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Enumerative geometry of Calabi–Yau 5-folds

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Abstract.

Gromov–Witten theory is used to define an enumerative geometry of curves in Calabi–Yau 5-folds. We find recursions for meeting numbers of genus 0 curves, and we determine the contributions of moving multiple covers of genus 0 curves to the genus 1 Gromov–Witten invariants. The resulting invariants, conjectured to be integral, are analogous to the previously defined BPS counts for Calabi–Yau 3 and 4-folds. We comment on the situation in higher dimensions where new issues arise.

Two main examples are considered: the local Calabi–Yau $\mathbb{P}^2$ with normal bundle $\oplus_{i=1}^3 \mathcal{O}(-1)$ and the compact Calabi–Yau hypersurface $X_7 \subset \mathbb{P}^6$. In the former case, a closed form for our integer invariants has been conjectured by G. Martin. In the latter case, we recover in low degrees the classical enumeration of elliptic curves by Ellingsrud and Strömme.

§0. Introduction

0.1. Overview

Let X be a nonsingular projective variety over $\mathbb{C}$. Let $\overline{\mathfrak{M}}_{g,k}(X,\beta)$ be the moduli space of genus g, k pointed stable maps to X representing the class $\beta \in H_2(X,\mathbb{Z})$. Let

$$\mathrm{ev}_i\colon \overline{\mathfrak{M}}_{g,k}(X,\beta) \longrightarrow X$$

be the evaluation morphism at the i^{th} marking. The Gromov–Witten theory of primary fields concerns the invariants

$$(0.1) \qquad N_{g,\beta}(\gamma_1,\ldots,\gamma_k) = \int_{[\overline{\mathfrak{M}}_{g,k}(X,\beta)]^{vir}} \prod_{i=1}^{k} \mathrm{ev}_i^*(\gamma_i) \ \in \mathbb{Q},$$

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where $\gamma_i \in H^*(X, \mathbb{Z})$. The relationship between the Gromov–Witten invariants and the actual enumerative geometry of curves in X is subtle. An overview of the subject in low dimensions can be found in the introduction of [10].

For Calabi–Yau 3-folds, the Aspinwall–Morrison formula [1] is conjectured to produce integer invariants in genus 0. A full integrality conjecture for the Gromov–Witten theory of Calabi–Yau 3-folds was formulated by Gopakumar and Vafa in [5, 6] in terms of BPS states with geometric motivation partially provided by [14]. The Aspinwall–Morrison prediction has been extended to all Calabi–Yau n-folds in [10]: the numbers $n_{0,\beta}(\gamma_1, \ldots, \gamma_k)$ defined by

$$\sum_{\beta \neq 0} N_{0,\beta}(\gamma_1, \ldots, \gamma_k) q^\beta = \sum_{\beta \neq 0} n_{0,\beta}(\gamma_1, \ldots, \gamma_k) \sum_{d=1}^{\infty} \frac{1}{d^{3-k}} q^{d\beta} \tag{0.2}$$

are conjectured to be integers.

Let X be a Calabi–Yau of dimension $n \geq 4$. Since Gromov–Witten invariants of genus $g \geq 2$ of X vanish for dimensional reasons, only integrality predictions for genus 1 invariants of X remain to be considered. The analogue of the genus 1 Gopakumar–Vafa integrality prediction for Calabi–Yau 4-folds has been formulated in [10]. Here, we find complete formulas in dimension 5 and reinterpret the dimension 4 predictions. The geometry becomes significantly more complicated in each dimension. We discuss new aspects of the higher dimensional cases.

The relationship between Gromov–Witten theory and enumerative geometry in dimensions greater than 3 is simplest in the Calabi–Yau case. The Fano case, even in dimension 4, involves complicated higher genus phenomena which have not yet been understood.

0.2. Elliptic invariants

If X is Calabi–Yau, the virtual moduli cycle for $\overline{\mathfrak{M}}_1(X, \beta)$ is of dimension 0. We denote the associated Gromov–Witten invariant by $N_{1,\beta}$,

$$N_{1,\beta} = \int_{[\overline{\mathfrak{M}}_1(X,\beta)]^{vir}} 1 \; \in \mathbb{Q}.$$

Integrality predictions for Calabi–Yau n-folds are obtained by relating curve counts to Gromov–Witten invariants in an ideal Calabi–Yau X. All genus 1 curves in X are assumed to be nonsingular, super-rigid[1],

[1] A nonsingular curve $E \subset X$ with normal bundle $\mathcal{N}_E$ is super-rigid if, for every dominant stable map $f : C \to E$, the vanishing $H^0(C, f^*\mathcal{N}_E) = 0$ holds.

and disjoint from other curves. Each genus 1 degree β curve then contributes $\sigma(d)/d$ to $N_{1,d\beta}$ for every $d \in \mathbb{Z}^+$ via étale covers, where

$$\sigma(d) = \sum_{i|d} i.$$

The genus 1 to genus 1 multiple cover contribution is independent of dimension.

If X is an ideal Calabi–Yau 3-fold, the genus 0 curves in X are also nonsingular, super-rigid, and disjoint. The contribution of a genus 0 degree β curve to $N_{1,d\beta}$ is then the integral of an Euler class of an obstruction bundle on $\overline{\mathfrak{M}}_1(\mathbb{P}^1, d)$,

$$\int_{[\overline{\mathfrak{M}}_1(\mathbb{P}^1,d)]^{vir}} e(\mathrm{Obs}) = \frac{1}{12d},$$

calculated in [14]. Thus, if X is an ideal Calabi–Yau 3-fold,

$$\sum_{\beta \neq 0} N_{1,\beta} q^\beta = \sum_{\beta \neq 0} n_{1,\beta} \sum_{d=1}^{\infty} \frac{\sigma(d)}{d} q^{d\beta} - \frac{1}{12} \sum_{\beta \neq 0} n_{0,\beta} \log(1 - q^\beta)\,, \tag{0.3}$$

where the enumerative invariant $n_{1,\beta}$ is defined by (0.3) and the genus 0 invariant $n_{0,\beta}$ is defined by the Aspinwall–Morrison formula (0.2). The invariants $n_{1,\beta}$ are then conjectured to be integers for all Calabi–Yau 3-folds.

If X is an ideal Calabi–Yau 4-fold, embedded genus 0 degree β curves in X form a nonsingular, compact, 1-dimensional family $\overline{\mathcal{M}}_\beta$. The moving multiple cover calculation of Section 2 of [10] shows that $\overline{\mathcal{M}}_\beta$ contributes $\chi(\overline{\mathcal{M}}_\beta)/24d$ to $N_{1,d\beta}$ for every $d \in \mathbb{Z}^+$. The calculation is done in two steps. First, the moving multiple cover integral is done assuming every genus 0 degree β curve is nonsingular. Second, the contribution from the nodal curves is determined for a particular, but sufficiently representative, Calabi–Yau 4-fold X by localization. For an ideal Calabi–Yau 4-fold X,

$$\sum_{\beta \neq 0} N_{1,\beta} q^\beta = \sum_{\beta \neq 0} n_{1,\beta} \sum_{d=1}^{\infty} \frac{\sigma(d)}{d} q^{d\beta} - \frac{1}{24} \sum_{\beta \neq 0} \chi(\overline{\mathcal{M}}_\beta) \log(1 - q^\beta)\,. \tag{0.4}$$

The topological Euler characteristic $\chi(\overline{\mathcal{M}}_\beta)$ is determined by

$$\chi(\overline{\mathcal{M}}_\beta) = -n_{0,\beta}(c_2(X)) + \sum_{\beta_1+\beta_2=\beta} m_{\beta_1,\beta_2},$$

where m_{β_1,β_2} is the number of ordered pairs $(\mathcal{C}_1, \mathcal{C}_2)$ of rational curves of classes β_1 and β_2 meeting at point, see Section 1.2 of [10].

The meeting numbers m_{β_1,β_2} can be expressed in terms of the invariants $n_{0,\beta}(\gamma)$ through a recursion on the total degree $\beta_1+\beta_2$ by computing the excess contribution to the topological Kunneth decomposition of $m_{\beta_1\beta_2}$, see Sections 0.3 and 1.2 of [10]. Along with these recursions, relations (0.2) and (0.4) effectively determine the numbers $n_{1,\beta}$ in terms of the genus 0 and genus 1 Gromov–Witten invariants of X. For arbitrary Calabi–Yau 4-folds, equation (0.4) is taken to be the definition of the numbers $n_{1,\beta}$ which are conjectured always to be integers.

If X is an ideal Calabi–Yau 5-fold, embedded genus 0 degree β curves in X form a nonsingular, compact, 2-dimensional family $\overline{\mathcal{M}}_\beta$. However, as the nodal curves are more complicated, the localization strategy of [10] does not appear possible. By viewing $N_{1,d\beta}$ as the number of solutions, counted with appropriate multiplicities, of a perturbed $\bar\partial$-equation as in [4, 11], we show in Section 2 that $\overline{\mathcal{M}}_\beta$ contributes

$$\frac{1}{24d}\int_{\overline{\mathcal{M}}_\beta}\big(2c_2(\overline{\mathcal{M}}_\beta)-c_1^2(\overline{\mathcal{M}}_\beta)\big)$$

to $N_{1,d\beta}$ for every $d\in\mathbb{Z}^+$. Thus, for an ideal Calabi–Yau 5-fold X,

$$\begin{aligned}\sum_{\beta\neq 0} N_{1,\beta}q^\beta &= \sum_{\beta\neq 0} n_{1,\beta}\sum_{d=1}^{\infty}\frac{\sigma(d)}{d}q^{d\beta}\\ &\quad -\frac{1}{24}\sum_{\beta\neq 0}\int_{\overline{\mathcal{M}}_\beta}\big(2c_2(\overline{\mathcal{M}}_\beta)-c_1^2(\overline{\mathcal{M}}_\beta)\big)\cdot\log(1-q^\beta)\,.\end{aligned} \tag{0.5}$$

The last term in (0.5) may be written in terms of various meeting numbers of total degree β via a Grothendieck–Riemann–Roch computation applied to the deformation characterization of the tangent bundle $T\overline{\mathcal{M}}_\beta$. We pursue a more efficient strategy in Sections 1 and 2. Degree 1 maps from genus 0 curves to degree β curves in X are regular. Thus, equation (2.15) in [23] expresses their contribution to $N_{1,\beta}$ in terms of counts of m-tuples of 1-marked curves with cotangent ψ-classes meeting at the marked point. The ψ-classes can be easily eliminated using the topological recursion relation at the cost of introducing counts of arbitrary meeting configurations of rational curves in X. The latter can be recursively defined as in the case of $m_{\beta 1,\beta 2}$ in dimension 4. Relations (0.2) and (0.5) then reduce the numbers $n_{1,\beta}$ to functions of genus 0 and genus 1 Gromov–Witten invariants.

Let X be an arbitrary Calabi–Yau 5-fold. Equation (0.5) together with the rules provided in Sections 1 and 2 for the calculation of

$$\int_{\overline{\mathcal{M}}_\beta} (2c_2(\overline{\mathcal{M}}_\beta) - c_1^2(\overline{\mathcal{M}}_\beta))$$

in terms of the Gromov–Witten invariants of X *define* the invariants $n_{1,\beta}$. We view $n_{1,\beta}$ as virtually enumerating elliptic curves in X.

Conjecture 1. *For all Calabi–Yau 5-folds X and curve classes $\beta \neq 0$, the invariants $n_{1,\beta}$ are integers.*

0.3. Examples

If the Gromov–Witten invariants of X are known, equation (0.5) provides an effective determination of the elliptic invariants $n_{1,\beta}$. We consider two representative examples.

The most basic local Calabi–Yau 5-fold is the total space of the bundle

$$\mathcal{O}(-1) \oplus \mathcal{O}(-1) \oplus \mathcal{O}(-1) \longrightarrow \mathbb{P}^2. \tag{0.6}$$

The balanced property of the bundle is analogous to the fundamental local Calabi–Yau 3-fold

$$\mathcal{O}(-1) \oplus \mathcal{O}(-1) \longrightarrow \mathbb{P}^1.$$

As in the 3-fold case, we find very simple closed forms in Section 3.1 for the genus 0 and 1 Gromov–Witten invariants of the local Calabi–Yau 5-fold (0.6).

We have computed the invariants $n_{1,d}$ via equation (0.5) up to degree 200. All are integers. Even the first 60, shown in Table 0.3, suggest intriguing patterns. For example, $n_{1,d} = 0$ for all multiples of 8. G. Martin has proposed an explicit formula for $n_{1,d}$ which holds for all the numbers we have computed. We state Martin's conjecture in Section 3.2.

The Calabi–Yau septic hypersurface $X_7 \subset \mathbb{P}^6$ is a much more complicated example. Using the closed formulas for the genus 1 and 2-pointed genus 0 Gromov–Witten invariants provided by [23] and [22] respectively, we have computed $n_{1,d}$ for $d \leq 100$. All are integers. The values of $n_{1,d}$ for $d \leq 10$ are shown in Table 0.3.

The invariants $n_{1,d}$ for $d \leq 4$ agree with known enumerative results for X_7. The invariants $n_{1,1}$ and $n_{1,2}$ vanish by geometric considerations. Since every genus 1 curve of degree 3 in $\mathbb{P}^6$ is planar, the number of elliptic cubics on a general X_7 can be computed classically via Schubert

d	$n_{1,d}$	d	$n_{1,d}$	d	$n_{1,d}$	d	$n_{1,d}$	d	$n_{1,d}$	d	$n_{1,d}$
1	0	11	-225	21	3025	31	-14400	41	-44100	51	105625
2	0	12	-19	22	3870	32	0	42	-51590	52	-7119
3	-1	13	-441	23	-4356	33	18496	43	-53361	53	-123201
4	0	14	630	24	0	34	22140	44	-3645	54	0
5	-9	15	784	25	0	35	23409	45	0	55	142884
6	20	16	0	26	7560	36	0	46	74250	56	0
7	-36	17	-1296	27	0	37	-29241	47	-76176	57	164836
8	0	18	0	28	-594	38	34560	48	0	58	187740
9	0	19	-2025	29	-11025	39	36100	49	0	59	-189225
10	162	20	-153	30	-13412	40	0	50	0	60	12628

Table 1. Invariants $n_{1,d}$ for $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \oplus \mathcal{O}(-1) \longrightarrow \mathbb{P}^2$

calculus. The classical calculation agrees with $n_{1,3}$. Using the expression of non-planar genus 1 curves of degree 4 as complete intersections of quadrics, Ellingsrud and Strömme have enumerated elliptic quartics on X_7 in Theorem 1.3 of [3]. The result agrees with $n_{1,4}$. To our knowledge, the numbers $n_{1,d}$ are inaccessible by classical techniques for $d \geq 5$.

0.4. BPS states

The integer expansion (0.5) can be alternatively written as

$$\begin{aligned}\sum_{\beta\neq 0} N_{1,\beta} q^\beta &= -\sum_{\beta\neq 0} \tilde{n}_{1,\beta} \cdot \log(1-q^\beta) \\ &\quad - \frac{1}{24} \sum_{\beta\neq 0} \int_{\overline{\mathcal{M}}_\beta} \big(2c_2(\overline{\mathcal{M}}_\beta) - c_1^2(\overline{\mathcal{M}}_\beta)\big) \cdot \log(1-q^\beta)\,. \end{aligned} \tag{0.7}$$

The integrality condition for the invariants $\tilde{n}_{1,\beta}$ is equivalent to the conjectured integrality for $n_{1,\beta}$. We view the invariants $\tilde{n}_{1,\beta}$ as analogous to the BPS state counts in dimensions 3 and 4.

0.5. Higher dimensions

The family $\overline{\mathcal{M}}_\beta$ of embedded genus 0 degree β curves in X is nonsingular and compact for ideal Calabi–Yau n-folds for $n=3,4,5$. The moving multiple cover results for $n=3,4,5$ can be summarized by the following equation. The contribution of $\overline{\mathcal{M}}_\beta$ to the genus 1 degree $d\beta$ Gromov–Witten invariant is

$$\mathsf{C}_\beta(d\beta) = \frac{1}{24d} \int_{\overline{\mathcal{M}}_\beta} \big(2c_{n-3}(\overline{\mathcal{M}}_\beta) - c_1(\overline{\mathcal{M}}_\beta) c_{n-4}(\overline{\mathcal{M}}_\beta)\big)\,. \tag{0.8}$$

For dimension 6 and higher, the family of embedded genus 0 degree β curves in X is not compact (multiple covers can occur as limits) even in

d	$n_{1,d}$
1	0
2	0
3	26123172457235
4	81545482364153841075
5	117498479295762788677099464
6	126043741686161819224278666855602
7	1172934624228244311229748659336872062 94
8	1009452959553443758790412274821747352135466 36
9	828985893486136257123874729446895764032159698397 72
10	660741465833356418077455400883338572507725675268489515 26

Table 2. Invariants $n_{1,d}$ for a degree 7 hypersurface in $\mathbb{P}^6$

ideal cases. Nevertheless, we expect a contribution equation of the form of (0.8) to hold. The result should yield integrality predictions in higher dimensions.

Since the complexity of the Gromov–Witten approach increases so much in every dimension, an alternate method for dimensions 6 and higher is preferable. It is hoped a connection to newer sheaf enumeration and derived category techniques will be made [15, 16].

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§1. Genus 0 invariants

1.1. Configuration spaces of genus 0 curves

Let X be a Calabi–Yau 5-fold. We specify here what conditions an ideal X is to satisfy with respect to genus 0 curves. We denote by

$$H_+(X) \subset H_2(X, \mathbb{Z}) - 0$$

the cone of effective curve classes. If $\beta, \beta' \in H_+(X)$, we write $\beta' < \beta$ if $\beta - \beta'$ is an element of $H_+(X)$.

If J is a finite set and $\beta \in H_+(X)$, we denote by $\overline{\mathfrak{M}}_{0,J}(X,\beta)$ the moduli space of genus 0, J-marked stable maps to X representing the class β. For $j \in J$, let

$$L_j \longrightarrow \overline{\mathfrak{M}}_{0,J}(X,\beta)$$

be the universal tangent line bundle at the jth marked point. Denote by

$$\mathcal{D}_j \in \Gamma\big(\overline{\mathfrak{M}}_{0,J}(X,\beta), \mathrm{Hom}(L_j, \mathrm{ev}_j^* TX)\big)$$

the bundle section induced by the differential of the stable maps at the j^{th} marked point.

If Σ is a curve, a map $u\colon \Sigma \longrightarrow X$ is called *simple* if u is injective on the complement of finitely many points and of the components of Σ on which u is constant. We will call a tuple $(u_1, \ldots, u_m)$ of maps $u_i\colon \Sigma_i \longrightarrow X$ *simple* if the map

$$\bigsqcup_{i=1}^{m} \Sigma_i \longrightarrow X, \qquad z \longrightarrow u_i(z) \text{ if } z \in \Sigma_i,$$

is simple. If J is a finite set and $\beta \in H_+(X)$, let

$$\mathfrak{M}^*_{0,J}(X,\beta) \subset \overline{\mathfrak{M}}_{0,J}(X,\beta)$$

be the open subspace of stable maps $[\Sigma, u]$ such that Σ is a $\mathbb{P}^1$ and u is a simple map.

If J_1 and J_2 are two finite sets and $\beta_1, \beta_2 \in H_+(X)$, we denote by

$$\mathfrak{M}^*_{0,(J_1,J_2)}\big(X,(\beta_1,\beta_2)\big) \subset$$
$$\big\{(b_1,b_2) \in \mathfrak{M}^*_{0,\{0\}\sqcup J_1}(X,\beta_1) \times \mathfrak{M}^*_{0,\{0\}\sqcup J_2}(X,\beta_2) \colon \mathrm{ev}_0(b_1) = \mathrm{ev}_0(b_2)\big\}$$

the subset of simple pairs of maps. Similarly, if $\beta_1, \beta_2, \beta_3 \in H_+(X)$, let

$$\mathfrak{M}^*_{0,\emptyset}\big(X,(\beta_1,\beta_2,\beta_3)\big) \subset$$
$$\big\{(b_1,b_2,b_3) \in \mathfrak{M}^*_{0,(\emptyset,\{1\})}(X,(\beta_1,\beta_2)) \times \mathfrak{M}^*_{0,\{0\}}(X,\beta_3) \colon \mathrm{ev}_1(b_2) = \mathrm{ev}_0(b_3)\big\}$$

be the subset of simple triples of maps. If X is an ideal Calabi–Yau 5-fold satisfying Conditions 1 and 2 below, there are no other configurations of simple genus 0 curves in X, see Figure 1.

Denote by $\overline{\mathfrak{M}}^*_{0,J}(X,\beta) \subset \overline{\mathfrak{M}}_{0,J}(X,\beta)$ and

$$\overline{\mathfrak{M}}^*_{0,(J_1,J_2)}\big(X,(\beta_1,\beta_2)\big) \subset \overline{\mathfrak{M}}_{0,\{0\}\sqcup J_1}(X,\beta_1) \times \overline{\mathfrak{M}}_{0,\{0\}\sqcup J_2}(X,\beta_2),$$

the closures of $\mathfrak{M}^*_{0,J}(X,\beta)$ and $\mathfrak{M}^*_{0,(J_1,J_2)}\big(X,(\beta_1,\beta_2)\big)$. Let

(1.1)
$$\pi_1, \pi_2 \colon \overline{\mathfrak{M}}^*_{0,(J_1,J_2)}\big(X,(\beta_1,\beta_2)\big) \to \overline{\mathfrak{M}}_{0,\{0\}\sqcup J_1}\big(X,\beta_1\big), \overline{\mathfrak{M}}_{0,\{0\}\sqcup J_2}\big(X,\beta_2\big),$$

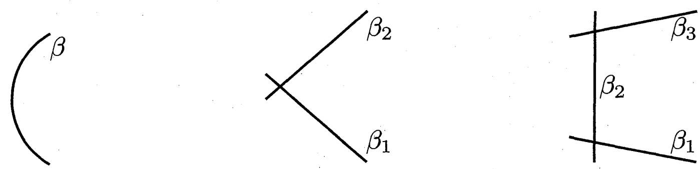


Fig. 1. The three possible configurations of rational curves in an ideal Calabi–Yau 5-fold. The label next to each component indicates the degree.

be the component projection maps.

Condition 1. *If* $u:\mathbb{P}^1\longrightarrow X$ *is a simple holomorphic map,* $H^1(\mathbb{P}^1,u^*TX)=0$.

By Condition 1, $\mathfrak{M}^*_{0,J}(X,\beta)$ is a nonsingular variety of the expected dimension $2+|J|$.

Condition 2. *For all* $\beta_1,\ldots,\beta_k\in H_+(X)$, *finite sets* $J_1,\ldots,J_k$, *and a partition of* $J_1\sqcup\ldots\sqcup J_k$ *into nonempty disjoint subsets* $I_1,\ldots,I_m$, *the restriction of the total evaluation map*[2]

$$\mathrm{ev}\colon \prod_{p=1}^k \mathfrak{M}^*_{0,J_p}(X,\beta_p)\longrightarrow \prod_{p=1}^k X^{J_p},$$
$$\mathrm{ev}\big((b_p)_{p\in[k]}\big)_{(p,j)}=\mathrm{ev}_j(b_p),\quad \forall\, p\in[k],\ j\in J_p,$$

to the open subspace of simple tuples is transverse to the diagonal

$$\big\{(x_{(p,j)})_{p\in[k],j\in J_p}: x_{(p,j)}=x_{(p',j')}\ \textit{if}\ (p,j),(p',j')\in I_q\ \textit{for some}\ q\big\}.$$

By Condition 2, $\mathfrak{M}^*_{0,(\emptyset,\emptyset)}\big(X,(\beta_1,\beta_2)\big)$ and $\mathfrak{M}^*_{0,\emptyset}\big(X,(\beta_1,\beta_2,\beta_3)\big)$ are nonsingular of dimensions 1 and 0, respectively. Furthermore, all simple genus 0 maps with reducible domains deform to curves with nonsingular domains. Furthermore, for all $\beta\in H_+(X)$, the open subspace of $\overline{\mathfrak{M}}_{0,J}(X,\beta)$ consisting of simple maps is nonsingular.

Condition 3. *For all* $\beta\in H_+(X)$, *the restriction of the bundle section* $\mathcal{D}_1$ *to* $\mathfrak{M}^*_{0,1}(X,\beta)$ *is transverse to the zero set. For all* $\beta_1,\beta_2\in H_+(X)$, *the bundle section*

$$\pi_1^*\mathcal{D}_0+\pi_2^*\mathcal{D}_0\in\Gamma\big(\mathbb{P}(\pi_1^*L_0\oplus\pi_2^*L_0)\big|_{\mathfrak{M}^*_{0,(\emptyset,\emptyset)}(X,(\beta_1,\beta_2))},\mathrm{Hom}(\gamma,\mathrm{ev}_0^*TX)\big),$$

[2]We denote the elements of $J_1\sqcup\ldots\sqcup J_k$ by pairs (p,j), where $j\in J_p$, and let $[k]=\{1,2,\ldots,k\}$.

where $\gamma \longrightarrow \mathbb{P}(\pi_1^* L_0 \oplus \pi_2^* L_0)$ *is the tautological line bundle, is transverse to the zero set.*

By Condition 1 and the first part of Condition 3, every simple holomorphic map $u\colon \mathbb{P}^1 \longrightarrow X$ is an immersion. By Condition 2, u is injective. Thus, every irreducible genus 0 curve $C \subset X$ is nonsingular. The normal bundle to such a curve must split as

$$\mathcal{N} = \mathcal{O}(a_1) \oplus \mathcal{O}(a_2) \oplus \mathcal{O}(a_3) \oplus \mathcal{O}(a_4) \to \mathbb{P}^1, \text{with } a_i \in \mathbb{Z}, \sum_{i=1}^{i=4} a_i = -2, \ a_i \geq -1,$$

the last restriction follows from Condition 1. By the first part of Condition 4 below, $a_i \in \{0, -1\}$ for all i. The second part of Condition 3 implies that every node of a reducible genus 0 curve in X is simple.

Condition 4. *For all* $\beta \in H_+(X)$, *the bundle section*

$$d\mathrm{ev}_1 \in \Gamma\big(\mathbb{P}(T\mathfrak{M}_{0,1}^*(X,\beta)), \mathrm{Hom}(\gamma, \mathrm{ev}_1^* TX)\big),$$

where $\gamma \longrightarrow \mathbb{P}(T\mathfrak{M}_{0,1}^*(X,\beta))$ *is the tautological line bundle, is transverse to the zero set. For all* $\beta_1, \beta_2 \in H_+(X)$, *the bundle section*

$$\begin{aligned} &\pi_1^* d\mathrm{ev}_0 + \pi_2^* \mathcal{D}_0 \in \\ &\Gamma\big(\mathbb{P}(\pi_1^* T\mathfrak{M}_{0,\{0\}}^*(X,\beta_1) \oplus \pi_2^* L_0)\big|_{\mathfrak{M}_{0,(\emptyset,\emptyset)}^*(X,(\beta_1,\beta_2))}, \mathrm{Hom}(\gamma, \mathrm{ev}_0^* TX)\big), \end{aligned}$$

where $\gamma \longrightarrow \mathbb{P}(\pi_1^* T\mathfrak{M}_{0,\{0\}}^*(X,\beta_1) \oplus \pi_2^* L_0)$ *is the tautological line bundle, is transverse to the zero set.*

By Condition 4, neither of the two bundle sections vanishes anywhere. In the case of the first bundle section, the dimension of the base space and the rank of the vector bundle both equal 5. On the other hand, the vanishing of the bundle section here implies the differential of the evaluation map

$$\mathrm{ev}_1 \colon \mathfrak{M}_{0,1}^*(X,\beta) \longrightarrow X$$

is not injective at some simple, degree β, 1-marked map $[\mathbb{P}^1, x_1, u]$. Hence, the normal bundle must split as

$$\mathcal{N} \approx \mathcal{O}(1) \oplus \mathcal{O}(-1) \oplus \mathcal{O}(-1) \oplus \mathcal{O}(-1).$$

Therefore $d\mathrm{ev}_1$ is not injective at $[\mathbb{P}^1, x, u]$ for all $x \in \mathbb{P}^1$. The zero set of the first bundle section in Condition 4 must be at least of dimension one. So by transversality, no vanishing is possible.

The non-vanishing of the second bundle section is clear from transversality since the base space is of dimension 4 and bundle is of rank 5.

Lemma 1.1. *Let X be an ideal Calabi–Yau 5-fold. If $\beta \in H_+(X)$ and J is a finite set, the space $\overline{\mathfrak{M}}^*_{0,J}(X,\beta)$ is nonsingular of dimension $2+|J|$ and consists of simple maps. Furthermore, the evaluation map*

$$\mathrm{ev}_1 : \overline{\mathfrak{M}}^*_{0,1}(X,\beta) \longrightarrow X$$

is an immersion. If $\beta_1, \beta_2 \in H_+(X)$ and J_1, J_2 are finite sets, $\overline{\mathfrak{M}}^*_{0,(J_1,J_2)}\big(X,(\beta_1,\beta_2)\big)$ is smooth of dimension $1+|J_1|+|J_2|$ and consists of simple maps.

Proof. By Condition 4, the restriction of ev_1 to the open subset

$$\mathfrak{M}^*_{0,J}(X,\beta) \subset \overline{\mathfrak{M}}^*_{0,J}(X,\beta)$$

is an immersion for every $\beta \in H_+(X)$. Therefore, by the argument given in Section 2.4, if

$$u : \Sigma \to X$$

is not simple, then no deformation of u is simple. Hence, u cannot lie in the closure of $\mathfrak{M}^*_{0,J}(X,\beta)$. We conclude $\overline{\mathfrak{M}}^*_{0,J}(X,\beta)$ consists of simple maps and therefore nonsingular of expected dimension. The proof of the claim for $\overline{\mathfrak{M}}^*_{0,(J_1,J_2)}\big(X,(\beta_1,\beta_2)\big)$ is the same. Q.E.D.

Conditions 1–4 can be extended to define an ideal Calabi–Yau n-fold for any n. However, Lemma 1.1, which depends on the dimension counting argument in the preceding paragraph, does not apply in dimensions 6 and higher. For example, if $X_8 \subset \mathbb{P}^7$ is the degree 8 Calabi–Yau hypersurface,

$$\overline{\mathfrak{M}}^*_{0,1}(X_8,1) = \overline{\mathfrak{M}}_{0,1}(X_8,1)$$

certainly consists of simple maps. However, a computation on $G(2,8)$ shows the evaluation map ev_1 is not an immersion along 133430226944 fibers of the forgetful morphism

$$\overline{\mathfrak{M}}^*_{0,1}(X_8,1) \longrightarrow \overline{\mathfrak{M}}^*_{0,0}(X_8,1).$$

A separate computation in a projective bundle over $G(3,8)$ shows the space of conics in X_8 contains 133430226944 double lines. In both cases the degenerate loci correspond to the 133430226944 lines in X_8 whose normal bundle splits as $\mathcal{O}(1)\oplus\mathcal{O}\oplus 3\mathcal{O}(-1)$, instead of the expected $3\mathcal{O}\oplus 2\mathcal{O}(-1)$. While the Calabi–Yau 6-fold X_8 is not ideal, low-degree curves in projective hypersurfaces do behave as expected. The appearance multiple covers as limits of simple maps is to be expected in dimensions 6 and higher, making a full enumerative treatment more complicated (and likely drastically so).

1.2. Genus 0 counts

We define here integer forms of the genus 0 Gromov–Witten invariants of Calabi–Yau 5-folds by considering all possible distributions of constraints and ψ-classes between the marked points. The 13 relevant types of invariants are indicated in Figure 2. We state relations motivated by ideal geometry which reduce all 13 to genus 0 Gromov–Witten invariants. These relations are taken to be the definition of 13 invariants for arbitrary Calabi–Yau 5-folds.

If J is a finite set, $J' \subset J$, and $\beta \in H_+(X)$, let

$$f_{J,J'}\colon \overline{\mathfrak{M}}_{0,J}(X,\beta) \longrightarrow \overline{\mathfrak{M}}_{0,J-J'}(X,\beta)$$

be the forgetful map dropping the marked points indexed by the set J'. If $j \in J$, let

$$\tilde{\psi}_j = f^*_{J,J-j}\psi_j \in H^2\big(\overline{\mathfrak{M}}_{0,J}(X,\beta)\big),$$

where ψ_j is the first chern class of the universal cotangent line bundle for the marked point on $\overline{\mathfrak{M}}_{0,\{j\}}(X,\beta)$.

If X is an ideal Calabi–Yau 5-fold and $\beta \in H_+(X)$, the dimension of $\overline{\mathfrak{M}}^*_{0,0}(X,\beta)$ is 2. There are 7 invariants of the form

$$n_\beta(\tilde{\psi}^a\mu_1,\mu_2,\ldots,\mu_k) = \int_{\overline{\mathfrak{M}}^*_{0,k}(X,\beta)} \tilde{\psi}_1^a \prod_{j=1}^{k} \mathrm{ev}_j^*\mu_j, \qquad a \geq 0,\ \mu_j \in H^{2*}(X),$$

which we require:

(1A) $n_\beta(\mu)$ where $\mu \in H^6(X)$ counting curves through μ,
(1B) $n_\beta(\mu_1,\mu_2)$ where $\mu_1,\mu_2 \in H^4(X)$ counting curves through μ_1 and μ_2,
(1C) $n_\beta(\tilde{\psi}\mu)$ where $\mu \in H^4(X)$,
(1D) $n_\beta(\tilde{\psi}\mu_1,\mu_2)$ where $\mu_1 \in H^2(X)$ and $\mu_2 \in H^4(X)$,
(1E) $n_\beta(\tilde{\psi}^2\mu)$ where $\mu \in H^2(X)$,
(1F) $n_\beta(\tilde{\psi}^2,\mu)$ where $\mu \in H^4(X)$,
(1G) $n_\beta(\tilde{\psi}^3)$.

Let $\overline{\mathcal{M}}_\beta$ denote the unpointed space $\overline{\mathfrak{M}}^*_{0,0}(X,\beta)$. We will need the Chern number

(1H) $\gamma_1(\beta) = \int_{\overline{\mathcal{M}}_\beta} \big(c_1^2(\overline{\mathcal{M}}_\beta) - c_2(\overline{\mathcal{M}}_\beta)\big)$.

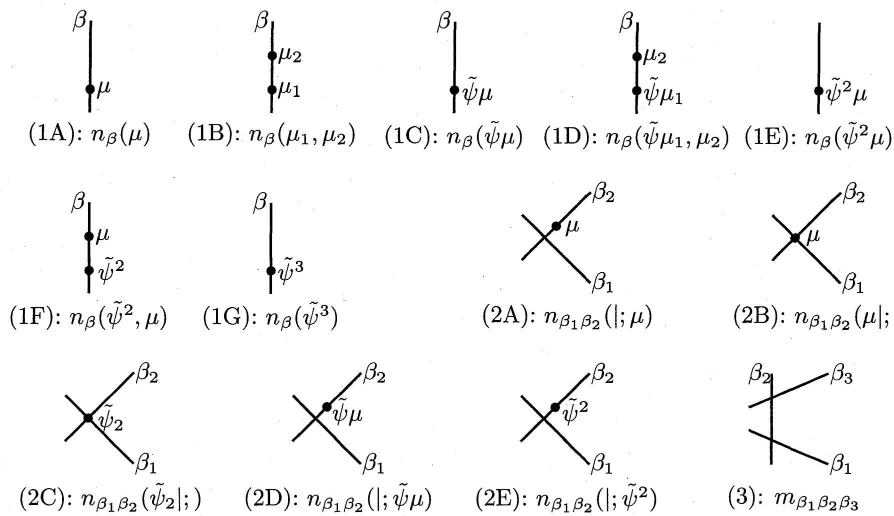


Fig. 2. Counts for Calabi–Yau 5-folds

There are 5 types of relevant counts of connected 2-component curves which we require,

$$
\begin{aligned}
&n_{\beta_1\beta_2}\big(\tilde{\psi}_1^{a_1}\tilde{\psi}_2^{a_2}\mu_0|\tilde{\psi}^{b_1}\mu_{1,1}, \mu_{1,2}, \ldots, \mu_{1,k_1}; \tilde{\psi}^{b_2}\mu_{2,1}, \mu_{2,2}, \ldots, \mu_{2,k_2}\big)\\
&= \int\limits_{\overline{\mathfrak{M}}^*_{0,([k_1],[k_2])}(X,(\beta_1,\beta_2))} \pi_1^*\bigg(\tilde{\psi}_0^{a_1}\tilde{\psi}_1^{b_1}\mathrm{ev}_0^*\mu_0\prod_{j=1}^{k_1}\mathrm{ev}_j^*\mu_{1,j}\bigg)\pi_2^*\bigg(\tilde{\psi}_0^{a_2}\tilde{\psi}_1^{b_2}\prod_{j=1}^{k_2}\mathrm{ev}_j^*\mu_{2,j}\bigg),
\end{aligned}
$$

where π_1, π_2 are the component projection maps as in (1.1), $a_i, b_i \geq 0$, and $\mu_0, \mu_{i,j} \in H^{2*}(X)$. The 5 types are represented by the following counts of (β_1, β_2)-curves:

(2A) $n_{\beta_1\beta_2}(|;\mu)$ where $\mu \in H^4(X)$,
(2B) $n_{\beta_1\beta_2}(\mu|;)$ where $\mu \in H^2(X)$,
(2C) $n_{\beta_1\beta_2}(\tilde{\psi}_2|;)$,
(2D) $n_{\beta_1\beta_2}(|;\tilde{\psi}\mu)$ where $\mu \in H^2(X)$,
(2E) $n_{\beta_1\beta_2}(|;\tilde{\psi}^2)$.

Finally, we denote the cardinality of the compact 0-dimensional space $\mathfrak{M}^*_{0,\emptyset}(\beta_1, \beta_2, \beta_3)$ for triples $\beta_1, \beta_2, \beta_3 \in H_+(X)$ by $m_{\beta_1\beta_2\beta_3}$:

(3) $m_{\beta_1\beta_2\beta_3}$ is the number of connected 3-component curves of tridegree $\beta_1, \beta_2, \beta_3$.

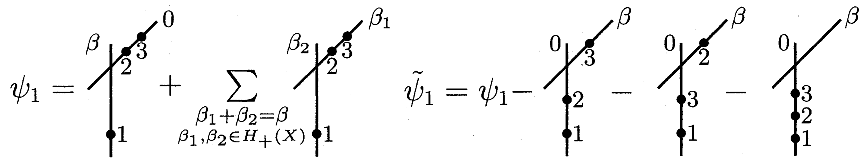


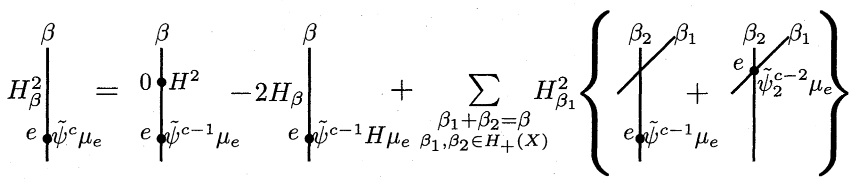
Fig. 3. Relations for ψ_1 and $\tilde{\psi}_1$ on $\overline{\mathfrak{M}}^*_{0,3}(X,\beta)$. Each curve represents the divisor in $\overline{\mathfrak{M}}^*_{0,3}(X,\beta)$ whose general element has the domain and the degree distribution specified by the curve.

The numbers (1A) and (1B) are determined from 1- and 2-pointed Gromov–Witten invariants via (0.2). The topological recursion relation for ψ_1 can be used to express $\tilde{\psi}_1$ in terms of boundary divisors on $\overline{\mathfrak{M}}^*_{0,3}(X,\beta)$, see Figure 3. The divisor relation then gives rise to the relations between the invariants (1C)–(1G) indicated in Figure 4, see also Section 3 in [13]. We now describe these relations formally. If H is a divisor on X and $H_\beta=(H,\beta)$, then

(1.2)

$$\begin{aligned}
H_\beta^2\, n_\beta(\tilde{\psi}\mu) &= n_\beta(\mu,H^2) - 2H_\beta\, n_\beta(H\mu) + \sum_{\beta_1+\beta_2=\beta} H_{\beta_1}^2 n_{\beta_1\beta_2}(|;\mu),\\
H_\beta^2\, n_\beta(\tilde{\psi}\mu_1,\mu_2) &= (\mu_1,\beta)\, n_\beta(\mu_2,H^2) - 2H_\beta\, n_\beta(H\mu_1,\mu_2)\\
&\quad + \sum_{\beta_1+\beta_2=\beta} \big((\mu_1,\beta_1)H_{\beta_2}^2 + (\mu_1,\beta_2)H_{\beta_1}^2\big) n_{\beta_1\beta_2}(|;\mu_2),\\
H_\beta^2\, n_\beta(\tilde{\psi}^2\mu) &= n_\beta(\tilde{\psi}\mu,H^2) - 2H_\beta\, n_\beta(\tilde{\psi}H\mu)\\
&\quad + \sum_{\beta_1+\beta_2=\beta} H_{\beta_1}^2 \big(n_{\beta_1\beta_2}(|;\tilde{\psi}\mu) + n_{\beta_1\beta_2}(\mu|;)\big),\\
n_\beta(\tilde{\psi}^2,\mu) &= -\sum_{\beta_1+\beta_2=\beta} n_{\beta_1\beta_2}(|;\mu),\\
H_\beta^2\, n_\beta(\tilde{\psi}^3) &= n_\beta(\tilde{\psi}^2,H^2) - 2H_\beta\, n_\beta(\tilde{\psi}^2 H)\\
&\quad + \sum_{\beta_1+\beta_2=\beta} H_{\beta_1}^2 \big(n_{\beta_1\beta_2}(|;\tilde{\psi}^2) + n_{\beta_1\beta_2}(\tilde{\psi}_2|;)\big),
\end{aligned}$$

the fourth identity above is obtained by applying the relation of Figure 4 twice. We can similarly remove ψ-classes from 2-component curves:



$H \subset X$ divisor, $H_\beta = H\cdot\beta$, $H_{\beta_1} = H\cdot\beta_1$

Fig. 4. Reducing the power of $\tilde{\psi}$ at marked point e in the absence of ψ-classes at other marked points.

(1.3)

$$H_{\beta_2}^2\, n_{\beta_1\beta_2}(\tilde{\psi}_2|;) = n_{\beta_1\beta_2}(|;H^2) - 2H_{\beta_2}\, n_{\beta_1\beta_2}(H|;) + \sum_{\beta+\beta'=\beta_2} H_\beta^2\, m_{\beta_1\beta'\beta},$$

$$\begin{aligned} H_{\beta_2}^2\, n_{\beta_1\beta_2}(|;\tilde{\psi}\mu) &= (\mu,H)\, n_{\beta_1\beta_2}(|;H^2) - 2H_{\beta_2}\, n_{\beta_1\beta_2}(|;H\mu) \\ &\quad + \sum_{\beta+\beta'=\beta_2} \big((\mu,\beta)H_{\beta'}^2 + (\mu,\beta')H_\beta^2\big) m_{\beta_1\beta'\beta}, \end{aligned}$$

$$n_{\beta_1\beta_2}(|;\tilde{\psi}^2) = -\sum_{\beta+\beta'=\beta_2} m_{\beta_1\beta'\beta}\,,$$

the last identity above is obtained by applying the relation of Figure 4 twice. On the other hand, by (1.15) and some manipulation,

$$\begin{aligned} \gamma_1(\beta) = \frac{1}{2}\Big(& n_\beta\big(c_3(X)\big) + n_\beta\big(\tilde{\psi}c_2(X)\big) + n_\beta\big(\tilde{\psi}^3\big) \\ & + n_\beta\big(c_2(X),c_2(X)\big) + 4\, n_\beta\big(\tilde{\psi}^2, c_2(X)\big)\Big) \\ & - \sum_{\beta_1+\beta_2=\beta} \Big(2\, n_{\beta_1\beta_2}(|;\tilde{\psi}^2) + \frac{5}{2}\, n_{\beta_1\beta_2}(\tilde{\psi}_2|;)\Big). \end{aligned} \tag{1.4}$$

The meeting numbers (2A), (2B), and (3) are computed via degree reducing recursions analogous to Rules (i)–(iv) of Section 0.3 of [10] for the 4-dimensional case. Let

$$\{\omega_1,\ldots,\omega_N\}, \{\omega_1^\# \ldots, \omega_N^\#\} \subset H^4(X)\oplus H^6(X)$$

be dual bases normalized so that

$$\mathrm{PD}_{X^2}\Delta_X - \sum_{l=1}^N \omega_l \times \omega_l^\# \in \bigoplus_{k=0,1,4,5} H^{2k}(X)\otimes H^{2(5-k)}(X) \oplus H^{odd}(X)\otimes H^{odd}(X),$$

where $\Delta_X \subset X^2$ is the diagonal. Then,

$$n_{\beta_1\beta_2}(|;\mu) = \sum_{l=1}^{N} n_{\beta_1}(\omega_l)\, n_{\beta_2}(\omega_l^{\#},\mu) + \begin{cases} n_{\beta_1,\beta_2-\beta_1}(|;\mu) + n_{\beta_2-\beta_1,\beta_1}(|;\mu), & \text{if } \beta_2 > \beta_1, \\ n_{\beta_1-\beta_2,\beta_2}(|;\mu), & \text{if } \beta_2 < \beta_1, \\ n_{\beta_1}(c_2(X),\mu) + 2n_{\beta_1}(\tilde{\psi}^2,\mu), & \text{if } \beta_2 = \beta_1. \end{cases} \tag{1.5}$$

In light of the fourth identity in (1.2), the relation differs from the 4-dimensional case only by the expected adjustment for the constraint μ.

The corresponding recursions for the numbers (2B) and (3) are more complicated. For classes $\beta_1, \beta_2 \in H_+(X)$, let

$$\gamma_2(\beta_1,\beta_2) = n_{\beta_1\beta_2}\big(|;c_2(X)\big) + 2\,n_{\beta_1\beta_2}(|;\tilde{\psi}^2) + n_{\beta_1\beta_2}(\tilde{\psi}_2|;) + n_{\beta_2\beta_1}(\tilde{\psi}_2|;). \tag{1.6}$$

For $\mu \in H^2(X)$, we define

$$\mathsf{C}_{\beta_1\beta_2}(\mu) = \begin{cases} n_{\beta_2-\beta_1,\beta_1}(|;\tilde{\psi}\mu) + n_{\beta_2-\beta_1,\beta_1}(\mu|;) \\ \quad + (\mu,\beta_1)\big(\gamma_2(\beta_2-\beta_1,\beta_1) + \frac{1}{2}\sum\limits_{\beta+\beta'=\beta_2-\beta_1} m_{\beta\beta_1\beta'}\big), & \text{if } \beta_2 > \beta_1; \\ \mathsf{C}_{\beta_2\beta_1}(\mu), & \text{if } \beta_2 < \beta_1; \\ n_{\beta_1}(c_2(X)\mu) + n_{\beta_1}(\tilde{\psi}^2\mu) \\ \quad + n_{\beta_1}(c_2(X),\tilde{\psi}\mu) + (\mu,\beta_1)\gamma_1(\beta_1) & \text{if } \beta_1 = \beta_2. \\ \quad - \sum\limits_{\beta+\beta'=\beta_2} \big(2n_{\beta\beta'}(|;\tilde{\psi}\mu) + \frac{5}{2}n_{\beta\beta'}(\mu|;)\big), \end{cases} \tag{1.7}$$

For $\beta_1, \beta_2, \beta_3 \in H_+(X)$, let

$$\mathsf{C}^{(1)}_{\beta_1\beta_2\beta_3} = \begin{cases} m_{\beta_3-\beta_1,\beta_1,\beta_2}, & \text{if } \beta_3 > \beta_1; \\ m_{\beta_1-\beta_3,\beta_3,\beta_2}, & \text{if } \beta_3 < \beta_1; \\ \gamma_2(\beta_2,\beta_1), & \text{if } \beta_3 = \beta_1; \end{cases}$$

$$\mathsf{C}^{(2)}_{\beta_1\beta_2\beta_3} = -\begin{cases} m_{\beta_1,\beta_2,\beta_3-\beta_2}, & \text{if } \beta_3 > \beta_2; \\ m_{\beta_1,\beta_3,\beta_2-\beta_3} + m_{\beta_1,\beta_2-\beta_3,\beta_3}, & \text{if } \beta_3 < \beta_2; \\ n_{\beta_1\beta_2}\big(|;c_2(X)\big) + 2\,n_{\beta_1\beta_2}(|;\tilde{\psi}^2), & \text{if } \beta_3 = \beta_2; \end{cases} \tag{1.8}$$

$$\mathsf{C}^{(12)}_{\beta_1\beta_2\beta_3} = -\begin{cases} m_{\beta_3-\beta_1-\beta_2,\beta_1,\beta_2}, & \text{if } \beta_3 > \beta_1+\beta_2; \\ m_{\beta_1+\beta_2-\beta_3,\beta_3-\beta_2,\beta_2}, & \text{if } \beta_2 < \beta_3 < \beta_1+\beta_2; \\ \gamma_2(\beta_2,\beta_1), & \text{if } \beta_3 = \beta_1+\beta_2; \\ 0, & \text{otherwise.} \end{cases}$$

$n_{d_1 d_2}(H\|;)$	$d_2 = 1$	$d_2 = 2$
$d_1 = 1$	145366465734	17628837973096812
2	17628837973096812	2134616449608028257452
3	4403307962301366086458	533112594803936499402982169

Table 3. Meeting invariants $n_{d_1 d_2}(H|;)$ for a degree 7 hypersurface in $\mathbb{P}^6$ counting the virtual number of (d_1, d_2)-curves with node on a fixed hyperplane.

Then,

$$n_{\beta_1\beta_2}(\mu|;) = \sum_{l=1}^{N} n_{\beta_1}(\omega_l\mu)\, n_{\beta_2}(\omega_l^{\#}) - \sum_{\beta<\beta_1,\beta_2} (\mu,\beta)\, m_{\beta_1-\beta,\beta,\beta_2-\beta} - \mathsf{C}_{\beta_1\beta_2}(\mu), \tag{1.9}$$

$$m_{\beta_1\beta_2\beta_3} = \sum_{l=1}^{N} n_{\beta_1\beta_2}(|;\omega_l)\, n_{\beta_3}(\omega_l^{\#}) - \mathsf{C}^{(1)}_{\beta_1\beta_2\beta_3} - \mathsf{C}^{(2)}_{\beta_1\beta_2\beta_3} - \mathsf{C}^{(12)}_{\beta_1\beta_2\beta_3}. \tag{1.10}$$

A few low degree 2-component meeting numbers for a degree 7 hypersurface in $\mathbb{P}^6$ are given in Table 1.2. The number $n_{1,1}(H|;)$ can be confirmed via a Schubert computation similar to Section 3 in [9].

Configurations of rational curves in a Calabi–Yau n-fold can be studied for any n. If $n \geq 6$, such configurations include curves with non-simple nodes (several components sharing a node). While describing such curves is just notationally involved, specifying degree reducing recursions for them (following the approach of Section 1.3 below) presents new difficulties. In particular, curves with unbalanced splittings of the normal bundle will effect excess contributions via the loci of non-simple tuples of maps in the closures of simple tuples of maps, see the end of Section 1.1. Thus, separate counts must be set up for such curves, and their multiple-cover contributions to the appropriate topological intersection numbers (represented by the first terms on the right-hand side of (1.5), (1.9), and (1.10)) must be determined.

1.3. Justification of degree reducing recursions

1.3.1. *Overview* Each curve $\mathcal{C}$ of type (2A), (2B), and (3) determines a pair $(\bar{\mathcal{C}}, \mathcal{C}^*)$ of curves, where $\mathcal{C}^*$ is the last component of $\mathcal{C}$ and $\bar{\mathcal{C}}$ consists of the remaining component(s) of $\mathcal{C}$. The curve $\bar{\mathcal{C}}$ has 1 component in the first two cases and 2 components in the last case. The curves $\bar{\mathcal{C}}$ and

$\mathcal{C}^*$ carry marking $x_e \in \bar{\mathcal{C}}$ and $y_e \in \mathcal{C}^*$ satisfying $x_e = y_e$. We denote by $\overline{\mathcal{M}}$ and $\mathcal{M}^*$ the corresponding compactified spaces of curves/maps:

	$\overline{\mathcal{M}}$	$\mathcal{M}^*$
Case (2A):	$\overline{\mathfrak{M}}^*_{0,\{e\}}(X,\beta_1)$	$\{\phi \in \overline{\mathfrak{M}}^*_{0,\{e\}}(X,\beta_2) : (\mathrm{Im}\,\phi) \cap \mu \neq \emptyset\}$,
Case (2B):	$\{\phi \in \overline{\mathfrak{M}}^*_{0,\{e\}}(X,\beta_1) : \mathrm{ev}_e(\phi) \in \mu\}$	$\overline{\mathfrak{M}}^*_{0,\{e\}}(X,\beta_2)$,
Case (3):	$\overline{\mathfrak{M}}^*_{0,(\emptyset,\{e\})}(X,(\beta_1,\beta_2))$	$\overline{\mathfrak{M}}^*_{0,\{e\}}(X,\beta_3)$,

where μ above denotes a generic representative for the Poincaré dual of $\mu \in H^*(X)$. The evaluation map

$$\mathrm{ev}_{e,e} : \overline{\mathcal{M}} \times \mathcal{M}^* \longrightarrow X \times X, \quad \big((\bar{\mathcal{C}}, x_e), (\mathcal{C}^*, y_e)\big) \longrightarrow (x_e, y_e),$$

is then a cycle of (complex) dimension 5. The relevant meeting number is the cardinality of the subset of

$$\mathcal{Z} = \mathrm{ev}_{e,e}^{-1}(\Delta_X) = \big\{\big((\bar{\mathcal{C}}, x_e), (\mathcal{C}^*, y_e)\big) \in \overline{\mathcal{M}} \times \mathcal{M}^* : x_e = y_e\big\}$$

consisting of simple pairs of maps.

The homological intersection number of the cycle $\mathrm{ev}_{e,e}$ with the class of the diagonal $\Delta_X \subset X^2$ in X^2 is given by the diagonal-splitting term on the right-hand side of (1.5), (1.9), and (1.10). The homological intersection is the number of points, counted with sign, in the preimage of Δ_X under a small deformation of the map $\mathrm{ev}_{e,e}$. All such points must lie near $\mathcal{Z}$. The points of $\mathcal{Z}$ at which $\mathrm{ev}_{e,e}$ is transverse to Δ_X contribute 1 each to the homology intersection. These points include all tuples as above such that the curves $\bar{\mathcal{C}}$ and $\mathcal{C}^*$ do not have any components in common. Thus, the relevant meeting number is the diagonal-splitting term in (1.5), (1.9), and (1.10) minus the contribution to the homology intersection number of $\mathrm{ev}_{e,e}$ with Δ_X from the subset $\mathcal{Z}'$ of $\mathcal{Z}$ consisting of tuples as above such that $\bar{\mathcal{C}}$ and $\mathcal{C}^*$ have at least one component in common. In the rest of this subsection, we determine these tuples and their excess contributions.[3]

If X is an ideal Calabi–Yau 5-fold and $\beta \in H_+(X)$, the space

$$\overline{\mathcal{M}}_{\beta,1} = \overline{\mathfrak{M}}^*_{0,1}(X,\beta)$$

of simple maps to X of degree β with 1 marking is nonsingular of dimension 3, and the evaluation map

$$\mathrm{ev} : \overline{\mathcal{M}}_{\beta,1} \longrightarrow X$$

[3] As in the 4-dimensional case considered in [10], all contributions in case (2A) are degenerate contributions arising from loci of dimensions 1 and 2. However, in cases (2B) and (3), $\mathcal{Z}'$ includes regular points with respect to the evaluation condition which are isolated and nondegenerate.

is an immersion, see Lemma 1.1. We denote by T_β the tangent bundle of $\overline{\mathcal{M}}_{\beta,1}$ and by $\mathcal{N}_\beta$ the normal bundle to the immersion ev. Let $\mathcal{N}_\Delta \longrightarrow \Delta$ be the normal bundle to the diagonal in X^2. If $\mathcal{C} \subset X$ is a curve, let $|\mathcal{C}|$ denote the number of irreducible components of $\mathcal{C}$.

1.3.2. *Chern classes* Let X be an ideal Calabi–Yau 5-fold, and let $\beta \in H_+(X)$. We relate here the Chern classes of the normal bundle $\mathcal{N}_\beta$ to the immersion

$$\mathrm{ev}_1 : \overline{\mathcal{M}}_{\beta,1} \longrightarrow X$$

to meeting numbers. Denote by

$$f : \overline{\mathcal{M}}_{\beta,1} \longrightarrow \overline{\mathcal{M}}_\beta \tag{1.11}$$

the forgetful map to the nonsingular 2-dimensional moduli space $\overline{\mathcal{M}}_\beta = \overline{\mathfrak{M}}^*_{0,0}(X,\beta)$.

Using the bundle homomorphism $df : T\overline{\mathcal{M}}_{\beta,1} \longrightarrow f^*T\overline{\mathcal{M}}_\beta$ over $\overline{\mathcal{M}}_{\beta,1}$, we obtain

$$\begin{aligned} c_1(T_\beta) &= -\psi + f^*c_1(\overline{\mathcal{M}}_\beta), \\ c_2(T_\beta) &= \Delta - \psi\, f^*c_1(\overline{\mathcal{M}}_\beta) + f^*c_2(\overline{\mathcal{M}}_\beta), \end{aligned} \tag{1.12}$$

where ψ is the first chern class of the cotangent line bundle on $\overline{\mathcal{M}}_{\beta,1}$ viewed as a 1-pointed moduli space and $\Delta \subset \overline{\mathcal{M}}_{\beta,1}$ is the locus of singular points of f (points at which df is not surjective). On the other hand, since $c_1(X)=0$,

$$\begin{aligned} c_1(\mathcal{N}_\beta) &= -c_1(T_\beta), \\ c_2(\mathcal{N}_\beta) &= \mathrm{ev}^*c_2(X) + c_1^2(T_\beta) - c_2(T_\beta). \end{aligned} \tag{1.13}$$

Combining (1.12) and (1.13), we find

(1.14)

$$\begin{aligned} &c_1(\mathcal{N}_\beta) = \psi - f^*c_1(\overline{\mathcal{M}}_\beta), \\ &c_2(\mathcal{N}_\beta) = \mathrm{ev}^*c_2(X) + \psi^2 - \Delta - \psi\, f^*c_1(\overline{\mathcal{M}}_\beta) + f^*\big(c_1^2(\overline{\mathcal{M}}_\beta) - c_2(\overline{\mathcal{M}}_\beta)\big). \end{aligned}$$

If $\beta_1+\beta_2=\beta$ and $\beta_1 \neq \beta_2$, let $D_{\beta_1,\beta_2} \subset \overline{\mathcal{M}}_\beta$ be the closure of the locus consisting of β-curves split into a β_1-curve and a β_2-curve. If $2\beta_1=\beta$, let $D_{\beta_1\beta_1} \subset \overline{\mathcal{M}}_\beta$ be twice the closure of the locus of consisting of β-curves split into two β_1-curves. In particular,

$$f_*\Delta = \frac{1}{2} \sum_{\substack{\beta_1+\beta_2=\beta \\ \beta_1,\beta_2 \in H_+(X)}} D_{\beta_1,\beta_2}\,.$$

Denote by $(\psi_1+\psi_2)D_{\beta_1,\beta_2} \in H^4(\overline{\mathcal{M}}_\beta)$ the class obtained by capping Δ with the first chern class of the cotangent line bundle at the chosen node for each of the two curves. From a Grothendieck–Riemann–Roch computation applied to the deformation characterization of $T\overline{\mathcal{M}}_\beta$, we find

$$\begin{aligned} c_1(\overline{\mathcal{M}}_\beta) &= -f_*\mathrm{ev}^*c_2(X) + \sum_{\substack{\beta_1+\beta_2=\beta \\ \beta_1,\beta_2\in H_+(X)}} D_{\beta_1,\beta_2}\,, \\ 2c_2(\overline{\mathcal{M}}_\beta) - c_1^2(\overline{\mathcal{M}}_\beta) &= -f_*\big(\mathrm{ev}^*c_3(X) + \psi\,\mathrm{ev}_2^*c_2(X) + \psi^3\big) \\ &\quad + \frac{1}{2}\sum_{\substack{\beta_1+\beta_2=\beta \\ \beta_1,\beta_2\in H_+(X)}} (\psi_1+\psi_2)D_{\beta_1,\beta_2}\,. \end{aligned} \tag{1.15}$$

The 4-dimensional case of the first equation above appears in Section 1.2.4 of [10] and is also an immediate consequence of the $n=4$ analogue of (2.5) below. The second identity in (1.15) is (2.5) itself.

1.3.3. *The numbers (2A)* Suppose $\big((\bar{\mathcal{C}},x_e),(\mathcal{C}^*,x_e)\big)$ is an element of $\mathcal{Z}'$. Since the curve $\bar{\mathcal{C}}\cup\mathcal{C}^*$ passes through μ, $\bar{\mathcal{C}}\cup\mathcal{C}^*$ has at most two components. We have three possibilities for $\mathcal{Z}'$.

Case 0 *($\bar{\mathcal{C}}=\mathcal{C}^*$):* Here $\beta_1=\beta_2$ and

$$\mathcal{Z}' = \big\{\big((\mathcal{C}^*,x_e),(\mathcal{C}^*,x_e)\big) : (\mathcal{C}^*,x_e)\in\mathcal{M}^*\big\}.$$

The normal bundle of $\mathcal{Z}'$ in $\overline{\mathcal{M}}\times\mathcal{M}^*$ is isomorphic to $T_{\beta_1}\longrightarrow\mathcal{M}^*$ and the differential

$$d\mathrm{ev}_{e,e} = d\mathrm{ev}_e : \mathcal{N} \longrightarrow \mathrm{ev}_{e,e}^*\mathcal{N}_\Delta$$

is injective over $\mathcal{M}^*$. Thus, the contribution of $\mathcal{Z}'$ to the homology intersection number is given by

$$\big\langle e(\mathcal{N}_\Delta/\mathcal{N}),\mathcal{Z}'\big\rangle = \big\langle c_2(\mathcal{N}_{\beta_1}),\mathcal{M}^*\big\rangle.$$

Using the second equation in (1.14), the first equation in (1.15), and the fourth equation in (1.2), we obtain the $\beta_1=\beta_2$ case of (1.5).

Case 1*A ($\bar{\mathcal{C}}\subsetneq\mathcal{C}^*$):* Here $\beta_1<\beta_2$ and

$$\mathcal{Z}' = \big\{\big((\bar{\mathcal{C}},x_e),(\bar{\mathcal{C}}\vee\mathcal{C}',x_e)\big) : (\bar{\mathcal{C}},x_e)\in\overline{\mathcal{M}},\ (\bar{\mathcal{C}}\vee\mathcal{C}',x_e)\in\mathcal{Z}^*\big\},$$

where $\mathcal{Z}^*\subset\mathcal{M}^*$ is the locus consisting of 2-component curves with the marked point on the first component. Thus, $\mathcal{Z}'$ is the union of the first components of the finitely many $(\beta_1,\beta_2-\beta_1)$-curves passing through the constraint μ. The normal bundle $\mathcal{N}$ of $\mathcal{Z}'$ in $\overline{\mathcal{M}}\times\mathcal{M}^*$ contains the subbundle $\pi_1^*T_{\beta_1}$ and $\mathcal{N}/\pi_1^*T_{\beta_1}$ is isomorphic to the normal bundle $\mathcal{N}\mathcal{Z}^*$ of $\mathcal{Z}^*$ in $\mathcal{M}^*$. Since the differential

$$d\mathrm{ev}_{e,e}\colon \mathcal{N} \longrightarrow \mathrm{ev}_{e,e}^*\mathcal{N}_\Delta$$

is injective over $\mathcal{Z}'$, the contribution of $\mathcal{Z}'$ to the homology intersection number is given by

$$\langle e(\mathcal{N}_\Delta/\mathcal{N}),\mathcal{Z}'\rangle = \langle c_1(\mathcal{N}_{\beta_1}) - c_1(\mathcal{N}\mathcal{Z}^*),\mathcal{Z}^*\rangle.$$

Since the degrees of the restrictions of $\mathcal{N}_{\beta_1}$ and $\mathcal{N}\mathcal{Z}^*$ to each curve $\bar{\mathcal{C}}$ are -2 and -1, respectively, we obtain the $\beta_1<\beta_2$ case of (1.5).

Case 1*B* *($\bar{\mathcal{C}}\supsetneq\mathcal{C}^*$):* Here $\beta_1>\beta_2$ and

$$\mathcal{Z}' = \big\{\big((\mathcal{C}'\vee\mathcal{C}^*,x_e),(\mathcal{C}^*,x_e)\big)\colon (\mathcal{C}'\vee\mathcal{C}^*,x_e)\in\overline{\mathcal{M}},\ (\mathcal{C}^*,x_e)\in\mathcal{Z}^*\big\},$$

where $\mathcal{Z}^*\subset\mathcal{M}^*$ is the locus of curves meeting a $(\beta_1-\beta_2)$-curve. Thus, $\mathcal{Z}'$ is the union of the second components of the finitely many $(\beta_1-\beta_2,\beta_2)$-curves whose second component passes through the constraint μ. The normal bundle $\mathcal{N}$ of $\mathcal{Z}'$ in $\overline{\mathcal{M}}\times\mathcal{M}^*$ contains the subbundle $\pi_1^*T_{\beta_1}$ and $\mathcal{N}/\pi_1^*T_{\beta_1}$ is isomorphic to the normal bundle $\mathcal{N}\mathcal{Z}^*$ of $\mathcal{Z}^*$ in $\mathcal{M}^*$. The latter is trivial. Since the differential

$$d\mathrm{ev}_{e,e}\colon \mathcal{N} \longrightarrow \mathrm{ev}_{e,e}^*\mathcal{N}_\Delta$$

is injective over $\mathcal{Z}'$, the contribution of $\mathcal{Z}'$ to the homology intersection number is given by

$$\langle e(\mathcal{N}_\Delta/\mathcal{N}),\mathcal{Z}'\rangle = \langle c_1(\mathcal{N}_{\beta_1}) - c_1(\mathcal{N}\mathcal{Z}^*),\mathcal{Z}^*\rangle.$$

The $\beta_1>\beta_2$ case of (1.5) now follows from the first equation in (1.14).

1.3.4. *The numbers (2B)* Suppose $\big((\bar{\mathcal{C}},x_e),(\mathcal{C}^*,x_e)\big)$ is an element of $\mathcal{Z}'$. The curve $\bar{\mathcal{C}}\cup\mathcal{C}^*$ then has one, two, or three components and carries a marked point e lying on the divisor μ. The 6 possibilities for the connected components of $\mathcal{Z}'$ are indicated in Figure 5.

Case 0 *($\bar{\mathcal{C}}=\mathcal{C}^*$):* Here $\beta_1=\beta_2$ and $\big((\bar{\mathcal{C}},x_e),(\mathcal{C}^*,x_e)\big)$ is an element of

$$\overline{\mathcal{S}} = \big\{\big((\bar{\mathcal{C}},x_e),(\bar{\mathcal{C}},x_e)\big)\colon (\bar{\mathcal{C}},x_e)\in\overline{\mathcal{M}}\big\}\subset\mathcal{Z}'.$$

The normal bundle of $\overline{\mathcal{S}}$ in $\overline{\mathcal{M}}\times\mathcal{M}^*$ is isomorphic to $T_{\beta_2}\longrightarrow\overline{\mathcal{M}}$, and the contribution of $\overline{\mathcal{S}}$ to the homology intersection number is given by

$$\langle e(\mathcal{N}_\Delta/\mathcal{N}),\overline{\mathcal{S}}\rangle = \langle c_2(\mathcal{N}_{\beta_2}),\overline{\mathcal{M}}\rangle.$$

Using the second equation in (1.14) and the first equation in (1.15), we obtain the $\beta_1 = \beta_2$ case of the last term in (1.9).

Case 1*A* ($|\mathcal{C}^*| = 2$, $\bar{\mathcal{C}} \subsetneq \mathcal{C}^*$): Here $\beta_1 < \beta_2$ and $\big((\bar{\mathcal{C}}, x_e), (\mathcal{C}^*, x_e)\big)$ is an element of

$$\overline{\mathcal{S}} = \big\{\big((\bar{\mathcal{C}}, x_e), (\bar{\mathcal{C}} \vee \mathcal{C}', x_e)\big) : (\bar{\mathcal{C}}, x_e) \in \overline{\mathcal{Z}},\ (\bar{\mathcal{C}} \vee \mathcal{C}', x_e) \in \mathcal{M}^*\big\} \subset \mathcal{Z}',$$

where $\overline{\mathcal{Z}} \subset \overline{\mathcal{M}}$ is the locus consisting of curves meeting a $(\beta_2 - \beta_1)$-curve. The normal bundle $\mathcal{N}$ of $\overline{\mathcal{S}}$ in $\overline{\mathcal{M}} \times \mathcal{M}^*$ contains the subbundle $\pi_2^* T_{\beta_2}$ and $\mathcal{N}/\pi_2^* T_{\beta_2}$ is isomorphic to the normal bundle $\mathcal{N}\overline{\mathcal{Z}}$ of $\overline{\mathcal{Z}}$ in $\overline{\mathcal{M}}$. Since the differential

$$d\mathrm{ev}_{e,e} : \mathcal{N} \longrightarrow \mathrm{ev}_{e,e}^* \mathcal{N}_\Delta$$

is injective over $\overline{\mathcal{S}}$, the contribution of $\overline{\mathcal{S}}$ to the homology intersection number is given by

$$\big\langle e(\mathcal{N}_\Delta/\mathcal{N}), \overline{\mathcal{S}}\big\rangle = \big\langle c_1(\mathcal{N}_{\beta_2}) - \big(c_1(\mathcal{N}_{\beta_2 - \beta_1}) + \tilde{\psi}_1\big), \overline{\mathcal{Z}}\big\rangle,$$

where $\tilde{\psi}_1$ is the untwisted ψ-class at the node of the $(\beta_2 - \beta_1)$-component of a curve in $\overline{\mathcal{Z}}$.

Using the first equations in (1.14) and in (1.15) and the fourth equation in (1.2), we obtain the $\beta_1 < \beta_2$ case of the last term in (1.9) minus the last term in (1.7). The latter arises from *Case* 2*A* below.

Case 1*B* ($|\bar{\mathcal{C}}| = 2$, $\bar{\mathcal{C}} \supsetneq \mathcal{C}^*$): Here $\beta_1 > \beta_2$ and $\big((\bar{\mathcal{C}}, x_e), (\mathcal{C}^*, x_e)\big)$ is an element of

$$\overline{\mathcal{S}} = \big\{\big((\mathcal{C}' \vee \mathcal{C}^*, x_e), (\mathcal{C}^*, x_e)\big) : (\mathcal{C}' \vee \mathcal{C}^*, x_e) \in \overline{\mathcal{Z}},\ (\mathcal{C}^*, x_e) \in \mathcal{M}^*\big\} \subset \mathcal{Z}',$$

where $\overline{\mathcal{Z}} \subset \overline{\mathcal{M}}$ is the locus of $(\beta_2, \beta_1 - \beta_2)$-curves with the marked point e lying on the first component. The normal bundle $\mathcal{N}$ of $\overline{\mathcal{S}}$ in $\overline{\mathcal{M}} \times \mathcal{M}^*$ contains the subbundle $\pi_2^* T_{\beta_2}$, $\mathcal{N}/\pi_2^* T_{\beta_2}$ is isomorphic to the normal bundle $\mathcal{N}\overline{\mathcal{Z}}$ of $\overline{\mathcal{Z}}$ in $\overline{\mathcal{M}}$, and the contribution of $\mathcal{Z}'$ to the homology intersection number is given by

$$\big\langle e(\mathcal{N}_\Delta/\mathcal{N}), \overline{\mathcal{S}}\big\rangle = \big\langle c_1(\mathcal{N}_{\beta_2}) + \big(\psi_1 + \psi_2\big), \overline{\mathcal{Z}}\big\rangle,$$

where ψ_1 and ψ_2 are the ψ-classes of the first and second components at the node of a curve in $\overline{\mathcal{Z}}$. We obtain the $\beta_1 > \beta_2$ analogue of the *Case* 1*A* contribution in (1.9).

Case 2*A* ($|\mathcal{C}^*| = 3$, $\bar{\mathcal{C}} \subsetneq \mathcal{C}^*$): Here $\beta_1 < \beta_2$. If $|\bar{\mathcal{C}}| = 2$, $\big((\bar{\mathcal{C}}, x_e), (\mathcal{C}^*, x_e)\big)$ is an element of the space $\bar{\mathcal{S}}$ in *Case* 1*A* above. This is also the case if

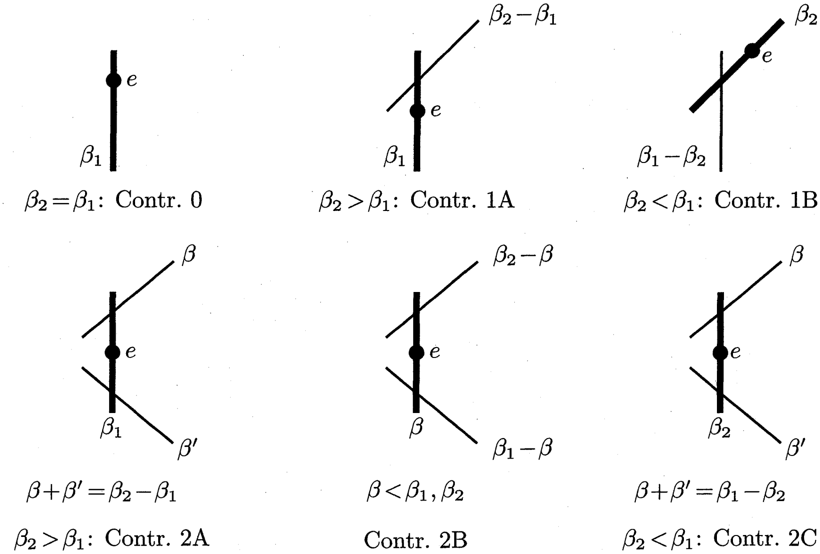


Fig. 5. Excess contributions for the meeting number $n_{\beta_1\beta_2}(\mu|;)$. The labels refer to the cases described in Section 1.3.4. The marked point e corresponds to the (former) node and lies on the divisor μ. The thicker lines indicate the multiple component. The space of curves in the first diagram in the top row is 2-dimensional. The other two spaces in the first row are 1-dimensional. All spaces in the bottom row are 0-dimensional.

$|\bar{\mathcal{C}}|=1$ and the curve $\mathcal{C}^*-\bar{\mathcal{C}}$ is connected. In the remaining case, $\bar{\mathcal{C}}$ is the middle component of the 3-component curve $\mathcal{C}^*$ and carries the marked point e, which lies on the divisor μ. Each such pair $\big((\bar{\mathcal{C}},x_e),(\mathcal{C}^*,x_e)\big)$ is a regular element of $\mathcal{Z}$ and therefore contributes 1 to the homology intersection. The contribution of such pairs is accounted for by the last term in (1.7).

Case 2B ($|\bar{\mathcal{C}}|=|\mathcal{C}^*|=2$, $\bar{\mathcal{C}}\neq\mathcal{C}^*$): Here the curve $\bar{\mathcal{C}}\cup\mathcal{C}^*$ consists of three components, with the middle component meeting the hyperplane μ at the marked point e. Such pairs $\big((\bar{\mathcal{C}},x_e),(\mathcal{C}^*,x_e)\big)$ are regular elements of $\mathcal{Z}$, and their contribution is accounted for by the middle term on the right side of (1.9).

Case 2*C* ($|\mathcal{C}^*|=3,\ \bar{\mathcal{C}}\supsetneq\mathcal{C}^*$): The analysis is the same as *Case* 2*A* with β_1 and β_2 interchanged.

1.3.5. *The numbers (3)* If $\big((\bar{\mathcal{C}},x_e),(\mathcal{C}^*,x_e)\big)$ is an element of $\mathcal{Z}'$, $\bar{\mathcal{C}}$ consists of two sets of components, $\bar{\mathcal{C}}_1$ and $\bar{\mathcal{C}}_2$, with the second component carrying the marked point e. Either $\bar{\mathcal{C}}_1$ or $\bar{\mathcal{C}}_2$ may consist of two components, while the other curve must consist of one component. The total number of components in $\bar{\mathcal{C}}\cup\mathcal{C}^*$ is either two or three. The 12 possibilities for the connected components of $\mathcal{Z}'$ are indicated in Figure 6.

Case 0 ($|\bar{\mathcal{C}}\cup\mathcal{C}^*|=2,\ \bar{\mathcal{C}}_2\subset\mathcal{C}^*$): If $\mathcal{C}^*=\bar{\mathcal{C}}_2$, then $\beta_2=\beta_3$ and $\big((\bar{\mathcal{C}},x_e),(\mathcal{C}^*,x_e)\big)$ is an element of

$$\overline{\mathcal{S}}=\big\{\big((\bar{\mathcal{C}}_1\vee\mathcal{C}^*,x_e),(\mathcal{C}^*,x_e)\big):(\bar{\mathcal{C}}_1\vee\mathcal{C}^*,x_e)\in\overline{\mathcal{M}}\big\}\subset\mathcal{Z}'.$$

Similarly to *Case* 0 in Sections 1.3.3 and 1.3.4, the normal bundle of $\overline{\mathcal{S}}$ in $\overline{\mathcal{M}}\times\mathcal{M}^*$ is isomorphic to $T_{\beta_2}\longrightarrow\overline{\mathcal{M}}$, and the contribution of $\overline{\mathcal{S}}$ to the homological intersection number is given by

$$\big\langle e(\mathcal{N}_\Delta/\mathcal{N}),\overline{\mathcal{S}}\big\rangle=\big\langle c_2(\mathcal{N}_{\beta_2}),\overline{\mathcal{M}}\big\rangle.$$

Using the second equation in (1.14), the first equation in (1.15), and the fourth equation in (1.2), we obtain the $\beta_3=\beta_2$ case of the term $\mathsf{C}^{(2)}_{\beta_1\beta_2\beta_3}$ in (1.10).

If $\mathcal{C}^*=\bar{\mathcal{C}}_1\cup\bar{\mathcal{C}}_2$, then $\beta_1+\beta_2=\beta_3$ and $\big((\bar{\mathcal{C}},x_e),(\mathcal{C}^*,x_e)\big)$ is an element of

$$\overline{\mathcal{S}}=\big\{\big((\mathcal{C}^*,x_e),(\mathcal{C}^*,x_e)\big):(\mathcal{C}^*,x_e)\in\overline{\mathcal{M}}\big\}\subset\mathcal{Z}'.$$

The normal bundle of $\overline{\mathcal{S}}$ in $\overline{\mathcal{M}}\times\mathcal{M}^*$ is isomorphic to $T_{\beta_1+\beta_2}\longrightarrow\overline{\mathcal{M}}$, and the contribution of $\overline{\mathcal{S}}$ to the homological intersection number is given by

$$\big\langle e(\mathcal{N}_\Delta/\mathcal{N}),\overline{\mathcal{S}}\big\rangle=\big\langle c_2(\mathcal{N}_{\beta_1+\beta_2}),\overline{\mathcal{M}}\big\rangle.$$

Using the second equation in (1.14), the first equation in (1.15), and the fourth equation in (1.2), we obtain the $\beta_3=\beta_1+\beta_2$ case of the term $\mathsf{C}^{(12)}_{\beta_1\beta_2\beta_3}$ in (1.10).

Case 0′ ($|\bar{\mathcal{C}}\cup\mathcal{C}^*|=2,\ \bar{\mathcal{C}}_2\not\subset\mathcal{C}^*$): Here $\beta_1=\beta_3$ and $\big((\bar{\mathcal{C}},x_e),(\mathcal{C}^*,x_e)\big)$ is an element of

$$\overline{\mathcal{S}}=\big\{\big((\mathcal{C}^*\vee\bar{\mathcal{C}}_2,x_e),(\bar{\mathcal{C}},x_e)\big):(\mathcal{C}^*\vee\bar{\mathcal{C}}_2,x_e)\in\overline{\mathcal{Z}}\big\}\subset\mathcal{Z}',$$

where $\overline{\mathcal{Z}}\subset\overline{\mathcal{M}}$ consists of the pairs of 1-marked curves with the marked point at the node of the two curves. The normal bundle $\mathcal{N}$ of $\overline{\mathcal{S}}$ in

$\overline{\mathcal{M}}\times\mathcal{M}^*$ contains T_{β_1} as a subbundle, and $\mathcal{N}/T_{\beta_1}$ is isomorphic to the normal bundle of $\overline{\mathcal{Z}}$ in $\overline{\mathcal{M}}$. The latter is the universal tangent line bundle at the marked point. Since the homomorphism

$$d\mathrm{ev}_{e,e}\colon \mathcal{N} \longrightarrow \mathrm{ev}_{e,e}^*\mathcal{N}_\Delta$$

is injective over $\mathcal{Z}'$, the contribution of $\mathcal{Z}'$ to the homological intersection number is given by

$$\big\langle e(\mathcal{N}_\Delta/\mathcal{N}), \overline{\mathcal{S}}\big\rangle = \big\langle c_1(\mathcal{N}_{\beta_2}) + \psi_2, \overline{\mathcal{Z}}\big\rangle.$$

Using the first equations in (1.14) and (1.15) and the fourth equation in (1.2), we obtain the $\beta_3\!=\!\beta_1$ case of the term $\mathrm{C}^{(1)}_{\beta_1\beta_2\beta_3}$ in (1.10).

Case 1*A* ($|\bar{\mathcal{C}}\cup\mathcal{C}^*|=3,\ \mathcal{C}^*\not\subset\bar{\mathcal{C}},\ \bar{\mathcal{C}}_2\subset\mathcal{C}^*$): If $\bar{\mathcal{C}}_1\not\subset\mathcal{C}^*$, then $\beta_2<\beta_3$ and $\big((\bar{\mathcal{C}},x_e),(\mathcal{C}^*,x_e)\big)$ is an element of

$$\overline{\mathcal{S}} = \big\{\big((\bar{\mathcal{C}}_1\vee\bar{\mathcal{C}}_2,x_e),(\bar{\mathcal{C}}_2\vee\mathcal{C}',x_e)\big) : (\bar{\mathcal{C}}_1\vee\bar{\mathcal{C}}_2,x_e)\in\overline{\mathcal{Z}}\big\}\subset\mathcal{Z}',$$

where $\overline{\mathcal{Z}}\subset\overline{\mathcal{M}}$ consists of the pairs of (β_1,β_2)-curves such that the second component meets a $(\beta_3-\beta_2)$-curve. We see $\overline{\mathcal{S}}$ is the union of the middle components of $(\beta_1,\beta_2,\beta_3-\beta_2)$-curves in X, with each curve contributing -1 to the homological intersection number. The contribution accounts for the $\beta_3>\beta_2$ case of the term $\mathrm{C}^{(2)}_{\beta_1\beta_2\beta_3}$ in (1.10).

If $\bar{\mathcal{C}}_1\subset\mathcal{C}^*$, then $\beta_1+\beta_2<\beta_3$ and $\big((\bar{\mathcal{C}},x_e),(\mathcal{C}^*,x_e)\big)$ is an element of

$$\overline{\mathcal{S}} = \big\{\big((\bar{\mathcal{C}}_1\vee\bar{\mathcal{C}}_2,x_e),(\bar{\mathcal{C}}_1\vee\bar{\mathcal{C}}_2\vee\mathcal{C}',x_e)\big) : (\bar{\mathcal{C}}_1\vee\bar{\mathcal{C}}_2,x_e)\in\overline{\mathcal{Z}}\big\}\subset\mathcal{Z}',$$

where $\overline{\mathcal{Z}}\subset\overline{\mathcal{M}}$ consists of the pairs (β_1,β_2)-curves meeting a $(\beta_3-\beta_1-\beta_2)$-curve with the β_2-component carrying the marked point e. Here, $\bar{\mathcal{S}}$ is the union of the last components of the $(\beta_3-\beta_1-\beta_2,\beta_1,\beta_2)$-curves and the middle components of $(\beta_1,\beta_2,\beta_3-\beta_1-\beta_2)$-curves. By reasoning analogous to *Case* 1*A* in Section 1.3.4, each of the former contributes -1 to the homological intersection number, while each of the latter contributes 0. We obtain the $\beta_3>\beta_1+\beta_2$ case of the term $\mathrm{C}^{(12)}_{\beta_1\beta_2\beta_3}$ in (1.10).

Case 1*A'* ($|\bar{\mathcal{C}}\cup\mathcal{C}^*|=3,\ \mathcal{C}^*\not\subset\bar{\mathcal{C}},\ \bar{\mathcal{C}}_2\not\subset\mathcal{C}^*$): Here $\beta_3>\beta_1$ and $(\bar{\mathcal{C}}\cup\mathcal{C}^*,x_e)$ is a $(\beta_3-\beta_1,\beta_1,\beta_2)$-curve with the marked point lying on the node joining the last two components. Each such pair $\big((\bar{\mathcal{C}},x_e),(\mathcal{C}^*,x_e)\big)$ is a regular element of $\mathcal{Z}$, contributing 1 to the homology intersection number. We obtain the $\beta_3>\beta_1$ case of the term $\mathrm{C}^{(1)}_{\beta_1\beta_2\beta_3}$ in (1.10).

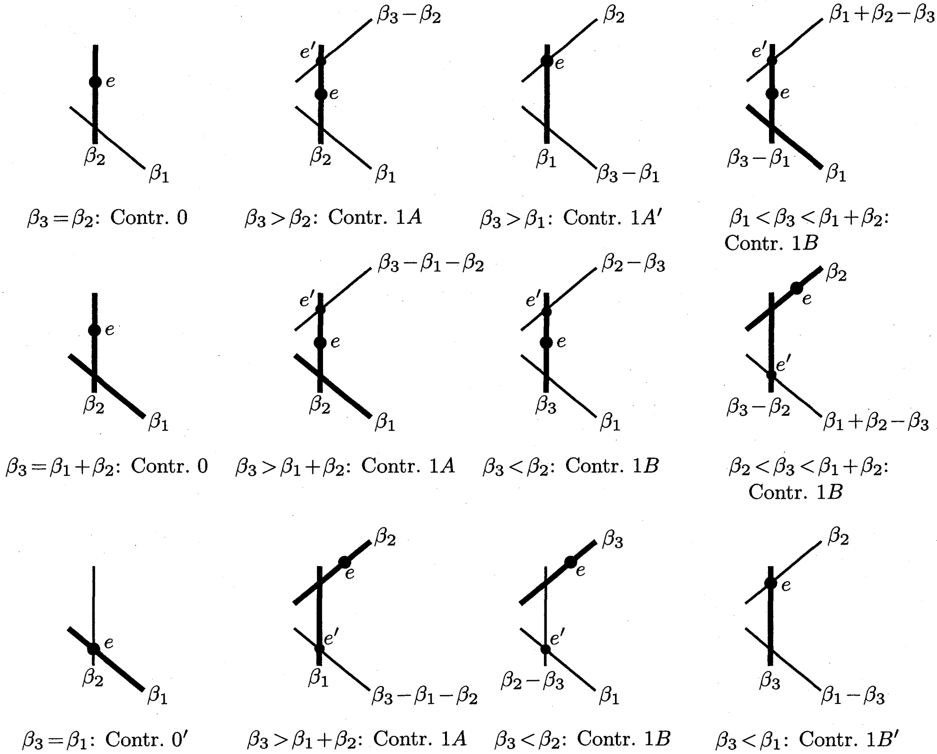


Fig. 6. Excess contributions for the meeting number $m_{\beta_1\beta_2\beta_3}$. The labels refer to the cases described in Section 1.3.5. The marked point e corresponds to the (former) node joining the β_2 and β_3 curves. For the curves of types 1A and 1B, e' indicates the new node on the (leftover) (β_1, β_2)-curve. The thicker lines represent the multiple component(s). The excess loci corresponding to *Contr.* 0 are 2-dimensional. The loci corresponding to *Contr.* $1A'$ and $1B'$ are 0-dimensional. The remaining loci are 1-dimensional.

Case $1B$ *($|\bar{\mathcal{C}}\cup\mathcal{C}^*|=3$, $\mathcal{C}^*\subset\bar{\mathcal{C}}$, $\mathcal{C}^*\not\subset\bar{\mathcal{C}}_1$):* If $\mathcal{C}^*=\bar{\mathcal{C}}_2$ or $\mathcal{C}^*=\bar{\mathcal{C}}$, $\big((\bar{\mathcal{C}},x_e),(\mathcal{C}^*,x_e)\big)$ is an element of one of the spaces $\bar{\mathcal{S}}$ defined in *Case* 0 above. Hence, we can assume that $\mathcal{C}^*\neq\bar{\mathcal{C}}_2,\bar{\mathcal{C}}$. If $\mathcal{C}^*\subset\bar{\mathcal{C}}_2$, then $\beta_2>\beta_3$ and $\big((\bar{\mathcal{C}},x_e),(\mathcal{C}^*,x_e)\big)$ is an element of

$$\overline{\mathcal{S}}=\big\{\big((\bar{\mathcal{C}}_1\vee\mathcal{C}^*\vee\mathcal{C}',x_e),(\mathcal{C}^*,x_e)\big):(\bar{\mathcal{C}}_1\vee\mathcal{C}^*\vee\mathcal{C}',x_e)\in\overline{\mathcal{Z}}\big\}\subset\mathcal{Z}',$$

where $\overline{\mathcal{Z}} \subset \overline{\mathcal{M}}$ is the locus of the pairs (β_1, β_2)-curves with the second component broken into two. As in *Case 1B* of Section 1.3.4, $\overline{\mathcal{S}}$ is the union of the middle components of $(\beta_1, \beta_3, \beta_2 - \beta_3)$-curves and the last components of $(\beta_1, \beta_2 - \beta_3, \beta_3)$, with each curve contributing -1 to the homological intersection number. The contribution accounts for the $\beta_3 < \beta_2$ case of the term $\mathsf{C}^{(2)}_{\beta_1\beta_2\beta_3}$ in (1.10).

If $\mathcal{C}^* \not\subset \bar{\mathcal{C}}_2$, then $\beta_1 + \beta_2 > \beta_3$ and $\big((\bar{\mathcal{C}}, x_e), (\mathcal{C}^*, x_e)\big)$ is an element of

$$\overline{\mathcal{S}} = \big\{\big((\bar{\mathcal{C}}_1 \vee \bar{\mathcal{C}}_2 \vee \mathcal{C}', x_e), (\bar{\mathcal{C}}_1 \vee \bar{\mathcal{C}}_2, x_e)\big) : (\bar{\mathcal{C}}_1 \vee \bar{\mathcal{C}}_2 \vee \mathcal{C}', x_e) \in \overline{\mathcal{Z}}\big\} \subset \mathcal{Z}',$$

where $\overline{\mathcal{Z}} \subset \overline{\mathcal{M}}$ is the locus of the pairs (β_1, β_2)-curves with one of the components broken into two. Here, $\overline{\mathcal{S}}$ is the union of the middle components of $(\beta_1, \beta_3 - \beta_1, \beta_1 + \beta_2 - \beta_3)$-curves, if $\beta_3 > \beta_1$, and the last components of $(\beta_1 + \beta_2 - \beta_3, \beta_3 - \beta_2, \beta_2)$-curves, if $\beta_3 > \beta_2$. Each of the latter curves contributes -1 to the homological intersection number, while each of the former contributes 0. We obtain the $\beta_3 < \beta_2$ case of the term $\mathsf{C}^{(12)}_{\beta_1\beta_2\beta_3}$ in (1.10).

Case $1B'$ *($|\bar{\mathcal{C}} \cup \mathcal{C}^*| = 3$, $\mathcal{C}^* \subset \bar{\mathcal{C}}_1$):* If $\mathcal{C}^* = \bar{\mathcal{C}}_1$, then $\big((\bar{\mathcal{C}}, x_e), (\mathcal{C}^*, x_e)\big)$ is an element of the space $\bar{\mathcal{S}}$ defined in *Case* $0'$ above. Hence, we can assume that $\mathcal{C}^* \neq \bar{\mathcal{C}}_1$. Then, $\beta_1 > \beta_3$ and $\big((\bar{\mathcal{C}}, x_e), (\mathcal{C}^*, x_e)\big)$ is an element of

$$\overline{\mathcal{S}} = \big\{\big((\mathcal{C}' \vee \mathcal{C}^* \vee \bar{\mathcal{C}}_2, x_e), (\mathcal{C}^*, x_e)\big) : (\mathcal{C}' \vee \mathcal{C}^* \vee \bar{\mathcal{C}}_2, x_e) \in \overline{\mathcal{Z}}\big\} \subset \mathcal{Z}',$$

where $\overline{\mathcal{Z}} \subset \overline{\mathcal{M}}$ is the locus of the pairs of 1-marked (β_1, β_2)-curves represented by a $(\beta_1 - \beta_3, \beta_3, \beta_2)$-curve in X with the marked point on the node of the last two components. Each such pair $\big((\bar{\mathcal{C}}, x_e), (\mathcal{C}^*, x_e)\big)$ is a regular element of $\mathcal{Z}$, contributing 1 to the homological intersection number and accounting for the $\beta_3 < \beta_1$ case of the term $\mathsf{C}^{(1)}_{\beta_1\beta_2\beta_3}$ in (1.10).

§2. Genus 1 counts

2.1. Overview

For each $\beta \in H_+(X)$, $N_{1,\beta}$ is the number of automorphism-weighted stable C^∞-maps

$$u \colon \Sigma \longrightarrow X$$

from prestable curve of genus 1 to X of degree β solving a perturbed Cauchy–Riemann equation,

$$\bar{\partial} u + \nu(u) = 0, \tag{2.1}$$

for a small generic multi-valued perturbation ν, see Section 1.3 of [21] for more details. If X is an ideal Calabi–Yau n-fold, $\overline{\mathfrak{M}}_1(X,\beta)$ decomposes into strata $\mathcal{Z}_{\mathcal{T}}$ which each have well-defined contribution to $N_{1,\beta}$ in following sense:

> *For every stratum $\mathcal{Z}_{\mathcal{T}}$, there exist $\mathrm{C}_{\mathcal{T}}(\beta)\in\mathbb{Q}$, $\epsilon_\nu\in\mathbb{R}^+$, and a compact subset K_ν of $\mathcal{Z}_{\mathcal{T}}$ with the following property. For every compact subset K of $\mathcal{Z}_{\mathcal{T}}$ and an open neighborhood U of K in the space of stable C^∞-maps, there exist an open neighborhood $U_\nu(K)$ of K and $\epsilon_\nu(U)\in(0,\epsilon_\nu)$, respectively,[4] such that*

$$^{\pm}\big|\{\bar{\partial}+t\nu\}^{-1}(0)\cap U\big| = \mathrm{C}_{\mathcal{T}}(\beta) \quad \textit{if } t\in(0,\epsilon_\nu(U)),\ K_\nu\subset K\subset U\subset U_\nu(K).$$

While there are many different strata, it turns out that $\mathrm{C}_{\mathcal{T}}(\beta)\neq 0$ only for strata of the three simplest types.

If X is an ideal Calabi–Yau n-fold, there are finitely many genus 1 curves in each homology class of X. Furthermore, every genus 1 curve $\mathcal{C}$ in X is embedded and super-rigid: if $\mathcal{N}$ is the normal bundle of $\mathcal{C}$ and

$$u\colon \Sigma\longrightarrow\mathcal{C}$$

is an unramified cover, then $H^0(\Sigma,u^*\mathcal{N})=0$. Hence, $H^1(\Sigma,u^*\mathcal{N})=0$ and for every $d\in\mathbb{Z}^+$

$$\mathcal{Z}_{(1,\beta/d)} = \bigcup_{[\mathcal{C}]=\beta/d} \overline{\mathfrak{M}}_1(\mathcal{C},d)$$

is a finite set of isolated regular points of $\overline{\mathfrak{M}}_1(X,\beta)$. Each such point u contributes $|1/\mathrm{Aut}(u)|$ to $N_{1,\beta}$. If $n_{1,\beta}$ is the number of genus 1 curves in the homology class β, then

$$\mathrm{C}_{(1,\beta/d)}(\beta) = \frac{\sigma(d)}{d}n_{1,\beta/d}, \tag{2.2}$$

where $\sigma(d)$ is the number of degree d unbranched covers of a genus 1 curve by connected genus 1 curves. The integral number $n_{1,\beta/d}$ is zero unless $d|\beta$, or equivalently, β/d is an integral homology class.

The remaining elements $u\colon\Sigma\longrightarrow X$ of $\overline{\mathfrak{M}}_1(X,\beta)$ are maps to genus 0 curves in X. They split into strata $\mathcal{Z}_{\mathcal{T}}$ indexed by combinatorial data described in Section 2.2. We will call a stratum $\mathcal{Z}_{\mathcal{T}}$ *basic* if either of the following conditions holds:

---

[4] $U_\nu(K)$ depends on K, while $\epsilon_\nu(U)$ depends on U.

(B1) the domain Σ of every element $[\Sigma, u]$ of $\mathcal{Z}_{\mathcal{T}}$ is a nonsingular genus 1 curve, or

(B2) the domain Σ of every element $[\Sigma, u]$ of $\mathcal{Z}_{\mathcal{T}}$ is a union of a nonsingular genus 1 curve Σ_P and a $\mathbb{P}^1$ and u is constant on Σ_P.

In both cases, the restriction of u to the non-contracted component must be a $d\!:\!1$ cover of a curve in the homology class β/d, for some $d\in\mathbb{Z}^+$. We will write $\mathcal{T}_{\text{eff}}(\beta/d, d)$ and $\mathcal{T}_{\text{gh}}(\beta/d, d)$ for the corresponding types of strata (B1) and (B2), with *eff* and *gh* standing for *effective* and *ghost* (principal component).

Theorem 2.1. *Suppose X is an ideal Calabi–Yau 5-fold.*

(i) If $\mathcal{Z}_{\mathcal{T}}$ is a stratum of $\overline{\mathfrak{M}}_1(X,\beta)$ consisting of maps to rational curves in X and is not basic, $\mathsf{C}_{\mathcal{T}}(\beta)=0$.

(ii) For $\beta\in H_+(X)$ and $d\in\mathbb{Z}^+$,

(2.3)
$$\mathsf{C}_{\mathcal{T}_{\text{eff}}(\beta,d)}(d\beta)=\frac{d-1}{d^2}\mathsf{C}_{\mathcal{T}_{\text{gh}}(\beta,1)}(\beta), \qquad \mathsf{C}_{\mathcal{T}_{\text{gh}}(\beta,d)}(d\beta)=\frac{1}{d^2}\mathsf{C}_{\mathcal{T}_{\text{gh}}(\beta,1)}(\beta).$$

In Section 2.3, we will prove

$$\mathsf{C}_{\mathcal{T}_{\text{gh}}(\beta,1)}(\beta)=\frac{1}{24}\int_{\overline{\mathcal{M}}_\beta}\big(2c_2(\overline{\mathcal{M}}_\beta)-c_1^2(\overline{\mathcal{M}}_\beta)\big). \tag{2.4}$$

On the other hand, the space $\overline{\mathcal{M}}_\beta=\overline{\mathfrak{M}}_0^{*}(X,\beta)$ consists of regular maps to X. Thus, the contribution to $N_{1,\beta}$ is given by the right side of equation (2.15) in [23]:

$$\mathsf{C}_{\mathcal{T}_{\text{gh}}(\beta,1)}(\beta)=\frac{1}{24}\bigg(-n_\beta\Big(\frac{c(X)}{1-\psi}\Big)+\frac{1}{2}\sum_{\substack{\beta_1+\beta_2=\beta\\ \beta_1,\beta_2\in H_+(X)}} n_{\beta_1\beta_2}(\psi_1+\psi_2|;)\bigg).$$

Comparing the above identity with (2.4), we find that

(2.5)
$$\int_{\overline{\mathcal{M}}_\beta}\big(2c_2(\overline{\mathcal{M}}_\beta)-c_1^2(\overline{\mathcal{M}}_\beta)\big)=-n_\beta\Big(\frac{c(X)}{1-\psi}\Big)+\frac{1}{2}\sum_{\substack{\beta_1+\beta_2=\beta\\ \beta_1,\beta_2\in H_+(X)}} n_{\beta_1\beta_2}(\psi_1+\psi_2|;).$$

We calculate the left side in terms of the Gromov–Witten invariants of X by expanding the right side via the equations of Section 1.

Our proof of Theorem 2.1 applies also in dimensions 3 and 4. In particular, the result provides a direct explanation of the $1/d$-scaling in the latter case discovered by other means in Section 2 of [10]. Many aspects of the proof are applicable in dimensions 6 and higher as well.

2.2. Preliminaries

Let X be an ideal Calabi–Yau 5-fold. The strata of $\overline{\mathfrak{M}}_1(X,\beta)$ consisting of maps to rational curves can be described by *decorated graphs*

$$\mathcal{T} = \big(\mathrm{Ver}, \mathrm{Edg}, \mathfrak{d}, \underline{\beta}, \kappa, i^*\big),$$

where

(D1) $\Gamma = (\mathrm{Ver}, \mathrm{Edg})$ is a connected graph containing either exactly one loop or a distinguished vertex, but not both,

(D2) $\underline{\beta} = (\beta_i)_{i\in[m]}$ is an m-tuple of elements of $H_+(X)$, with $m \in \{1,2,3\}$,

(D3) $\mathfrak{d} : \mathrm{Ver} \longrightarrow \mathbb{Z}^{\ge 0}$ is a map, $\kappa : \mathfrak{d}^{-1}(\mathbb{Z}^+) \longrightarrow [m]$ is a surjective map,

(D4) $i^* \in \{\star\} \cup [m]$.

The irreducible components and the nodes of the domain Σ of every element $[\Sigma, u]$ of $\mathcal{Z}_{\mathcal{T}}$ correspond to the sets Ver and Edg respectively. If $v \in \mathrm{Ver}$ is not the distinguished vertex of Γ, the corresponding component Σ_v of Σ is a $\mathbb{P}^1$. Otherwise, Σ_v is nonsingular of genus 1. If $v \in \mathrm{Ver}$, the restriction of u to Σ_v is constant if $\mathfrak{d}(v) = 0$. If $\mathfrak{d}(v) \neq 0$, $u|_{\Sigma_v}$ is a $\mathfrak{d}(v){:}1$ cover of the component $\mathcal{C}_{\kappa(v)}$ of $\mathcal{C}$. If $\mathfrak{d}$ does not vanish identically of the loop in the graph (Ver, Edg) or on the distinguished vertex, i^* is set to $\star$. If $\mathfrak{d}$ vanishes identically on the loop or on the distinguished vertex, the corresponding components of Σ are mapped by u to a point on the i^*-component of $\mathcal{C}$. Since u is continuous, $\mathcal{Z}_{\mathcal{T}} = \emptyset$ unless κ satisfies certain combinatorial conditions.[5]

Given a generic deformation of ν of the $\bar{\partial}$-operator as in (2.1) and sufficiently small $t \in \mathbb{R}^+$, we will determine the number of solutions $[\Sigma, u]$ of

$$\bar{\partial} u + t\nu(u) = 0, \tag{2.6}$$

with u close to the stratum $\mathcal{Z}_{\mathcal{T}}$. The assumption that ν is generic implies that all solutions of (2.6) are maps from nonsingular genus 1 curves. The arguments follow [19, 20]. In particular, the gluing construction for $\mathcal{Z}_{\mathcal{T}}$ will be performed on a family of representatives (Σ, u) for the elements $[\Sigma, u]$ in $\mathcal{Z}_{\mathcal{T}}$, see Section 2.2 of [20]. Our treatment here is less explicit in order to streamline the discussion.

[5] The strata $\mathcal{Z}_{\mathcal{T}}$ as defined above intersect if $m \ge 2$ and $\mathfrak{d}$ vanishes on the loop or the distinguished vertex of (Ver, Edg). The issue can be easily addressed by allowing i^* to take values in $\{\star\} \cup [m] \cup \{(1,2),(2,3)\}$. However, equation (2.6) will be shown to have no solutions near $\mathcal{Z}_{\mathcal{T}}$ for a good choice of ν if $m \ge 2$, so further discussion is not needed.

For the rest of Section 2, we fix a decorated graph $\mathcal{T}$ as above. We define

$$|\underline{\beta}| = \sum_{i=1}^{m} \beta_i \in H_+(X).$$

With notation as in Section 1.1, let

$$\mathcal{M}_{\underline{\beta}} = \mathfrak{M}^*_{0,\emptyset}(X,\underline{\beta}) \qquad \text{and} \qquad \overline{\mathcal{M}}_{\underline{\beta}} = \overline{\mathfrak{M}}^*_{0,\emptyset}(X,\underline{\beta}).$$

We denote by $\mathcal{M}_{\underline{\beta},1}$ and $\overline{\mathcal{M}}_{\underline{\beta},1}$ the spaces of pairs $(\mathcal{C},x)$ such that $\mathcal{C}\in\mathcal{M}_{\underline{\beta}}$ and $x\in\mathcal{C}$ is a nonsingular point of $\mathcal{C}$ in the first case and $\mathcal{C}\in\overline{\mathcal{M}}_{\underline{\beta}}$ and $x\in\mathcal{C}$ is any point of $\mathcal{C}$ in the second case.

Let $\mathcal{S}\longrightarrow\mathcal{M}_{\underline{\beta}}$ be a family of deformations in X of curves in $\mathcal{M}_{\underline{\beta}}$. In other words, the fiber $\mathcal{S}_{\mathcal{C}}$ of $\mathcal{S}$ over $\mathcal{C}\in\mathcal{M}_{\underline{\beta}}$ contains $\mathcal{C}$ and

$$\dim\mathcal{S}_{\mathcal{C}} = \dim\mathcal{M}_{|\underline{\beta}|,1} - \dim\mathcal{M}_{\underline{\beta}} = m.$$

There is a fibration

$$\pi_{\mathcal{C}} : \mathcal{S}_{\mathcal{C}} \longrightarrow \Delta\subset\mathbb{C}^{m-1} \tag{2.7}$$

giving the universal family of deformations of $\mathcal{C}$. If $m=1$, then $\mathcal{S}=\mathcal{M}_{|\underline{\beta}|,1}$. If $m=3$, $\mathcal{S}$ is a small neighborhood of $\mathcal{M}_{\underline{\beta},1}$ in $\overline{\mathcal{M}}_{|\underline{\beta}|,1}$.

If $\mathrm{ev}\colon\mathcal{M}_{\underline{\beta},1}\longrightarrow X$ is the evaluation map at the marked point, the bundle

$$Q = \mathrm{ev}^*TX/T\mathcal{S} \longrightarrow \mathcal{M}_{\underline{\beta},1} \tag{2.8}$$

extends naturally over $\overline{\mathcal{M}}_{\underline{\beta},1}$ so that there is an exact sequence

$$0\longrightarrow f^*T\overline{\mathcal{M}}_{\underline{\beta}}\longrightarrow Q\longrightarrow\mathcal{N}_{|\underline{\beta}|}\longrightarrow 0, \tag{2.9}$$

where $f:\overline{\mathcal{M}}_{\underline{\beta},1}\longrightarrow\overline{\mathcal{M}}_{\underline{\beta}}$ is the forgetful map and $\mathcal{N}_{|\underline{\beta}|}$ is the normal bundle to the family of simple curves of class $|\underline{\beta}|$.

Similarly to Section 3.3 in [12], we choose a family of "exponential" maps

$$\exp^{\mathcal{C}}\colon TX\longrightarrow X \text{ such that } \exp^{\mathcal{C}}_x(v)\in\mathcal{S}_{\mathcal{C}} \text{ if } x\in\mathcal{C},\ v\in T_x\mathcal{S}_{\mathcal{C}},\ |v|<\delta(\mathcal{C}), \tag{2.10}$$

for some $\delta\in C^{\infty}(\mathcal{M}_{\bar{\Gamma}};\mathbb{R}^+)$. Below we will place additional assumptions on $\exp^{\mathcal{C}}$ as needed.

For an ideal Calabi–Yau n-fold with $n\geq 6$, the above stratification would need to be refined further based on the deviation of the normal bundles of curves in $\mathcal{M}_{\underline{\beta},1}$ from balanced splitting. The arguments in Sections 2.3–2.5 below apply to the strata with balanced splitting with minor changes. The main change here is that the map ev is no longer an immersion, and one would need to pass to a blowup of $\overline{\mathcal{M}}_{\underline{\beta},1}$ to obtain analogues of the vector bundle Q and the short exact sequence (2.10). The strata with unbalanced splittings need to be treated separately, with the conclusion that they do not contribute to the genus 1 Gromov–Witten invariants under certain assumptions on X.

2.3. Strata with ghost principal component I

Here we describe the contribution to $N_{1,*}$ from a stratum $\mathcal{Z}_{\mathcal{T}}$ consisting of maps $u\colon \Sigma \longrightarrow X$ that are constant on the principal, genus-carrying, component(s) Σ_P of Σ. We show $\mathcal{Z}_{\mathcal{T}}$ does not contribute to $N_{1,*}$ unless $\mathcal{Z}_{\mathcal{T}}$ is of type (B2).

For each $m\in\mathbb{Z}^+$, let $\overline{\mathcal{M}}_{1,m}$ be the moduli space of stable curves of genus 1 with m marked points. Let $\mathbb{E}\longrightarrow\overline{\mathcal{M}}_{1,m}$ be the Hodge line bundle of holomorphic differentials. For each $i\in[m]$, denote by $L_i \longrightarrow \overline{\mathcal{M}}_{1,m}$ the universal tangent line bundle at the i^{th} marked point. Let

$$s_i \in \Gamma\big(\overline{\mathcal{M}}_{1,m}, \mathrm{Hom}(L_i, \mathbb{E}^*)\big)$$

be the homomorphism induced by the natural pairing of tangent and cotangent vectors at the ith marked point. Denote by

$$\mathcal{M}_{1,m}, \mathcal{M}_{1,m}^{\mathrm{eff}} \subset \overline{\mathcal{M}}_{1,m}$$

the subspaces consisting of nonsingular curves and of curves $\mathcal{C}$ with no bubble components ($\mathcal{C}$ is either a nonsingular genus 1 curve or is a circle of rational curves).

Let $L_1 \longrightarrow \overline{\mathfrak{M}}_{0,1}(X,\beta)$ be the universal tangent line bundle at the marked point. Denote by

$$\mathcal{D}_1 \in \Gamma\big(\overline{\mathfrak{M}}_{0,1}(X,\beta), \mathrm{Hom}(L_1, \mathrm{ev}_1^*TX)\big)$$

the natural homomorphism induced by the derivative of the map at the marked point. For $m\in\mathbb{Z}^+$, let

$$\overline{\mathfrak{M}}_{(0,m)}(X,\beta) = \Big\{(b_i)_{i\in[m]} \in \prod_{i=1}^{m} \overline{\mathfrak{M}}_{0,\{0\}}(X,\beta_i) : \beta_i \in H_+(X),\ \sum_{i=1}^{m}\beta_i = \beta,$$
$$\mathrm{ev}_0(b_i) = \mathrm{ev}_0(b_{i'})\ \forall i,i' \in [m]\Big\}.$$

There is a well-defined evaluation map

$$\mathrm{ev}_0\colon \overline{\mathfrak{M}}_{(0,m)}(X,\beta) \longrightarrow X, \qquad (b_i)_{i\in[m]} \longrightarrow \mathrm{ev}_0(b_i),$$

which is independent of the choice of i. Let

$$\pi_i\colon \overline{\mathfrak{M}}_{(0,m)}(X,\beta) \longrightarrow \bigsqcup_{\beta_i\in H_+(X)} \overline{\mathfrak{M}}_{0,\{0\}}(X,\beta_i)$$

be the projection onto the i^{th} component. Denote by

$$\mathfrak{M}^{\mathrm{eff}}_{(0,m)}(X,\beta) \subset \overline{\mathfrak{M}}_{(0,m)}(X,\beta)$$

the subset consisting of the tuples $(u_i)_{i\in[m]}$ such that for each $i\in[m]$ the restriction of u_i to the domain component carrying the marked point 0 is not constant.

The stratum $\mathcal{Z}_{\mathcal{T}}$ admits a decomposition

$$\mathcal{Z}_{\mathcal{T}} = \big(\mathcal{Z}_{\mathcal{T},P}\times\mathcal{Z}_{\mathcal{T},PB}\times\mathcal{Z}_{\mathcal{T},B}\big)\big/S_{m_B}, \tag{2.11}$$

where $\mathcal{Z}_{\mathcal{T},P}$ is a stratum of $\mathcal{M}^{\mathrm{eff}}_{1,m_P}$ for some $m_P\in\mathbb{Z}^+$, $\mathcal{Z}_{\mathcal{T},B}$ is a stratum of $\mathfrak{M}^{\mathrm{eff}}_{(0,m_B)}(X,\beta)$ for some $m_B\in\mathbb{Z}^+$, and $\mathcal{Z}_{\mathcal{T},PB}$ is a product of moduli spaces of irreducible stable genus 0 curves. The stratum $\mathcal{Z}_{\mathcal{T},P}$ consists of curves of a fixed topological type, while the elements of $\mathcal{Z}_{\mathcal{T},B}$ are tuples of stable maps from domains of fixed topological types so that the image of the restriction of the map to each component is of a specified homology class and multiplicity. The requirement that

$$\mathcal{Z}_{\mathcal{T},P}\subset\mathcal{M}^{\mathrm{eff}}_{1,m_P} \qquad\text{and}\qquad \mathcal{Z}_{\mathcal{T},B}\subset\mathfrak{M}^{\mathrm{eff}}_{(0,m_B)}(X,\beta)$$

implies that the decomposition (2.11) is well-defined. Let

$$\pi_P,\pi_B\colon \mathcal{Z}_{\mathcal{T},P}\times\mathcal{Z}_{\mathcal{T},PB}\times\mathcal{Z}_{\mathcal{T},B}\longrightarrow\mathcal{Z}_{\mathcal{T},P},\mathcal{Z}_{\mathcal{T},B}$$

denote the projection maps. The quotient is by the automorphism groups S_{m_B} of the data.

If X is an ideal CY 5-fold, $\mathcal{Z}_{\mathcal{T},B}$ is smooth. The cokernels of the linearizations D_b of the $\bar\partial$-operator along $\mathcal{Z}_{\mathcal{T}}$ form the obstruction bundle

$$\mathfrak{O} = \mathfrak{O}_{PB}\oplus\pi_B^*\mathfrak{O}_B = \pi_P^*\mathbb{E}^*\otimes\pi_B^*\mathrm{ev}_0^*TX\oplus\pi_B^*\mathfrak{O}_B, \tag{2.12}$$

where $\mathfrak{O}_B\longrightarrow\mathcal{Z}_{\mathcal{T},B}$ is the obstruction bundle associated with the moduli space $\overline{\mathfrak{M}}_{(0,m)}(X,\beta)$. Let $\bar\nu\in\Gamma(\mathcal{Z}_{\mathcal{T}},\mathfrak{O})$ be the section induced by ν: $\bar\nu(b)$ is the projection of $\nu(b)$ to the cokernel of D_b. We write

$$\bar\nu_{PB},\bar\nu_B\in\Gamma\big(\mathcal{Z}_{\mathcal{T}},\mathfrak{O}_{PB}\big),\Gamma\big(\mathcal{Z}_{\mathcal{T}},\mathfrak{O}_B\big)$$

for the two components of $\bar\nu$.

There is a natural projection map

$$\bar{\pi}: \mathcal{Z}_{\mathcal{T},B} \longrightarrow \mathcal{M}_{\underline{\beta},1},$$

sending $\pi_B([\Sigma, u])$ to $(u(\Sigma), u(\Sigma_P))$. Denote by

$$\bar{\nu}_{PB}^{\perp} \in \Gamma\big(\mathcal{Z}_{\mathcal{T}}, \pi_P^*\mathbb{E}^* \otimes \pi_B^*\bar{\pi}^* Q\big)$$

the image of $\bar{\nu}_{PB}$ under the natural projection map. Let

$$f: \overline{\mathcal{M}}_{1,m_P} \longrightarrow \overline{\mathcal{M}}_{1,1}$$

be the forgetful map, dropping all but the first marked point. The restriction of the bundle

$$\pi_1^*\mathbb{E}^* \otimes \pi_2^* Q \longrightarrow \overline{\mathcal{M}}_{1,1} \times \overline{\mathcal{M}}_{\underline{\beta},1} \tag{2.13}$$

to any boundary stratum $\mathcal{Z}_\Gamma$ contains a subbundle $\mathfrak{O}_\Gamma$ such that

$$\operatorname{rk} \mathfrak{O}_\Gamma - \dim \mathcal{Z}_\Gamma > \operatorname{rk}\big(\pi_1^*\mathbb{E}^* \otimes \pi_2^* Q\big) - \dim\big(\overline{\mathcal{M}}_{1,1} \times \overline{\mathcal{M}}_{\underline{\beta},1}\big) = 0 \tag{2.14}$$

and $\big\{(f \circ \pi_P) \times (\bar{\pi} \circ \pi_B)\big\}^* \mathfrak{O}_\Gamma$ is a quotient of the cokernel bundle over a boundary stratum of $\overline{\mathcal{Z}}_{\mathcal{T}}$. Thus, we can choose a section $\bar{\nu}_{\underline{\beta}}$ of (2.13) with all zeros transverse and contained in $\mathcal{M}_{1,1} \times \mathcal{M}_{\underline{\beta},1}$ and such that there exists ν as above satisfying

$$\bar{\nu}_{PB}^{\perp} = \big\{(f \circ \pi_P) \times (\bar{\pi} \circ \pi_B)\big\}^* \bar{\nu}_{\underline{\beta}}.$$

It is shown in the next section that the contribution of $\mathcal{Z}_{\mathcal{T}}$ to $N_{1,*}$ comes from $\bar{\nu}_{PB}^{\perp\,-1}(0)$. Thus, if $\mathcal{Z}_{\mathcal{T},P} \not\subset \mathcal{M}_{1,m_P}$, then $\bar{\nu}_{PB}^{\perp\,-1}(0)$ is empty for a good choice of ν by (2.14) and the stratum $\mathcal{Z}_{\mathcal{T}}$ does not contribute to $N_{1,*}$. Otherwise, $\bar{\nu}_{PB}^{\perp\,-1}(0)$ is the preimage of a finite subset in $\mathcal{M}_{1,1} \times \mathcal{M}_{\underline{\beta},1}$. It decomposes into connected components

$$\bar{\nu}_{PB}^{\perp\,-1}(0) = \bigsqcup_{(\mathcal{C},x) \in \pi_2(\bar{\nu}_{\underline{\beta}}^{-1}(0))} \mathcal{Z}_{\mathcal{C},x}, \tag{2.15}$$

where $\mathcal{C}$ is a $\underline{\beta}$-curve and x is a nonsingular point of $\mathcal{C}$. Then, $\mathsf{C}_{\mathcal{T}}(\beta)$ is the number of zeros of a map $\varphi_{t\nu}$ from the vector bundle F of gluing parameters to $\mathfrak{O}$ over each of the components $\mathcal{Z}_{\mathcal{C},x}$. The projection of $\varphi_{t\nu}$ in the decomposition (2.12) onto $\pi_P^*\mathbb{E} \otimes T_x\mathcal{S}_{\mathcal{C}}/T_x\mathcal{C}$ is essentially the same as the projection of $t\nu$, which we denote by $t\tilde{\nu}$. Since $\tilde{\nu}$ is a section of a trivial bundle over $\mathcal{Z}_{\mathcal{C},x}$, it can be chosen not to vanish if $m > 1$. Thus, $\mathsf{C}_{\mathcal{T}}(\beta) = 0$ if $m > 1$. On the other hand, the second component of $\varphi_{t\nu}$ with respect to the decomposition (2.12) is essentially $t\bar{\nu}_B$. It does not vanish on $\bar{\nu}_{PB}^{\perp\,-1}(0)$ for dimensional reasons if $m=1$, but $|\mathrm{Ver}|>2$. Thus, $\mathrm{C}_{\mathcal{T}}(\beta)=0$ if $\mathcal{T}$ is not basic.

Finally, if $\mathcal{T}$ is basic, the principal component of every element of $\mathcal{Z}_{\mathcal{C},x}$ is a fixed nonsingular genus 1 curve Σ_P with one special point z_1 and

$$\mathcal{Z}_{\mathcal{C},x} \approx \mathfrak{M}_{0,1}(\mathbb{P}^1_p, d),$$

where $\mathfrak{M}_{0,1}(\mathbb{P}^1_p, d) \subset \overline{\mathfrak{M}}_{0,1}(\mathbb{P}^1, d)$ is the subspace of elements $[\Sigma, u]$ such that Σ is nonsingular and $\mathrm{ev}_1([\Sigma, u])=p$ for a fixed $p\in\mathbb{P}^1$. Let

$$\mathfrak{D} \in \Gamma\big(\overline{\mathcal{M}}_{1,1}\times\overline{\mathfrak{M}}_{0,1}(\mathbb{P}^1, d), \mathrm{Hom}(\pi_1^*L_1\otimes\pi_2^*L_1, \pi_1^*\mathbb{E}^*\otimes\pi_2^*\mathrm{ev}_1^*T\mathbb{P}^1)\big)$$

be given by

$$\mathfrak{D}(v\otimes w) = s_1(v)\otimes\mathcal{D}_1(w)\,. \tag{2.16}$$

The first component of $\varphi_{t\nu}$ with respect to the decomposition (2.12) is essentially

$$F = \pi_1^*L_1|_{z_1}\otimes\pi_2^*L_1 \longrightarrow \mathfrak{O}_{PB} = \mathbb{E}^*_{\Sigma_P}\otimes T_x\mathcal{C}, \qquad v \longrightarrow \mathfrak{D}(v)+t\bar{\nu}_{PB}. \tag{2.17}$$

Let $\mathfrak{U}$ be the universal curve over $\overline{\mathfrak{M}}_{0,1}(\mathbb{P}^1, d)$, with structure map π and evaluation map ev:

$$\begin{array}{ccc} \mathfrak{U} & \xrightarrow{\mathrm{ev}} & \mathbb{P}^1 \\ \big\downarrow{\scriptstyle\pi} & & \\ \overline{\mathfrak{M}}_{0,1}(\mathbb{P}^1, d). & & \end{array} \tag{2.18}$$

The restriction of $\bar{\nu}_B$ to $\bar{\nu}_{PB}^{\perp\,-1}(0)$ is a section of

$$\mathfrak{O}_B = R^1\pi_*\mathrm{ev}_*\big(\mathcal{O}(-1)\oplus\mathcal{O}(-1)\big) \longrightarrow \mathfrak{M}_{0,1}(\mathbb{P}^1_p, d).$$

Thus, by the Aspinwall–Morrison and divisor formulas, as in Section 1.1 in [10],

$$^{\pm}\big|\bar{\nu}_B^{-1}(0)\big| = \frac{1}{d^2}. \tag{2.19}$$

On the other hand, $\bar{\nu}_{PB}^{\perp}$ is a section of

$$\pi_1^*\mathbb{E}^*\otimes\pi_2^*\big(f^*T\overline{\mathcal{M}}_\beta\oplus\mathcal{N}_\beta\big) \longrightarrow \overline{\mathcal{M}}_{1,1}\times\overline{\mathcal{M}}_{\beta,1},$$

see (2.8). Therefore,

$$^{\pm}\big|\bar{\nu}_{PB}^{\perp\,-1}(0)\big| = -\frac{1}{24}\int_{\overline{\mathcal{M}}_{\beta,1}}\big(f^*c_2(\overline{\mathcal{M}}_\beta)\mathrm{ev}_1^*c_1(\mathcal{N}_\beta)+f^*c_1(\overline{\mathcal{M}}_\beta)\,c_2(\mathcal{N}_\beta)\big). \tag{2.20}$$

Since (2.17) has a unique zero in every fiber of F over $\bar{\nu}_{PB}^{\perp\,-1}(0)\cap\bar{\nu}_B^{-1}(0)$ and the restriction of $\mathcal{N}_\beta$ to a fiber of f is of degree -2, equations (2.19) and (2.20) imply
(2.21)

$$\mathsf{C}_{\mathcal{T}_{\mathrm{gh}}(\beta,d)}(d\beta)=\frac{1}{24d^2}\left(\int_{\overline{\mathcal{M}}_\beta}2c_2(\overline{\mathcal{M}}_\beta)-\int_{\overline{\mathcal{M}}_{\beta,1}}f^*c_1(\overline{\mathcal{M}}_\beta)c_2(\mathcal{N}_\beta)\right).$$

We have proved the second scaling identity in (2.3). The equation

$$\mathsf{C}_{\mathcal{T}_{\mathrm{gh}}(\beta,1)}(\beta)=\frac{1}{24}\int_{\overline{\mathcal{M}}_\beta}2c_2(\overline{\mathcal{M}}_\beta)-c_1^2(\overline{\mathcal{M}}_\beta) \tag{2.22}$$

is obtained from (2.21) from relations (1.14) and (1.15).

2.4. Strata with ghost principal component II

We continue with the setup of Section 2.3. For each $[\Sigma_u,u]\in\mathcal{Z}_\mathcal{T}$, denote by $\Sigma_u^0\subset\Sigma_u$ the largest union of irreducible components of Σ_u that contains the principal component(s) of Σ_u and on which u is constant. The topological types of Σ_u and Σ_u^0 are independent of the choice of $[\Sigma_u,u]\in\mathcal{Z}_\mathcal{T}$.

The bundle of gluing parameters (or smoothing of the nodes) over $F\longrightarrow\mathcal{Z}_\mathcal{T}$ is a direct sum of line bundles (up to a quotient by a finite group). Let $F^\emptyset\subset F$ be the subspace of smoothings with all components nonzero, smoothings that do not leave any nodes. If $\upsilon\in F_u^\emptyset$ is sufficiently small, there is a C^∞-map

$$q_\upsilon\colon\Sigma_\upsilon\longrightarrow\Sigma_u,$$

where Σ_u is the domain of u and Σ_υ is a genus 1 Riemann surface with thin necks replacing the nodes of Σ_u, see Section 2.2 of [18]. This map determines Riemannian metrics and weights on Σ_υ which induce the L_1^p- and L^p Sobolev norms, $\|\cdots\|_{\upsilon,p,1}$ and $\|\cdots\|_{\upsilon,p}$, with $p>2$, appearing below, see Section 3.3 in [18]. These norms are equivalent to the ones used in Section 3 of [11]. Let $\Sigma_\upsilon^0=q_\upsilon^{-1}(\Sigma_u^0)$.

We take the approximately holomorphic map corresponding to $\upsilon\in F_u$ to be

$$u_\upsilon=u\circ q_\upsilon\colon\Sigma_\upsilon\longrightarrow X.$$

The map satisfies

$$\big\|\bar\partial u_\upsilon\big\|_{\upsilon,p}\le C(u)|\upsilon|^{1/p}. \tag{2.23}$$

Let

$$D_\upsilon\colon\Gamma(\upsilon)=\Gamma(\Sigma_\upsilon,u_\upsilon^*TX)\longrightarrow\Gamma^{0,1}(\upsilon)=\Gamma(\Sigma_\upsilon,T^*\Sigma_\upsilon^{0,1}\otimes u_\upsilon^*TX)$$

be the linearization of the $\bar{\partial}$-operator at u_υ defined using the Levi-Civita connection of a Kähler metric $g_{X,u}$ on X. As in Sections 2 and 4.1 in [20], we can construct splittings

$$\Gamma(\upsilon)=\Gamma_-(\upsilon)\oplus\Gamma_+(\upsilon) \text{ and } \Gamma^{0,1}(\upsilon)=\Gamma^{0,1}_{-;PB}(\upsilon)\oplus\Gamma^{0,1}_{-;B}(\upsilon)\oplus\Gamma^{0,1}_{+}(\upsilon), \tag{2.24}$$

and isomorphisms

$$R_\upsilon:\mathfrak{O}_{PB}\oplus\pi_B^*\mathfrak{O}_B\longrightarrow\Gamma^{0,1}_{-;PB}(\upsilon)\oplus\Gamma^{0,1}_{-;B}(\upsilon) \tag{2.25}$$

with the following properties:

(G1) $D_\upsilon:\Gamma_+(\upsilon)\longrightarrow\Gamma^{0,1}_+(\upsilon)$ is an isomorphism with the norm of the inverse bounded independently of $\upsilon\in F_u^\emptyset$ (but depending on $[\Sigma,u]$),

(G2) the elements of $\Gamma^{0,1}_{-;PB}(\upsilon)$ are supported on a small neighborhood of Σ^0_υ,

(G3) if $\pi^{0,1}_{-;PB}:\Gamma^{0,1}(\upsilon)\longrightarrow\Gamma^{0,1}_{-;PB}(\upsilon)$ is the projection in the second decomposition (2.24),

$$\big\|\pi^{0,1}_{-;PB}D_\upsilon\xi\big\|_{\upsilon,2}\le C(u)|\upsilon|\|\xi\|_{\upsilon,p,1},\ \forall\xi\in\Gamma(\upsilon), \tag{2.26}$$

(G4) if $|\mathrm{Ver}|=2$ (and thus $\mathcal{Z}_\mathcal{T}$ is basic),

$$\pi^{0,1}_{-;PB}\bar{\partial}u_\upsilon=R_\upsilon\mathfrak{D}\upsilon, \tag{2.27}$$

with $\mathfrak{D}$ as in (2.16),

(G5) every map $\tilde{u}:\Sigma\longrightarrow X$, where Σ is a smooth genus-one Riemann surface, that lies in a small neighborhood of $\mathcal{Z}_\mathcal{T}$ can be written uniquely as $\tilde{u}=\exp_{u_\upsilon}\xi$ for small $\upsilon\in F^\emptyset$ and $\xi\in\Gamma_+(\upsilon)$.

Let

$$\pi^{0,1}_+:\Gamma^{0,1}(\upsilon)\longrightarrow\Gamma^{0,1}_+(\upsilon)\qquad\text{and}\qquad\pi^{0,1}_{-;B}:\Gamma^{0,1}(\upsilon)\longrightarrow\Gamma^{0,1}_{-;B}(\upsilon)$$

be the component projections in the second decomposition (2.24).

The relation (2.6) for $\tilde{u}=\exp_{u_\upsilon}\xi$ is equivalent to

$$\bar{\partial}u_\upsilon+D_\upsilon\xi+t\nu_\upsilon+N_\upsilon(\xi)+tN_{\nu,\upsilon}(\xi)=0, \tag{2.28}$$

with N_υ and $N_{\nu,\upsilon}$ satisfying

$$\begin{aligned}\big\|N_\upsilon(\xi)-N_\upsilon(\xi')\big\|_{\upsilon,p}&\le C(u)\big(\|\xi\|_{\upsilon,p,1}+\|\xi'\|_{\upsilon,p,1}\big)\big\|\xi-\xi'\big\|_{\upsilon,p,1},\\ \big\|N_{\nu,\upsilon}(\xi)-N_{\nu,\upsilon}(\xi')\big\|_{\upsilon,p}&\le C(u)\big\|\xi-\xi'\big\|_{\upsilon,p,1},\end{aligned} \tag{2.29}$$

if $v\in F_u^{\emptyset}$. For a good choice of identifications,

$$\pi^{0,1}_{-;PB} N_v\xi = 0,\ \forall \xi \in \Gamma(v). \tag{2.30}$$

By the Contraction Principle and (G1), the equation

$$\pi^{0,1}_{+}\big(\bar\partial u_v + D_v\xi + t\nu_v + N_v(\xi) + tN_{\nu,v}(\xi)\big) = 0$$

has a unique small solution $\xi_{t\nu}(v)\in\Gamma_+(v)$. By (2.23), it satisfies

$$\big\|\xi_{t\nu}(v)\big\|_{v,p,1} \le C(u)\big(|v|^{1/p}+t\big). \tag{2.31}$$

Thus, the number of solutions of (2.6) near $\mathcal{Z}_\mathcal{T}$ is the number of solutions of the equation

$$\bar\partial u_v + D_v\xi_{t\nu}(v) + t\nu_v + N_v\big(\xi_{t\nu}(v)\big) + tN_{\nu,v}\big(\xi_{t\nu}(v)\big) = 0 \in \Gamma^{0,1}_{-;PB}(v)\oplus\Gamma^{0,1}_{-;B}(v). \tag{2.32}$$

This is an equation on $v\in F_u^{\emptyset}$ with $|v|<\delta(u)$ for some $\delta\in C^\infty(\mathcal{Z}_\mathcal{T};\mathbb{R}^+)$.

For each $[\Sigma,u]\in\mathcal{Z}_\mathcal{T}$, let $\mathcal{C}_u = u(\Sigma)$ and $\mathcal{S}_u = \mathcal{S}_{\mathcal{C}_u}$, see the end of Section 2.2. The pregluing map u_v satisfies

$$u_v(\Sigma_v) \subset \mathcal{C}_u \subset \mathcal{S}_u.$$

We can choose the splittings (2.24) so that they restrict to splittings for vector fields and $(0,1)$-forms along u_v with values in $T\mathcal{S}_u$ and (G1) holds when restricted to $T\mathcal{S}_u$. If $\exp_{u_v}\xi$ is defined using the "exponential" $\exp^{\mathcal{C}_u}$, the operators D_v and N_v in (2.28) preserve $T\mathcal{S}_u$ as well. Therefore,

$$\xi_0(v)\in\Gamma(\Sigma_v, u_v^*T\mathcal{S}_u),\ D_v\xi_0(v), N_v\big(\xi_0(v)\big)\in\Gamma\big(\Sigma_v, T^*\Sigma_v^{0,1}\otimes u_v^*T\mathcal{S}_u\big). \tag{2.33}$$

On the other hand, by (G1) and (2.29),

$$\big\|\xi_{t\nu}(v)-\xi_0(v)\big\|_{v,p,1} \le C(u)t, \tag{2.34}$$

if $v\in F_u$ is sufficiently small. Taking the projection $\pi^{\perp}_{-;PB}$ of (2.32) to $\pi_P^*\mathbb{E}^*\otimes\bar\pi^*Q$, we thus find that any solution $v\in F_u$ of (2.32) satisfies

$$\big\|\bar\nu^{\perp}_{PB}(u)\big\| \le \varepsilon(t,v)$$

for some function $\varepsilon\colon \mathbb{R}\times F^{\emptyset}\longrightarrow\mathbb{R}^+$ approaching 0 as (t,v) approaches 0. Therefore, all solutions of (2.6) lie in a small neighborhood of $\bar\nu^{\perp\,-1}_{PB}(0)\subset\mathcal{Z}_\mathcal{T}$, as claimed in Section 2.3.

If $m=1$, for a good choice of R_v on $\pi_B^*\mathfrak{O}_B$

$$(2.35)\quad \langle\!\langle \eta,\eta_-\rangle\!\rangle_{v,2}=0,\ \forall\ \eta\in\Gamma\big(\Sigma_v,T^*\Sigma_v^{0,1}\otimes u_v^*TS_u\big),\ \eta_-\in\Gamma^{0,1}_{-;B}(v).^6$$

Taking the projection of (2.32) onto $\Gamma^{0,1}_{-;B}(v)$ and using (2.29), (2.33), (2.34), and (2.35), we obtain

$$(2.36)\qquad t\bar{\nu}_B(u)+t\eta(t,v)=0,$$

for some $\eta(t,v)$ approaching 0 as (t,v) approaches 0. Since u is $d\!:\!1$ cover of the smooth curve $\mathcal{C}_u$, the dimension of the projection of $\bar{\nu}_{PB}^{\perp\,-1}(0)$ onto the third component in the decomposition (2.11) is of dimension at most $2d-2m_B$.[7] Since the rank of $\mathfrak{O}_B$ is $2d-2$, (2.36) has no solutions for a generic choice of ν unless $\mathcal{Z}_{\mathcal{T}}$ is described by (B2) of Section 2.1. If $\mathcal{Z}_{\mathcal{T}}$ is of type (B2), the number of solutions of (2.32) is the same as the number of small solutions of

$$(2.37)\qquad \mathfrak{D}v+t\bar{\nu}_{PB}(u)+\eta(t,v)=0\in\mathbb{E}^*\otimes T_{u(\Sigma_P)}\mathcal{C}_u$$
$$v\in\pi_1^*L_1\otimes\pi_2^*L_1|_u,\quad u\in\bar{\nu}_{PB}^{\perp\,-1}(0)\cap\bar{\nu}_B^{-1}(0)\subset\mathcal{M}_{1,1}\times\mathfrak{M}_{0,1}(\mathbb{P}^1;d),$$

with the error term $\eta(t,v)$ satisfying

$$\big\|\eta(t,v)\big\|_{v,2}\le\varepsilon(t,v)\big(t+|v|\big);$$

see (2.26), (2.27), (2.29), and (2.31). If ν is generic, $\mathcal{D}_1$ and thus $\mathfrak{D}$ are nowhere zero on the finite set $\bar{\nu}_{PB}^{\perp\,-1}(0)\cap\bar{\nu}_B^{-1}(0)$. By the same rescaling and cobordism argument as in Section 3.1 of [19], the number of small solutions of (2.37) is the same as the number of solutions of

$$\mathfrak{D}v+\bar{\nu}_{PB}(u)=0,\qquad v\in\pi_1^*L_1\otimes\pi_2^*L_1\big|_{\bar{\nu}_{PB}^{\perp\,-1}(0)\cap\bar{\nu}_B^{-1}(0)}.$$

There is one solution for each of the elements of $\bar{\nu}_{PB}^{\perp\,-1}(0)\cap\bar{\nu}_B^{-1}(0)$. This concludes the consideration of the $m=1$ case.

We will next show that (2.32) has no solution if $m>1$. Let $\mathcal{Z}_{\mathcal{C},x}$ be as in (2.15). Since x is a nonsingular point of $\mathcal{C}$, on a neighborhood U of x in $\mathcal{C}$ there is an orthogonal decomposition

$$(2.38)\qquad TS_{\mathcal{C}}|_U=T\mathcal{C}|_U\oplus\mathcal{N}\mathcal{C}|_U.$$

[6] In this case, $\mathfrak{O}_B$ is isomorphic to the cokernel of a $\bar{\partial}$-operator on $\mathcal{N}_{|\beta|}$.

[7] This is the dimension of the space of degree-d covers of $\mathbb{P}^1$ by m_B copies of $\mathbb{P}^1$. The dimension is less than $2d-2m_B$ unless $\mathcal{Z}_{\mathcal{T},B}$ is the main stratum of $\overline{\mathfrak{M}}_{(0,m_B)}(X,\beta)$.

We can assume that the "exponential" map $\exp^{\mathcal{C}}$ satisfies

$$\pi_{\mathcal{C}}\big(\exp_y^{\mathcal{C}} v\big) = d\pi_{\mathcal{C}}|_x v, \ \forall y \in U, \, v \in T_y\mathcal{S}_{\mathcal{C}}, \, |v| < \delta, \tag{2.39}$$

with $\pi_{\mathcal{C}}$ as in (2.7). For any $[\Sigma, u] \in \mathcal{Z}_{\mathcal{T}}$ in a small neighborhood of $\mathcal{Z}_{\mathcal{C},x}$, let $W_u = u^{-1}(U)$ be an open neighborhood of Σ_u^0 in Σ_0. We can assume that every element η of $\Gamma^{0,1}_{-;PB}(\upsilon)$ is supported in $W_\upsilon = q_\upsilon^{-1}(W_u)$, whenever $\upsilon \in F_u^{\emptyset}$ is sufficiently small. With D_υ^* denoting the formal adjoint of D_υ with respect to the inner-product $\langle\!\langle \cdot, \cdot \rangle\!\rangle_{\upsilon,2}$, let

$$\begin{aligned}
\Gamma_{+-}(\upsilon) &= \big\{\xi \in \Gamma_+(\upsilon) \colon \langle\!\langle \xi, D_\upsilon^* R_\upsilon \eta \rangle\!\rangle = 0, \forall \eta \in \mathbb{E}^*_{\Sigma_P} \otimes \mathcal{N}_{u(\Sigma_P)}\mathcal{C}_u\big\} \subset \Gamma(\upsilon),\\
\Gamma_{++}(\upsilon) &= \big\{D_\upsilon^* R_\upsilon \eta \colon \eta \in \mathbb{E}^*_{\Sigma_P} \otimes \mathcal{N}_{u(\Sigma_P)}\mathcal{C}_u\big\} \subset D_\upsilon^*\Gamma^{0,1}_{-;PB}(\upsilon) \subset \Gamma(\upsilon).
\end{aligned} \tag{2.40}$$

An explicit expression for $D_\upsilon^* R_\upsilon \eta$ is given in the proof of Lemma 2.2 in [19]. Section 2.3 of [19] implies that we can take

$$\Gamma_+(\upsilon) = \Gamma_{++}(\upsilon) \oplus \Gamma_{+-}(\upsilon). \tag{2.41}$$

In particular, the proof of Lemma 2.6 shows that the limits of the spaces $\Gamma_{++}(\upsilon)$ as $\upsilon \longrightarrow 0$ are orthogonal to the limits of the spaces $\Gamma_-(\upsilon)$. The decomposition (2.41) is L^2-orthogonal by (2.40) and

$$\|\xi\|_{\upsilon,p,1} \le C(u)\|\xi\|_{\upsilon,2}, \ \forall \xi \in \Gamma_{++}(\upsilon), \tag{2.42}$$

see the proof of Lemma 2.2 in [19].

Let $\xi^+_{t\nu}(\upsilon)$ and $\xi^-_{t\nu}(\upsilon)$ be the components of $\xi_{t\nu}(\upsilon)$ with respect to the decomposition (2.41). Denote by $\tilde{\nu}(u) \in \mathbb{E}^*_{\Sigma_P} \otimes \mathcal{N}_{u(\Sigma_P)}\mathcal{C}_u$ the projection of $\nu(u)$ to $\mathbb{E}^*_{\Sigma_P} \otimes \mathcal{N}_{u(\Sigma_P)}\mathcal{C}_u$. Since $\tilde{\nu}$ is a section of a trivial bundle near $\mathcal{Z}_{\mathcal{C},x}$, we can assume that it has no zeros on $\mathcal{Z}_{\mathcal{C},x}$. In the next paragraph we will show

$$\big\|\xi^+_{t\nu}(\upsilon)\big\|_{\upsilon,p,1} \le C(u)t. \tag{2.43}$$

Assuming this is the case, we project both sides of (2.32) onto

$$R_\upsilon\big(\mathbb{E}^*_{\Sigma_P} \otimes \mathcal{N}_{u(\Sigma_P)}\mathcal{C}_u\big) \subset \Gamma^{0,1}_{-;PB}(\upsilon)$$

and take the preimage under R_υ. Since the projections of $\bar{\partial} u_\upsilon$ and $N_\upsilon(\xi_{t\nu(\upsilon)})$ vanish, using the first equation in (2.40), (2.43), (2.27), and (2.29), we obtain

$$t\tilde{\nu}(u) + \eta(t, \upsilon) = 0$$

with $\eta(t, \upsilon)$ satisfying

$$\big\|\eta(t, \upsilon)\big\|_{\upsilon,2} \le \varepsilon(t, u)t.$$

However, this is impossible if t and v are sufficiently small ("small" depending continuously on u), since $\tilde{\nu}$ has no zeros over $\mathcal{Z}_{\mathcal{C},x}$.

We now verify (2.43). Let

$$\tilde{\xi}_{t\nu}(v) = \pi_{\mathcal{C}_u} \circ \exp_{u_v} \xi_{t\nu}(v) \colon \Sigma_v \longrightarrow \mathbb{C}^{m-1}, \qquad \tilde{\xi}^{\pm}_{t\nu}(v) = d\pi_{\mathcal{C}_u} \circ \xi^{\pm}_{t\nu}(v).$$

Since $\xi^{+}_{t\nu}(v)$ is supported on W_u, by (2.39)

$$(2.44) \qquad \big\langle \tilde{\xi}^{+}_{t\nu}(v), \tilde{\xi}^{\pm}_{t\nu}(v) \big\rangle_z = \big\langle \xi^{+}_{t\nu}(v), \xi^{\pm}_{t\nu}(v) \big\rangle_z, \ \forall\, z \in \Sigma_v .$$

By (2.39), we also have

$$(2.45) \qquad \tilde{\xi}_v|_{W_v} = \tilde{\xi}^{+}_v|_{W_v} + \tilde{\xi}^{-}_v|_{W_v} .$$

Since (2.32) is equivalent to (2.6) for $\tilde{u} = \exp_{u_v} \xi_{t\nu}(v)$,

$$(2.46) \qquad \big\| \bar{\partial} \tilde{\xi}_{t\nu}(v) \big\|_{v,p} \le C(u) t.$$

Since the operator

$$L^p_1(\Sigma_v, \mathbb{C}^{m-1}) \to L^p(\Sigma_v, T^*\Sigma^{0,1}_v \mathbb{C}^{m-1}) \oplus \mathbb{C}^{m-1}, \ \tilde{\xi} \to \Big(\bar{\partial}\tilde{\xi}, \int_{\Sigma_v} \tilde{\xi}\, dvol_{\Sigma_v} \Big),$$

is an isomorphism with the norm of the inverse bounded independently of v (but depending on u), (2.46) implies that

$$(2.47) \qquad \big\| \tilde{\xi}_{t\nu}(v) - A_{t\nu}(v) \big\|_{v,p,1} \le C(u) t$$

for some $A_{t\nu}(v) \in \mathbb{C}^{m-1}$. Since $\xi^{+}_{t\nu}(v)$ is supported on W_u, by (2.39) and (2.40),

$$\big\langle\!\big\langle \tilde{\xi}^{+}_{t\nu}(v), A_{t\nu}(v) \big\rangle\!\big\rangle_{v,2} = 0.$$

Thus, by (2.44), (2.45), and (2.47),

$$\begin{aligned} \big\| \xi^{+}_{t\nu}(v) \big\|_{v,2} = \big\| \tilde{\xi}^{+}_{t\nu}(v)|_{W_v} \big\|_{v,2} &\le \big\| (\tilde{\xi}_{t\nu}(v) - A_{t\nu}(v))|_{W_v} \big\|_{v,2} \\ &\le \big\| \tilde{\xi}_{t\nu}(v) - A_{t\nu}(v) \big\|_{v,p,1} \le C'(u) t. \end{aligned}$$

The estimate (2.43) now follows from (2.42).

Finally, we comment on the choices made in (2.24) and (2.25). Choosing the splittings (2.24) so that (G1) and (G5) hold is essentially equivalent to choosing approximate kernel and cokernel for D_v that vary smoothly with v. This is easily accomplished in many possible ways, including via the construction in Section 3 of [11]. In order to ensure that (G2)–(G4) hold, R_v on $\mathfrak{O}_{PB}$ is constructed by pushing harmonic forms on Σ_P over a small neighborhood of Σ^0_v, see Section 2.2 of [19]. Finally, in order to obtain (2.30), define $\exp_{u_v}$ and parallel transport using a Kähler metric which is flat near $u(\Sigma_P)$, as in Section 2.1 of [19].

2.5. Strata with effective principal component

We determine here the contribution to $N_{1,*}$ from a stratum $\mathcal{Z}_{\mathcal{T}}$ consisting of maps $u\colon \Sigma \longrightarrow X$ that are not constant on the principal, genus-carrying, component(s) Σ_P of Σ. We show that $\mathcal{Z}_{\mathcal{T}}$ does not contribute to $N_{1,*}$ unless $\mathcal{Z}_{\mathcal{T}}$ is of type (B1).

Let $\mathfrak{O} \longrightarrow \mathcal{Z}_{\mathcal{T}}$ and $F \longrightarrow \mathcal{Z}_{\mathcal{T}}$ be the obstruction bundle and the bundle of gluing parameters as before. The projection map $\mathrm{ev}^*TX \longrightarrow Q$ induces a surjective homomorphism

$$\pi^{\perp}\colon \mathfrak{O} \longrightarrow \mathfrak{O}^{\perp}, \tag{2.48}$$

where $\mathfrak{O}^{\perp}|_{[\Sigma,u]}$ is the cokernel of the $\bar{\partial}$-operator on Q induced by the $\bar{\partial}$-operator D_b on TX. By a gluing and obstruction bundle analysis similar to Section 2.4, $\mathrm{C}_{\mathcal{T}}(*)$ is the number of zeros of a bundle map

$$\varphi_{t\nu}\colon F \longrightarrow \mathfrak{O}$$

over $\mathcal{Z}_{\mathcal{T}}$ for t sufficiently small. As in the previous case, all zeros of $\varphi_{t\nu}$ arise from the zeros of

$$\bar{\nu}^{\perp} = \pi^{\perp} \circ \bar{\nu},$$

where $\bar{\nu} \in \Gamma(\mathcal{Z}_{\mathcal{T}}; \mathfrak{O})$ is the section induced by ν. The homomorphism (2.48) extends to a surjective homomorphism from the cokernel bundles over $\bar{\mathcal{Z}}_{\mathcal{T}}$. In the next two paragraphs, we show that $\mathfrak{O}^{\perp} \longrightarrow \bar{\mathcal{Z}}_{\mathcal{T}}$ contains a trivial C^{∞}-subbundle unless $|\mathrm{Ver}| = 1$. Therefore, $\mathrm{C}_{\mathcal{T}}(*) = 0$ if $\mathcal{Z}_{\mathcal{T}}$ is not of type (B1).

Suppose first that (Ver, Edg) contains a loop $L \subset \mathrm{Ver}$ and κ is not constant on L. Then, the image of the principal components Σ_P of any element $[\Sigma, u]$ of $\mathcal{Z}_{\mathcal{T}}$ contains at least two curves in X. Then, $\mathfrak{O}^{\perp}$ contains a pull-back of the bundle

$$\mathbb{E}^* \otimes f^*T\overline{\mathcal{M}}_{\underline{\beta}} \oplus s^*\mathcal{N}_{|\underline{\beta}|} \longrightarrow \overline{\mathcal{M}}_{1,1} \times \overline{\mathcal{M}}_{\underline{\beta}},$$

where $s\colon \overline{\mathcal{M}}_{\underline{\beta}} \longrightarrow \overline{\mathcal{M}}_{\underline{\beta},1}$ is the bundle section taking each curve $\mathcal{C}$ to one of the nodes. The bundle contains a trivial C^{∞}-subbundle for dimensional reasons.

We next consider the remaining cases. Let $P \in [m]$ be the component of the curves in $\overline{\mathcal{M}}_{\underline{\beta}}$ containing the image of the principal component Σ_P of any element $[\Sigma, u]$ of $\mathcal{Z}_{\mathcal{T}}$. Denote by $\overline{\mathcal{M}}_{\underline{\beta},1}^{P} \subset \overline{\mathcal{M}}_{\underline{\beta},1}$ the component consisting of the curves $\mathcal{C}_P$, with $\mathcal{C} \in \overline{\mathcal{M}}_{\underline{\beta}}$. If X is an ideal Calabi–Yau 5-fold and $m > 1$ (implying $\mathcal{C} \neq \mathcal{C}_P$), the restriction of $\mathcal{N}_{|\underline{\beta}|}$ to $\mathcal{C}_P \approx \mathbb{P}^1$ splits as either $\mathcal{O} \oplus \mathcal{O}$ or $\mathcal{O} \oplus \mathcal{O}(-1)$. If $m = 1$, the splitting is $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$.

Case 1: $m>1$ and the restriction of $\mathcal{N}_{|\underline{\beta}|}$ to $\mathcal{C}_P$ splits as $\mathcal{O}\oplus\mathcal{O}$. Here, $Q|_{\overline{\mathcal{M}}^P_{\underline{\beta},1}}=f^*\bar{Q}$ for a bundle $\bar{Q}\longrightarrow\overline{\mathcal{M}}_{\underline{\beta}}$. Since the restriction of $f^*\bar{Q}$ to $u(\Sigma_P)$ is trivial, $\mathfrak{O}^\perp$ contains the subbundle $\mathbb{E}^*\otimes f^*\bar{Q}$, where $\mathbb{E}\longrightarrow\overline{\mathfrak{M}}_1(X,\beta)$ is the Hodge line bundle. The subbundle is a pull-back of the bundle

$$\mathbb{E}^*\otimes\bar{Q}\longrightarrow\overline{\mathcal{M}}_{1,1}\times\overline{\mathcal{M}}_{\underline{\beta}},$$

which contains a trivial C^∞-subbundle for dimensional reasons by (2.8).

Case 2: $m>1$ and the restriction of $\mathcal{N}_{|\underline{\beta}|}$ to $\mathcal{C}_P$ splits as $\mathcal{O}\oplus\mathcal{O}(-1)$. Here, $Q|_{\overline{\mathcal{M}}^P_{\underline{\beta},1}}$ contains a subbundle $f^*\bar{Q}'$ of co-rank 1 for a bundle $\bar{Q}'\longrightarrow\overline{\mathcal{M}}_{\underline{\beta}}$. Since the restriction of $f^*\bar{Q}'$ to $u(\Sigma_P)$ is trivial, $\mathfrak{O}^\perp$ contains the subbundle $\mathbb{E}^*\otimes f^*\bar{Q}'$, which is a pull-back of the bundle

$$\mathbb{E}^*\otimes\bar{Q}'\longrightarrow\overline{\mathcal{M}}_{1,1}\times\overline{\mathcal{M}}_{\underline{\beta}}.$$

Thus, the subbundle admits a section s such that $s^{-1}(0)$ is contained in the union of the spaces $\overline{\mathfrak{M}}_1(\mathcal{C}_i,A_i)$ taken over finitely many $\underline{\beta}$-curves $\mathcal{C}_i$. Since the restriction of $\mathcal{N}_{|\underline{\beta}|}$ to $\mathcal{C}_P$ contains $\mathcal{O}(-1)$, $\mathfrak{O}^\perp$ also contains a line subbundle is isomorphic to $\mathrm{ev}^*_{z_1}L$, where $\mathrm{ev}_{z_1}:\bar{\mathcal{Z}}_\mathcal{T}\longrightarrow X$ is the evaluation map sending $[\Sigma,u]$ to the value of u at a node of Σ taken to a node of $\mathcal{C}_P$. The restriction of this subbundle to $s^{-1}(0)$ is trivial.

Case 3: $m=1$. Here, $\mathfrak{O}^\perp=\mathfrak{O}$ is a bundle of the same rank as the dimension of $\overline{\mathfrak{M}}_1(\overline{\mathcal{M}}_{\beta,1},d)$ for some $d\in\mathbb{Z}^+$. Thus, if $\mathcal{Z}_\mathcal{T}$ is not the main stratum of $\overline{\mathfrak{M}}_1(\overline{\mathcal{M}}_{\beta,1},d)$, the restriction of $\mathfrak{O}^\perp$ to $\bar{\mathcal{Z}}_\mathcal{T}$ contains a trivial C^∞-subbundle.

It remains to consider the case $\mathcal{Z}_\mathcal{T}=\mathcal{Z}_{\mathcal{T}_{\mathrm{eff}}(\beta,d)}$ with $|\mathrm{Ver}|=1$. Then, $\varphi_{t\nu}$ is a generic section of

$$\mathfrak{O}=\mathfrak{O}^\perp=\mathbb{E}^*\otimes f^*T\overline{\mathcal{M}}_\beta\oplus R^1\pi_*\mathrm{ev}^*\mathcal{N}_\beta\longrightarrow\overline{\mathfrak{M}}{}^0_1(\overline{\mathcal{M}}_{\beta,1},d),$$

where $\overline{\mathfrak{M}}{}^0_1(\overline{\mathcal{M}}_{\beta,1},d)\subset\overline{\mathfrak{M}}_1(\overline{\mathcal{M}}_{\beta,1},d)$ is the closure of the space of maps with smooth domains, π is the structure map for the universal curve over $\overline{\mathfrak{M}}_1(\overline{\mathcal{M}}_{\beta,1},d)$, and ev is the corresponding evaluation map, see (2.18).

Thus,

(2.49)
$$\begin{aligned} \mathsf{C}_{\mathcal{T}_{\mathrm{eff}}(\beta,d)}(d\beta) &= \big\langle e\big(\mathbb{E}^*\otimes f^*T\overline{\mathcal{M}}_\beta\big)e\big(R^1\pi_*\mathrm{ev}^*\mathcal{N}_\beta\big), \overline{\mathfrak{M}}_1^0(\overline{\mathcal{M}}_{\beta,1},d)\big\rangle \\ &= \big\langle c_2\big(\overline{\mathcal{M}}_\beta\big),\overline{\mathcal{M}}_\beta\big\rangle \int_{\overline{\mathfrak{M}}_1^0(\mathbb{P}^1,d)} e\big(R^1\pi_*\mathrm{ev}^*\big(\mathcal{O}(-1)\oplus\mathcal{O}(-1)\big)\big) \\ &\quad - \big\langle \lambda\, f^*c_1\big(\overline{\mathcal{M}}_\beta\big)e\big(R^1\pi_*\mathrm{ev}^*\mathcal{N}_\beta\big), \overline{\mathfrak{M}}_1^0(\overline{\mathcal{M}}_{\beta,1},d)\big\rangle, \end{aligned}$$

where $\lambda = c_1(\mathbb{E})$. Using the Atiyah–Bott Localization Theorem of [2] as in Section 27.5 of [8], we find

(2.50)
$$\int_{\overline{\mathfrak{M}}_1^0(\mathbb{P}^1,d)} e\big(R^1\pi_*\mathrm{ev}^*\big(\mathcal{O}(-1)\oplus\mathcal{O}(-1)\big)\big) = \frac{d-1}{12d^2}\,.$$[8]

In the next paragraph, we will obtain

(2.51)
$$\begin{aligned} &\big\langle \lambda\, f^*c_1\big(\overline{\mathcal{M}}_\beta\big)e\big(R^1\pi_*\mathrm{ev}^*\mathcal{N}\big), \overline{\mathfrak{M}}_1^0(\overline{\mathcal{M}}_{\beta,1},d)\big\rangle \\ &\qquad\qquad = \frac{d-1}{24d^2}\int_{\overline{\mathcal{M}}_{\beta,1}} c_2(\mathcal{N}_\beta)\, f^*c_1(\overline{\mathcal{M}}_\beta). \end{aligned}$$

Along with (2.49), (2.50), and (2.21), we conclude the first identity in (2.3).

With $\mathfrak{M}_{0,2}(\overline{\mathcal{M}}_{\beta,1},d)\subset\overline{\mathfrak{M}}_{0,2}(\overline{\mathcal{M}}_{\beta,1},d)$ denoting the locus of maps with nonsingular domains, let

$$\mathfrak{M}_{0,1=2}(\overline{\mathcal{M}}_{\beta,1},d) = \big\{b\in\mathfrak{M}_{0,2}(\overline{\mathcal{M}}_{\beta,1},d)\colon \mathrm{ev}_1(b)=\mathrm{ev}_2(b)\big\}.$$

Denote by $\overline{\mathfrak{M}}^0_{0,1=2}(\overline{\mathcal{M}}_{\beta,1},d)$ the closure of $\mathfrak{M}^0_{0,1=2}(\overline{\mathcal{M}}_{\beta,1},d)$ in $\overline{\mathfrak{M}}_{0,2}(\overline{\mathcal{M}}_{\beta,1},d)$. Let

$$\overline{\Delta}_1^0(\overline{\mathcal{M}}_{\beta,1},d)\subset\overline{\mathfrak{M}}_1^0(\overline{\mathcal{M}}_{\beta,1},d)$$

[8] The space $\overline{\mathfrak{M}}_1^0(\mathbb{P}^1,d)$ is singular, and thus [2] is not generally applicable. However, with the setup of Section 27.5 in [8], the restrictions of $e\big(R^1\pi_*\mathrm{ev}^*\big(\mathcal{O}(-1)\oplus\mathcal{O}(-1)\big)\big)$ to all fixed loci, with the exception of the simplest 1-edged ones, vanish. Furthermore, $\overline{\mathfrak{M}}_1^0(\mathbb{P}^1,d)$ is smooth along the 1-edged fixed loci. Therefore, the usual Atiyah–Bott localization formula applies. The normal bundles to the only contributing loci are the same as the normal bundles in the desingularization $\widetilde{\mathfrak{M}}_1^0(\mathbb{P}^1,d)$ of $\overline{\mathfrak{M}}_1^0(\mathbb{P}^1,d)$ constructed in [17] and described in Section 1.4 in [17].

be the subspace consisting of the stable maps $[\Sigma, u]$ such that the principal component Σ_P of Σ is singular. There is a natural node-identifying immersion

$$\iota\colon \overline{\mathfrak{M}}^0_{0,1=2}(\overline{\mathcal{M}}_{\beta,1}, d)/S_2 \longrightarrow \overline{\Delta}^0_1(\overline{\mathcal{M}}_{\beta,1}, d),$$

which is an embedding outside of a divisor. Note

$$\iota^* e\big(R^1\pi_*\mathrm{ev}^*\mathcal{N}_\beta\big) = c_2(\mathcal{N}_\beta)\, e\big(R^1\pi_*\mathrm{ev}^*\mathcal{N}_\beta\big).$$

If $\Delta_1 \subset \overline{\mathcal{M}}_{1,1}$ is the locus of the nodal elliptic curve,

$$\lambda = \frac{1}{12}\Delta_1 \in H^2\big(\overline{\mathcal{M}}_{1,1}\big).$$

Therefore,

$$\begin{aligned}
&\big\langle \lambda\, f^* c_1\big(\overline{\mathcal{M}}_\beta\big) e\big(R^1\pi_*\mathrm{ev}^*\mathcal{N}_\beta\big), \overline{\mathfrak{M}}^0_1(\overline{\mathcal{M}}_{\beta,1}, d)\big\rangle \\
&\quad = \frac{1}{24}\big\langle f^* c_1\big(\overline{\mathcal{M}}_\beta\big)\, c_2(\mathcal{N}_\beta) e\big(R^1\pi_*\mathrm{ev}^*\mathcal{N}_\beta\big), \overline{\mathfrak{M}}^0_{0,1=2}(\overline{\mathcal{M}}_{\beta,1}, d)\big\rangle \\
&\quad = \frac{1}{24}\big\langle c_2(\mathcal{N}_\beta) f^* c_1\big(\overline{\mathcal{M}}_\beta\big), \overline{\mathcal{M}}_{\beta,1}\big\rangle \\
&\qquad\qquad \times \int_{\overline{\mathfrak{M}}^0_{0,1=2}(\mathbb{P}^1_p, d)} e\big(R^1\pi_*\mathrm{ev}^*\big(\mathcal{O}(-1)\oplus\mathcal{O}(-1)\big)\big).
\end{aligned} \tag{2.52}$$

Using the localization as in Section 27.5 of [8] once again, we find

$$\int_{\overline{\mathfrak{M}}^0_{0,1=2}(\mathbb{P}^1_p, d)} e\big(R^1\pi_*\mathrm{ev}^*\big(\mathcal{O}(-1)\oplus\mathcal{O}(-1)\big)\big) = \frac{d-1}{d^2}.$$ [9]

Along with (2.52), this identity implies (2.51).

§3. Local $\mathbb{P}^2$

3.1. Gromov–Witten invariants

We consider here the local Calabi–Yau 5-fold given by the total space

$$X = \mathcal{O}(-1) \oplus \mathcal{O}(-1) \oplus \mathcal{O}(-1) \longrightarrow \mathbb{P}^2. \tag{3.1}$$

There are only two primary Gromov–Witten invariants in each degree d:

$$N_{0,d} = N_{0,d}(H^2, H^2) \quad \text{and} \quad N_{1,d},$$

[9]Similarly to the situation discussed for (2.50), $\overline{\mathfrak{M}}^0_{0,1=2}(\mathbb{P}^1_p, d)$ has singularities but is nonsingular along the only fixed locus to which $e\big(R^1\pi_*\mathrm{ev}^*\big(\mathcal{O}(-1)\oplus\mathcal{O}(-1)\big)\big)$ restricts non-trivially.

where H is the hyperplane class in $H^2(X,\mathbb{Z}) = H^2(\mathbb{P}^2,\mathbb{Z})$. We compute both Gromov–Witten invariants by localization[10] and then state a conjectural formula found by Martin for the integer counts $n_{1,d}$.

Lemma 3.1. *For $d\in\mathbb{Z}^+$,*

$$N_{0,d} = \frac{(-1)^{d-1}}{d} \qquad \textit{and} \qquad N_{1,d} = \frac{(-1)^d}{8d}.$$

Proof. Let (a,b,c) be the weights of the torus action on the vector space $\mathbb{C}^3$. The weights of the torus action on $T\mathbb{P}^2$ at the fixed points are then

$$\begin{aligned} P_1=[1,0,0]: &\qquad b-a, c-a,\\ P_2=[0,1,0]: &\qquad a-b, c-b,\\ P_3=[0,0,1]: &\qquad a-c, b-c. \end{aligned}$$

We choose linearizations on the 3 bundles $\mathcal{O}(-1)$ with the following weights at the fixed points:

$$\begin{array}{cccc} & \mathcal{O}(-1) & \mathcal{O}(-1) & \mathcal{O}(-1)\\ P_1: & 0 & a-b & a-c\\ P_2: & b-a & 0 & b-c\\ P_3: & c-a & c-b & 0. \end{array}$$

In order to compute the numbers $N_{0,d}$, we choose the points P_1 and P_2 for the insertions and integrate over

$$\overline{\mathfrak{M}} = \big\{b\in\overline{\mathfrak{M}}_{0,2}(\mathbb{P}^2,d) : \mathrm{ev}_1(b)=P_1,\ \mathrm{ev}_2(b)=P_2\big\}.$$

By the choice of the weights and the points, there is a unique fixed locus with non-zero contribution, see Section 27.5 in [8] for a similar situation. The locus consists of the d-fold cover u of the line

$$\mathbb{P}^1_{12}=\overline{P_1P_2}$$

branched over only P_1 and P_2 and with the marked points 1 and 2 mapped to P_1 and P_2, respectively. The weights of the fibers of the

[10]In the genus 0 case, the moduli space $\overline{\mathfrak{M}}_{0,2}(\mathbb{P}^2,d)$ is a nonsingular stack and the usual Atiyah–Bott localization formula applies. In the genus 1 case, the virtual localization formula of [7] is used.

relevant bundles at the fixed locus are given by

$$\begin{aligned}
&H^1(u^*\mathcal{O}(-1)): && \tfrac{(-1)^{d-1}(d-1)!}{d^{d-1}}(a-b)^{d-1}\\
&H^1(u^*\mathcal{O}(-1)): && \tfrac{(-1)^{d-1}(d-1)!}{d^{d-1}}(b-a)^{d-1}\\
&H^1(u^*\mathcal{O}(-1)): && (-1)^{d-1}\textstyle\prod_{r=1}^{d-1}\left(c-\tfrac{(d-r)a+rb}{d}\right)\\
&T\overline{\mathfrak{M}}: && \tfrac{(-1)^{d-1}(d-1)!^2}{d^{2(d-1)}}(a-b)^{2(d-1)}\textstyle\prod_{r=1}^{d-1}\left(c-\tfrac{(d-r)a+rb}{d}\right),
\end{aligned}$$

see Section 27.2 in [8]. The number $N_{0,d}$ is the ratio of the product of the first three expressions and the last expression, divided by d for the stack automorphism factor.

We next compute the number $N_{1,d}$. There are now 6 fixed loci with nonzero contribution: the three d-fold Galois covers of the three lines together with a choice of vertex for the contracted elliptic component. By symmetry, the contribution of the d-fold cover of $\mathbb{P}^1_{12}$ with the contracted component at P_1 determines the other cases. The weights of the fibers of the relevant bundles at the d-fold cover of $\mathbb{P}^1_{12}$ are given by

$$\begin{aligned}
&H^1(u^*\mathcal{O}(-1)): && \tfrac{(-1)^{d-1}(d-1)!}{d^{d-1}}(a-b)^{d-1}(-\lambda)\\
&H^1(u^*\mathcal{O}(-1)): && \tfrac{(-1)^{d-1}(d-1)!}{d^{d-1}}(b-a)^{d-1}(a-b-\lambda)\\
&H^1(u^*\mathcal{O}(-1)): && (-1)^{d-1}\textstyle\prod_{r=1}^{d-1}\left(c-\tfrac{(d-r)a+rb}{d}\right)(a-c-\lambda)\\
&Obs(\mathbb{P}^2): && (b-a-\lambda)(c-a-\lambda)\\
&T\overline{\mathfrak{M}}_1(\mathbb{P}^2,d): && \tfrac{(-1)^{d}d!^2}{d^{2d-1}}(a-b)^{2d-1}\textstyle\prod_{r=0}^{r=d}\left(c-\tfrac{(d-r)a+rb}{d}\right)\left(\tfrac{b-a}{d}-\psi\right),
\end{aligned}$$

where λ is the first chern class of the Hodge line bundle $\mathbb{E}\longrightarrow\overline{\mathcal{M}}_{1,1}$. The contribution of the locus to $N_{1,d}$ is the ratio of the product of the first four expressions and the last expression, divided by the stack factor d, and integrated over $\overline{\mathcal{M}}_{1,1}$,

$$\mathrm{Cont}(a,b)=\frac{(-1)^d}{24d}\,\frac{c-a}{c-b}.$$

Symmetrizing over a, b, and c, we obtain $N_{1,d}$. Q.E.D.

By Lemma 3.1 and the $n=2$ case of (0.2), the genus 0 counts for X are given by

$$n_{0,d}=n_{0,d}(H^2,H^2)=\begin{cases}1, & \text{if } d=1;\\ -1, & \text{if } d=2;\\ 0, & \text{if } d\geq 3.\end{cases}$$

Using the algorithm of Section 1.2, we have computed the genus 1 count $n_{1,d}$ for X for $d \leq 200$. All are integers.

3.2. Martin's conjecture

Recall the definition of the Möbius μ-function,

$$\mu\colon \mathbb{Z}^+ \to \{0, \pm 1\},$$
$$\mu(d) = \begin{cases} (-1)^r, & \text{if } d \text{ is the product of } r \text{ distinct primes,} \\ 0, & \text{otherwise.} \end{cases}$$

Define a sign function $S(d)$ and an absolute value function $V(d)$ as follows:

$$S(d) = \begin{cases} \mu(d), & \text{if } d \not\cong 4 \pmod 8, \\ \mu(d/4), & \text{if } d \cong 4 \pmod 8, \end{cases}$$

$$V(d) = \frac{k^2-1}{8} \times \begin{cases} \frac{k^2-1}{8}, & \text{if } d = k,\ 2 \nmid k, \\ \frac{17k^2+7}{8}, & \text{if } d = 2k,\ 2 \nmid k, \\ 2k^2+1, & \text{if } d = 4k,\ 2 \nmid k. \end{cases}$$

Conjecture 2 (G. Martin). *For every $d \in \mathbb{Z}^+$, the genus 1 degree d count for the local Calabi–Yau 5-fold $\mathbb{P}^2$ is given by*

$$n_{1,d} = S(d)V(d). \tag{3.2}$$

If $8|d$, then $S(d)$ vanishes and a definition of $V(d)$ is not required for (3.2). As our method for computing the numbers $n_{1,d}$ from $n_{0,d}$ and $N_{1,d}$ is completely explicit and the starting data is fairly simple, a verification of Conjecture 2 by elementary identities may be possible. Unfortunately, the algorithm involves a significant number of simultaneous recursions.[11]

Geometric consequences are easily obtained from the conjecture. For example, since $n_{1,d}$ is predicted to vanish whenever $8|d$, we expect Calabi–Yau 5-folds obtained from suitably generic deformations of the local $\mathbb{P}^2$ geometry to contain *no embedded elliptic curves of degrees divisible by 8.* Is there a simple symplectic reason for this?

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