(D_o + \sigma^*D_o - D). \end{aligned}$$

Hence, by Lemma 2.1, R is a splitting divisor with respect to f. Moreover, if there is no 2-torsion in $\mathrm{Pic}(Y)$, f is determined by $B + \sigma^*B$. Hence $B + \sigma^*B$ is a quadratic residue divisor $\mathrm{mod} R$. Q.E.D.

Proposition 2.1. *Let $p_2 : S_2 \to \Sigma_d$ and $p_1 : S_1 \to \Sigma_d$ be the double covers as in the Introduction. Under the assumption of Proposition 0.1, if there exists a σ_{p_2}-invariant divisor D on S_2 such that $B_1^+ + D$ is 2-divisible in* $\mathrm{Pic}(S_2)$, *then B_2 is a splitting curve with respect to p_1.*

Proof. Let ψ_1 and ψ_2 be rational function on Σ_d such that $\mathbb{C}(S_1')(= \mathbb{C}(S_1)) = \mathbb{C}(\Sigma_d)(\sqrt{\psi_1})$ and $\mathbb{C}(S_2')(= \mathbb{C}(S_2)) = \mathbb{C}(\Sigma_d)(\sqrt{\psi_2})$, respectively. Note that $(\psi_1) = \Delta_0 + B_1 + 2D_1$ and $(\psi_2) = \Delta_0 + B_2 + 2D_2$ for some divisors D_1 and D_2 on Σ_d. Let X' be the $\mathbb{C}(\Sigma_d)(\sqrt{\psi_1}, \sqrt{\psi_2})$-normalization of Σ_d and let $q : X \to S_2$ be the canonical resolution of the induced double cover of S_2 by the quadratic extension $\mathbb{C}(\Sigma_d)(\sqrt{\psi_1}, \sqrt{\psi_2}) / \mathbb{C}(\Sigma_d)(\sqrt{\psi_2})$.



Put

$$R := \overline{(p_2^*B_2)_{red} \setminus (\text{the exceptinonal set of } S_2 \to S_2')},$$

where $\overline{\bullet}$ denotes the closure of $\bullet$. Note that R is smooth as $\mu_2 : S_2 \to S_2'$ is the canonical resolution. We infer that B_2 is a splitting curve with respect to p_1 if and only if R is a splitting curve with respect to q. Now by Lemma 2.2, our statement follows. Q.E.D.

We are now in position to prove Proposition 0.1. We first note that the algebraic equivalence $\approx$ and the linear equivalence $\sim$ coincides on S_i by Lemma 1.2.

The case of $g_2 \geq 2$. Let s_0 be an element in $\mathrm{MW}(\mathcal{J}_{S_2})$ such that $2s_0 = s(B_1^+)$ on $\mathrm{MW}(\mathcal{J}_{S_2})$. By [8], there exists a divisor D on S_2 such that $s(D) = s_0$. By [8], D satisfies the following relation

$$2D \sim B_1^+ + (2D\mathfrak{f}_2 - 2g_1 - 1)O_2 + \left\{2DO_2 + \frac{d}{2}(2D\mathfrak{f}_2 - 2g_1 - 1)\right\}\mathfrak{f}_2 + \Xi,$$

where $\mathfrak{f}_2$ denotes a fiber of φ_2 and Ξ is a divisor whose irreducible components consist of those of singular fibers not meeting O_2. By our assumption on the singularity of B_2, we can infer that any irreducible component of Ξ is σ_{p_2}-invariant. As $\sigma_{p_2}^* O_2 = O_2$, $\sigma_{p_2}^*\mathfrak{f}_2 = \mathfrak{f}_2$, by Proposition 2.1, our statement follows.

The case of $g_2 = 1$. Let s_0 be an element in $\mathrm{MW}(\mathcal{J}_{S_2})$ such that $2s_0 = s(B_1^+)$.

By Theorem 1.1 and Corollary 1.1, we have

$$2s_0 - s(B_1^+) \in \mathrm{Tr}(\varphi_2).$$

Let $\phi : \mathrm{MW}(\mathcal{J}_{S_2}) \to \mathrm{NS}_{\mathbb{Q}}(:= \mathrm{NS}(S_2)\otimes\mathbb{Q})$ be the homomorphism given in [7, Lemmas 8.1 and 8.2]. Note that there will be no harm in considering $\mathrm{NS}_{\mathbb{Q}}$ since $\mathrm{NS}(S_2)$ is torsion free. By [7, Lemmas 8.1 and 8.2], $\phi(s)$ satisfies the following properties:

(i) $\phi(s) \equiv s \bmod \mathrm{Tr}(\varphi_2)_{\mathbb{Q}}(:= \mathrm{Tr}(\varphi_2) \otimes \mathbb{Q})$.
(ii) $\phi(s)$ is orthogonal to $\mathrm{Tr}(\varphi_2)$.

Explicitly $\phi(s)$ is given by

$$\phi(s) = s - O_2 - (sO_2 + \chi(\mathcal{O}_{S_2}))\mathfrak{f}_2 + \text{the contribution terms.}$$

The contribution terms is a $\mathbb{Q}$-divisor arising from reducible singular fiber in the following way:

Let $\mathfrak{f}_v$ be a singular fiber over $v \in \mathbb{P}^1$ and let $\Theta_{v,0}$ be the irreducible component with $O_2\Theta_{v,0} = 1$.

- If s meets $\Theta_{v,0}$, then there is no correction term from $\mathfrak{f}_v$.

- If s does not meet $\Theta_{v,0}$, the contribution term from $\mathfrak{f}_v$ is as follows:

  Let $\Theta_{v,1},\dots,\Theta_{v,r_v-1}$ denote irreducible components of $\mathfrak{f}_v$ other than $\Theta_{v,0}$ and let $A_v := ((\Theta_{v,i}\Theta_{v,j}))$ be the intersection matrix of $\Theta_{v,1},\dots,\Theta_{v,r_v-1}$. With these notation, the contribution term is

$$\sum_i (\Theta_{v,1},\dots,\Theta_{v,r_v-1})(-A_v^{-1})\begin{pmatrix} s\Theta_{v,1} \\ \cdot \\ s\Theta_{v,r_v-1}\end{pmatrix}.$$

By our assumption on $B_1 \cap B_2$, both of $B_1^{\pm}$ meet any $\Theta_{v,0}$ only and so does $s(B_1^+)$ by [6, Theorem 9.1]. By [7, Lemma 5.1], we have

$$B_1^+ \sim s(B_1^+) + 2g_1O_2 + n\mathfrak{f}_2$$

for some integer n, and

$$\phi(s(B_1^+)) = s(B_1^+) - O_2 - (s(B_1^+)O_2 + \chi(\mathcal{O}_{S_2}))\mathfrak{f}_2.$$

Put

$$\phi(s_0) = s_0 - O_2 - (s_oO_2 + \chi(\mathcal{O}_{S_2}))\mathfrak{f}_2 + \sum_{v\in \mathrm{Red}(\varphi_2)} \mathrm{Contr}_v,$$

where $\mathrm{Red}(\varphi_2) = \{v \in \mathbb{P}^1 | \varphi_2^{-1}(v) \text{ is reducible}\}$ and Contr_v denotes the contribution term arising from the singular fiber $\varphi_2^{-1}(v)$. Since $2s_0 - s(B_1^+) \in \mathrm{Tr}(\varphi_2)$, $\phi(2s_0) - \phi(s(B_1^+)) = 0$ in $\mathrm{NS}_{\mathbb{Q}}$. Hence

$$(*)\quad 2s_0 - B_1^+ \quad \sim_{\mathbb{Q}} \quad (1-2g_1)O_2 + (2s_0O_2 - s(B_1^+)O_2 + \chi(\mathcal{O}_{S_2}) - n)\mathfrak{f}_2 - 2\sum_{v\in \mathrm{Red}(\varphi_2)} \mathrm{Contr}_v.$$

Thus

$$2\sum_{v\in \mathrm{Red}(\varphi_2)} \mathrm{Contr}_v \sim_{\mathbb{Q}} E,$$

for some element $E \in \mathrm{Tr}(\varphi_2)$.

Claim. $2\sum_{v\in \mathrm{Red}(\varphi_2)} \mathrm{Contr}_v \in \mathrm{Tr}(\varphi_2)$.

Proof of Claim. We first note that $2\sum_{v\in \mathrm{Red}(\varphi_2)} \mathrm{Contr}_v = E$ in $\mathrm{Tr}(\varphi_2)_{\mathbb{Q}}$. Since $O_2, \mathfrak{f}_2$ and all the irreducible components of reducible singular fibers which do not meet O_2 form a basis of the free $\mathbb{Z}$-module $\mathrm{Tr}(\varphi_2)$ as well as the $\mathbb{Q}$-vector space $\mathrm{Tr}(\varphi_2)_{\mathbb{Q}}$, E is expressed as a $\mathbb{Z}$-linear combination of these divisors. As Contr_v is a $\mathbb{Q}$-linear combination of the irreducible components of reducible singular fibers which do not meet O_2, if $2\sum_{v\in \mathrm{Red}(\varphi_2)} \mathrm{Contr}_v \notin \mathrm{Tr}(\varphi_2)$, then we have a nontrivial relation among $O_2, \mathfrak{f}_2$ and all the irreducible components of reducible singular fibers which do not meet O_2. This leads us to a contradiction. Q.E.D.

By Claim, we have

(i) $\mathrm{Contr}_v = 0$ if the singular fiber over v is of type either I_n (n: odd), IV or IV^* and

(ii) if $\mathrm{Contr}_v \neq 0$, one can write Contr_v in such a way that

$$\mathrm{Contr}_v = \frac{1}{2}D_{1,v} + D_{2,v},$$

where $D_{1,v}, D_{2,v} \in \mathrm{Tr}(\varphi_2)$ and $D_{1,v}$ is reduced.

Since $s_0 + \sigma_{p_2}^* s_0 \in \mathrm{Tr}(\varphi_2)$, we have

$$\frac{1}{2}(D_{1,v} + \sigma_{p_2}^* D_{1,v}) \in \mathrm{Tr}(\varphi_2).$$

Therefore we infer that we can rewrite $D_{1,v}$ in such a way that

$$D_{1,v} = D'_{1,v} + \sigma_{p_2}^* D'_{1,v} + D''_{1,v},$$

where

- $D'_{1,v} \neq \sigma_{p_2}^* D'_{1,v}$ and there is no common component between $D'_{1,v}$ and $\sigma_{p_2}^* D'_{1,v}$, and
- each irreducible component of $D''_{1,v}$ is σ_{p_2}-invariant.

In particular, $D_{1,v}$ is σ_{p_2}-invariant. Now put

$$D := O_2 + \sum_{v\in \mathrm{Red}(\varphi_2)} D_{1,v} + \left((2s_0O_2 - s(B_1^+)O_2 + \chi(\mathcal{O}_{S_2}) - n) - 2\left[\frac{(2s_0O_2 - s(B_1^+)O_2 + \chi(\mathcal{O}_{S_2}) - n)}{2}\right]\right)\mathfrak{f}_2$$

$$D_o := s_0 + g_1 O_2 - \left[\frac{(2s_0O_2 - s(B_1^+)O_2 + \chi(\mathcal{O}_{S_2}) - n)}{2}\right]\mathfrak{f}_2 + \sum_{v\in \mathrm{Red}(\varphi_2)} (D_{1,v} + D_{2,v}),$$

where $[\bullet]$ means the greatest integer not exceeding $\bullet$. Then the relation $(*)$ becomes

$$B_1^+ + D \sim 2D_o.$$

As $\sigma_{p_2}^* O_2 = O_2$, $\sigma_{p_2}^* \mathfrak{f}_2 = \mathfrak{f}_2$, by Proposition 2.1, our statement follows.

§3. Proof of Proposition 0.2.

We first note that $\mathrm{NS}(S_1) = \mathrm{Tr}(\varphi_1)$ by Remark 1.1. Choose an affine open subset U of Σ_d as follows:

- $U := \Sigma_d \setminus (\Delta_0 \cup F) \cong \mathbb{C}^2$.
- Let (t, x) denote an affine coordinate of U. B_1 and B_2 are given by

$$\begin{aligned} B_1 : f_1(t,x) &= x^{2g_1+1} + a_1^{(1)} x^{2g_1} + \ldots + a_{2g_1+1}^{(1)}(t) \in \mathbb{C}[t,x], \\ B_2 : f_2(t,x) &= x^{2g_2+1} + a_1^{(2)} x^{2g_2} + \ldots + a_{2g_2+1}^{(2)}(t) \in \mathbb{C}[t,x], \end{aligned}$$

where $\deg a_k^{(i)}(t) \le dk$ $(i = 1, 2)$.

Under these circumstances, $(p_1')^{-1}(U)$ is given by

$$(p_1')^{-1}(U) = \mathrm{Spec}(\mathbb{C}[t, x, \zeta_1]), \quad \zeta_1^2 = f_1.$$

By our assumption,

$$\mathrm{NS}(S_1) = \mathrm{Tr}(\varphi_1) = \mathbb{Z}O_1 \oplus \mathbb{Z}\mathfrak{f}_1 \oplus \bigoplus_{v \in \mathrm{Red}(\varphi_1)} T_v,$$

where

- $\mathfrak{f}_1$ denotes a fiber of $\varphi_1 : S_1 \to \mathbb{P}^1$,
- $\mathrm{Red}(\varphi_1) := \{v \in \mathbb{P}^1 \mid \varphi_1^{-1}(v) \text{ is reducible}\}$, and
- $T_v :=$ the subgroup of $\mathrm{NS}(S_1)$ generated by irreducible components of $\varphi_1^{-1}(v)$, $v \in \mathrm{Red}(\varphi_1)$, not meeting O_1.

Since $B_2^+\Theta = 0$ for any irreducible component of $\varphi_1^{-1}(v)$, $v \in \mathrm{Red}(\varphi_1)$, not meeting O_1, and T_v is a negative definite lattice with respect to the intersection pairing, we may assume $B_2^+ \sim aO_1 + b\mathfrak{f}_1$ for some $a, b \in \mathbb{Z}$. Since $B_2^- = \sigma_{p_1}^* B_2^+ \sim a\sigma_{p_1}^* O_1 + b\sigma_{p_1}^* \mathfrak{f}_1 = aO_1 + b\mathfrak{f}_1$ and $B_2^+ + B_2^- \sim p_1^* B_2 \sim (2g_2 + 1)(2O_1 + d\mathfrak{f}_1)$, we have

$$B_2^+ \sim B_2^- \sim (2g_2+1)\left(O_1 + \frac{d}{2}\mathfrak{f}_1\right).$$

Let $\psi^+ \in \mathbb{C}(S_1)(= \mathbb{C}(S_1'))$ such that

$$\begin{aligned} (\psi^+) &= B_2^+ - (2g_2+1)\left(O_1 + \frac{d}{2}\mathfrak{f}_1\right) \\ (\sigma_{p_1}^* \psi^+) &= B_2^- - (2g_2+1)\left(O_1 + \frac{d}{2}\mathfrak{f}_1\right). \end{aligned}$$

By choosing $\mathfrak{f}_1 = p_1^* F$, we may assume that both rational functions ψ^+ and $\sigma_{p_1}^* \psi^+$ are regular on $p_1^{-1}(U)$. Hence by [5, Theorem 2.29, p.147], they are also regular on $p_1'^{-1}(U)$. This means that

$$\begin{aligned} \psi^+|_U &= g(t,x) + h(t,x)\zeta_1, \\ \sigma_{p_1}^* \psi^+|_U &= g(t,x) - h(t,x)\zeta_1, \end{aligned}$$

for some $g, h \in \mathbb{C}[t,x]$. On the other hand, one can choose a rational function $\psi \in \mathbb{C}(\Sigma_d)$ in such a way that

$$(\psi) = B_2 - (2g_2+1)(\Delta_0 + dF_0) \quad \text{and} \quad \psi|_U = f_2(t,x).$$

Since $(p_1^*\psi) = (\psi^+ \sigma_{p_1}^* \psi^+)$, we infer that $p_1^*\psi = (\text{non-zero constant}) \times \psi^+ \sigma_{p_1}^* \psi^+$. Hence we may assume that $p_1^*\psi|_U = \psi^+|_U \sigma_{p_1}^* \psi^+|_U$, i.e.,

$$f_2(t,x) = g^2 - h^2 f_1.$$

From this equation, we infer that B_1 is a splitting curve with respect to p_2. Since the generic fiber of $S_{2,\eta}$ is given by

$$\zeta_2^2 - f_2(t,x) = 0,$$

we may assume $B_1^+|_{S_{2,\eta}}$ is given by $\zeta_2 - g = 0$ and $f_1 = 0$. If we put $D_2 :=$ the divisor given by $\zeta_2 - g = 0$ and $h = 0$, then the divisor of the rational function $\zeta_2 - g$ on $S_{2,\eta}$,

$$B_1^+|_{S_{2,\eta}} + 2D_2|_{S_{2,\eta}} - (2g_2+1)O_2|_{S_{2,\eta}}.$$

Hence $s(B_1^+) + 2s(D_2) = 0$ in $\mathrm{MW}(\mathcal{J}_{S_2})$.

Q.E.D.

§4. Proof of Theorem 0.1

Under the assumption, we first note that

- B_1 is a section of Σ_d, i.e., B_1 is smooth and isomorphic to $\mathbb{P}^1$,
- $S_1 \cong \Sigma_{d/2}$ and $\mathrm{NS}(S_1) = \mathrm{Tr}(\varphi_1)$ (i.e., $\mathrm{MW}(\mathcal{J}_{S_1}) = \{0\}$ by Remark 1.1), and
- B_1 is a splitting curve with respect to p_2.

Hence if $s(B_1^+)(= B_1^+)$ is 2-divisible in $\mathrm{MW}(\mathcal{J}_{S_2})$, then B_2 is a splitting curve with respect to p_1 by Proposition 0.1. Conversely, if B_2 is a splitting curve with respect to p_1, $s(B_1^+)$ is 2-divisible by Proposition 0.2. As p_1 is determined by $\Delta_0 + B_1$, our statement follows.

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