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Varieties of lines on Fermat hypersurfaces

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Abstract.

Let X be a hypersurface in the projective space $\mathbf{P}^{n+1}$ and $G(n+1,1)$ be the Grassmann variety $G(n+1,1)$ of lines in $\mathbf{P}^{n+1}$. The subvariety $F(X)$ of $G(n+1,1)$ consisting of lines contained in X is called the Fano variety of X. We study a detailed structure of the Fano variety of the Fermat hypersurface X of degree d for $n \geq d$. More precisely, we show that a certain open subset $F^0(X)$ of $F(X)$ has a fibration structure over a moduli space of marked pointed rational curves and that the fibers are complete intersections of Fermat hypersurfaces introduced in [T]. We also study singularities of $F(X)$.

§1. Introduction

In this paper, all algebraic varieties are considered over the complex number field $\mathbf{C}$ unless otherwise stated. Let n, d be integers such that $n \geq 2$, $d \geq 2$, and $\mathbf{P}^{n+1}$ be the $(n+1)$-dimensional projective space and X a hypersurface of degree d in $\mathbf{P}^{n+1}$. Let $G(n+1,1)$ be the Grassmann variety of lines in $\mathbf{P}^{n+1}$ defined by

$$G(n+1,1) = \{l \mid l \text{ is a line in } \mathbf{P}^{n+1}\}.$$

The variety consisting of lines contained in X is denoted by $F(X)$ and called the Fano variety of lines of X, i.e.

$$F(X) = \{l \in G(n+1,1) \mid l \subset X\}.$$

Let $Sym^d(\mathbf{C}^{n+2})$ be the vector space of homogeneous polynomials of degree d on $\mathbf{C}^{n+2}$ and $\mathbf{P}(Sym^d(\mathbf{C}^{n+2}))$ the projective space associated to $Sym^d(\mathbf{C}^{n+2})$. The class of a non-zero element f of $Sym^d(\mathbf{C}^{n+2})$ in

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$\mathbf{P}(Sym^d(\mathbf{C}^{n+2}))$ is denoted by $[f]$, and the hypersurface in $\mathbf{P}^{n+1}$ defined by the zeros of f is denoted by $V(f)$.

The following theorem is due to Barth-Van de Ven [BV].

Theorem 1 (Generic smoothness). *Assume that $2n > d+1$. There exists a non-empty open set U of $\mathbf{P}(Sym^d(\mathbf{C}^{n+2}))$ such that the Fano variety of lines of $F(V(f))$ is smooth of dimension $2n-d-1$ if $[f] \in U$.*

The open set U in the above theorem is described as follows. Let $\mathcal{U}$ be the universal family of lines contained in the universal family of hypersurfaces defined by

$$\mathcal{U} = \{([f], l) \in \mathbf{P}(Sym^d(\mathbf{C}^{n+2})) \times G(n+1,1) \mid l \subset V(f)\}.$$

Let

$$pr_1 : \mathcal{U} \to \mathbf{P}(Sym^d(\mathbf{C}^{n+2})), \quad pr_2 : \mathcal{U} \to G(n+1,1)$$

be the restriction to $\mathcal{U}$ of the first and second projections of the product $\mathbf{P}(Sym^d(\mathbf{C}^{n+2})) \times G(n+1,1)$. The fiber of pr_2 at a line l is the set of hypersurfaces containing the line l, which is isomorphic to a linear subspace in $\mathbf{P}(Sym^d(\mathbf{C}^{n+2}))$ of codimension $d+1$. Therefore the map pr_1 is smooth and the variety $\mathcal{U}$ is smooth. By the generic smoothness of pr_1, there exists a non-empty open set U of $\mathbf{P}(Sym^d(\mathbf{C}^{n+2}))$ such that the restriction of pr_1 to U is smooth.

On the other hand, there exists a smooth hypersurface X such that the Fano variety $F(X)$ of X is not smooth even if $2n > d+1$. In this paper, we give a description of $F(X)$ of a Fermat hypersurface for $n \geq d$ using a moduli space of marked rational curves. The main theorems are Theorem 2 (the case $n = d$) and Theorem 4 (the case $n \geq d$). In Section 5, we study the singular loci of the Fano varieties $F(X)$ of Fermat hypersurfaces for $n = d$.

§2. Transversal Fano varieties of Fermat hypersurfaces

Let $G(n+1,1)$ be the Grassmann variety of lines in $\mathbf{P}^{n+1}$. The coordinates of the projective space is written as $(X_0 : \cdots : X_{n+1})$. The coordinate hyperplane defined by $\{X_i = 0\}$ is denoted by H_i. We define the subset $G(n+1,1)^0$ of $G(n+1,1)$ by

$$\begin{aligned} &G(n+1,1)^0 \\ =&\{l \in G(n+1,1) \mid \text{ the intersections } l \cap H_0, \dots, l \cap H_{n+1} \text{ are distinct}\}. \end{aligned}$$

The transversal part $F^0(X)$ of $F(X)$ is defined by $F(X) \cap G(n+1,1)^0$.

Definition 1 (Moduli space of rational curves with $(n+2)$ marked points).

(1) *The moduli space of rational curves with distinct $(n+2)$ marked points is denoted by $\mathcal{M}_{0,n+2}$.*

(2) *We define the moduli space of rigidified rational curves with $(n+2)$ marked points by*

$$\widetilde{\mathcal{M}_{0,n+2}} = (\mathbf{P}^1)^{n+2} - D,$$

where D is the big diagonal defined by

$$D = \cup_{0 \le i < j \le n+1} \{(p_0, \dots, p_{n+1}) \mid p_i = p_j\}.$$

The group $PGL(2)$ acts on $(\mathbf{P}^1)^{n+2}$ diagonally and the open set $\widetilde{\mathcal{M}_{0,n+2}}$ is stable and fixed point free under this action. The variety $\mathcal{M}_{0,n+2}$ is isomorphic to $\widetilde{\mathcal{M}_{0,n+2}}/PGL(2)$. The class of $(p_0, \dots, p_{n+1})$ in $\mathcal{M}_{0,n+2}$ is denoted by $[p_0, \dots, p_{n+1}]$. The dimension of $\mathcal{M}_{0,n+2}$ is $n-1$.

Let $l \in F^0(X)$. By the condition of transversality, the intersections $l \cap H_0, \dots, l \cap H_{n+1}$ are distinct $n+2$ points, and they define a point $conf(l) = [l \cap H_0, \dots, l \cap H_{n+1}] \in \mathcal{M}_{0,n+2}$. Thus we have a map

$$conf : F^0(X) \to \mathcal{M}_{0,n+2}.$$

We define the Fermat hypersurface X of degree d in $\mathbf{P}^{n+1}$ by

$$X : X_0^d + \cdots + X_{n+1}^d = 0.$$

Let G be the group $\mu_d^{n+2}/\Delta(\mu_d)$, where $\Delta(\mu_d) = \{(\zeta, \cdots, \zeta) \mid \zeta \in \mu_d\} \subset \mu_d^{n+2}$. Then G acts on X by

$$(X_0 : \cdots : X_{n+1}) = (\zeta_0 X_0 : \cdots : \zeta_{n+1} X_{n+1})$$

for $(\zeta_0, \dots, \zeta_{n+1}) \in G$. The action of G on the Fermat hypersurface X induces that on the Fano variety $F(X)$ of lines on X.

Let l be an element in $F^0(X)$ and $g = (\zeta_0, \dots, \zeta_{n+1})$ an element in G. Since the image of the point $l \cap H_i$ under the map $g : l \to g(l)$ is equal to $g(l) \cap H_i$, the condition of transversality is stable under the action of G. Therefore G induces an action on $F(X)^0$ and $conf(g(l)) = conf(l)$. Thus we have a map

$$\overline{conf} : F^0(X)/G \to \mathcal{M}_{0,n+2}.$$

Our first main theorem is the following.

Theorem 2. *If $d = n$, then the action of G on $F(X)^0$ is fixed point free and the map $\overline{conf}$ is an isomorphism.*

§3. Construction of the family $\mathcal{F}$

3.1. Trivialized Kummer covering

Let k be a field of characteristic zero, S a smooth connected scheme over k, and $\pi : \mathcal{C} \to S$ a geometrically connected smooth projective curve over S. Let $D = \sum_i a_i D_i$ be a divisor on $\mathcal{C}$ such that $\mathcal{O}_{\mathcal{C}}(D) \simeq \mathcal{O}_{\mathcal{C}}$. Let $p_\infty : S \to \mathcal{C}$ be a section of π such that $Im(p_\infty) \cap Supp(D) = \emptyset$.

Definition 2 (Trivialized Kummer covering). *Let d be a positive integer and f a rational function on $\mathcal{C}$ such that*

(1) *The divisor (f) of f is equal to D, and*
(2) $f\,|_{Im(p_\infty)} = 1$.

Then the projective curve D over S defined by $y^d = f$ is called the Kummer covering with the branch divisor D trivialized at p_∞. Since there is a unique rational function f with the properties in Definition 2, the trivialized Kummer covering is uniquely determined by D and p_∞.

3.2. Definition of Kum^q

Let $\widetilde{\mathcal{U}} = \mathbf{P}^1 \times \widetilde{M_{0,n+2}}$ be the universal rational curve over the rigidified moduli space $\widetilde{M_{0,n+2}}$ and $\widetilde{p_i} : \widetilde{\mathcal{M}_{0,n+2}} \to \widetilde{\mathcal{U}}$ be the universal section defined by

$$\widetilde{p_i}(\lambda_0, \dots, \lambda_{n+1}) = (\lambda_i, (\lambda_0, \dots, \lambda_{n+1})) \in \widetilde{\mathcal{U}}.$$

The group $PGL(2)$ acts diagonally on $\widetilde{\mathcal{U}}$ freely and the quotient variety is denoted by $\mathcal{U}$. Since the action of $PGL(2)$ on $\widetilde{\mathcal{U}}$ and that on $\widetilde{\mathcal{M}_{0,n+2}}$ are equivariant and the sections $\widetilde{p_i}$ are stable under its action, we have a map $\mathcal{U} \to \mathcal{M}_{0,n+2}$ and sections $p_i : \mathcal{M}_{0,n+2} \to \mathcal{U}$.

We fix $q \in \mathbf{P}^1$. The affine line defined by $\mathbf{P}^1 - q$ is denoted by $\mathbf{A}^1_q$. We choose an inhomogeneous coordinate x of $\mathbf{P}^1$ such that $x(q) = \infty$. The open set $(\mathbf{A}^1_q)^{n+2} \cap \widetilde{\mathcal{M}_{0,n+2}}$ of $\widetilde{\mathcal{M}_{0,n+2}}$ is denoted by $\widetilde{\mathcal{M}_q} = \widetilde{\mathcal{M}_{0,n+2,q}}$ and the restriction of the universal curve $\widetilde{\mathcal{U}}$ to $\widetilde{\mathcal{M}_q}$ is denoted by $\widetilde{\mathcal{U}_q}$. The section of $\widetilde{\mathcal{U}_q} \to \widetilde{\mathcal{M}_q}$ defined by the point q is also denoted by q. The three sections p_i, p_j and q do not intersect to each other. The Kummer covering of $\widetilde{\mathcal{U}_q}$ branching at $(\widetilde{p_i}) - (\widetilde{p_j})$ trivialized at q is denoted by $Kum^q_{i,j}$. Using the above inhomogeneous coordinate x of $\mathbf{P}^1$, we have the coordinates $(x, \lambda_0, \dots, \lambda_{n+1})$ of $\widetilde{\mathcal{U}}$. Then the covering $Kum^q_{i,j}$ is defined by

$$y^d_{i,j} = \frac{x - \lambda_i}{x - \lambda_j}.$$

We set

$$Kum^q = Kum^q_{0,1} \times_{\widetilde{\mathcal{U}}} Kum^q_{0,2} \times_{\widetilde{\mathcal{U}}} \cdots \times_{\widetilde{\mathcal{U}}} Kum^q_{0,n+1}.$$

An element $(\zeta_0, \dots, \zeta_{n+1})$ in the group G acts on Kum^q by $y_{0,i} \mapsto \dfrac{\zeta_0}{\zeta_i} y_{0,i}$. Under this action of G, $Kum^q \to \widetilde{\mathcal{U}}$ becomes a G-covering.

3.3. Definition of $\widehat{\mathcal{M}_q}$

We define $\Delta_2 = \{(\lambda, \nu) \in \mathbf{A}^1_q \times \mathbf{A}^1_q \mid \lambda \neq \nu\}$. We set

$$\pi : \mathbf{P}^1 \times \Delta_2 \to \Delta_2 : (x, \lambda, \nu) \mapsto (\lambda, \nu).$$

The sections of π defined by $(\lambda, \nu) \mapsto (\lambda, \lambda, \nu)$ and $(\lambda, \nu) \mapsto (\nu, \lambda, \nu)$ are also denoted as λ and ν and the Kummer covering branching at $(\lambda) - (\nu)$ trivialized at q is denoted as $\widetilde{\Delta_3} \to \mathbf{P}^1 \times \Delta_2$. The pull back of $\widetilde{\Delta_3}$ by the map

$$\widetilde{\mathcal{M}_q} \to \mathbf{P}^1 \times \Delta_2 : (\lambda_0, \dots, \lambda_{n+2}) \mapsto (\lambda_i, \lambda_j, \lambda_k)$$

is denoted by $\Delta^i_{j,k}$. Then the μ_d-covering $\Delta^i_{j,k} \to \widetilde{\mathcal{M}_q}$ is defined by $\delta^d_{i,j,k} = \frac{\lambda_i - \lambda_j}{\lambda_i - \lambda_k}$. The Galois group of the covering

$$\prod_{i \neq j,k \; \widetilde{\mathcal{M}_q}} \Delta^i_{j,k} \to \widetilde{\mathcal{M}_q}$$

is isomorphic to μ_d^n and the covering corresponding to the group $\mathrm{Ker}(\mu_d^n \to \mu_d : (\zeta_i)_i \mapsto \sum_i \zeta_i)$ is denoted by $\widehat{\mathcal{M}_{j,k}}$. Here $\prod_{\widetilde{\mathcal{M}_q}}$ is the fiber product over $\widetilde{\mathcal{M}_q}$. The μ_d-covering $\widehat{\mathcal{M}_{j,k}}$ of $\widetilde{\mathcal{M}_q}$ is defined by

$$\delta^d_{j,k} = \prod_{i \neq j,k} \frac{\lambda_i - \lambda_j}{\lambda_i - \lambda_k}.$$

We define a G-covering $\widehat{\mathcal{M}_q} \to \widetilde{\mathcal{M}_q}$ by

$$\widehat{\mathcal{M}_q} = \prod_{i=1 \; \widetilde{\mathcal{M}_q}}^{n+1} \widehat{\mathcal{M}_{0,i}}.$$

3.4. The family $\mathcal{F}_q$

We set $k = n - d$. We define the variety $\widetilde{\mathcal{F}}_q$ by the following fiber product:

$$\widetilde{\mathcal{F}}_q = \underbrace{Kum^q \times_{\widetilde{M}_q} Kum^q \times_{\widetilde{M}_q} \cdots \times_{\widetilde{M}_q} Kum^q}_{k\text{-times}} \times_{\widetilde{M}_q} \widetilde{\mathcal{M}_q}.$$

Then $\widetilde{\mathcal{F}}_q$ is a $G^k \times G$ covering of

$$\widetilde{\mathcal{U}}_q^k = \underbrace{\widetilde{\mathcal{U}}_q \times_{\widetilde{\mathcal{M}}_q} \widetilde{\mathcal{U}}_q \times_{\widetilde{\mathcal{M}}_q} \cdots \times_{\widetilde{\mathcal{M}}_q} \widetilde{\mathcal{U}}_q}_{k\text{-times}}.$$

We define the variety $\mathcal{F}_q$ by the G-covering of $\widetilde{\mathcal{U}}_q^k$ corresponding to the subgroup of the Galois group:

$$\operatorname{Ker}\big(G^k \times G \to G : (g_1, \dots, g_k, g) \mapsto -g + \sum_{i=1}^{k} g_i\big).$$

Let $x_1, \dots, x_k$ be the coordinates for the fibers of $\widetilde{\mathcal{U}}_q^k \to \widetilde{\mathcal{M}_q}$. Then $\mathcal{F}_q$ is defined by

$$\xi_j^d = \prod_{i=1}^{k} \frac{(x_i - \lambda_j)}{(x_i - \lambda_0)} \prod_{p \neq 0, j} \frac{(\lambda_0 - \lambda_p)}{(\lambda_j - \lambda_p)} \quad (j = 1, \dots, n+1). \tag{1}$$

3.5. Patching and an action of $PGL(2)$

We define a fiber space $\mathcal{F}_q \to \widetilde{\mathcal{M}_q}$ of relative dimension k for each $q \in \mathbf{P}^1$. In this subsection, we patch them into a variety $\mathcal{F}$. Let $q_1, \dots, q_{n+3}$ be distinct elements in $\mathbf{P}^1$. Then we have $\widetilde{\mathcal{M}_{0,n+2}} = \bigcup_{i=1}^{n+3} \widetilde{\mathcal{M}_{q_i}}$.

Proposition 3. *There exists a patching data*

$$\varphi_{ij} : \mathcal{F}_{q_i} \mid_{\widetilde{\mathcal{M}_{q_i}} \cap \widetilde{\mathcal{M}_{q_j}}} \xrightarrow{\simeq} \mathcal{F}_{q_j} \mid_{\widetilde{\mathcal{M}_{q_i}} \cap \widetilde{\mathcal{M}_{q_j}}} \tag{2}$$

for $\mathcal{F}_{q_i}$ and $\mathcal{F}_{q_j}$ such that the glued variety $\mathcal{F} = \bigcup_{i=1}^{n+3} \mathcal{F}_{q_i}$ admits a lifting of the action of $PGL(2)$ on $\widetilde{\mathcal{M}_{0,n+2}}$.

Proof. We choose coordinates x and x' of $\mathbf{P}^1$ such that $x(q_i) = \infty$ and $x'(q_j) = \infty$. We introduce an isomorphism φ_{ij} in (2). We write

$$x' = \frac{ax + b}{cx + d}.$$

We use these coordinates to express points in $\widetilde{\mathcal{M}_{q_i}}$ and $\widetilde{\mathcal{M}_{q_j}}$. Then a point $(\lambda_0, \dots, \lambda_{n+1}) \in \widetilde{\mathcal{M}_{q_i}}$ corresponds to $(\lambda'_0, \dots, \lambda'_{n+1}) \in \widetilde{\mathcal{M}_{q_i}}$, where

$$\lambda'_i = \frac{a\lambda_i + b}{c\lambda_i + d}.$$

We write coordinates of fibers for $\mathcal{F}_{q_i} \to \widetilde{\mathcal{M}_{q_i}}$ and $\mathcal{F}_{q_j} \to \widetilde{\mathcal{M}_{q_j}}$ as $x_1, \dots, x_k$ and $x'_1, \dots, x'_k$. Then we have

$$x'_i - \lambda'_j = \frac{(ad-bc)(x_i - \lambda_j)}{(cx_i + d)(c\lambda_j + d)}, \quad \lambda'_i - \lambda'_j = \frac{(ad-bc)(\lambda_i - \lambda_j)}{(cx_i + d)(c\lambda_j + d)}.$$

On $\widetilde{\mathcal{M}_{q_i}}$, the equation of $\mathcal{F}_{q_j}$ is given by

$$\begin{aligned} {\xi'_j}^d &= \prod_{i=1}^{k} \frac{(x'_i - \lambda'_j)}{(x'_i - \lambda'_0)} \prod_{p \neq 0,j} \frac{(\lambda'_0 - \lambda'_p)}{(\lambda'_j - \lambda'_p)} \\ &= \prod_{i=1}^{k} \frac{(x_i - \lambda_j)}{(x_i - \lambda_0)} \prod_{p \neq 0,j} \frac{(\lambda_0 - \lambda_p)}{(\lambda_j - \lambda_p)} \left(\frac{c\lambda_j + d}{c\lambda_0 + d} \right)^d. \end{aligned}$$

By setting

$$\xi'_i = \xi_i \cdot \frac{c\lambda_j + d}{c\lambda_0 + d},$$

we have an isomorphism $\varphi : \mathcal{F}_{q_1} \to \mathcal{F}_{q_2}$. To show that the family of isomorphisms $\{\varphi_{ij}\}$ actually gives a patching data, we check its 1-cocyle condition. Let x, x', x'' be inhomogeneous coordinates of $\mathbf{A}^1_{q_i}, \mathbf{A}^1_{q_j}, \mathbf{A}^1_{q_k}$ and write

$$x' = \frac{ax + b}{cx + d}, \quad x'' = \frac{a'x' + b'}{c'x' + d'}, \quad x'' = \frac{a''x + b''}{c''x + d''}.$$

We set

$$g' = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad g' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}, \quad g'' = \begin{pmatrix} a'' & b'' \\ c'' & d'' \end{pmatrix},$$

then we have $g'' = g'g$ in $PGL(2)$. Since the rules of the isomorphisms φ_{ij} and φ_{jk} for ξ_p, ξ'_p, ξ''_p are given by

$$\xi'_p = \xi_p \cdot \frac{c\lambda_p + d}{c\lambda_0 + d}, \quad \xi''_p = \xi'_p \cdot \frac{c'\lambda'_p + d'}{c'\lambda'_0 + d'}.$$

Since $\lambda'_p = \dfrac{a\lambda_p + b}{c\lambda_p + d}$, we have $\xi''_p = \xi_p \cdot \dfrac{c''\lambda_p + d''}{c''\lambda_0 + d''}$. Therefore, we have $\varphi_{ik} = \varphi_{jk} \circ \varphi_{ij}$. This computation also shows that the action of $PGL(2)$ on $\widetilde{\mathcal{U}}^k$ extends to that on $\mathcal{F}$. Q.E.D.

§4. Proof of Theorem 2

We use the same notations as in the last section. We assume that $k = n - d \geq 0$. We have the following sequence of varieties

$$\mathcal{F} \xrightarrow{\alpha} \widetilde{\mathcal{U}}^k \xrightarrow{\beta} \widetilde{\mathcal{M}_{0,n+2}}.$$

We have the followings.

(1) The morphisms α and β are $PGL(2)$ equivariant.
(2) The morphism α is a finite etale Galois covering with Galois group G.
(3) The relative dimension of the morphism β is k.
(4) There are actions of the symmetric group $\mathfrak{S}_k$ of degree k on $\widetilde{\mathcal{U}}^k$ and $\mathcal{F}$ over $\widetilde{\mathcal{M}_{0,n+2}}$ and the morphism α is equivariant. These actions of $\mathfrak{S}_k$ commute with actions of $PGL(2)$.

Therefore we have the following sequence of homomorphisms:

$$\mathcal{F}/(\mathfrak{S}_k \times PGL(2)) \xrightarrow{\overline{\alpha}} \widetilde{\mathcal{U}}^k/(\mathfrak{S}_k \times PGL(2)) \xrightarrow{\overline{\beta}} \widetilde{\mathcal{M}_{0,n+2}}/PGL(2).$$

In this section, we prove the following theorem, which is a generalization of Theorem 2.

Theorem 4. *We assume that $n \geq d$ and set $k = n - d$. Then there is a G-equivariant isomorphism between the variety of transversal lines $F^0(X)$ in the Fermat hypersurface X of degree d in $\mathbf{P}^{n+1}$ and $\mathcal{F}/(\mathfrak{S}_k \times PGL(2))$.*

Remark 5. *Theorem 2 is the special case for $n = d$. In this case, the morphism $\overline{\beta}$ is an isomorphism and $\widehat{\mathcal{M}_{0,n+2}}$ is isomorphic to $\mathcal{F}$. Therefore $F^0(X) \to \mathcal{M}_{0,n+2}$ is an etale G-covering.*

Let l be an element of the Grassmann variety $G(n+1,1)$. Then by using homogeneous parameter $(t_0 : t_1)$, the line l is expressed as

$$l : (\alpha_0 t_0 - \beta_0 t_1 : \alpha_1 t_0 - \beta_1 t_1 : \cdots \alpha_{n+1} t_0 - \beta_{n+1} t_1). \tag{3}$$

Using this expression and the inhomogeneous coordinate $t = \frac{t_1}{t_0}$, the intersection of l and the coordinate hyperplane H^i is equal to

$$\lambda_i = \frac{\alpha_i}{\beta_i}. \tag{4}$$

The matrix

$$\begin{pmatrix} \alpha_0 & \cdots & \alpha_{n+1} \\ \beta_0 & \cdots & \beta_{n+1} \end{pmatrix}$$

is called a frame of the line l. A $(2 \times (n+2))$-matrix A is a frame of a line in $\mathbf{P}^{n+1}$ if and only if the rank A is equal to 2. The set of $2 \times (n+2)$-matrix of rank 2 is denoted as $M(2, n+2)^0$. Then the space of non-constant maps $Map(\mathbf{P}^1, \mathbf{P}^{n+1}) = \{f : \mathbf{P}^1 \to \mathbf{P}^{n+1} \mid f \text{ is a non-constant map}\}$ is identified with $M(2, n+2)^0/\mathbf{C}^\times$ and the Grassmann variety $G(n+1, 1)$ is isomorphic to $M(2, n+2)^0/GL(2)$, where $GL(2)$ acts on $M(2, n+2)^0$ from the right. The line l given by (3) is contained in the Fermat hypersurface if the equality

$$(\alpha_0 t_0 + \beta_0 t_1)^d + \cdots + (\alpha_{n+1} t_0 + \beta_{n+1} t_1)^d = 0$$

is satisfied for all $(t_0 : t_1)$. Therefore the condition of $\alpha_0, \dots, \alpha_{n+1}$, $\beta_0, \dots, \beta_{n+1}$ is

$$\text{(5)} \qquad \begin{cases} \alpha_0^d + \cdots + \alpha_{n+1}^d = 0, \\ \beta_0 \alpha_0^{d-1} + \cdots + \beta_{n+1} \alpha_{n+1}^{d-1} = 0, \\ \vdots \\ \beta_0^{d-1} \alpha_0 + \cdots + \beta_{n+1}^{d-1} \alpha_{n+1} = 0, \\ \beta_0^d + \cdots + \beta_{n+1}^d = 0. \end{cases}$$

We assume that the coordinate $(t_0 : t_1)$ of l satisfies the condition $\beta_i \neq 0$, i.e. $\lambda_i \neq \infty$. Then $(\lambda_0, \dots, \lambda_{n+1})$ is an element of $\widetilde{\mathcal{M}_{0,n+1,q}}$.

We set

$$\mathcal{F}' = \{A \in M(2, n+2)^0 \mid (1)\ A \text{ satisfies the condition (5), and } (2) \text{ all the } 2 \times 2 \text{ minors of } A \text{ are non-zero}\}/\mathbf{C}^\times.$$

Then we have the following maps

$$\begin{array}{ccc} \{f \in Map(\mathbf{P}^1, \mathbf{P}^{n+1} \mid f(\infty) \notin H_i \text{ for all } i\} & \xrightarrow{\psi} & \widetilde{\mathcal{M}_{0,n+2,q}} \\ \cap & & \cap \\ \mathcal{F}' & \xrightarrow{\psi} & \widetilde{\mathcal{M}_{0,n+2}}. \end{array}$$

Using the relation (4), the fiber of the map ψ at the point $(\lambda_0, \dots, \lambda_{n+1}) \in \widetilde{\mathcal{M}_q}$ is equal to the subvariety of $\mathbf{P}^{n+1} = \{(\beta_0 : \cdots : \beta_{n+1})\}$ defined by

$$\text{(6)} \qquad \begin{cases} \lambda_0^d \beta_0^d + \cdots + \lambda_{n+1}^d \beta_{n+1}^d = 0, \\ \lambda_0^{d-1} \beta_0^d + \cdots + \lambda_{n+1}^{d-1} \beta_{n+1}^d = 0, \\ \vdots \\ \lambda_0 \beta_0^d + \cdots + \lambda_{n+1} \beta_{n+1}^d = 0, \\ \beta_0^d + \cdots + \beta_{n+1}^d = 0. \end{cases}$$

Remark 6. *This variety is a special complete intersection of Fermat hypersurfaces defined in* [T].

As a consequence, we have the following proposition.

Proposition 7. *The variety $\mathcal{F}'$ is isomorphic to the variety:*

$$\begin{aligned}\mathcal{F}'' = \{(\beta,\lambda) =&((\beta_0 : \cdots : \beta_{n+1}), (\lambda_0, \dots, \lambda_{n+1})) \in \mathbf{P}^{n+1} \times \widetilde{\mathcal{M}_{0,n+2}} \mid \\ &\beta \text{ and } \lambda \text{ satisfies the relation (6)}\}.\end{aligned}$$

The action of $PLG(2)$ on $\mathcal{F}''$ is induced by the action on $\mathcal{F}''$ via this isomorphism. This action is described as follows. Let g be an element of $PGL(2)$ such that $g^*(\lambda_i) = \dfrac{a\lambda_i + b}{c\lambda_i + d}$. Then the action on the variable β_i is given by $g^*(\beta_i) = \beta_i \cdot (c\lambda_i + d)$. We can check that the subvariety $\mathcal{F}''$ is stable under this action. Since $\mathcal{F}'/PGL(2)$ is isomorphic to $F^0(X)$ by the definition of $F(X)$, Proposition 7 implies the following proposition.

Proposition 8. *$\mathcal{F}''/PGL(2)$ is isomorphic to $F^0(X)$.*

Therefore Theorem 4 is a consequence of the following proposition.

Proposition 9. *There exists a $G \times PGL(2)$ equivariant isomorphism between $\mathcal{F}''$ and $\mathcal{F}/\mathfrak{S}_k$.*

Proof. We fix a coordinate t of $\mathbf{P}^1$. Let $x_1, \dots, x_k$ and $\lambda_0, \dots, \lambda_{n+2}$ be coordinates of $\widetilde{\mathcal{U}}^k$ as in §3.4. Then the covering $\mathcal{F}$ of $\widetilde{\mathcal{U}}^k$ is defined by (1). Let ξ_i be the rational function of $\mathcal{F}$ defined in (1). We define the map $\phi : \mathcal{F} \to \mathcal{F}''$ by

$$\begin{aligned}\phi :&(\lambda_0, \dots, \lambda_{n+1}, x_1, \dots, x_k, \xi_1, \dots, \xi_{n+1}) \\ &\mapsto ((1 : \xi_1 : \cdots : \xi_{n+1}), \lambda_0, \dots, \lambda_{n+1}) \in \mathbf{P}^{n+1} \times \widetilde{\mathcal{M}_{0,n+2}}.\end{aligned}$$

We show that the image of this map ϕ is contained in $\mathcal{F}''$. We set $(\beta_0 : \cdots : \beta_{n+1}) = (1 : \xi_1 : \cdots : \xi_{n+1})$. Then by equality (1), we have

$$\begin{aligned}&(\beta_0^d : \cdots : \beta_{n+1}^d) \\ =&\left(\frac{\prod_{i=1}^k (x_i - \lambda_0)}{\prod_{p \neq 0} (\lambda_0 - \lambda_p)} : \frac{\prod_{i=1}^k (x_i - \lambda_1)}{\prod_{p \neq 0} (\lambda_1 - \lambda_p)} : \cdots : \frac{\prod_{i=1}^k (x_i - \lambda_{n+1})}{\prod_{p \neq 0} (\lambda_{n+1} - \lambda_p)}\right).\end{aligned}$$

Therefore by Lagrange interpolation formula, we have

$$\lambda_0^s \beta_0^d + \cdots \lambda_{n+1}^s \beta_{n+1}^d = 0 \quad \text{for } s = 0, \dots, d.$$

Therefore the image of ψ is contained in $\mathcal{F}''$. Since the map ϕ is invariant under the action of $\mathfrak{S}_k$, it factors through the map $\overline{\phi} : \mathcal{F}/\mathfrak{S}_k \to \mathcal{F}''$. We show that the morphism ϕ is an isomorphism. It is enough to show that each fiber of $\overline{\phi}$ at $(\lambda_0, \ldots, \lambda_{n+1})$ is an isomorphism.

The restriction of the morphism $\overline{\phi}$ to the fibers of $\mathcal{F}/\mathfrak{S}_k$ and $\mathcal{F}''$ are G-equivariant and the quotients by the group G are isomorphic to

$$L_1 = \{(\frac{L_0(s_0, \ldots, s_k)}{\prod_{j \neq 0}(\lambda_j - \lambda_0)} : \cdots : \frac{L_{n+1}(s_0, \ldots, s_k)}{\prod_{j \neq n+1}(\lambda_j - \lambda_{n+1})}) \mid s_0, \ldots, s_k \in \mathbf{C}\}$$

and

$$L_2 = \{(b_0 : \cdots : b_{n+1}) \mid \sum_{i=0}^{n+1} \lambda_i^s b_i = 0 \quad \text{for } s = 0, \ldots, d\},$$

respectively. Here $L_i(s_0, \ldots, s_k) = \sum_{p=0}^{k} (-\lambda_i)^p s_{k-p}$ is a linear form on $s_0, \ldots, s_k$ and the quotient map is given by the map

$$s_p \mapsto \text{the elementary symmetric function of } x_i \text{ of degree } p,$$
$$b_i \mapsto \beta_i^d.$$

On the other hand, the two linear spaces L_1 and L_2 are equal. Thus we have the proposition. Q.E.D.

Example 10. *We consider the case $n = d = 3$. In this case the Fano variety $F(X)$ becomes a surface, which is called the Fano surface of the Fermat cubic three fold X. The surface $F(X)^0$ is a $G = (\mathbf{Z}/3\mathbf{Z})^4$-covering of the moduli space $\mathcal{M}_5 = (\mathbf{A}^1 - \{0, 1\})^2 - \Delta$, where Δ is the diagonal. In this case, it is known (*[CG]*) that the Fano variety $F(X)$ is smooth. We refer to* [R] *for other properties in this case.*

§5. Singularities

In this section, we study the singular locus of $F(X)$ in the case $d = n$. Similar computations are not difficult for the case $n > d$. Let l be an element in $F(X)$. Then the divisor $l \cap (\cup_i H_i)$ defines a partition

$$P(l) = \{P_x = \{i \mid x \in H_i\} \mid x \in l\}$$

of the set $[0, n+1] = \{0, \ldots, n+1\}$. The variety $F(X)$ is stratified by the type of partition $P(l)$:

$$F(X) = \coprod_{P:\text{partition of } [0,n+1]} F(X)^P,$$

where $F(X)^P = \{l \in F(X) \mid P(l) = P\}$. For example,

$$F^0(X) = \{\{0\}, \ldots, \{n+1\}\}.$$

Let l be an element of $F(X)$ and $i_1, \dots, i_k$ be the cardinalities of the elements in $P(l)$. Then $i_1 + \cdots + i_k = n+2$. We compute the dimension of the tangent space of $F(X)$ at the point l. By changing the coordinates of the line, we may assume that the frame of l is of the form $A = (A_1, \dots, A_k)$, where

$$A_p = \begin{pmatrix} \lambda_p b_{p1} & \dots & \lambda_p b_{pi_p} \\ b_{p1} & \dots & b_{pi_p} \end{pmatrix}.$$

By changing coordinates of the line l, we may assume that $\lambda_i \neq 0$ and $b_{ij} \neq 0$ for all i and j. We consider a deformation of $A(\epsilon) = (A_1 + E_1, \dots, A_k + E_k)$, where

$$E_p = \begin{pmatrix} \epsilon\xi_{p1} & \dots & \epsilon\xi_{pi_p} \\ \epsilon\eta_{p1} & \dots & \epsilon\eta_{pi_p} \end{pmatrix},$$

with $\epsilon^2 = 0$. Since the line corresponding to the frame $A(\epsilon)$ is contained in the Fermat hyper surface, we have

$$\begin{cases} \sum_{p=1}^k \sum_{i=1}^{i_p} (\lambda_p b_{pi} + \epsilon\xi_{pi})^d = 0, \\ \sum_{p=1}^k \sum_{i=1}^{i_p} (\lambda_p b_{pi} + \epsilon\xi_{pi})^{d-1}(b_{pi} + \epsilon\eta_{pi}) = 0, \\ \vdots \\ \sum_{p=1}^k \sum_{i=1}^{i_p} (\lambda_p b_{pi} + \epsilon\xi_{pi})(b_{pi} + \epsilon\eta_{pi})^{d-1} = 0, \\ \sum_{p=1}^k \sum_{i=1}^{i_p} (b_{pi} + \epsilon\eta_{pi})^d = 0. \end{cases}$$

By looking at the coefficients of ϵ, we have
(7)

$$\begin{cases} \sum_{p=1}^k d\lambda_p^{d-1}(\sum_{i=1}^{i_p} b_{pi}^{d-1}\xi_{pi}) = 0, \\ \sum_{p=1}^k \left[(d-1)\lambda_p^{d-2}(\sum_{i=1}^{i_p} b_{pi}^{d-1}\xi_{pi}) + \lambda_p^{d-1}(\sum_{i=1}^{i_p} b_{pi}^{d-1}\eta_{pi})\right] = 0, \\ \vdots \\ \sum_{p=1}^k \left[(\sum_{i=1}^{i_p} b_{pi}^{d-1}\xi_{pi}) + (d-1)\lambda_p(\sum_{i=1}^{i_p} b_{pi}^{d-1}\eta_{pi})\right] = 0, \\ \sum_{p=1}^k d(\sum_{i=1}^{i_p} b_{pi}^{d-1}\eta_{pi}) = 0. \end{cases}$$

We define linear functions L_p and M_p by

$$L_p = \sum_{i=1}^{i_p} b_{pi}^{d-1}\xi_{pi}, \quad M_p = \sum_{i=1}^{i_p} b_{pi}^{d-1}\eta_{pi}.$$

Then L_p and M_p are independent linear functions for $\xi_0, \dots, \xi_{n+1}, \eta_0, \dots, \eta_{n+1}$. Therefore the linear equation (7) can be written as

$$N \,{}^t\begin{pmatrix} L_1 & \dots & L_k & M_1 & \dots & M_k \end{pmatrix} = 0,$$

where

$$N = \begin{pmatrix} d\lambda_1^{d-1} & \dots & d\lambda_k^{d-1} & 0 & \dots & 0 \\ (d-1)\lambda_1^{d-2} & \dots & (d-1)\lambda_k^{d-2} & \lambda_1^{d-1} & \dots & \lambda_k^{d-1} \\ \vdots & & \vdots & \vdots & & \vdots \\ 1 & \dots & 1 & (d-1)\lambda_1 & \dots & (d-1)\lambda_k \\ 0 & \dots & 0 & d & \dots & d \end{pmatrix}.$$

Lemma 11. *The rank of N is equal to* $\max(n+1, 2k)$.

Proof. We consider Vandermonde matrix for $\lambda_0, \lambda_1, \dots, \lambda_n$ and consider the limit where λ_0 tends to λ_1. We set $\lambda_0 = \lambda_1 + \epsilon$ with $\epsilon^2 = 0$. Then we have

$$\det \begin{pmatrix} 1 & 1 & \dots & 1 \\ \lambda_1 + \epsilon & \lambda_1 & \dots & \lambda_n \\ \vdots & \vdots & & \vdots \\ (\lambda_1 + \epsilon)^n & \lambda_1^n & \dots & \lambda_n^n \end{pmatrix} = \epsilon \det \begin{pmatrix} 0 & 1 & \dots & 1 \\ 1 & \lambda_1 & \dots & \lambda_n \\ 2\lambda_1 & \lambda_1^2 & \dots & \lambda_n^2 \\ \vdots & \vdots & & \vdots \\ n\lambda_1^{n-1} & \lambda_1^n & \dots & \lambda_n^n \end{pmatrix}.$$

It is equal to

$$-\epsilon \cdot \prod_{1 \le i < j \le n} (\lambda_j - \lambda_i) \cdot \prod_{2 \le i} (\lambda_i - \lambda_1).$$

Using this procedure, we can prove that the determinant

$$\det \begin{pmatrix} 0 & \dots & 0 & 1 & \dots & 1 \\ 1 & \dots & 1 & \lambda_1 & \dots & \lambda_q \\ 2\lambda_1 & \dots & 2\lambda_p & \lambda_1^2 & \dots & \lambda_q^2 \\ \vdots & & \vdots & \vdots & & \vdots \\ n\lambda_1^{n-1} & \dots & n\lambda_p^{n-1} & \lambda_1^n & \dots & \lambda_q^n \end{pmatrix}$$

is non-zero if $p+q = n+1$ and $p \le q$. Thus we have the lemma. Q.E.D.

Definition 3. *Let $P(l)$ be the partition and $i_1, \dots, i_k$ be the cardinalities of the elements in $P(l)$. The number $2k$ is called the rank of l and is denoted by $rk(l)$.*

By the computation of the dimension of the tangent space of $F(X)$ at l, we have the following theorem.

Theorem 12. *Assume that $n = d$. The point $l \in F(X)$ is singular if and only if $rk(l) \le n$.*

Corollary 13. *Assume that $d = n$. The variety $F(X)$ is singular if and only if $4 \le d$.*

§6. Period integrals for Fano varieties

6.1. Arrangement of hyperplanes

We apply Theorem 4 to study period integrals for Fano varieties of Fermat hypersurfaces. We study the case where $n = d$. Then $\widehat{\mathcal{M}_{0,n+2}} \to \widetilde{\mathcal{M}_{0,n+2}}$ is a $(\mathbf{Z}/d\mathbf{Z})^{n+1}$-covering. We take the quotient by $GPL(2)$ and normalize $\lambda_0, \dots, \lambda_{n+1}, \lambda_{n+2}$ so that

$$\lambda_0 = \infty, \lambda_1 = 0, \lambda_2 = 1.$$

The normalized parameter will be written as $\mu_3, \dots, \mu_{n+1}$. Therefore

$$\mu_p = (\lambda_p, \lambda_1; \lambda_2, \lambda_0)$$

for $p = 3, \dots, n+1$. Here we used the notation of the cross ratio

$$(x, y; z, w) = \frac{(x-y)(z-w)}{(x-w)(z-y)}.$$

Therefore $\mathcal{M}_{0,n+2} = \widetilde{\mathcal{M}_{0,n+2}}/PGL(2)$ is isomorphic to

$$\{(\mu_3, \dots, \mu_{n+1}) \mid \mu_i \neq 0, 1, \ \mu_i \neq \mu_j \text{ for } i \neq j\}.$$

We set

$$\Delta_1 = \prod_{i=3}^{n+1} \mu_i = \xi_1^{-d} \left(\frac{\lambda_2 - \lambda_0}{\lambda_2 - \lambda_1} \right)^d,$$

$$\Delta_2 = \prod_{i=3}^{n+1} (1 - \mu_i) = \xi_2^{-d} \left(\frac{\lambda_0 - \lambda_1}{\lambda_2 - \lambda_1} \right)^d,$$

$$\begin{aligned} \Delta_i &= \mu_i(\mu_i - 1) \prod_{j \neq i, 3 \leq j \leq n+1} (\mu_i - \mu_j) \\ &= \xi_i^{-d} \left(\frac{(\lambda_0 - \lambda_1)(\lambda_2 - \lambda_0)}{(\lambda_j - \lambda_0)(\lambda_2 - \lambda_1)} \right)^d, \qquad \text{for } i = 3, \dots, n+1. \end{aligned}$$

Then the covering $F^0(X) \simeq \widehat{\mathcal{M}_{0,n+2}}/PGL(2)$ over $\mathcal{M}_{0,n+2}$ is defined by

$$\eta_i^d = \Delta_i \text{ for } i = 1, \dots, n+1. \tag{8}$$

Therefore $F^0(X)$ is an open set of a ramified covering of $\mathbf{A}^{n-1} = \{\mu_3, \dots, \mu_{n+1}\}$ branching at the arrangement of hyperplanes defined by

$$\mathcal{B}_n = \{\mu_i, 1 - \mu_i\}_{3 \leq i \leq n+1} \cup \{\mu_i - \mu_j\}_{3 \leq i < j \leq n+1}.$$

It is called the Selberg arrangement.

6.2. Selberg integral

Using the main theorem, we can show that the period integrals are written down using certain kind of Selberg integrals. The subjects in this subsection are compilations of well known facts, which might be useful for readers. Let $\pi : F^0(X) \to \mathcal{M}_{0,n+2}$ be the covering introduced in the main theorem, G be the Galois group of π, and $K = \mathbf{Q}(\mu_d) \subset \mathbf{C}$. Let

$$G^* = \mathrm{Ker}((\mathbf{Z}/d\mathbf{Z})^{n+2} \xrightarrow{\Sigma} (\mathbf{Z}/d\mathbf{Z}))$$

be the character group of G and $\mathcal{L}(\chi)$ be the χ-part of the direct image sheaf $\pi_*\mathbf{K}$ of $\mathbf{K}$ under the map π. The i-th singular cohomology with the local coefficient $\mathcal{L}(\chi)$ is denoted by $H(\mathcal{M}_{0,n+2}, \mathcal{L}(\chi))$. Then we have a decomposition

$$H^i(F^0(X), \mathbf{K}) \simeq \oplus_{\chi \in G^*} H^i(\mathcal{M}_{0,n+2}, \mathcal{L}(\chi)).$$

Let $j : \mathcal{M}_{0,n+2} \to \mathbf{A}^{n-1}$ be the open immersion defined by $\{\mu_i\}_{3 \leq i \leq n+1}$ and set $\partial\mathcal{M}_{0,n+2} = \mathbf{A}^{n-1} - \mathcal{M}_{0,n+2}$. Let $j_!\mathcal{L}(\chi)$ be the zero extension of the local system $\mathcal{L}(\chi)$ to $\mathbf{A}^{n-1}$. Then we have a natural homomorphism

$$\varphi^i(\chi) : H^i(\mathbf{A}^{n-1}, j_!\mathcal{L}(\chi)) \to H^i(\mathcal{M}_{0,n+2}, \mathcal{L}(\chi)). \tag{9}$$

Since $F^0(X)$ is an affine variety, the algebraic de Rham cohomology $H^i(F^0(X)/\mathbf{C})$ is the cohomology of the following complex

$$0 \to \Gamma(\mathcal{O}_{F^0(X)}) \to \Gamma(\Omega^1_{F^0(X)}) \to \cdots \to \Gamma(\Omega^{n-1}_{F^0(X)}) \to 0.$$

By de Rham Theorem and taking χ-part, we have an isomorphism

$$H^{n-1}(F^0(X)/\mathbf{C})(\chi) \xrightarrow{\simeq} H^{n-1}(\mathcal{M}_{0,n+2}, \mathcal{L}(\chi)) \otimes_{\mathbf{K}} \mathbf{C}.$$

We construct an element $\widetilde{\Delta_\sigma}$ in the relative homology $H_{n-1}(\mathbf{A}^{n-1}; \partial\mathcal{M}_{0,n+2}, \mathcal{L}(\chi))$ which can be identified with the dual vector space of $H^{n-1}(\mathbf{A}^{n-1}, j_!\mathcal{L}(\chi))$. Let $\mathfrak{S}[3, n+1]$ be the symmetric group of $[3, n+2]$. For an element $\sigma \in \mathfrak{S}[3, n+1]$, we define a simplex $\Delta_\sigma \subset \mathbf{R}^{n-1}$ by

$$\Delta_\sigma = \{0 \leq \mu_{\sigma(3)} \leq \cdots \leq \mu_{\sigma(n+1)} \leq 1 \mid \mu_i \in \mathbf{R} \text{ for } 3 \leq i \leq n+1\}.$$

By choosing a section of $\mathcal{L}(\chi)$ over Δ_σ, we have a chain $\widetilde{\Delta}_\sigma$ with value in $\mathcal{L}(\chi)$, which defines an element in $H_{n-1}(\mathbf{A}^{n-1}; \partial\mathcal{M}_{0,n+2}, \mathcal{L}(\chi))$.

Let $\eta_1, \ldots, \eta_{n+1}$ be rational functions defined in (8). Then a character χ of G can be written as the Kummer character of $f = \prod_{i=1}^{n+1} \eta_i^{\chi_i}$ with $0 \leq \chi_i < d$. Then the differential forms

$$\omega = \prod_{i=1}^{n+1} \eta^{\gamma_i} d\mu_3 \wedge \cdots \wedge d\mu_{n+1}, \quad \gamma_i \equiv \chi_i (\mathrm{mod}\ d), \gamma_i \geq 0 \text{ for } i = 1, \ldots, n+1$$

define elements in the image of φ^{n-1} of (9) in the space $H^{n-1}(F^0(X)/\mathbf{C})(\chi)$. By choosing a suitable choice of branch in the integrand, we have

$$(\widetilde{\Delta}_\sigma, \omega) = \int_{0<\mu_{\sigma(3)}<\cdots<\mu_{\sigma(n+1)}<1} \Delta_i^{\frac{\gamma_i}{d}} d\mu_3 \cdots d\mu_{n+1}.$$

The integral in the right hand side is called a Selberg integral.

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